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RUPRECHT SCHATTNER
MICHAEL STREUBEL

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Section A :
Physique théorique.

Properties of extended bodies in spacetimes admitting isometries

by

Ruprecht SCHATTNER and Michael STREUBEL

Max-Planck-Institut für Physik und Astrophysik
Institut für Astrophysik, Karl-Schwarzschild-Strasse 1
D-8046 Garching b. München, FRG

ABSTRACT. — Dixon's centre-of-mass description of extended bodies is analyzed for spacetimes admitting isometries. In particular, we treat stationary, axisymmetric and stationary and axisymmetric spacetimes. In the last case, a new coordinate system is constructed which is based on the centre-of-mass world line l_0 and we obtain in the case that l_0 is a geodesic (a sufficient condition for this is given) a new restriction on the energy-momentum tensor.

RÉSUMÉ. — On analyse la description des corps étendus selon W. G. Dixon pour des espaces-temps admettant des isométries. On considère particulièrement des espaces-temps stationnaires, axisymétriques ainsi que stationnaires et axisymétriques. Dans ce dernier cas on construit un nouveau système de coordonnées fondé sur la ligne d'univers du centre de masse l_0 et on trouve une nouvelle restriction sur le tenseur d'énergie-impulsion si l_0 est géodésique (une condition suffisante pour cela est obtenue).

§ 1. INTRODUCTION

In [*1*], we started a programme aimed at establishing an energy-momentum balance between material sources and the gravitational radiation field for isolated systems of extended bodies in General Relativity. Our approach uses only covariantly defined concepts so as to render possible going beyond merely formal approximations. One such exact framework employed in our approach is Dixon's theory of the local structure and dynamics of extended bodies [2]. A review of its basic structures will be given in this paper.

By principal reasons, in [1] we were also led to consider stationary as well as axisymmetric spacetimes representing isolated bodies within Dixon's framework. In this first of a series of papers we complete and extend these investigations to spacetimes which are both stationary and axisymmetric, representing e. g., isolated rotating stars. In the second part, these results will be applied to such spacetimes which are also asymptotically flat to obtain a connection between the asymptotic and the local description of isolated symmetrical bodies.

As far as this paper is concerned, besides several new local results presented in the following sections, in Section 7 we obtain a global condition restricting the multipole structure of objects of the type considered here if their center-of-mass moves on a geodesic. Up to this point no field equations are used, all results follow from the local balance equation $\nabla \cdot T = 0$ for the matter distribution together with several technical assumptions.

To be more precise, in Section 3 we present a short review of Dixon's theory. In Section 4, we state our assumptions in detail and prove local geometrical properties of an isolated system in a spacetime admitting a group of motions. In the following three sections, these results are then applied to the special cases of spacetimes which are stationary, axisymmetric and both stationary and axisymmetric, respectively. In addition, Section 7 contains the geometrical construction of a new coordinate system based on the centre-of-mass world line of the body. In particular, there is a canonical time function in these spacetimes (up to the choice of origin).

Finally, the last section treats « dynamical » properties of such bodies, starting from the introduction of one further assumption which ensures the centre-of-mass world line to be geodesic. We first show that force and torque (in the sense of Dixon) exerted on the body vanish and then derive an integral condition on the energy-momentum tensor which would have been hard to obtain without the use of Dixon's description of isolated bodies.

§ 2. NOTATION AND CONVENTIONS

We put $c = 1 = G$. Signature of the metric $g : (+, -, -, -)$. Round and square brackets stand for symmetrization and antisymmetrization, respectively. ∇ denotes the covariant derivative. The curvature tensors of Riemann and Ricci are defined by

$$2\nabla_{[c}\nabla_{d]}V^a =: R^a{}_{bcd}V^b, \; R_{ab} := R^c{}_{acb}.$$

η_{abcd} denotes the completely antisymmetric tensor (volume form) satisfying $\eta_{0123} = 1$ in an oriented orthonormal tetrad.

The Dixon theory uses bitensors (two point tensors) extensively. For a bitensor function of the pair of points (x, z) one must distinguish indices at x and those at z. We shall use $a, b, c, \ldots$ for indices at x and $i, j, k, \ldots$ for indices at z, also with respect to ordinary tensors.

§ 3. SHORT REVIEW OF DIXON'S THEORY

A detailed account of the theory is given in [2]. We will be very concise here.

Basic auxiliary quantities of the framework are the world function biscalar $\sigma(z, x)$ and the Jacobi propagators $K^a{}_k$ and $H^a{}_k$. The world function is defined as follows: let $x(u)$ be a geodesic joining the points $x(0) = z$ and $x(u) = x$. Then

$$\sigma(z, x) := \frac{1}{2} u \int_0^u g_{ab}(x(u'))\dot{x}^a \dot{x}^b du'.$$

Covariant derivatives of σ are denoted by the corresponding suffixes, e. g. $\sigma_{abk} = \nabla_k \nabla_b \nabla_a \sigma$. It is easy to see that

$$\sigma^a(z, x) = u\dot{x}^a(u), \quad \sigma^k(z, x) = -\, u\dot{x}^k(0).$$

The Jacobi propagators arise by considering solutions of the equation of geodesic deviation (Jacobi equation) along $x(u)$:

$$D_u^2 \xi^a - R^a{}_{bcd}\dot{x}^b \dot{x}^c \xi^d = 0$$

where $D_u = \dot{x}^a \nabla_a$ is absolute differentiation along $x(u)$. The solutions of this differential equation are determined by the values of ξ and $D_u \xi$ at the point z. Since the equation is linear, the dependence of the solution on the data is linear. Therefore there exist bitensors K and H such that

$$\xi^a(x(u)) = K^a{}_k(z, x(u))\xi^k(z) + uH^a{}_k(z, x(u))D_u \xi^k(z).$$

It can be shown (see e. g. [2]) that they satisfy

$$H^a{}_k(z, x)\sigma^k{}_b(z, x) = -\, \delta^a_b(x)$$
$$K^a{}_k(z, x) = H^a{}_l(z, x)\sigma^l{}_k(z, x).$$

Geometrically, H represents the differential of the exponential map

$$\frac{\partial}{\partial X^k}(\exp_z X)^a = H^a{}_k(z, \exp_z X)$$

where $X \in T_z M$.

All these bitensors are well defined in a neighbourhood N of the diagonal set of $M \times M$ such that $(x, y) \in N$ iff there is a unique geodesic joining x and y.

Now, it is easy to see that

$$\xi^a(x) := K^a{}_k(z, x)A^k - H^a{}_k(z, x)\sigma^l(z, x)B^k{}_l \tag{3.1}$$

defines the unique vector field which is a Jacobi field along all geodesics emanating from z and satisfies

$$\xi^k(z) = A^k,\ \nabla_l\xi^k(z) = B^k{}_l.$$

If the spacetime M admits an isometry generated by a Killing vector field ξ^a, then ξ^a is *a fortiori* a Jacobi field along all geodesics and satisfies therefore (3.1) with respect to all $z \in M$.

Next consider a system of isolated (1) bodies in a spacetime admitting an isometry generated by a Killing vector field ξ^a. Let Σ be a spacelike hypersurface and dS_a the vector element of volume on Σ. T^{ab} denotes the energy momentum tensor of the matter satisfying $T^{[ab]} = 0 = \nabla_a T^{ab}$. Then

$$E_\Sigma(\xi) := \int_\Sigma \xi_a T^{ab} dS_b \tag{3.2}$$

is independent of the hypersurface Σ. In view of (3.1) we rewrite E_Σ as a bilinear functional on the (Jacobi) data of the Killing field in a point $z \in \Sigma$ (the « generators of the isometry group »):

$$E_\Sigma(\xi) = \xi_k P^k + \frac{1}{2}\nabla_k\xi_l S^{kl} \tag{3.3}$$

where

$$P^k(z, \Sigma) := \int_\Sigma K_a{}^k T^{ab} dS_b \tag{3.4}$$

and

$$S^{kl}(z, \Sigma) := 2\int_\Sigma H_a{}^{[k}\sigma^{l]} T^{ab} dS_b. \tag{3.5}$$

In analogy with flat spacetime P^k and S^{kl} are called (Dixon's) momentum and angular momentum. The essential point is that these quantities are independent of the existence of a Killing field ξ^a and may therefore be chosen as definitions of momentum and angular momentum in a general spacetime where no symmetries are present. The quantities defined by (3.4) and (3.5) depend on both the hypersurface Σ and the point $z \in \Sigma$. Given a point $z \in M$ and a timelike future directed unit vector $u^k \in T_zM$, we define momentum and angular momentum with respect to (z, u) by choosing Σ to be the geodesic hypersurface orthogonal to u, i. e.

$$\Sigma := \Sigma(z, u) := \{ x \in M \mid u_k\sigma^k(z, x) = 0 \}.$$

(1) By this we mean supp $(T^{ab}) \cap S$ is compact for all spacelike hypersurfaces S.

Now, to reduce arbitrariness further, we impose the condition

$$P^{[k}u^{l]} = 0. \tag{3.6}$$

It has been shown in [3] that an energy condition together with weak conditions on the curvature of spacetime and the size of the bodies ensure the existence of an unique timelike future directed u^k satisfying (3.6) in a neighbourhood of supp (T). These conditions are also sufficient to show (see [3]) that there exists exactly one timelike curve $l_0 \equiv z(s)$, the centre-of-mass, such that the mass dipole moment m^k, defined by

$$m^k := u_l S^{kl} \tag{3.7}$$

vanishes. If we introduce a spin vector S^k by

$$S^k = \frac{1}{2}\eta^{klmn}S_{lm}u_n, \tag{3.8}$$

we find from the centre-of-mass condition that

$$S^{kl} = \eta^{klmn}u_m S_n.$$

We normalize the parametrization of l_0 by

$$u_k \dot{z}^k = 1.$$

In virtue of (3.6) there exists of course a scalar function M_D, the Dixon mass, along l_0 such that

$$P^k = M_D u^k.$$

M_D is positive. Note that u^k is not necessarily tangent to l_0.

As a consequence of the local law of motion,

$$\nabla_a T^{ab} = 0, \tag{3.9}$$

P^k and S^{kl} have to satisfy (ordinary) differential equations along l_0, Dixon's integral laws of motion:

$$D_s P^i = \frac{1}{2}R_{lmk}{}^{i}S^{lm}\dot{z}^k + F^i \tag{3.10}$$

$$D_s S^{ij} = 2P^{[i}\dot{z}^{j]} + L^{ij}. \tag{3.11}$$

The form of equations (3.10) and (3.11) does not depend on the particular choice of l_0 and $\Sigma(s)$, but F^i and L^{ij} do. They are well defined only if the world-line, its parametrization and the family of hypersurfaces are specified. Therefore we will in the future always adopt the above choices of $l_0 \equiv z(s)$ and $\Sigma(s)$, i. e. a centre-of-mass description.

In order to evaluate the gravitational force F^k and torque L^{kl} explicitly, one needs another structure, the so-called « energy momentum skeleton ». Together with P^k and S^{kl} it provides a complete description of the bodies (i. e. it is equivalent with T^{ab}) but it is *not* subject to any differential equations *along* l_0 as a consequence of (3.9). The details are as follows: T^{ab} is completely determined knowing the linear functional

$$\phi_{ab} \to \int T^{ab}\phi_{ab}\eta$$

for all smooth symmetric tensor fields of compact support. Dixon has shown that for any smooth T^{ab} with spatially compact support satisfying (3.9) there exists a tensor distribution on the tangent space $\widehat{T}_z{}^{kl}$, the energy momentum skeleton, such that

$$\int T^{ab}\phi_{ab}\eta = \int ds \left\{ P^k \dot{z}^l(x)\phi_{kl}(z(s)) + S^{kl}\dot{z}^m \nabla_k \phi_{lm}(z(s)) + \int_{T_{z(s)}M} \widehat{T}^{kl}_{z(s)}(\Phi_{kl} + G^m_{kl}\Lambda_m) DX \right\}. \quad (3.12)$$

$\widehat{T}^{kl}_z$ is zero unless $z \in l_0$ and has compact support contained in the hypersurface orthogonal to u^k in $T_z(M)$.

$DX = \sqrt{-g(z)}dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3$ is the volume element on $T_z(M)$, Φ_{kl} is the pull back of ϕ_{ab} to $T_z(M)$ by the exponential map, i. e.

$$\Phi_{kl}(z, X) = H^a{}_k H^b{}_l \phi_{ab}(\exp_z X). \quad (3.13)$$

G^m_{kl} is the pull back of the linear connexion, i. e.

$$G^m_{kl}(z, X) = H^a{}_k H^b{}_l \sigma^m{}_{ab}(z, \exp_z X) \quad (3.14)$$

and Λ_m is the pull back of the two point co-vector field $\lambda_a(z, x)$ which satisfies

$$D^2_u \lambda_a - R^d{}_{bca}\dot{x}^b \dot{x}^c \lambda_d = \dot{x}^b \dot{x}^c (\nabla_b \phi_{ac} - \nabla_a \phi_{cb} + \nabla_c \phi_{ba}) \quad (3.15)$$

along all affinely parametrized geodesics $x(u)$ through z. It is fully determined by imposing the initial conditions

$$\lim_{x\to z} \lambda_a = 0 \quad , \quad \lim_{x \to z} (\nabla_a \lambda_b - \phi_{ab}) = 0. \quad (3.16)$$

It has been proven by Dixon that $\widehat{T}^{kl}_z$ is unique if one imposes the further conditions

$$\frac{\partial}{\partial X^l}\widehat{T}^{kl}_z = 0 \quad , \quad (u_k X^k) X^{[l}\widehat{T}^{m][n}X^{p]} = 0. \quad (3.17)$$

One may define an extended skeleton by

$$\check{T}_z{}^{kl} := P^{(k}\dot{z}^{l)}\delta(X) - S^{m(k}\dot{z}^{l)}\frac{\partial}{\partial X^m}\delta(X) + \widehat{T}^{kl}_z$$

and the 2^n-pole moments at $z(s) \in l_0$

$$I^{m_1 \dots m_n kl}(s) := \int_{T_{z(s)}M} X^{m_1} X^{m_n} \check{T}_{z(s)}(X) DX.$$

It can be shown that

$$M_D = \int \check{T}^{kl} u_k u_l DX$$

$$S^{kl} = 2\int X^{[k}\check{T}^{l]m} u_m DX.$$

$\widehat{T}^{kl}$ contains only quadrupole and higher order information about the body.

Using the energy momentum skeleton, Dixon gave explicit formulae for F^k and L^{kl}: given a Jacobi field ξ, the conventions being as in (3.1), these representations of F^k and L^{kl} are in fact equivalent with

$$\int \widehat{T}^{kl} \mathscr{L}_\xi G_{kl} DX = 2\xi_k F^k + \nabla_k \xi_l L^{kl} \tag{3.18}$$

for all initial values ξ_k, $\nabla_k \xi_l = -\nabla_l \xi_k$. G_{kl} is the pull back of the metric,

$$G_{kl}(z, X) = H^a{}_k H^b{}_l g_{ab}(\exp_z X). \tag{3.19}$$

It is important to note that these explicit expressions do *not* contain $\dot{z}^k$ so that, surprising as it may seem, (3.10) and (3.11) together with (3.6) can be solved with respect to $\dot{z}$ algebraically. This has been achieved by Ehlers and Rudolph [*4*]. Finally we quote a theorem which shows that there are no further restrictions on (P, S, $\widehat{T}$) in view of (3.9):

THEOREM (Dixon [*2*]). — If a smooth (²) symmetric tensor field satisfies (3.12) for some choice of (P, S, $\widehat{T}$) satisfying (3.17), then

$$\nabla_b T^{ab} = 0. \tag{3.9}$$

In other words, one has succeeded in finding variables describing the bodies completely, which are such that they satisfy ten ordinary differential equations being equivalent to the four partial differential equations $\nabla_b T^{ab} = 0$! Thus the situation for a given spacetime metric is very similar (³) to Newtonian dynamics: momentum and angular momentum have to be given at an instant of time, the higher order moments at all times (i. e. in general knowledge about the internal structure of the bodies is required in addition). These data determine the motion.

For the following it will be useful to have a formula which expresses integrals of T^{ab} over the hypersurfaces $\Sigma(s)$ by the skeleton. To obtain it, we define a time function t by

$$t(x) = s \Leftrightarrow x \in \Sigma(s) \tag{3.20}$$

(²) This condition may be weakened, cf. [*2*] and [*5*]. For the question of existence of such a T see [*5*].

(³) In fact in this regard they are the same. All preceeding theorems may be proven using only structures common to both theories ([*6*]).

and a vector field v^a the local flow of which maps the hypersurfaces $\Sigma(s)$ onto other such hypersurfaces,

$$v^a \partial_a t = 1.$$

Then

$$dS := v^c dS_c$$

is a scalar 3-form (volume element) associated with the foliation $\{ \Sigma(s) \}$, and

$$dS_a = \partial_a t dS.$$

We shall use the auxiliary fields t, v^a, dS throughout this paper.

In particular, it has been shown that there exist tensor valued distributions $A^{ab}_{(\nu)}(s)$, $\nu = 1, 2, 3$, on the hypersurfaces $\Sigma(s)$ depending upon T^{ab} and g^{ab} only such that

$$\begin{aligned}\int_{\Sigma(s)} T^{ab}\phi_{ab} dS = {} & P^k \dot{z}^l \phi_{kl}(z(s)) \\ & + S^{kl}\dot{z}^m \nabla_k \phi_{lm} + \int_{T_{z(s)}M} \widehat{T}^{kl}_{z(s)}(\Phi_{kl} + G^m_{kl}\Lambda_m) DX \qquad (3.21) \\ & + \sum_{\nu=1}^{3} \left(\frac{d}{ds}\right)^{\nu} \langle A^{ab}_{(\nu)}(s), \phi_{ab} \rangle\end{aligned}$$

for all smooth tensor fields ϕ_{ab} with compact support.

We shall finish this section with a discussion of angular velocity and the relativistic inertia tensor.

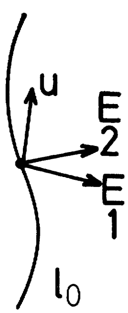

FIG. 0.

Let $(u, \underset{\alpha}{E})$ be an orthonormal tetrad along l_0. Then the tensor

$$D^{kl} := u^k \dot{u}^l - \Sigma \underset{\alpha}{\dot{E}}{}^k \underset{\alpha}{E}{}^l$$

obeys the equation

$$(\delta^k_l - u^k u_l)\underset{\alpha}{\dot{E}}{}^l = D^k{}_l \underset{\alpha}{E}{}^l \qquad (3.22)$$

and is characterized by it. D^{kl} is, from (3.22), the angular velocity of $(\underset{\alpha}{E}{}^k)$ with respect to any quasiparallel base, quasiparallelity being defined by

$$(\delta^k_l - u^k u_l)\dot{\underset{\alpha}{E}}{}^l = 0.$$

(3.1) LEMMA. — D^{kl} is antisymmetric and spacelike, i. e.

$$D^{(kl)} = 0$$
$$D^{kl}u_l = 0.$$

Proof. — Since $(u, \underset{\alpha}{E})$ is an orthonormal tetrad along l_0,

$$g^{kl} = u^k u^l - \underset{\alpha}{\Sigma}\underset{\alpha}{E}{}^k\underset{\alpha}{E}{}^l \tag{3.23}$$

Therefore

$$0 = \frac{1}{2}\nabla_{\dot{z}} g^{kl} = u^{(k}\dot{u}^{l)} - \underset{\alpha}{\Sigma}\dot{\underset{\alpha}{E}}{}^{(k}\underset{\alpha}{E}{}^{l)},$$

which proves the first assertion. Furthermore

$$D^{kl}u_l = u^k\dot{u}^l u_l - \underset{\alpha}{\Sigma}\dot{\underset{\alpha}{E}}{}^k\underset{\alpha}{E}{}^l u_l = 0. \qquad \square$$

Using the relativistic inertia tensor M^{kl}, defined by

$$M^{kl} = \int [X^k X^l - (g^{kl} - u^k u^l)X^r X_r]\hat{T}^{ij}u_i u_j DX, \tag{3.24}$$

a dynamical angular velocity of the body may be introduced as follows: it can be shown under conditions similar to those guaranteeing existence and uniqueness of the centre-of-mass line that

$$M^{kl} = \underset{\alpha}{\Sigma}\underset{\alpha}{\theta}\underset{\alpha}{E}{}^k\underset{\alpha}{E}{}^l \tag{3.25}$$

and

$$M^{kl}\underset{\alpha}{E}{}_l = -\underset{\alpha}{\theta}\underset{\alpha}{E}{}^k \tag{3.26}$$

with

$$g_{kl}\underset{\alpha}{E}{}^k\underset{\beta}{E}{}^l = -\delta_{\alpha\beta}$$

and

$$u_k\underset{\alpha}{E}{}^k = 0 \qquad \text{for} \qquad \alpha = 1, 2, 3,$$

and

$$\underset{\alpha}{\theta} > 0.$$

Therefore, M^{kl} has matrix rank 3 and a solution Ω^k of

$$S^k = M^{kl}\Omega_l, \ \Omega_l u^l = 0$$

always exists. Ω_l will be called the dynamical angular velocity vector. The dynamical angular velocity is then defined by

$$\Omega_{kl} = \eta_{klmn} u^m \Omega^n.$$

Now, we may define an orthonormal tetrad $(u^k, \underset{\alpha}{e}{}^k)$ along l_0 whose timelike vector is the dynamical velocity u^k and whose spatial vectors rotate with the dynamical angular velocity Ω^{kl}. A body is said to be dynamically rigid (quasirigid in the sense of [4]) if its multipole moments of order $n \geq 2$ have constant components w. r. t. such a tetrad field. Formally, this may be expressed as follows: the motion is quasirigid, iff there exists a vector field Y satisfying

$$Y^a(x) = K^a{}_k(z(s), x)\dot{z}^k(s) + H^a{}_k(z(s), x)\sigma_l(z(s), x)(2u^{[k}\dot{u}^{l]} - \Omega^{kl})$$

for $x \in \Sigma(s)$ and

$$\mathscr{L}_Y \widehat{T}^{kl} = 0. \tag{3.27}$$

If the body is quasirigid, the principal moments of inertia $(\underset{\alpha}{E}{}^k)$ can be chosen to rotate with the dynamical velocity Ω, if the $\underset{\alpha}{\theta}$ are mutually distinct, the $\underset{\alpha}{(E)}{}^k$ are unique (except for enumeration) and rotate with the dynamical velocity

§ 4. PROPERTIES OF AN ISOLATED SYSTEM IN A SPACETIME ADMITTING A GROUP OF MOTIONS

Now, as in the following chapters, we consider a spacetime (M, g_{ab}) and a symmetric energy-momentum tensor T^{ab} with the following properties:

(A1) M is a connected Hausdorff C^∞-manifold which is spatially and temporally-oriented;

(A2) (g_{ab}, T^{ab}) are such that a unique centre-of-mass line $l_0 : z(s)$ exists and the (Dixon-) momentum P is timelike (4).

In addition we assume throughout this chapter:

(S1) There exists a group of local motions G leaving (M, g) invariant, which is generated by a vector field Z^a on M such that $\mathscr{L}_Z g = 0$.

(S2) T^{ab} possesses the same symmetry as (M, g), i. e. $\mathscr{L}_Z T = 0$.

If (M, g, T) is a solution of Einstein's equation, then (S2) is of course a consequence of (S1).

(4) For a definition of these quantities see § 3. Sufficient conditions for (A2) to be valid may be found in [3].

(4.1) LEMMA. — Suppose condition (3.6), but not (3.7) holds, i. e. we have specified the moments completely apart from choosing the centre-of-mass description. This means that momentum P^k and angular momentum S^{kl} may be considered as tensor fields in a neighbourhood of the support of T^{ab}. Then

$$\mathscr{L}_Z P^k = 0 = \mathscr{L}_Z S^{kl}. \tag{4.1}$$

Proof. — The assertion follows immediately from the fact that P^k and S^{kl} are geometric objects constructed from g_{ab} and T^{ab}. □

(4.2) LEMMA. — The centre-of-mass line $z(s)$ is a group orbit or a line of fixed points of the group action, i. e. there exists a constant κ such that

$$Z^k = \kappa \dot{z}^k \tag{4.2}$$

and

$$\kappa = u_k Z^k. \tag{4.3}$$

Proof. — Let ϕ_t be the local flow of Z. The images $\phi_t(z(s))$ of the centre-of-mass line $l_0 : z(s)$ are again timelike curves which can be used as reference lines to construct P^k, S^{kl} along each of them (confer Lemma 4.1). On the other side we get, using (4.1),

$$\mathscr{L}_Z(P_l S^{kl}) = 0$$

which is equivalent to

$$\nabla_Z m^k = m^n \nabla_n Z^k.$$

Therefore

$$\nabla_Z(m^k m_k) = 2m^k m^l \nabla_k Z_l = 0,$$

i. e. $m^k m_k$ is constant along $\phi_t(z(s))$ for fixed s. Since $\phi_0(z(s)) = z(s)$ and $(m^k m_k)(z(s)) = 0$ for all s, we find that $(m^k m_k)(\phi_t(z(s))) = 0$ for all t and s, hence

$$m^k(\phi_t(z(s))) = 0,$$

i. e. for fixed but arbitrary t the curve $\phi_t(z(s))$ is a centre-of-mass line. By uniqueness it must coincide with l_0. This implies (4.2) and (4.3). It follows from Lemma 4.1 that

$$\mathscr{L}_Z u^k = 0$$

hence, in view of (4.3),

$$Z(\kappa) = \mathscr{L}_Z \kappa = 0.$$

(4.2) yields now

$$\frac{d}{ds}(\kappa)^2 = 2\kappa \frac{d}{ds}\kappa = 2Z(\kappa) = 0.$$

κ being continuous this implies $\kappa = \text{const.}$ □

Since the curve $z(s)$ is timelike, one immediately has

(4.3) COROLLARY. — A spacetime containing an isolated body does not admit any everywhere spacelike (« translational ») isometry. □

(4.4) LEMMA. — The image of the geodesic hypersurface $\Sigma(s)$ under the local flow of the Killing field Z, ϕ_r, is again a member of the family of hypersurfaces $\{ \Sigma(s) \mid s \in \mathbb{R} \}$. Furthermore one finds

$$Z(t) = \kappa \tag{4.4}$$

and

$$\phi_r(\Sigma(s)) = \Sigma(s + \kappa r) \tag{4.5}$$

Proof. — Lemmata (4.1) and (4.2) imply

$$\phi_r(l_0) = l_0,$$
$$\phi_r(u(z)) = u(\phi_r(z)).$$

ϕ_r is an isometry and maps therefore geodesics orthogonal to $u(z)$ into geodesics orthogonal to $u(\phi_r(z))$. This completes the first part of the proof. This, together with (4.2) and (4.3), yields

$$Z(t) \,|_{x \in \Sigma(s)} = Z(t) \,|_{z(s)} = \kappa \dot{z}^k \partial_k t \,|_{z(s)} = \kappa .$$

Finally, (4.2) tells us, that

$$\phi_r(z(s)) = z(s + \kappa r)$$

which implies (4.5). □

This is as far as we can get without further assumptions concerning the nature of the isometry group. In the following chapters we will treat some physically significant special cases.

§ 5. PROPERTIES OF AN ISOLATED SYSTEM IN A STATIONARY SPACETIME

We assume again (A1), (A2), (S1) and (S2). In addition we require the following:

(S3) (M, g) is invariant under an action $\phi : \mathbb{R}(1) \times M \to M$ of the one parameter group $\mathbb{R}(1)$, such that the trajectories are timelike curves.

The corresponding Killing vector field shall be denoted by ξ, i. e. the results of § 4 apply for $Z = \xi$.

The following are simple corollaries of Lemmata (4.1) and (4.2):

(5.1) COROLLARY. — The centre-of-mass line l_0 is an integral curve of ξ. □

(5.2) COROLLARY. — The Dixon-mass M_D and the parameter

$$\kappa := u_k \xi^k \neq 0$$

are constant along l_0. □

(5.3) PROPOSITION. — If the eigenvalues of the inertia tensor M^{kl} are mutually distinct, then

$$\frac{1}{\kappa}\nabla^k\xi^l = u^k\dot{u}^l - \sum_{\alpha=1}^{3} \underset{\alpha}{E}^k \underset{\alpha}{\dot{E}}^l \tag{5.1}$$

where the $\underset{\alpha}{E}^k$ are the eigendirections of the inertia tensor, as defined in (3.24).

Proof. — Since ξ is an isometry, $\mathscr{L}_\xi M^{kl} = 0$.
Applying $\mathscr{L}_\xi$ to (3.26) yields

$$M^{kl}\mathscr{L}_\xi \underset{\alpha}{E}_l = -\underset{\alpha}{\theta}\mathscr{L}_\xi \underset{\alpha}{E}^k - \xi(\underset{\alpha}{\theta})\underset{\alpha}{E}^k. \tag{5.2}$$

Transvection with $\underset{\alpha}{E}_k$ yields

$$\xi(\underset{\alpha}{\theta}) = 0.$$

Now equation (5.2) implies that $\mathscr{L}_\xi \underset{\alpha}{E}^k$ is an eigenvector of M to the eigenvalue $\underset{\alpha}{\theta}$. The non-degeneracy entails

$$\mathscr{L}_\xi \underset{\alpha}{E}^k \propto \underset{\alpha}{E}^k.$$

But $\underset{\alpha}{E}^k$ is normalized. Therefore

$$\mathscr{L}_\xi \underset{\alpha}{E}^k = 0.$$

Next we recall from (3.23) that

$$g^{kl} = u^k u^l - \sum_{\alpha=1}^{3} \underset{\alpha}{E}^k \underset{\alpha}{E}^l. \tag{5.3}$$

$\mathscr{L}_\xi u^k = 0 = \mathscr{L}_\xi \underset{\alpha}{E}^k$ together with (4.2) are equivalent with

$$\kappa \underset{\alpha}{E}^l \underset{\alpha}{\dot{E}}^k = \underset{\alpha}{E}^l \underset{\alpha}{E}^m \nabla_m \xi^k$$

$$\kappa u^l \dot{u}^k = u^l u^m \nabla_m \xi^k.$$

Therefore

$$\kappa(u^l\dot{u}^k - \Sigma \underset{\alpha}{E}^l \underset{\alpha}{\dot{E}}^k) = (u^l u^m - \Sigma \underset{\alpha}{E}^l \underset{\alpha}{E}^m)\nabla_m\xi^k = \nabla^l\xi^k. \qquad \square$$

(5.4) PROPOSITION. — If the eigenvalues of M^{kl} are mutually distinct and the motion is dynamically rigid (cf. § 3), then

$$\frac{1}{\kappa}\nabla^k\xi^l = 2u^{[k}\dot{u}^{l]} - \Omega^{kl}. \tag{5.4}$$

Proof. — (3.27) implies

$$\mathscr{L}_Y M^{kl} = 0,$$

therefrom we deduce as in the proof of Proposition (5.3) that

$$\nabla^k Y^l = 2u^{[k}\dot{u}^{l]} - \Omega^{kl} = u^k\dot{u}^l - \sum_{\alpha=1}^{3} \underset{\alpha}{E}^k \underset{\alpha}{\dot{E}}^l \tag{5.5}$$

(5.1) yields now (5.4). □

(5.5) REMARK. — (5.5) is equivalent to the relation

$$\Omega^{kl} = - u^l\dot{u}^k + \sum_{\alpha=1}^{3} \underset{\alpha}{E}^k \underset{\alpha}{\dot{E}}^l. \tag{5.6}$$

We may use (5.6) as definition of a « geometrical angular velocity » in general (in non-dynamically rigid non-stationary situations). This suggests the following definition:

(5.6) DEFINITION. — If the eigenvalues of the inertia tensor are mutually distinct we define the geometrical mean angular velocity by

$$W^{kl} := - u^l\dot{u}^k + \underset{\alpha}{\Sigma} \underset{\alpha}{E}^k \underset{\alpha}{\dot{E}}^l.$$

We know from § 3 that W^{kl} is a « reasonable » angular velocity satisfying the properties of Lemma (3.1). Propositions (5.3) and (5.4) may be reformulated in the following way:

(5.7) PROPOSITION. — If the eigenvalues of the inertia tensor are mutually distinct, then

$$\frac{1}{\kappa} \nabla^k \xi^l = 2u^{[k}\dot{u}^{l]} - W^{kl}.$$ □

(5.8) PROPOSITION. — If the eigenvalues of the inertia tensor are mutually distinct and if the motion is dynamically rigid, then dynamical and geometrical angular velocity are equal,

$$W^{kl} = \Omega^{kl}.$$ □

§ 6. PROPERTIES OF AN ISOLATED SYSTEM IN AN AXISYMMETRIC SPACETIME

We assume again (A1), (A2), (S1) and (S2). In addition we require the following:

(S4) (M, g) is invariant under an effective action $\psi : SO(2) \times M \to M$ of the one parameter cyclic group SO(2).

The Killing vector field corresponding to the action ψ shall be denoted by η. We normalize η such that $\psi_{2\pi}(x) = x$, i. e. we describe the elements of SO(2) by an angular coordinate running from 0 to 2π in the usual way.

(6.1) PROPOSITION. — The centre-of-mass line l_0 is a set of fixed points of ψ.

Proof. — Putting $Z = \eta$ in Lemma (4.2) we find that for some constant κ^*

$$\eta^k(z(s)) = \kappa^* \dot{z}^k(s).$$

But η^k is spacelike and $\dot{z}$ is timelike. Therefore

$$\kappa^* = 0. \qquad \square$$

(6.2) PROPOSITION. — There is an imbedded 2-surface A, the axis of ψ, which is the fixed point set of ψ.

Proof. — Carter [7] has shown, that (S4) implies Proposition (6.2), if the action ψ has fixed points, but this is guaranteed by Proposition (6.1). $\square$

(6.3) PROPOSITION. — A is timelike and totally geodesic.

Proof. — A proof of the first assertion is given in [7]. For the second part, we proceed as follows: let $z \in A$ and $V \in T_z(M)$, such that V is tangent to A. Then $\psi_{\theta *}V = V$ for all $\theta \in SO(2)$. Let V be such that $\exp_z V$ is well defined. Then

$$\psi_\theta(\exp_z V) = \exp_{\psi_\theta(z)} \psi_{\theta *}V = \exp_z V,$$

i. e. all geodesics starting from a point $z \in A$ tangentially to A remain within A.

(6.4) PROPOSITION. — There is a unit vector w^k orthogonal to u^k such that

$$\nabla^k \eta^l = \eta^{klmn} u_m w_n. \qquad (6.1)$$

Proof. — From Lemma (4.1) it follows in view of Proposition (6.1) that

$$\mathcal{L}_\eta u^k = -u^l \nabla_l \eta^k = 0.$$

Therefore there exists a unique vector w^k satisfying

$$w_k u^k = 0$$

such that (6.1) holds, namely

$$w^k = \frac{1}{2} \eta^{klmn} \nabla_l \eta_m u_n.$$

Now, let $x^a(r, s)$ be a geodesic emanating from $z(s) \in l_0$ such that

$$x^k(0, s) = z^k(s), \qquad (6.2)$$

$$\dot{x}^k(0, s) w_k = 0 \qquad (6.3)$$

and

$$\dot{x}^k(0, s)u_k = 0 \tag{6.4}$$

where $\cdot = \frac{\partial}{\partial r}$ and r being the arc-length. Then for all $\theta \in [0, 2\pi]$

$$\psi_\theta(x(r, s)) = : x(r, \theta, s)$$

is also a geodesic satisfying (6.2), (6.3) and (6.4). For fixed $(r, s) \in [0, \infty) \times \mathbb{R}$,

$$\theta \to x(r, \theta, s)$$

describes a circle of « radius » r. Its length is given by

$$\mathrm{L}(r, s) = \int_0^{2\pi} \left| g_{ab} \frac{\partial}{\partial\theta} x^a(r, \theta, x) \frac{\partial}{\partial\theta} x^b(r, \theta, s) \right|^{1/2} d\theta.$$

But, by definition,

$$\frac{\partial}{\partial\theta} x^a(r, \theta, s) = \eta^a(x(r, \theta, s)).$$

Furthermore it follows from (3.1) and Proposition (6.1) that

$$\eta^a(x, (r, \theta, s)) = r\mathrm{H}^a{}_k \dot{x}_l(0, \theta, s)\eta^{klmn} w_m u_n,$$

therefore

$$g_{ab}\eta^a\eta^b = r^2 \mathrm{G}_{kl} \dot{x}_m \dot{x}^n \eta^{kmij} w_i u_j \eta^l{}_{nrs} w^r u^s$$

where G_{kl} has been defined in (3.19). It may be shown, cf. [3], that there exists a regular bitensor field γ_{kl} depending on the curvature, such that

$$\mathrm{G}_{kl} = g_{kl} + r^2\gamma_{kl}$$

whence

$$g_{ab}\eta^a\eta^b = r^2 w^k w_k (1 + r^2\gamma),$$

γ being some regular function of r and s. Therefore

$$\mathrm{L}(r, s) = 2\pi r \sqrt{| w^k w_k |}\,(1 + r^1\gamma)^{1/2}.$$

Finally we get

$$\lim_{r \to 0} \frac{\mathrm{L}(r, s)}{2\pi r} = \sqrt{| w_k w^k |}.$$

Since (M, g) is locally euclidean this equals unity. □

(6.5) PROPOSITION. — Let

$$\mathrm{L} : = \mathrm{E}_\Sigma(\eta) = \int_\Sigma \mathrm{T}^{ab}\eta_a d\mathrm{S}_b$$

be the (conserved) angular momentum. Then

$$\mathrm{S}^k = \mathrm{L}w^k$$

Proof. — From the definition of Dixon's angular momentum tensor S it follows that

$$L = \frac{1}{2} \nabla_k \eta_l S^{kl}. \tag{6.5}$$

In view of (6.1) and (3.8) this is equivalent with

$$L = - w_k S^k.$$

On the other hand, we also have

$$\mathscr{L}_\eta w = \mathscr{L}_\eta S = \mathscr{L}_\eta u = 0$$

or

$$w^l \nabla_l \eta_k = S^l \nabla_l \eta_k = u^l \nabla_l \eta_k. \tag{6.6}$$

η_k is a Killing field, i. e. $\nabla_l \eta_k$ is antisymmetric. Looked at as a linear map it has therefore either rank 4 or rank 2 or rank 0. The first case is ruled out by (6.6), the last case cannot apply, since $\eta \neq 0$. Therefore $\nabla_k \eta_l$ has rank 2, which means that w, S, and u are linearly dependent. Since both w and S are orthogonal to u, we find for $\alpha \neq 0$

$$S^k = \alpha w^k.$$

Transvection with w_k yields

$$- \alpha = w_k S^k.$$

□

§ 7. PROPERTIES OF AN ISOLATED SYSTEM IN A STATIONARY AND AXISYMMETRIC SPACETIME

Now we assume (A1), (A2), (S1), (S2), (S3) and (S4). We also assume the two Killing fields to commute which is no restriction of generality as shown by Carter [7]. For the rest of this chapter we also assume

(S5) The action of G_2 is orthogonally transitive.

This means that the two-dimensional cylindrical ($\mathbb{R} \times \mathbb{S}^1$) group orbits are orthogonal to two-surfaces in M. It has been shown by Carter [*8*] and Schmidt [*9*] that (S5) is equivalent with

(S5′) There exists a discrete isometry σ which

i) maps each orbit of G_2 onto itself;
ii) satisfies $g\sigma g = \sigma$ for all $g \in G_2$;
iii) has a fixed point on each orbit 0 of G_2.

Kundt and Trümper [*10*] have shown that for a wide class of field equations the orbits of the isometry group defining axial symmetry and stationarity admit orthogonal 2-surfaces. The field equations covered by this result include those of a perfect fluid.

(7.1) PROPOSITION. — The 4-velocity $u^k\left(=\frac{1}{M_D}P^k\right)$ is tangent to the centre-of-mass line l_0.

Proof. — On the orbit l_0, let $z(0)$ be a fixed point of the discrete isometry σ. Then, since in view of Lemma (4.4)

$$\Sigma_{\kappa s} = \phi_s \Sigma_0,$$

we find

$$\sigma \circ \phi_s(\Sigma_0) = \phi_{-s} \circ \sigma(\Sigma_0) = \phi_{-s}(\Sigma_0) = \Sigma_{-\kappa s}$$

and also

$$\sigma \circ \phi_s(\Sigma_0) = \sigma(\Sigma_{\kappa s}),$$

hence

$$\sigma(\Sigma_s) = \Sigma_{-s}.$$

Next we notice that because of the presence of the axisymmetry, u^k has to be tangent to the 2-dimensional axis being a timelike 2-surface. Otherwise there would be a preferred spacelike direction not tangent to the axis. Finally, since, under σ, normal directions of Σ_s are mapped into those of Σ_{-s}, we have $\sigma_* : u^k \to -u^k$. From this it follows that

$$u^k a_k \,|_p = - u^k a_k \,|_{\sigma(p)}$$

where a^k is tangent to the « spatial axes » being the intersections of orthogonal 2-surfaces with the 2-dimensional axis.

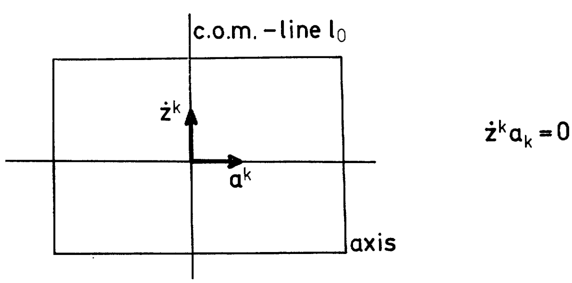

FIG. 1.

If $p = z(0)$ (the fixed point of σ), we obtain

$$u^k a_k \,|_{z(0)} = 0.$$

On the other hand,

$$0 = g_{z(0)}(u, a) = g_{\phi_s(z(0))}(\phi_{s*}u, \phi_{s*}a) = g_{\phi_s(z(0))}(u, a)$$

for all $s \in \mathbb{R}$. This shows that $u^k a_k = 0$ everywhere on l_0. □

(7.2) COROLLARY. — The vector w introduced in Proposition (6.4) is tangent to the intersections of orthogonal 2-surfaces with the axis A. □

(7.3) COROLLARY. — The hypersurfaces Σ_s contain the spacelike 2-surfaces which are orthogonal to the group orbits of G_2.

Proof. — Let V be any of the geodesic vector fields spanning Σ_s. The functions $g(\xi, V)$ and $g(\eta, V)$ are constant along V. They even have to be zero, since they vanish at l_0 by propositions (6.1) and (7.1). Hence V is orthogonal to all group orbits 0 of G_2. Therefore all geodesics starting orthogonal to ξ at a point of l_0 remain in orthogonal 2-surfaces. □

Another consequence of more general interest is

(7.4) LEMMA. — If the unit 4-velocity V^a of the matter stream (i. e. the unit timelike eigenvector of $T^a{}_b$) is tangent to the group orbits and invariant under the group action, then $V^k = u^k$ at l_0, whence l_0 is a streamline.

Remark. — For a perfect fluid distribution it can be deduced from the orthogonal transitivity assumption that V^a does satisfy these conditions. Thus the centre-of-mass line is a streamline of the fluid.

Proof of *Lemma* (7.4). — The assumptions imply that V^a is of the form

$$V^a = A(\xi^a + \Omega\eta^a)$$

with functions A, Ω being constant on the group orbits. Since $V^aV_a = 1$, and $u^k = \dot{z}^k$, $u^ku_k = 1$, the result follows. □

Next, we turn to the introduction of coordinate functions.

(7.5) LEMMA. — The time function t satisfies

$$\begin{aligned} t_{,[a}\xi_b\eta_{c]} &= 0, \\ t_{,a}\eta^a &= 0, \\ t_{,a}\xi^a &= \kappa. \end{aligned} \tag{7.1}$$

Locally, there exists also a function ϕ satisfying

$$\begin{aligned} \phi_{,[a}\xi_b\eta_{c]} &= 0, \\ \phi_{,a}\xi^a &= 0, \\ \phi_{,a}\eta^a &= 1. \end{aligned} \tag{7.2}$$

Proof. — $t_{,a}\eta^a = 0$ and $t_{,a}\xi^a = \kappa$ follow from Lemma (4.4). Corollary (7.3) implies that $t_{,[a}\xi_b\eta_{c]} = 0$. Secondly, dragging orthogonal 2-surfaces along ξ yields timelike hypersurfaces which are chosen to be $\phi = \text{const}$. Then it follows analogously that ϕ satisfies the properties stated above. □

(7.6) REMARK. — The functions t and ϕ are determined uniquely up to arbitrary additive constants.

Another two coordinate functions being constant on the group orbits are constructed as follows.

Let $p \in l_0$. Then $p \in \Sigma_{t(p)}$. We first construct a coordinate system $(x^\alpha) = (x, y, z)$ in $\Sigma_{t(p)}$ around p. For this purpose, we define an orthonormal triad of spacelike vectors $\overset{(\alpha)}{E}{}^k$ at p:

$\overset{(3)}{E}{}^k : = w^k$, the unit vector tangent to the geodesic $\Sigma_{t(p)} \cap A$ (cf. Lemma (6.3) !).

$\overset{(1)}{E}{}^k$: is tangent to the geodesic through p which starts orthogonal to w^k and lies in the orthogonal 2-surface $\phi = 0$ in $\Sigma_{t(p)}$ (cf. Corollary (7.3) !).

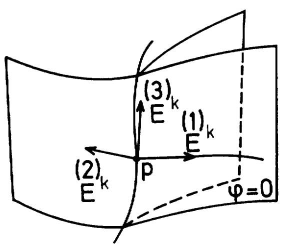

FIG. 2. — Orthogonal 2-surfaces in $\Sigma_{t(p)}$.

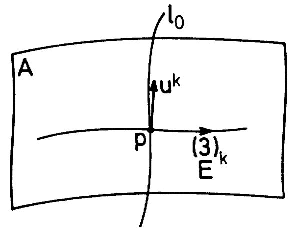

FIG. 3. — The axis A.

Let N_p be a normal neighbourhood of p and let $q \in N_p \cap \Sigma_{t(p)}$. Then we define

$$x^\alpha(q) : = g_p(\overset{(\alpha)}{E}, \exp_p^{-1}(q)) = - \overset{(\alpha)}{E}{}^k \sigma_k(p, q),$$

so x^α are normal coordinates.

By repeating the construction in each slice Σ_s, it is immediately clear that the triad field $\overset{(\alpha)}{E}$ obtained that way along l_0 satisfies

$$\mathscr{L}_\xi \overset{(\alpha)}{E} = 0.$$

Since, in particular, ξ is a Jacobi field, we have

$\mathscr{L}_\xi \sigma_k = 0$ (see [2], p. 184) and therefore.

$\mathscr{L}_\xi x^\alpha = 0$. It then follows together with assumption (A2) that the system (t, x^α) is well-defined within supp (T^{ab}).

We are now ready to prove

(7.7) LEMMA. — *i*) Define the function $\rho : = {}_+\sqrt{x^2 + y^2}$. The functions ρ and z are constant on the group orbits O of G_2.

ii) The inverse transformation is

$$x = \rho \cos \phi, \; y = \rho \sin \phi.$$

Proof. — *i*) We want to show

$$\mathscr{L}_\xi \rho = 0 = \mathscr{L}_\xi z$$

and

$$\mathscr{L}_\eta \rho = 0 = \mathscr{L}_\eta z.$$

The first two equations are obvious from the construction of (x^α). By the same reason as above, we also have

$$\mathscr{L}_\eta \sigma_k = 0.$$

Furthermore,

$$(\mathscr{L}_\eta \overset{(3)}{\mathrm{E}})^k = - \overset{(3)}{\mathrm{E}}{}^l \nabla_l \eta^k = - w_l \eta^{lkmn} u_m w_n = 0.$$

Hence we also have

$$\mathscr{L}_\eta z = 0.$$

Finally,

$$\rho^2 = x^2 + y^2 = (\overset{(1)}{\mathrm{E}}{}^k \overset{(1)}{\mathrm{E}}{}^l + \overset{(2)}{\mathrm{E}}{}^k \overset{(2)}{\mathrm{E}}{}^l)\sigma_k \sigma_l.$$

Since the geodesic 2-surface spanned by the bivector $\overset{(1)}{\mathrm{E}}{}^{[k} \overset{(2)}{\mathrm{E}}{}^{l]}$ is invariant under the action generated by η and since

$$g(\overset{(1)}{\mathrm{E}}, \overset{(1)}{\mathrm{E}}) = -1 = g(\overset{(2)}{\mathrm{E}}, \overset{(2)}{\mathrm{E}}),\ g(\overset{(1)}{\mathrm{E}}, \overset{(2)}{\mathrm{E}}) = 0,$$

it follows that

$$\begin{aligned} (\mathscr{L}_\eta \overset{(1)}{\mathrm{E}})^k &= - \overset{(2)}{\mathrm{E}}{}^k \overset{(2)}{\mathrm{E}}{}_l (\mathscr{L}_\eta \overset{(1)}{\mathrm{E}})^l, \\ (\mathscr{L}_\eta \overset{(2)}{\mathrm{E}})^k &= - \overset{(1)}{\mathrm{E}}{}^k \overset{(1)}{\mathrm{E}}{}_l (\mathscr{L}_\eta \overset{(2)}{\mathrm{E}})^l. \end{aligned} \tag{7.3}$$

From this it is easily deduced that

$$(\mathscr{L}_\eta (\overset{(1)}{\mathrm{E}}{}^k \overset{(1)}{\mathrm{E}}{}^l + \overset{(2)}{\mathrm{E}}{}^k \overset{(2)}{\mathrm{E}}{}^b) = 0,$$

whence

$$\mathscr{L}_\eta \rho = 0.$$

ii) Equations (7.3) can be written as

$$\mathscr{L}_\eta x = - \gamma y \quad \text{and} \quad \mathscr{L}_\eta y = \gamma x$$

with

$$\gamma := \overset{(2)}{\mathrm{E}}{}^k (\mathscr{L}_\eta \overset{(1)}{\mathrm{E}})_k.$$

From Lemma (7.5), $\eta = \partial_\phi$ with $\phi \in [0, 2\pi]$. Therefore the above system of differential equations has the solutions

$$x = \rho \cos \gamma\phi,\ y = \rho \sin \gamma\phi.$$

Since ϕ has to be periodic, we must have $\gamma = 1$. This completes the proof. □

Altogether we have shown that the metric can be written in the form

$$e^{2\phi} dt^2 - \rho^2 e^{2\psi} (d\phi - \omega dt)^2 - (e^{2\mu} d\rho^2 + e^{2\nu} dz^2 + 2\beta d\rho dz) \tag{7.4}$$

where all the functions depend on ρ, z only. They are well behaved within supp (T^{ab}) including the axis where $\rho = 0$ and at the center-of-mass line where $\rho = 0, z = 0$. The information about ρ and z to be normal coordinates on the hypersurfaces $t =$ const. is coded in the functions μ, ν and β in the following way.

(7.8) LEMMA. — In the interior of supp (T^{ab}), the necessary and sufficient condition for ρ and z to be normal coordinates on $t =$ const. hypersurfaces is that the following relations hold:

$$\rho e^{2\mu} + z\beta = \rho . \alpha$$

and

$$z e^{2\nu} + \rho\beta = z . \alpha$$

where α is a constant.

Proof. — During the proof, we will need spherical polar coordinates defined within supp (T^{ab}) by

$$r : = {}_{+}\sqrt{\rho^2 + z^2}, \qquad \theta : = \operatorname{arc\,tan} \frac{\rho}{z}.$$

The equations for the geodesics known to exist in the $t =$ const. hypersurfaces read

$$\beta \ddot{z} + f\ddot{\rho} + \frac{1}{2} f_\rho \dot{\rho}^2 + f_z \dot{\rho}\dot{z} - \frac{1}{2} g_\rho \dot{z}^2 + \beta_z \dot{z}^2 = 0$$

$$\beta \ddot{\rho} + g\ddot{z} + \frac{1}{2} g_z \dot{z}^2 + g_\rho \dot{\rho}\dot{z} - \frac{1}{2} f_z \dot{\rho}^2 + \beta_\rho \dot{\rho}^2 = 0$$

where we have put

$$f : = e^{2\mu}, \qquad g : = e^{2\nu}, \qquad f_\rho : = \frac{\partial f}{\partial \rho}, \text{ etc.}$$

They follow from (7.4) by noting that $\dot{\phi} - \omega \dot{t} = 0$ for all geodesics passing through the axis and that for the geodesics in question $\dot{t} = 0$ and hence $\dot{\phi} = 0$. Since ρ, z are normal coordinates, the geodesics starting at l_0 are given by $\rho(u) = \rho_0 u$, $z(u) = z_0 u$ with constants ρ_0, z_0 and they must solve the above equations. I. e. we must have

$$f_\rho(\rho_0 u, z_0 u)\rho_0^2 + 2f_z(\rho_0 u, z_0 u)\rho_0 z_0 - g_\rho(\rho_0 u, z_0 u) z_0^2 + 2\beta_z(\rho_0 u, z_0 u) z_0^2 = 0$$

and similarly for the second equation. Multiplying by u^2 and observing that any point in supp (T^{ab}) can be connected with l_0 by such a geodesic, we obtain two differential equations for the functions f, g, β. They can be written as

$$\begin{aligned} \rho(f_\rho \rho + f_z z) + z(f_z \rho - g_\rho z) + 2\beta_z z^2 &= 0 \\ z(g_\rho \rho + g_z z) + \rho(g_\rho z - f_z \rho) + 2\beta_\rho \rho^2 &= 0. \end{aligned} \tag{7.5}$$

They are solved as follows. Adding them after multiplication with ρ and z, respectively, yields

$$\rho^2(f_\rho\rho + f_z z) + z^2(g_\rho\rho + g_z z) + 2\beta_z\rho z^2 + 2\beta_\rho z\rho^2 = 0,$$

which, by using the coordinates r, θ, reads

$$\sin^2\theta f_r + \cos^2\theta g_r + 2\sin\theta\cos\theta\beta_r = 0.$$

Integration gives

$$\sin^2\theta f + \cos^2\theta g + 2\sin\theta\cos\theta\,\beta = \alpha(\theta). \qquad (7.6)$$

By rewriting the first equation (7.5) and by using (7.6) to eliminate θ-derivatives as well as g itself, one gets after several manipulations

$$\sin\theta\,(rf)_{,r} + \cos\theta\,(r\beta)_{,r} = \sin\theta\,\alpha + \frac{1}{2}\cos\theta\,\alpha',$$

which can again be integrated to yield

$$\sin\theta f + \cos\theta\,\beta = \sin\theta\,\alpha + \frac{1}{2}\cos\theta\,\alpha' + \frac{\gamma(\theta)}{r}$$

with some function γ. However, since f and β are regular at $r = 0$, we must have $\gamma = 0$. Moreover, β even vanishes at $r = 0$ and also by regularity, we have to have

$$\lim_{r\to 0} f(r, \theta) = \text{const.}$$

It follows that $\alpha + \frac{1}{2}\cot\theta\alpha' = \text{const.}$, the only regular solution of which is $\alpha = \text{const.}$ Together with (7.6), this is the result. □

In addition to the previous assumptions we now postulate the existence of another family of discrete isometries acting on the rotation axis. For this purpose, we recall from Proposition (6.4) the vector w^k given by

$$w^k = \frac{1}{2}\eta^{klmn}\nabla_l\eta_m u_n.$$

It is tangent to the geodesic $\Sigma_t \cap A$ representing the axis in $t = \text{const.}$

(S6) The spatial axes of rotation, the geodesics $\gamma_t(u) = \Sigma_t \cap A$ ($t \in \mathbb{R}$, u being an affine parameter such that $\gamma_t(0) \in l_0$) are reflection symmetric w. r. t. the centre-of-mass, i. e. for each $t \in \mathbb{R}$ there exists a discrete isometry $\alpha_t : \Sigma_t \cap A \to \Sigma_t \cap A$ such that $\alpha_t(\gamma_t(u)) = \gamma_t(-u)$.

In the canonical coordinates (t, x, y, z), α_t is given by $\alpha_t : (t, 0, 0, z) \to (t, 0, 0, -z)$. In the sequel, we write $\dot{} = \dot{z}^k\nabla_k = u^k\nabla_k$. We can now prove

(7.9) PROPOSITION. — The centre-of-mass line is a geodesic: $\dot{u}^k = 0$.

Proof. — From Section 2, we know that

$$(\mathcal{L}_\xi u)^k = 0, \text{ i. e. } \xi^l\nabla_l u^k = u^l\nabla_l\xi^k.$$

From Lemma (4.2) and Proposition (7.1) we have

$$\xi^k = \kappa \dot{z}^k = \kappa u^k \quad \text{with} \quad \xi^k \xi_k = \kappa^2.$$

Therefore we can write

$$\kappa^2 \dot{u}^k = \xi_l \nabla^l \xi^k = - \xi_l \nabla^k \xi^l = - \frac{1}{2} \nabla^k \kappa^2$$

or

$$\dot{u}^k = - \partial^k \log \kappa.$$

By construction, the coordinate functions x, y, z are smooth on the axis. By imposing axis regularity conditions on the metric functions in (7.4), the metric can be rewritten in a form which is manifestly regular on this axis. This will not be done explicitly here. Then one finds that $\partial^k \log \kappa = \partial^k \phi$.

Now the regularity of the axis implies $\partial_\rho \phi \,|_{\rho = 0} = 0$; the existence of the discrete isometry α_t implies $\partial_z \phi \,|_{z = 0} = 0$. Hence $\dot{u}^k = 0$. □

In order to deduce a first consequence of this, we also need the following results.

(7.10) PROPOSITION. — The spin vector S^k is parallel along the geodesic $l_0 : \dot{S}^k = 0$.

Proof. — The centre-of-mass condition $u_k S^k = 0$ together with the geodesy of l_0 (Proposition (7.9)) entail

$$u_k \dot{S}^k = 0.$$

Since the norm $S_k S^k$ is constant along l_0, we have

$$S_k \dot{S}^k = 0.$$

By a symmetry argument, $\dot{S}^k$ must be tangent to the two-dimensional axis A. Since it is orthogonal to u^k, it must be parallel to w^k and hence to S^k (Proposition (6.5)) whence $\dot{S}^k = 0$. □

We will now approach a statement about the structure of the body. It will first be expressed in terms of Dixon's notions of force and torque, quantities which depend on quadrupole and higher moments only (i. e. they are determined by the energy-momentum skeleton). Only in a second step, the statement will be reexpressed in termes of the energy-momentum tensor itself.

We recall Dixon's laws of motion

$$\dot{P}^i = \frac{1}{2} R_{lmk}{}^i S^{lm} \dot{z}^k + F^i \tag{3.10}$$

$$\dot{S}^{ij} = 2P^{[i} \dot{z}^{j]} + L^{ij}. \tag{3.11}$$

(7.11) THEOREM. — For a body embedded in a spacetime satisfying assumptions (A1), (A2), (S1) to (S6) the force and torque vanish:

$$F^i = 0 = L^{ij}.$$

Proof. — Proposition (7.1) implies $P^{[i}\dot{z}^{j]} = 0$. Proposition (7.9) and (7.10) imply that $\dot{S}^{ij} = 0$. (3.11) then yields $L^{ij} = 0$.

Corollary (5.2) together with Proposition (7.9) entail $\dot{P}^i = 0$. The vanishing of the Riemann tensor term in equation (3.10) follows from the axis regularity conditions as well as from (S6). The proof requires a calculation of the Riemann tensor of (7.4). Having done this, we also find $F^i = 0$. □

Theorem (7.11) imposes certain restrictions on the multipole structure of T^{ab}. They are most easily expressed in terms of the energy momentum skeleton.

First we formulate the following

(7.12) COROLLARY. — Let ζ be a Jacobi field along all geodesics emanating from the centre-of-mass z, such that $\nabla_{(k}\zeta_{l)} = 0$, let $\widehat{T}_z$ be the skeleton of T, and let $G := \exp^* g$. Then

$$\int \widehat{T}^{kl} \mathscr{L}_\zeta G_{kl} DX = 2 \int \widehat{T}^{kl} H_k{}^a H_l{}^b \nabla_{(a}\zeta_{b)} DX = 0$$

Proof. — This follows immediately from (3.18), (3.19). □

From this we deduce similar integral equalities for the energy-momentum tensor T itself:

(7.13) THEOREM. — Let ζ be a Jacobi field as in Corollary (7.12). Then

$$\int_{\Sigma(s)} T^{ab} \nabla_a \zeta_b dS = 0.$$

Proof. — It follows from the definition of $\widehat{T}_z$ (see e. g. [2], [5]) that

$$\int_{T_zM} \widehat{T}_z{}^{kl} \Phi_{kl} DX = \int_{\Sigma(s)} T^{ab} \varphi_{ab} dS - \int_{\Sigma(s)} T^{ab} \psi_a dS_b$$

if $X^k \Phi_{kl}(z, X) = 0$ for all $X \in T_zM$, where ψ is given by

$$\psi_a(z(s), x) = \frac{\partial}{\partial s} \lambda_a(z(s), x) + K_a{}^k \dot{z}^l \varphi_{kl}(z(s)) + 2H_a{}^{[k} \sigma^{l]} \dot{z}^m \nabla_k \varphi_{lm}(z(s)).$$

λ being determined from φ according to (3.15), (3.16). But

$$\mathscr{L}_\zeta X = 0,$$

whence

$$X^k \mathscr{L}_\zeta G_{kl} = \mathscr{L}_\zeta X^k G_{kl} = \mathscr{L}_\zeta X_l = 0.$$

Therefore, putting $\varphi_{ab} = \frac{1}{2}\mathcal{L}_\zeta g_{ab} = \nabla_{(a}\zeta_{b)}$,

$$0 = \int_{\Sigma(s)} \mathrm{T}^{ab}\nabla_a\zeta_b d\mathrm{S} - \int_{\Sigma(s)} \mathrm{T}^{ab}\psi_a d\mathrm{S}_b$$

where ψ is now to be constructed from $\nabla_{(a}\zeta_{b)}$: substituting $\nabla_{(a}\zeta_{b)} \to \varphi_{ab}$, we find

$$\lambda_a = \zeta_a - \theta_a$$

where

$$\theta_a = \mathrm{K}_a{}^k\zeta_k + \mathrm{H}_a{}^k\sigma^l\nabla_{[k}\zeta_{l]}$$

i. e. $\lambda_a = 0$. This implies

$$\psi_a = \mathrm{K}_a{}^k\dot{z}^l\nabla_{(k}\zeta_{l)} + 2\mathrm{H}_a{}^{[k}\sigma^{l]}\dot{z}^m\nabla_k\nabla_{(l}\zeta_{m)}$$

But, by definition,

$$\nabla_{(k}\zeta_{l)} = 0$$

and

$$\dot{x}^b\dot{x}^c[\nabla_{bc}\zeta_a - \mathrm{R}_{abc}{}^d\zeta_d] = 0$$

whence

$$\nabla_{(kl)}\zeta_m = \mathrm{R}_{m(kl)}{}^n\zeta_n.$$

The Ricci-identity yields

$$\nabla_{[kl]}\zeta_m = \frac{1}{2}\mathrm{R}_{klm}{}^n\zeta_n$$

whence

$$\nabla_{kl}\zeta_m = \frac{1}{2}(\mathrm{R}_{klm}{}^n + \mathrm{R}_{mkl}{}^n + \mathrm{R}_{mlk}{}^n)\zeta_n$$

and therefore

$$\nabla_{k(l}\zeta_{m)} = 0.$$

This implies $\psi_a = 0$, hence the conclusion. □

§ 8. SUMMARY

The results which we have obtained in the general, the stationary and the axisymmetric cases are in agreement with the behaviour that one would expect for momentum, angular momentum, centre-of-mass, etc. on the basis of plausibility arguments. One should not forget, however, that these terms are only names which have been chosen for certain mathematical objects, whose definition is analogous to the definition of these terms both in classical mechanics and special relativity. Therefore one should interpret these results as giving further support to Dixon's definitions. The last theorem restricting the multipole structure of a body in a stationary axisymmetric spacetime is somewhat unexpected and would have hardly been obtained without the use of Dixon's theory. This result as well as the coordinate system with its preferred time coordinate introduced in § 7 might be helpful in finding a realistic interior solution for the Kerr metric.

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REFERENCES

[1] M. STREUBEL and R. SCHATTNER, in *Proceedings of the Third Gregynog Relativity Workshop*, M. Walker (ed.), Max-Planck-Institut für Physik und Astrophysik, MPI-PAE/Astro, 204, Munich, 1979, p. 82-121.
[2] W. G. DIXON, in *Course 67 of the International School of Physics « Enrico Fermi »*, J. Ehlers (ed.), North Holland, Amsterdam, 1979, p. 156-219.
[3] R. SCHATTNER, *Gen. Rel. Grav.*, t. **10**, 1979, p. 377-395.
[4] J. EHLERS and E. RUDOLPH, *Gen. Rel. Grav.*, t. **8**, 1977, p. 197.
[5] R. SCHATTNER, *The Structure of Extended Bodies in General Relativity*, Max-Planck-Institut für Physik und Astrophysik, MPI-PAE/Astro 236, Munich, 1979.
[6] J. EHLERS, R. SCHATTNER and M. TRÜMPER (to appear), 1980.
[7] B. CARTER, *Comm. Math. Phys.*, t. **17**, 1970, p. 233.
[8] B. CARTER, *J. Math. Phys.*, t. **10**, 1969, p. 70.
[9] B. SCHMIDT, *Z. Naturf.*, t. **22 a**, 1967, p. 1351.
[10] W. KUNDT and M. TRÜMPER, *Z. Physik*, t. **192**, 1966, p. 419.

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