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The geometrical and gauge structure of a generalized theory of gravitation

by

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ABSTRACT. — A generalized theory of gravitation is constructed in a superspace base manifold of eight dimensions with an octad of gauge fields and a superspace fiber bundle connection. The gauge structure is a non-compact unitary group $U(i, j) \supset SU(3) \otimes SU(2) \otimes U(1)$ which can be used as a unification scheme. Field equations with uniquely determined sources are derived from an action principle.

I. INTRODUCTION

A new theory of gravity has been formulated on the basis of a non-symmetric Hermitian $g_{\mu\nu}$ [*1*]-[*3*]. As this theory has certain interesting consequences such as geodesically complete spherically symmetric solutions [*1*]-[*4*] and cosmological solutions [*5*], it is of interest to investigate whether there exists an extended theory in a higher dimensional space that contains a large enough fiber bundle connection to unify all the gauge fields of nature.

We recall that the n-dimensional real manifolds correspond to the holonomy group $SO(n)$. Any oriented Riemann manifold, described by

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a chart of real coordinates $(x^1, \ldots, x^n)$ belongs to this class. The scalar product in its tangent bundle $T_x(M)$ is real Euclidean and the elements of the holonomy group are described by the rotations of the orthonormal basis in $T_x(M)$. In $2n$-dimensional complex manifolds, the holonomy group is $U(n)$. These manifolds can be covered by the holomorphic charts of complex-analytic coordinates $(z^1, \ldots, z^n)$; the scalar product in the tangent bundle $T_z(M)$ describes the elements of $U(n)$. These structures correspond to complex Riemannian manifolds. An example is the complex Kähler manifold. In the following we shall concern ourselves with a $2n$-dimensional real manifold with the holonomy group $U(n)$. The metric of the manifold is Hermitian and the space has a non-vanishing torsion.

II. GAUGE THEORY CONCEPT AND GENERAL RELATIVITY

Let us begin by reviewing the concepts of gauge invariance and differential geometry in general relativity [6].

The gauge fields are characterized by the existence of tetrad or bein fields. They appear as coefficients in the definition of a Lie valued operator, which may be referred to as the fully covariant derivative. In general relativity the real tetrad fields $e^{\alpha}_{\dot{\mu}}(x)$ are simultaneously a representation of SO(3,1) and GL(4, R). Here the undotted suffixes denote the coordinates of the tangent space, while the dotted suffixes denote the coordinates of the real four-dimensional manifold. The tetrad transforms as SO(3,1) on the undotted suffix and as GL(4, R) on the dotted suffix.

When we define the fully covariant derivative, we need a set of gauge fields for both groups. The real manifold M is assumed to be C^∞. The principal bundle of linear coframes of M is defined by [7]

$$\mathscr{L}^*(M) = \langle L^*(M), \pi_{L^*}, M, GL(4) \rangle \tag{2.1}$$

The equivalence classes of Lorentz related linear coframes form a fiber bundle associated with the principal bundle (2.1), namely

$$\mathscr{L}\mathscr{L}^*(M) = \langle LL^*(M), \pi_{LL^*}, M, SO(3,1)/GL(4), \mathscr{L}^*(M) \rangle \tag{2.2}$$

where SO(3, 1)/GL(4) is the space of cosets with respect to the left action of SO(3, 1) on GL(4).

A differentiable cross section

$$e : M \to LL^*(M) \tag{2.3}$$

defines a Lorentz G-structure on M. For each $p \in M$ a cross section determines an equivalence class of Lorentz related linear coframes

$$e(p) = \{ \Lambda^{\alpha}_{\beta} e^{\beta}_{\dot{\mu}} dx^{\dot{\mu}}_p \mid \Lambda^{\alpha}_{\beta} \in SO(3,1) \} \tag{2.4}$$

where

$$e^{\alpha} = e^{\alpha}_{\dot{\mu}} dx^{\dot{\mu}} \tag{2.5}$$

The cross section (2.3) is then represented locally by differentiable functions $e^{\alpha}_{\dot\mu}$ and the metric $g(p)$ is determined by

$$g_{\dot\rho\dot\mu} = \eta_{\alpha\beta} e^{\alpha}_{\dot\rho} e^{\beta}_{\dot\mu} \tag{2.6}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric.

The functions $e^{\alpha}_{\dot\mu}$ define the tetrad fields and they must satisfy

$$\partial_{\dot\kappa} e^{\alpha}_{\dot\lambda} + (\omega_{\dot\kappa})^{\alpha}{}_{\beta} e^{\beta}_{\dot\lambda} - \Gamma^{\dot\mu}_{\dot\kappa\dot\lambda} e^{\alpha}_{\dot\mu} = 0 \tag{2.7}$$

where $(\omega_{\dot\kappa})^{\alpha}_{\beta}$ and $\Gamma^{\dot\mu}_{\dot\kappa\dot\lambda}$ are the tetrad connection and the torsion-free connection on the principal bundle of linear coframes (2.1), respectively. We can express Γ in terms of e and ω:

$$\begin{aligned}\Gamma_{\dot\kappa\dot\lambda\dot\mu} &= \Gamma^{\dot\rho}_{\dot\kappa\dot\lambda} g_{\dot\rho\dot\mu} \\ &= \eta_{\alpha\beta}\left[\partial_{\dot\kappa} e^{\alpha}_{\dot\lambda} + (\omega_{\dot\kappa})^{\alpha}{}_{\gamma} e^{\gamma}_{\dot\lambda}\right] e^{\beta}_{\dot\mu}\end{aligned} \tag{2.8}$$

If we perform the transformation

$$e^{\alpha}_{\dot\mu} = U^{\alpha}_{\beta} e^{\beta}_{\dot\mu} \tag{2.9}$$

then the metric (2.6) will remain invariant if U is an element of SO(3, 1). Requiring that Γ also be invariant under this transformation, leads to the equation

$$(\omega_{\dot\kappa})^{\alpha}_{\beta} \rightarrow [U\omega_{\dot\kappa}U^{-1} - (\partial_{\dot\kappa}U)U^{-1}]^{\alpha}_{\beta} \tag{2.10}$$

which is the familiar transformation law of a vector gauge field.

Since the $\omega_{\dot\kappa}$ transform like the generators of SO(3, 1), it follows that

$$\partial_{\dot\kappa} g_{\dot\lambda\dot\mu} - g_{\dot\rho\dot\mu}\Gamma^{\dot\rho}_{\dot\kappa\dot\lambda} - g_{\dot\lambda\dot\rho}\Gamma^{\dot\rho}_{\dot\kappa\dot\mu} = 0 \tag{2.11}$$

and the gauge field Γ acts as the connection coefficient for the metric in the sense of differential geometry. In terms of the relation $x'^{\dot\mu} = f^{\dot\mu}(x)$ we have that under the transformations

$$\begin{aligned} e^{\rho}_{\dot\lambda} &\rightarrow \frac{\partial f^{\dot\alpha}}{\partial x^{\dot\lambda}} e^{\rho}_{\dot\alpha} \\ \partial_{\dot\mu} &\rightarrow \frac{\partial f^{\dot\alpha}}{\partial x^{\dot\mu}} \partial_{\dot\alpha} \\ \omega_{\dot\kappa} &\rightarrow \frac{\partial f^{\dot\alpha}}{\partial x^{\dot\kappa}} \omega_{\dot\alpha} \end{aligned} \tag{2.12}$$

the connection satisfies the transformation law

$$\begin{aligned}\Gamma_{\dot\kappa\dot\lambda\dot\mu} \rightarrow\ & \frac{\partial f^{\dot\alpha}}{\partial x^{\dot\kappa}} \frac{\partial f^{\dot\beta}}{\partial x^{\dot\lambda}} \frac{\partial f^{\dot\gamma}}{\partial x^{\dot\mu}} \Gamma_{\dot\alpha\dot\beta\dot\gamma} \\ & + \frac{\partial^2 f^{\dot\alpha}}{\partial x^{\dot\kappa}\partial x^{\dot\lambda}} g_{\dot\alpha\dot\beta} \frac{\partial f^{\dot\beta}}{\partial x^{\dot\mu}}\end{aligned} \tag{2.13}$$

In general relativity the connection Γ is symmetric. If on the other hand a skew symmetric part

$$\Gamma_{[\dot\kappa\dot\lambda]\dot\mu} = \frac{1}{2}(\Gamma_{\dot\kappa\dot\lambda\dot\mu} - \Gamma_{\dot\lambda\dot\kappa\dot\mu}) \tag{2.14}$$

is defined, then it satisfies the transformation law of a tensor

$$\Gamma_{[\dot\kappa\dot\lambda]\dot\mu} \rightarrow \frac{\partial f^{\dot\alpha}}{\partial x^{\dot\kappa}}\frac{\partial f^{\dot\beta}}{\partial x^{\dot\lambda}}\frac{\partial f^{\dot\gamma}}{\partial x^{\dot\mu}}\Gamma_{[\dot\alpha\dot\beta]\dot\gamma} \tag{2.15}$$

Let us define the partially covariant operator by the equation

$$D_{\dot\kappa} = \partial_{\dot\kappa}\delta^{\alpha}_{\beta} + (\omega_{\dot\kappa})^{\alpha}_{\beta} \tag{2.16}$$

Then

$$\Gamma_{(\dot\kappa\dot\lambda)\dot\mu} = \frac{1}{2}(\Gamma_{\dot\kappa\dot\lambda\dot\mu} + \Gamma_{\dot\lambda\dot\kappa\dot\mu}) \tag{2.17}$$

and $\Gamma_{[\dot\kappa\dot\lambda]\dot\mu}$ can be expressed in terms of D and e:

$$\Gamma_{(\dot\kappa\dot\lambda)\dot\mu} = \frac{1}{2}\eta_{\alpha\beta}[D_{\dot\kappa}e^{\alpha}_{\dot\lambda} + D_{\dot\lambda}e^{\alpha}_{\dot\kappa}]e^{\beta}_{\dot\mu} \tag{2.18}$$

$$\Gamma_{[\dot\kappa\dot\lambda]\dot\mu} = \frac{1}{2}\eta_{\alpha\beta}[D_{\dot\kappa}e^{\alpha}_{\dot\lambda} - D_{\dot\lambda}e^{\alpha}_{\dot\kappa}]e^{\beta}_{\dot\mu} \tag{2.19}$$

A calculation shows that

$$([D_{\dot\kappa}, D_{\dot\lambda}])^{\alpha}_{\beta} = (R_{\dot\kappa\dot\lambda})^{\alpha}_{\beta} \tag{2.20}$$

where

$$(R_{\dot\kappa\dot\lambda})^{\alpha}_{\beta} = \partial_{\dot\kappa}(\omega_{\dot\lambda})^{\alpha}_{\beta} - \partial_{\dot\lambda}(\omega_{\dot\kappa})^{\alpha}_{\beta} + ([\omega_{\dot\kappa}, \omega_{\dot\lambda}])^{\alpha}_{\beta} \tag{2.21}$$

Under the transformations (2.12), it can be shown that $(R_{\dot\kappa\dot\lambda})^{\alpha}_{\beta}$ is a tensor.

We can now form the scalar curvature

$$R = \eta^{\alpha\gamma}e^{\dot\kappa}_{\alpha}e^{\dot\lambda}_{\beta}(R_{\dot\kappa\dot\lambda})^{\beta}_{\gamma} \tag{2.22}$$

and use this to form the Einstein Lagrangian

$$\mathscr{L} = eR \tag{2.23}$$

where $e = \det(e^{\alpha}_{\dot\mu})$.

In the above, we have seen how a few simple ideas which are derived from gauge theories and differential geometry lead directly to general relativity.

III. SUPERSPACE THEORY OF GRAVITATION

Let us introduce the superspace [8] coordinates

$$\{X^{\dot M} : X^{\dot M} = (x^{\dot\mu}, x^{\dot m}), \quad (\dot\mu = 1, \ldots, 4; \dot m = 5, \ldots, 8).$$

The coordinates $X^{\dot{M}}$ define an eight-dimensional real differentiable manifold which is assumed to be C^∞. The principal bundle of complex linear coframes of this manifold is defined to be

$$\mathscr{L}^*(M) = \langle L^*(M), \pi_{L^*}, M, GL(8, \mathbb{C}) \rangle \tag{3.1}$$

and the linear coframes define an equivalence class with the structure

$$\mathscr{L}\mathscr{L}^*(M) = \langle LL^*(M), \pi_{LL^*}, M, U(i, j)/GL(8, \mathbb{C}), \mathscr{L}^*(M) \rangle \tag{3.2}$$

where $U(i, j)/GL(8, \mathbb{C})$ is the space of cosets and $U(i, j)$ is a (local) noncompact unitary gauge group of transformations. Specific values of i and j are determined by the signature of the superspace. Two possibilities are $U(7, 1)$ and $U(6, 2)$. Of course everything that is derived in the following holds for $\dot{M} = 1, \ldots, 4$ with the tangent space gauge group $U(3, 1)$ corresponding to complex Lorentz transformations.

The equivalence class of $U(i, j)$ related linear coframes is now

$$e(p) = \{ U^K_L e^L_{\dot{M}} dX^{\dot{M}}_p \mid U^K_L \in U(i, j) \} \tag{3.3}$$

where

$$e^K = e^K_{\dot{M}} dX^{\dot{M}} \tag{3.4}$$

and $e^K_{\dot{M}}$ is a complex octad of gauge fields. The superspace consists of a manifold $\dot{M}$ with basis vectors $\xi_{\dot{\mu}}$ satisfying

$$[\xi_{\dot{\mu}}, \xi_{\dot{\nu}}] = [\partial_{\dot{\mu}}, \partial_{\dot{\nu}}] = 0 \tag{3.5}$$

and a manifold N with basis vectors $\xi_{\dot{m}}$ that obey

$$[\xi_{\dot{m}}, \xi_{\dot{n}}] = [\partial_{\dot{m}}, \partial_{\dot{n}}] = 0 \tag{3.6}$$

Then we can define the one forms

$$\omega_1 = \omega^{\dot{\mu}} d\xi_{\dot{\mu}}, \qquad \omega_2 = \omega^{\dot{m}} d\xi_{\dot{m}} \tag{3.7}$$

A coordinate transformation can be defined in superspace in terms of the relationship $X^{\dot{M}} = F^{\dot{M}}(Z)$, where $Z^{\dot{N}}$ denotes another set of superspace coordinates. Then we have

$$dX^{\dot{M}} = dZ^{\dot{K}} \frac{\partial F^{\dot{M}}}{\partial Z^{\dot{K}}} \tag{3.8}$$

and

$$dX^{\dot{M}} e^A_{\dot{M}} = dZ^{\dot{K}} \frac{\partial F^{\dot{M}}}{\partial Z^{\dot{K}}} e^A_{\dot{M}}(X) \tag{3.9}$$

Moreover, we define $e^A_{\dot{K}}(Z)$ so that

$$e^A_{\dot{K}}(Z) = \frac{\partial F^{\dot{M}}}{\partial Z^{\dot{K}}} e^A_{\dot{M}}(X) \big|_{X = F(Z)} \tag{3.10}$$

which leads to the equation

$$dX^{\dot{M}} e^A_{\dot{M}}(X) = dZ^{\dot{K}} e^A_{\dot{K}}(Z) \tag{3.11}$$

The superspace metric is determined by

$$g_{\dot{M}\dot{N}} = e_{\dot{M}}^{\ K}\eta_{KL}e_{\dot{N}}^{*L} \tag{3.12}$$

where η_{KL} is the flat superspace metric

$$\eta_{KL} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{mn} \end{pmatrix} \tag{3.13}$$

We assume that $\partial_M\eta_{KL} = 0$. Thus the supermanifold consists of the base manifold M (labelled by lower case Greek letters) and the manifold N (labelled by lower case Latin letters) and the signature of $\eta_{\mu\nu}$ is the Lorentz signature -2. There are two metrics $g_{\dot{\mu}\dot{\nu}}$ and $g_{\dot{m}\dot{n}}$ that can be defined in a basis invariant way:

$$g_{\dot{\mu}\dot{\nu}} = e_{\dot{\mu}}^{\ K}\eta_{KL}e_{\dot{\nu}}^{*L} \tag{3.14}$$

$$g_{\dot{m}\dot{n}} = e_{\dot{m}}^{\ K}\eta_{KL}e_{\dot{n}}^{*L} \tag{3.15}$$

and

$$g_{\dot{\mu}\dot{n}} = e_{\dot{\mu}}^{\ K}\eta_{KL}e_{\dot{n}}^{*L} = 0 \tag{3.16}$$

Under a superspace coordinate transformation, the metric $g_{\dot{K}\dot{L}}$ satisfies

$$g_{\dot{K}\dot{L}}(Z) = \frac{\partial F^{\dot{M}}}{\partial Z^{\dot{K}}} g_{\dot{M}\dot{N}}(X) \frac{\partial F^{\dot{N}}}{\partial Z} \tag{3.17}$$

We require that the octads satisfy the equation

$$\partial_{\dot{K}}e_{\dot{L}}^{\ A} + (\omega_{\dot{K}})_C^{\ A}e_{\dot{L}}^{\ C} - \Gamma_{\dot{K}\dot{L}}^{\dot{R}}e_{\dot{R}}^{\ A} = 0 \tag{3.18}$$

This equation can be solved for Γ in terms of e and ω:

$$\Gamma_{\dot{K}\dot{L}}^{\dot{R}}g_{\dot{R}\dot{M}} = \Gamma_{\dot{K}\dot{L}\dot{M}} = \eta_{AB}(D_{\dot{K}}e_{\dot{L}}^{\ A})e_{\dot{M}}^{*B} \tag{3.19}$$

where

$$D_{\dot{K}}e_{\dot{L}}^{\ A} = \partial_{\dot{K}}e_{\dot{L}}^{\ A} + (\omega_{\dot{K}})_C^{\ A}e_{\dot{L}}^{\ C} \tag{3.20}$$

A group of isometries is defined by the transformation

$$e_{\dot{M}}^{\ A} = e_{\dot{M}}^{\prime B}(U)_B^{\ A} \tag{3.21}$$

Here U will be an element of U(i, j) which contains O(3, 1) as a subgroup. Moreover, Γ will remain invariant under this set of transformations provided

$$(\omega_{\dot{K}})_M^{\ L} \rightarrow [U\omega_{\dot{K}}U^{-1} - (\partial_{\dot{K}}U)U^{-1}]_M^{\ L} \tag{3.22}$$

By differentiating the supermetric, we find the equation

$$\partial_{\dot{K}}g_{\dot{L}\dot{M}} - \Gamma_{\dot{K}\dot{L}}^{\dot{R}}g_{\dot{R}\dot{M}} - g_{\dot{L}\dot{R}}\Gamma_{\dot{K}\dot{M}}^{*\dot{R}} = 0 \tag{3.23}$$

To arrive at this result, we have required that ω satisfies the condition

$$(\omega_{\dot{K}})_A^{\ C}\eta_{CB} = -\eta_{AC}(\omega_{\dot{K}}^{*})_B^{\ C} \tag{3.24}$$

Let us now consider the transformation properties of the Γ's under general coordinate transformations in superspace. Using (3.10) in (3.20) we find

$$\Gamma_{\dot{K}\dot{L}\dot{M}} = \frac{\partial F^{\dot{A}}}{\partial Z^{\dot{K}}} \frac{\partial F^{\dot{B}}}{\partial Z^{\dot{L}}} \Gamma_{\dot{A}\dot{B}\dot{C}} \frac{\partial F^{\dot{C}}}{\partial Z^{\dot{M}}} + \frac{\partial^2 F^{\dot{A}}}{\partial Z^{\dot{K}} \partial Z^{\dot{L}}} g_{\dot{A}\dot{B}} \frac{\partial F^{\dot{B}}}{\partial Z^{\dot{M}}} \tag{3.25}$$

The $\Gamma_{\dot{K}\dot{L}\dot{M}}$ can be separated into symmetric and skew parts where $\Gamma_{[\dot{K}\dot{L}]\dot{M}}$ transforms as a tensor

$$\Gamma_{[\dot{K}\dot{L}]\dot{M}} = \frac{\partial F^{\dot{A}}}{\partial Z^{\dot{K}}} \frac{\partial F^{\dot{B}}}{\partial Z^{\dot{L}}} \Gamma_{[\dot{A}\dot{B}]\dot{C}} \frac{\partial F^{\dot{C}}}{\partial Z^{\dot{M}}} \tag{3.26}$$

We shall now use the partially covariant derivative operator D to define a curvature tensor

$$([D_{\dot{K}}, D_{\dot{L}}])^{A}_{B} = (R_{\dot{K}\dot{L}})^{A}_{B} \tag{3.27}$$

where

$$(R_{\dot{K}\dot{L}})^{A}_{B} = \partial_{\dot{K}}(\omega_{\dot{L}})^{A}_{B} - \partial_{\dot{L}}(\omega_{\dot{K}})^{A}_{B} + ([\omega_{\dot{K}}, \omega_{\dot{L}}])^{A}_{B} \tag{3.28}$$

In terms of the transformation (3.22) with

$$(U^{-1})^{M}_{B} = \eta_{BD}(U^*)^{D}_{E}\eta^{EM} \tag{3.29}$$

and

$$\eta_{AE}\eta^{EB} = \delta^{B}_{A} \tag{3.30}$$

we have the result

$$(R_{\dot{K}\dot{L}})^{A}_{B} = U^{A}_{M}(R'_{\dot{K}\dot{L}})^{M}_{N}(U^{-1})^{N}_{B} \tag{3.31}$$

The quantity $R_{\dot{K}\dot{L}\dot{M}\dot{N}}$ is given by

$$R_{\dot{K}\dot{L}\dot{M}\dot{N}} = \eta_{AC}(R_{\dot{K}\dot{L}})^{A}_{B} e^{B}_{\dot{M}} e^{C}_{\dot{N}} \tag{3.32}$$

and is invariant under local isometries in the tangent superspace. Under the transformations

$$\begin{aligned} e^{A}_{\dot{M}} &\to \frac{\partial F^{\dot{C}}}{\partial Z^{\dot{M}}} e^{A}_{\dot{C}} \\ \partial_{\dot{K}} &\to \frac{\partial F^{\dot{A}}}{\partial Z^{\dot{K}}} \partial_{\dot{A}} \\ \omega_{\dot{L}} &\to \frac{\partial F^{\dot{B}}}{\partial Z^{\dot{L}}} \omega_{\dot{B}} \end{aligned} \tag{3.33}$$

we can show that the superspace curvature tensor transforms as

$$R_{\dot{K}\dot{L}\dot{M}\dot{N}} \to \frac{\partial F^{\dot{A}}}{\partial Z^{\dot{K}}} \frac{\partial F^{\dot{B}}}{\partial Z^{\dot{L}}} \frac{\partial F^{\dot{C}}}{\partial Z^{\dot{M}}} R_{\dot{A}\dot{B}\dot{C}\dot{D}} \frac{\partial F^{\dot{D}}}{\partial Z^{\dot{N}}} \tag{3.34}$$

which proves that it is a true tensor in superspace. It satisfies the condition

$$(R_{\dot{K}\dot{L}\dot{M}\dot{N}})^* = R_{\dot{L}\dot{K}\dot{N}\dot{M}} \tag{3.35}$$

Moreover, we have that

$$R_{\dot{K}\dot{L}\dot{M}\dot{N}} = \partial_{\dot{K}}\Gamma_{\dot{L}\dot{M}\dot{N}} - \partial_{\dot{L}}\Gamma_{\dot{K}\dot{M}\dot{N}} + \Gamma_{\dot{K}\dot{M}\dot{R}}\Gamma^{*\dot{R}}_{\dot{K}\dot{N}} - \Gamma_{\dot{L}\dot{M}\dot{R}}\Gamma^{*\dot{R}}_{\dot{K}\dot{N}} \tag{3.36}$$

If we demand that

$$\frac{\partial F^{\dot{R}}}{\partial Z^{\dot{K}}}\frac{\partial Z^{\dot{M}}}{\partial F^{\dot{R}}} = \delta^{\dot{M}}_{\dot{K}} \tag{3.37}$$

then we can define the relationship

$$g_{\dot{K}\dot{L}}g^{\dot{L}\dot{M}} = g^{\dot{M}\dot{L}}g_{\dot{L}\dot{K}} = \delta^{\dot{M}}_{\dot{K}} \tag{3.38}$$

in a covariant fashion. We shall also introduce the inverse octad fields by the equations

$$\begin{aligned} e^{\dot{N}}_{C}e^{B}_{\dot{N}} &= \delta^{B}_{C} \\ e^{A}_{\dot{M}}e^{\dot{N}}_{A} &= \delta^{\dot{N}}_{\dot{M}} \end{aligned} \tag{3.39}$$

IV. THE SUPERSPACE FIELD EQUATIONS

A contracted superspace curvature tensor can be formed

$$R_{\dot{K}\dot{M}} = R_{\dot{K}\dot{L}\dot{M}\dot{N}}g^{\dot{N}\dot{L}} \tag{4.1}$$

and it can be verified to be a second rank tensor under superspace transformations.

We can also form the superspace scalar curvature

$$R = \eta^{AC}e^{*\dot{L}}_{A}e^{\dot{K}}_{B}(R_{\dot{K}\dot{L}})^{B}_{C} \tag{4.2}$$

As the Lagrangian of our theory we choose

$$\mathscr{L} = (ee^*)^{1/2}R \tag{4.3}$$

where $e = \det(e^{K}_{\dot{M}})$. The variational principle is

$$\delta\int d^8X(ee^*)^{1/2}R = 0 \tag{4.4}$$

The field equations of superspace are obtained by varying $\mathscr{L}$ with respect to ω and e. They are

$$\partial_{M}((ee^*)^{1/2}(e^{\dot{L}}e^{*\dot{M}} - e^{\dot{M}}e^{*\dot{L}})) + [\omega_{\dot{M}}, (ee^*)^{1/2}(e^{\dot{L}}e^{*\dot{M}} - e^{\dot{M}}e^{*\dot{L}})] = 0 \tag{4.5}$$

and

$$R_{\dot{L}\dot{M}}e^{\dot{M}} = 0 \tag{4.6}$$

We have supressed the tangent superspace indices for convenience. We shall now change equation (3.19) to the form

$$W^{\dot{K}}_{\dot{L}\dot{M}} = \eta_{AB}(D_{\dot{R}}e^{A}_{\dot{L}})e^{*B}_{\dot{M}}g^{\dot{R}\dot{K}} \tag{4.7}$$

where $W^{\dot{K}}_{L\dot{M}}$ is connected to $\Gamma^{\dot{K}}_{L\dot{M}}$ by a projective transformation

$$W^{\dot{K}}_{L\dot{M}} = \Gamma^{\dot{K}}_{L\dot{M}} + V_{\dot{L}}\delta^{\dot{K}}_{M} \tag{4.8}$$

Here V is an arbitrary vector field. We also require that Γ satisfies

$$\Gamma^{\dot{R}}_{[K\dot{R}]} = 0 \tag{4.9}$$

Then $g_{\dot{M}\dot{L}}$ obeys the compatibility equation

$$\partial_{\dot{K}}g_{\dot{M}\dot{L}} - \Gamma^{\dot{R}}_{K\dot{L}}g_{\dot{R}\dot{M}} - g_{\dot{L}\dot{R}}\Gamma^{*\dot{R}}_{KM} = 0 \tag{4.10}$$

because of (4.5), (4.7) and (4.8). By using (3.28), (4.7) and (4.8), we get

$$\eta_{AC}(R_{\dot{K}\dot{L}})^{A}_{B}e^{B}_{\dot{M}}e^{*C}_{\dot{R}}g^{\dot{R}\dot{N}} = R^{\dot{N}}_{\dot{K}\dot{L}\dot{M}} + (\partial_{\dot{K}}V_{\dot{L}} - \partial_{\dot{L}}V_{\dot{K}})\delta^{\dot{N}}_{\dot{M}} \tag{4.11}$$

where $R_{\dot{K}\dot{L}\dot{M}\dot{N}} = R^{\dot{A}}_{\dot{K}\dot{L}\dot{M}}g_{\dot{A}\dot{N}}$ is expressed in terms of (3.36). Eq. (4.6) is now equivalent to

$$R_{\dot{K}\dot{L}} = \partial_{\dot{K}}V_{\dot{L}} - \partial_{\dot{L}}V_{\dot{K}} \tag{4.12}$$

In the physical base manifold M the field equations are

$$B_{\dot{\mu}\dot{\nu}} + (\partial_{\dot{\nu}}V_{\dot{\mu}} - \partial_{\dot{\mu}}V_{\dot{\nu}}) = KT_{\dot{\mu}\dot{\nu}} \tag{4.13}$$

where K is a constant and

$$B_{\dot{\mu}\dot{\nu}} = \partial_{\dot{\beta}}\Gamma^{\dot{\beta}}_{\dot{\mu}\dot{\nu}} - \frac{1}{2}(\partial_{\dot{\nu}}\Gamma^{\dot{\beta}}_{(\dot{\mu}\dot{\beta})} + \partial_{\dot{\mu}}\Gamma^{\dot{\beta}}_{(\dot{\nu}\dot{\beta})}) - \Gamma^{\dot{\beta}}_{\dot{\alpha}\dot{\nu}}\Gamma^{\dot{\alpha}}_{\dot{\mu}\dot{\beta}} + \Gamma^{\dot{\beta}}_{\dot{\alpha}\dot{\beta}}\Gamma^{\dot{\alpha}}_{\dot{\mu}\dot{\nu}} \tag{4.14}$$

Moreover, $T_{\dot{\mu}\dot{\nu}}$ is a uniquely determined tensor source:

$$T_{\dot{\mu}\dot{\nu}} = -\partial_{\dot{k}}\Gamma^{\dot{k}}_{\dot{\mu}\dot{\nu}} + \frac{1}{2}(\partial_{\dot{\nu}}\Gamma^{\dot{k}}_{(\dot{\mu}\dot{k})} + \partial_{\dot{\mu}}\Gamma^{\dot{k}}_{(\dot{\nu}\dot{k})}) + \Gamma^{\dot{k}}_{\dot{m}\dot{\nu}}\Gamma^{\dot{m}}_{\dot{\mu}\dot{k}} - \Gamma^{\dot{k}}_{\dot{m}\dot{k}}\Gamma^{\dot{m}}_{\dot{\mu}\dot{\nu}} \tag{4.15}$$

In addition we also have the field equation that is a consequence of (4.9) and (4.10):

$$\partial_{\dot{B}}(\sqrt{-g}g^{[\dot{A}\dot{B}]}) = 0 \tag{4.16}$$

In the M manifold sector we get

$$\partial_{\dot{\nu}}(\sqrt{-g}g^{[\dot{\mu}\dot{\nu}]}) = \sqrt{-g}S^{\dot{\mu}} \tag{4.17}$$

where $\sqrt{-g}S^{\dot{\mu}}$ is a conserved four vector current density determined by

$$\sqrt{-g}S^{\dot{\mu}} = -\partial_{\dot{k}}(\sqrt{-g}g^{[\dot{\mu}\dot{k}]}) \tag{4.18}$$

Eqs. (4.13) and (4.17) comprise an extended version of the field equations of the nonsymmetric Hermitian theory of gravity [1]-[3] with the sources fixed by the new gauge fields. There are also additional gauge field equations in the $\dot{\mu}\dot{m}$ and $\dot{m}\dot{n}$ components of (4.12) that play the role of constraints. Further work must be done to analyse the particle spectrum and also determine the specific structure of the sources (4.15) and (4.17).

The present superspace formulation of the theory could form the basis for a grand unified gauge theory, because the gauge group of the fiber bundle could be chosen to be $U(7, 1) \supset SU(3)_c \otimes SU(2) \otimes U(1)$.

Thus the group is large enough to encompass the minimal particle structure for such a theory, including gravitation. Standard supergravity and supersymmetry theories [9] have as their largest gauge group SO(8) which does not contain $SU(3)_c \otimes SU(2) \otimes U(1)$. The present work can be extended to a supersymmetric framework by defining the N manifold in terms of four fermi coordinates [8]. We have formulated the theory in a $4 + n$-dimensional space-time, and it is necessary to compactify the spacelike n-dimensional domain [10].

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