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Canonical forms for separability structures with less than five Killing tensors

by

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ABSTRACT. — We review the general theory of separability structures in Riemannian manifolds of arbitrary dimension and signature. Canonical forms for the metric tensor and the Killing tensors associated to separability are computed for structures with at most four Killing tensors. Also the separated ordinary differential equations are listed for each case. This paper covers completely the general framework for dealing with separability structures in General Relativity.

1. INTRODUCTION

In previous papers [*1, 2, 3, 4*] one of us introduced the concept of separability structure for investigating the integrability by separation of variables of the Hamilton-Jacobi equation for the geodesics of a Riemannian manifold (V_n, g) (1):

$$\frac{1}{2} g^{ij} \partial_i S \partial_j S = e, \tag{1.1}$$

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(1) In this paper by *Riemannian* manifold (V_n, g) we mean a (connected paracompact Hausdorff) C^∞ manifold endowed with a non-degenerate metric tensor of whatever signature. The term *proper-Riemannian* will be used for signature $(0, n)$.

where $S : V_n \to \mathbb{R}$ is a real valued unknown function on V_n and e is a real parameter. A *complete integral* of (1.1) is a n-parameter family S_c,

$$c = (c_1, \ldots, c_n),$$

of solutions of (1.1) which depend on c's in an essential way according to the Hamilton-Jacobi theory. A *separable coordinate system* (for geodesics) at a point $x \in V_n$ is a coordinate system (x^i) $(i = 1, \ldots, n)$ such that a complete integral S_c exists which in the given coordinates assumes the separated form:

$$(1.2) \qquad S_c = \sum_{i=1}^{n} S_i(x^i; c),$$

with $\partial_j S_i = 0$ for $i \neq j$. A *separability structure* (for geodesics) at a point $x \in V_n$ is the equivalence class of all separable coordinate systems (at x) such that the corresponding separated complete solutions (1.2) are the coordinate representations of the same complete integral.

According to the theory of the separability structures ([*4*]) all separability structures on a Riemannian manifold (V_n, g) (i. e. concerning the HJ-equation of the geodesics) can be preliminarly classified by means of two non-negative integer numbers (r, d), with $0 \leqslant r \leqslant n$ and $0 \leqslant d \leqslant n - r$, which are called the *class* and the *index* of the separability structure respectively. A separability structure of class r and index d will be briefly denoted by the symbol $\mathscr{S}_{r;d}$ (2). When a coordinate system (x^i) is given, a single coordinate x^k is called a *first class coordinate* if:

$$(1.3) \qquad \partial_k g_{ij} = 0 \qquad \text{for} \quad i, j \neq k.$$

The class r of a $\mathscr{S}_{r;d}$ structure is the invariant number of first class coordinates existing in each separable coordinate system of the given separability structure (3). We recall that a single coordinate x^k is called an *ignorable coordinate* if

$$(1.4) \qquad \partial_k g_{ij} = 0, \qquad \forall i, j.$$

Hence an ignorable coordinate is in particular a first class coordinate. It can be proved that in each $\mathscr{S}_{r;d}$ structure there exist separable coordinate systems with exactly r ignorable coordinates.

In order to simplify the notations we agree to re-label any separable coordinate system (x^i) $(i = 1, \ldots, n)$ so that the first class coordinates are the last coordinates (x^α), labeled by Greek indices $\alpha, \beta, \ldots$, ranging from $n - r + 1$ to n. The remaining coordinates (up to the $(n - r)$-th) will be called *second class coordinates*. We agree to denote them by (x^a), by using Latin indices from the first part of the alphabet, $a, b, \ldots$, ranging from 1 to $n - r$.

The index d of a $\mathscr{S}_{r;d}$ structure is the invariant number of second class

(2) Further sub-classifications may be considered. A first example has been given in [*5*].
(3) The fact that r is an invariant is not trivial. For a proof see [*3*] and [*4*].

coordinates (x^a) such that $g^{aa} = 0$. Hence, in principle, d may range from 0 to $m = n - r$. However, non-degeneracy and signature of the metric imply in fact the following limitation:

$$0 \leqslant d \leqslant \min(m, p, q) \tag{1.5}$$

where (p, q) is the signature of g. We agree to re-label the second class coordinate (x^a) $(a = 1, \ldots, n - r)$ so that the last d ones are those satisfying $g^{aa} = 0$. These coordinates will be labeled by barred indices $\bar{a}, \bar{b}, \ldots$ ranging from $n - r - d + 1$ to $n - r$. The remaining coordinates will be labeled by twiddled indices $\tilde{a}, \tilde{b}, \ldots$ ranging from 1 to $n - r - d$.

It has been proved (cf. [*4*]) that in a given $\mathscr{S}_{r;d}$ structure there always exists at least one separable coordinate system (x^i) such that all the first class coordinate are ignorable and the metric tensor components take the following form:

$$\begin{cases} g^{\bar{a}i} = 0 \qquad (i \neq \tilde{a}), \qquad g^{\bar{a}\bar{b}} = 0, \\ g^{\tilde{a}\tilde{a}} = \underset{m}{u}^{\tilde{a}}, \quad g^{\tilde{a}\alpha} = \theta^{\alpha}_{\tilde{a}} \underset{m}{u}^{\tilde{a}} \quad (\tilde{a} \text{ n. s.}) \quad (^4) \\ g^{\alpha\beta} = \zeta^{\alpha\beta}_{a} \underset{m}{u}^{a} \end{cases} \tag{1.6}$$

$(m = n - r;\ \tilde{a} = 1, \ldots, m - d;\ \bar{a}, \bar{b} = m - d + 1, \ldots, m;\ \alpha, \beta = m + 1, \ldots, n)$ where: $(\underset{m}{u}^a)$ is the m-th row of a non singular $m \times m$ matrix of functions $\| \underset{b}{u}^a \|$ such that each element $\overset{b}{u}_a$ of the inverse matrix $\| \overset{b}{u}_a \|$ $(\overset{b}{u}_a \underset{c}{u}^a = \delta^b_c$ or equivalently $\overset{b}{u}_a \underset{b}{u}^c = \delta^c_a)$ is a function of the variable corresponding to the lower index only, i. e. $\partial_c \overset{b}{u}_a = 0$ for $c \neq a$; (θ^{α}_{a}) and $(\zeta^{\alpha\beta}_{a})$ are functions of the variable corresponding to the lower index only. Such coordinates have been called *normal separable coordinates* (for the given separability structure). In these coordinates the matrix $\| g^{ij} \|$ takes the following form (see, e. g., [*3, 4, 6*]):

$$\left\| \begin{array}{c|c|c} \begin{matrix} g^{aa} & 0 \\ 0 & \ddots \end{matrix} & 0 & 0 \\ \hline 0 & 0 & g^{a\beta} \\ \hline 0 & g^{\alpha a} & g^{\alpha\beta} \end{array} \right\| \begin{array}{l} \}\ m - d \\ \}\ d \\ \}\ r \end{array} \tag{1.7}$$

$$\underbrace{\quad}_{m-d} \quad \underbrace{\quad}_{d} \quad \underbrace{\quad}_{r}$$

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(4) Throughout this paper we adopt the summation convention for repeated indices, in the respective range, unless the symbol « n. s. » appear together with a distinguished index.

We emphasize that the meaning of class (and consequently of index) of a separability structure is more subtle than what might appear at a first sight. In fact let us assume that coordinates (x^i) are given in (V_n, g) so that they split naturally in three groups in such a way that, by suitable reordering, the metric tensor components take the form (1.6). It is certainly true that the HJ-equation of the geodesics is separable in the given coordinates, but one is not allowed to claim that the class and the index of the separability structure are respectively r and d. Actually, it might happen that one (or more) of the non-ignorable coordinates (x^a) are first class coordinates (say exactly s of them). Then the argument of [*4*], § 2 applies and shows that (1.6) can be further transformed, leaving unaltered the separability structure, to a similar form in which the number of ignorable coordinates is exactly $r + s$. This number is the true class of the separability structure. Thus, the condition to be imposed on (1.6) to assure that the coordinates (x^i) are normal separable coordinates of a $\mathcal{S}_{r;d}$ structure is that all the non-ignorable coordinates (x^a) $(a = 1, \ldots, m)$ are truly second class coordinates, which means that: *for each coordinate x^a there are indices $i, j \neq a$ such that*

$$(1.8) \qquad \partial_a g_{ij} \neq 0 .$$

In the sequel these conditions will be recalled as *class conditions*. They impose further restrictions to the functions $\underset{a}{u}^b$, $\zeta_a^{\alpha\beta}$ and θ_a^α appearing in (1.6). Unfortunately they cannot be easily written in terms of these functions. Nevertheless, since the reduction to normal separable coordinates makes the integration of the HJ-equation easier, in the applications one should always keep in mind the class conditions and apply them case by case, by direct computation over the covariant metric tensor components. If they are not satisfied, the transformations to normal separable coordinates can be obtained by applying the methods shown in [*4*].

It is convenient to introduce the following definition: a C^1 functional $m \times m$ matrix $\mathcal{U} = \| \overset{b}{u}_a \|$ is called a *Stackel matrix* (of order m) if it is regular, each *row* $(\overset{1}{u}_a, \overset{2}{u}_a, \ldots, \overset{m}{u}_a)$ is function of the corresponding variable x^a only and one of the rows of the inverse matrix $\mathcal{U}^{-1} = \| \underset{b}{u}^a \|$ is formed by nowhere vanishing functions; we assume that this privileged row is the last one (i. e. the m-th row). Therefore, a Stäckel matrix is characterized by the following conditions:

$$(1.9) \qquad \det \| \overset{b}{u}_a \| \neq 0 , \qquad \partial_c \overset{b}{u}_a = 0 \quad \text{if} \quad c \neq a , \qquad \underset{m}{u}^a \neq 0 .$$

The inverse $\mathcal{U}^{-1} = \| \underset{b}{u}^a \|$ of a Stäckel matrix is called an *inverse Stäckel matrix* (of order m) ([5]).

As we see from (1.6), Stäckel matrices enter the form of the metric compo-

([5]) In [*4*] the reverse terminology is adopted.

nents g^{ij} with respect to normal separable coordinates. We remark that, in general, the Stäckel matrix entering (1.6) for a given separable sistem of coordinates is not uniquely determined. Moreover we stress that condition $(1.9)_3$ is essential to assure the non-degeneracy of g.

In applications to Mathematical Physics, one deals often with metric coefficients g^{ij} which are known to be separable (this property can in fact be checked by applying the well known Levi-Civita's conditions [7]) and, more in particular, that the separable coordinates are normal (i. e. all the first class coordinates are ignorable and the matrix $\| g^{ij} \|$ has the form (1.7)). However, in general, one does not detect immediately the functions $(\overset{a}{u}{}^{b})$, $(\zeta_a^{\alpha\beta})$ and (θ_a^α) entering (1.6), while their knowledge is explicitly required for the reduction of the H-J equation into separated equations (cfr. (2.6) of [4]):

$$(1.10)\qquad \begin{cases} (\partial_{\bar a} S)^2 + \zeta_{\bar a}^{\alpha\beta} c_\alpha c_\beta = c_b \overset{b}{u}_{\bar a}, \\ 2 c_\alpha \theta_{\bar a}^\alpha \partial_{\bar a} S + \zeta_{\bar a}^{\alpha\beta} c_\alpha c_\beta = c_b \overset{b}{u}_{\bar a}. \end{cases}$$

where $c_m = e$.

Thus the problem is to find explicitly, first of all, a Stäckel matrix $\| \overset{b}{u}_a \|$ which allows to decompose g^{ij} according to (1.6). As already remarked in [4] and elsewhere, this (purely algebraic) problem can be simplified by knowing how to express, in a simple manner, the elements of a generic inverse Stäckel matrix (of any order m) in terms of functions depending on a single variable. We call such a representation a *canonical representation* of an inverse Stäckel matrix of order m. The corresponding form of the metric components has been referred as a *canonical form* for a $\mathscr{S}_{r;d}$ separable metric ([5, 6]).

In recent years the separability of the H-J equation for geodesics in space-time (V_4, g) has played a certain role in the field of General Relativity (see [5, 8, 9] and [11, 12] for detailed reviews). Therefore it is interesting to classify all possible separability structures which may exist in a four-dimensional Lorentzian manifold. This classification was undertaken, by relying on a different approach to separability, by Boyer *et al.* in [10], where separable metric forms have been derived. However, it seems interesting to investigate all the possible separable metric forms in the framework of the theory of separability structures, possibly without any *a priori* assumption on the dimension and signature of the manifold (V_n, g), but only relying on the classification given by the integer numbers $m = n - r$ and d, where r is the class and d the index of the separability structure (6), and on the standard form of the metric given by (1.6). In fact,

(6) We stress that the classification of coordinates proposed in [10] is based on another approach to separability of the HJ-equation, and it is only apparently similar to the classification based on the definition of class and index of a separability structure. A short comparison between the methods of [4] and [10] has been presented in [12].

the general properties of separability structures shown in [*4*] tell us, as we indicated above, that in considering separable metrics of the kind (1.6) we have no loss of generality.

On the other hand, for a fixed dimension n, it turns out that the canonical forms become increasingly more complicated the larger is m, due to the fact that the discussion of canonical representation of inverse Stäckel matrices becomes more involved for larger orders m. A short note presenting the general scheme will appear early [*13*]. Preliminary investigations, fully covering the cases of $\mathcal{S}_{n-2;0}$ and $\mathcal{S}_{n-3;0}$ structures (7) have been presented respectively in [*8*] and [*5*]. In this paper we are trying to mediate our interest for the most general cases with the aforementioned difficulties and the pourpose of applications to General Relativity (8). It is clear that the cases of separability structures of classes n and $n-1$ in a (V_n, g) are trivial. Therefore we shall present the canonical forms for the following three cases: $\mathcal{S}_{n-2;d}$ and $\mathcal{S}_{n-3;d}$ structures without any restriction to the index, and $\mathcal{S}_{n-4;0}$ structures. Our results are therefore given independently of the dimension n of the manifold. On the other hand, these cases cover fully the range of applications to General Relativity, since a separability structure of class $n-4$ for a four-dimensional manifold becomes of class 0, and a separability structure of class 0 has necessarily index 0.

We remark that Boyer *et al.* classified in [*14*] all separability structures of class 0 on a Lorentzian space-time satisfying Einstein vacuum field equations (with cosmological constant). The program of determining all solutions of Einstein vacuum field equations with an $\mathcal{S}_{r;d}$ structure is yet unfinished. We hope that our classification of canonical forms will help in looking for new such solutions, or at least in understanding geometric features of previously known ones. We have not investigated in detail other more general cases since we do not know relevant interest of them in problems arising from Mathematical Physics. In any case, we remark that the method presented here could be used (with some more efforts) to deal (if necessary) with cases of higher $m = n - r$.

Another reason which makes interesting the knowledge of a canonical form for separable metrics is the fact that to any $\mathcal{S}_{r;d}$ structure there is associated a linear $(n-r)$-dimensional space of commuting Killing tensors of order 2 (containing the metric itself), together with (of course) an r-dimensional space of commuting Killing vectors. In normal separable coordinates

(7) Separability structures with zero index have been previously (and improperly) called « regular ».

(8) We remark that for Lorentzian metrics (signature $(1, n-1)$) the index of a separability structure may assume only the values 0 or 1 (see (5.1)).

a basis of the space of K-tensors associated to the separability structure is given by (see [4]):

$$(1.11)\quad K_b = \underset{b}{u}^{\bar{a}}\partial_{\bar{a}} \otimes \partial_{\bar{a}} + \underset{b}{u}^{\bar{a}}\theta^{\alpha}_{\bar{a}}(\partial_{\bar{a}} \otimes \partial_{\alpha} + \partial_{\alpha} \otimes \partial_{\bar{a}}) + \underset{b}{u}^{a}\zeta^{\alpha\beta}_{a}\partial_{\alpha} \otimes \partial_{\beta} \qquad \left(\partial_i = \frac{\partial}{\partial x^i}\right)$$

if g (which coincides with K_m) is given in the form (1.6). As we can see, the determination of such a basis depend on the knowledge of the whole inverse Stäckel matrix whose m-th row appears in the metric tensor components (1.6). Hence, together with a canonical form of a separable metric, we shall give a canonical form of the remaining $n - r + 1$ non-trivial K-tensors, which will allow their computation, in the practical applications, through a simple algebraic process, starting from the knowledge of the metric.

2. TRIVIAL SEPARABILITY STRUCTURES

As we already pointed out in the Introduction, the cases $r = n$ and $r = n - 1$ are in a certain sense trivial cases and they do not require much discussion. For the sake of completeness we sketch below their behaviour.

If $r = n$, then (V_n, g) admits n commuting K-vectors (all the normal separable coordinates (x^i) are ignorable). Therefore (V_n, g) is flat (in the domain of the separability structure). The index of such a separability structure is of course zero.

A little more interesting is the case of separability structures of class $n - 1$. In this case (V_n, g) admits (at least in the domain of the separability structure) $n - 1$ commuting K-vectors and the space of K-tensors is trivially spanned by the metric itself. For the index we have two possibilities: *i)* $d = 0$, *ii)* $d = 1$.

i) If $d = 0$, expressions (1.6) reduce to:

$$g^{11} = u_1 \neq 0\,, \qquad g^{1\alpha} = 0\,, \qquad g^{\alpha\beta} = \zeta^{\alpha\beta}_1\,,$$

where: $\alpha, \beta = 2, \ldots, n$; u_1 and $\zeta^{\alpha\beta}_1$ are functions of the non-ignorable coordinate x^1 only.

The class conditions (1.8), which assure that the class of the separability structure represented in (2.1) is exactly $n - 1$, reduce in this case to the following conditions:

$$(2.2)\qquad \exists \alpha, \beta : \partial_1 g_{\alpha\beta} \neq 0\,,$$

which are equivalent to:

$$\exists \alpha, \beta : \partial_1 \zeta^{\alpha\beta}_1 \neq 0\,.$$

If on the contrary we have a metric in the form (2.1) with the matrix $\| \zeta_1^{\alpha\beta} \|$ constant, it is clear that the change of coordinates

$$x^{1'} = \int \frac{1}{\sqrt{u_1}} dx^1 , \qquad x^{\alpha'} = x^\alpha ,$$

transforms (2.1) to the new components

$$g^{1'1'} = 1 , \qquad g^{1'\alpha'} = 0 , \qquad g^{\alpha'\beta'} = \zeta_1^{\alpha\beta} .$$

The coordinates $(x^{i'})$ belong to the same separability structure determined by (x^i) (we can say briefly that the two coordinate systems are $\mathscr{S}$-compatible [*4*]); moreover, they are all ignorable. This shows that, also if the metric tensor components (2.1) (with $\zeta_1^{\alpha\beta}$ constant) have the standard form (1.6) with $r = n - 1$ and $d = 0$, the coordinates (x^i) are not normal separable coordinates of a $\mathscr{S}_{n-1;0}$ structure but define a $\mathscr{S}_{n;0}$ structure, whose normal separable coordinates are just the $(x^{i'})$. Of course, this example is rather trivial. However, it is instructive, because it is one of the simplest examples in which one can see that the mere knowledge of the form (1.6) is not enough to derive the class of the separability structure.

ii) We now turn to the case of a $\mathscr{S}_{n-1;1}$ structure. In this case (1.6) give the metric components:

$$g^{11} = 0 , \qquad g^{1\alpha} = \theta_1^\alpha , \qquad g^{\alpha\beta} = \zeta_1^{\alpha\beta} , \tag{2.3}$$

where: $\alpha, \beta = 2, \ldots, n$; θ_1^α and $\zeta_1^{\alpha\beta}$ are functions of x^1 only. Of course for a metric like (2.3) the class is exactly $n - 1$ if and only if conditions (1.8) are satisfied, i. e. there exist indices α, β such that $\partial_1 g_{\alpha\beta} \neq 0$. These conditions have not a simple expression in terms of functions θ_1^α and $\zeta_1^{\alpha\beta}$.

3. SEPARABILITY STRUCTURES OF CLASS $n - 2$

In order to compute canonical forms for all separability structures of class $n - 2$ in a (V_n, g), we first need canonical representation for an inverse Stäckel matrix of order 2. This has been already given in [*8*], in implicit form. The result of a very simple analysis is the following: there are functions Ψ_a, φ_a, of x^a only ($a = 1, 2$), with $\Psi_a \neq 0$ and $\varphi_1 + \varphi_2 \neq 0$, such that $\mathscr{U}^{-1}$ takes the form:

$$\| \underset{a}{u}^b \| = \frac{1}{\varphi_1 + \varphi_2} \begin{Vmatrix} \Psi_1 \varphi_2 & \overline{\Psi}_1 \varphi_2 \\ \Psi_1 & \Psi_2 \end{Vmatrix} , \tag{3.1}$$

where the index a (resp. b) is the index of row (resp. of column).

Provided the limitation (1.5) holds, on the index of a separability structure of class $m = n - 2$, we have in general three cases: *i)* $d = 0$, *ii)* $d = 1$, *iii)* $d = 2$.

i) $d = 0$. $\mathcal{S}_{n-2;0}$ structures have been already treated in [8]. We recall here the results, in a more compact form. The metric components in normal separable coordinates take the following form:

$$(3.2) \qquad \mathcal{S}_{n-2;0} \qquad \boxed{\begin{array}{ll} g^{11} = \dfrac{\Psi_1}{\varphi_1 + \varphi_2}, & g^{22} = \dfrac{\Psi_2}{\varphi_1 + \varphi_2}, \\ g^{1\alpha} = g^{2\alpha} = 0 & (\alpha, \beta = 3, \dots, n) \\ g^{\alpha\beta} = \dfrac{1}{\varphi_1 + \varphi_2}(\Psi_1^{\alpha\beta} + \Psi_2^{\alpha\beta}), & \end{array}}$$

where $\Psi_1^{\alpha\beta}$ (resp. $\Psi_2^{\alpha\beta}$) are functions of x^1 (resp. x^2) only (9). From (1.11), (3.1) and (3.2) we obtain immediately the components of the associated non-trivial K-tensor:

$$(3.3) \qquad \mathcal{S}_{n-2;0} \qquad \boxed{\begin{array}{ll} K^{11} = \dfrac{\Psi_1\varphi_2}{\varphi_1 + \varphi_2}, & K^{22} = \dfrac{-\Psi_2\varphi_1}{\varphi_1 + \varphi_2}, \\ K^{1\alpha} = K^{2\alpha} = 0 & (\alpha, \beta = 3, \dots, n) \\ K^{\alpha\beta} = \dfrac{1}{\varphi_1 + \varphi_2}(\varphi_2\Psi_1^{\alpha\beta} - \varphi_1\Psi_2^{\alpha\beta}). & \end{array}}$$

It is easy to recognize (from (1.10)) that the separated equations arising from the HJ-equation are the following:

$$(3.4) \quad \mathcal{S}_{n-2;0} \qquad \boxed{\Psi_a(\partial_a S)^2 + c_\alpha c_\beta \Psi_a^{\alpha\beta} - c_2\varphi_a + (-1)^a c_1 = 0 \quad (a = 1, 2)}$$

where $c_2 = e$, together with the trivial ones: $\partial_\alpha S = c_\alpha$ $(\alpha = 3, \dots, n)$.

ii) $d = 1$. In this case (1.6) and (3.1) imply:

$$(3.5) \quad \mathcal{S}_{n-2;1} \qquad \boxed{\begin{array}{lll} g^{11} = \dfrac{\Psi_1}{\varphi_1 + \varphi_2}, & g^{1i} = 0 & (i \neq 1), \\ g^{22} = g^{21} = 0, & g^{2\alpha} = \dfrac{\Psi_2^\alpha}{\varphi_1 + \varphi_2}, & \\ g^{\alpha\beta} = \dfrac{1}{\varphi_1 + \varphi_2}(\Psi_1^{\alpha\beta} + \Psi_2^{\alpha\beta}) & & (\alpha, \beta = 3, \dots, n), \end{array}}$$

(9) With respect to (1.6) (compare also with the expressions given in [8]) we set $\Psi_a^{\alpha\beta} = \Psi_a \zeta_a^{\alpha\beta}$.

where Ψ_2^α, $\Psi_2^{\alpha\beta}$ (resp. $\Psi_1^{\alpha\beta}$) are functions of x^2 (resp. x^1) only ([10]). Then the K-tensor assumes the canonical form:

$$(3.6)\quad \mathscr{S}_{n-2;1}\qquad \boxed{\begin{array}{l} \mathrm{K}^{11} = \dfrac{\Psi_2\varphi_1}{\varphi_1+\varphi_2}, \qquad \mathrm{K}^{1i} = 0 \quad (i \neq 1), \\ \mathrm{K}^{22} = \mathrm{K}^{21} = 0, \qquad \mathrm{K} = \dfrac{-\Psi_2^\alpha\varphi_1}{\varphi_1+\varphi_2}, \\ \mathrm{K}^{\alpha\beta} = \dfrac{1}{\varphi_1+\varphi_2}(\varphi_2\Psi_1^{\alpha\beta} - \varphi_1\Psi_2^{\alpha\beta}) \quad (\alpha, \beta = 3, \ldots, n). \end{array}}$$

The two non-trivial separated equations are now of different kind:

$$(3.7)\quad \mathscr{S}_{n-2;1}\qquad \boxed{\begin{array}{l} \Psi_1(\partial_1\mathrm{S})^2 + c_\alpha c_\beta\Psi_1^{\alpha\beta} - c_2\varphi_1 - c_1 = 0, \\ 2c_\alpha\Psi_2^\alpha\partial_2\mathrm{S} + c_\alpha c_\beta\Psi_2^{\alpha\beta} - c_2\varphi_1 + c_1 = 0. \end{array}}$$

where $c_2 = e$.

iii) $d = 2$. This case has no relevance to General Relativity, since in a Lorentzian manifold separability structures can have index 0 or 1 only. From (1.6) and (3.1) it follows:

$$(3.8)\quad \mathscr{S}_{n-2;2}\qquad \boxed{\begin{array}{l} g^{ab} = 0 \qquad (a, b = 1, 2), \\ g^{a\alpha} = \dfrac{\Psi_a^\alpha}{\varphi_1+\varphi_2} \qquad (\alpha, \beta = 3, \ldots, n) \\ g^{\alpha\beta} = \dfrac{1}{\varphi_1+\varphi_2}(\Psi_1^{\alpha\beta} + \Psi_2^{\alpha\beta}). \end{array}}$$

Moreover, by (1.11) we find for the non-trivial K-tensor components:

$$(3.9)\quad \mathscr{S}_{n-2;2}\qquad \boxed{\begin{array}{l} \mathrm{K}^{ab} = 0 \qquad (a, b = 1, 2), \\ \mathrm{K}^{a\alpha} = (-1)^{a+1}\dfrac{\varphi_{a+1}\Psi_a^\alpha}{\varphi_1+\varphi_2} \qquad (\alpha, \beta = 3, \ldots, n), \\ \mathrm{K}^{\alpha\beta} = \dfrac{1}{\varphi_1+\varphi_2}(\varphi_2\Psi_1^{\alpha\beta} - \varphi_1\Psi_2^{\alpha\beta}) \end{array}}$$

where Latin indices a, b are taken modulo 2. The separated equations turn out to be:

$$(3.10)\quad \mathscr{S}_{n-2;2}\qquad \boxed{2c_\alpha\Psi_a^\alpha\partial_a\mathrm{S} + c_\alpha c_\beta\Psi_a^{\alpha\beta} - c_2\varphi_a + (-1)^a c_1 = 0 \quad (a = 1, 2)}$$

where $c_2 = e$.

([10]) With respect to (1.6) we set $\Psi_a^\alpha = \Psi_a\theta_a^\alpha$.

As clearly follows from the general formulae (1.6) and (1.11), in the canonical forms given above, for both g and K, the components corresponding to the ignorable coordinates $(x^\alpha, x^\beta, \ldots)$ have always the same structure for all values of the index d.

The freedom of rescaling second class coordinates implied by the theory of the separability structures (see [4], Prop. (4.24)), allows one to rescale some of the coordinates (x^a) in such a way that some of the functions Ψ_a, φ_a, Ψ_a^α and $\Psi_a^{\alpha\beta}$ become constants equal to ± 1 (according to signature). For example, in the case of $\mathcal{S}_{n-2;0}$ structure one could always choose $\mathcal{S}$-compatible coordinates in which g has the form (3.2) with $\Psi_1 = \pm 1$ and $\Psi_2 = \pm 1$ (see [8]). However, we remark that it is better to give the canonical forms without imposing such normalizations, for the following reasons. First, because in applications one is generally faced with non-normalized metrics (in the sense we have just explained). Second, because the freedom in rescaling second class coordinates may in general be used more conveniently at the level of explicit integration of the separated equations. In fact, it might happen that normalizations which make for example $|\Psi_1| = |\Psi_2| = 1$ make more involved the integration of the separated equations, while other choices could be suggested by the equations themselves. This remark has to be taken into account in the sequel.

Now, let us compare the canonical forms for separable metrics of class $n-2$ given above with the metric components given in [10] for the cases with 2 ignorable coordinates. Case C in [10] is analogous to the case of $\mathcal{S}_{n-2;0}$ structures (we remark that the choice $a = b = c = d = 0$ in [10] is nothing but the transformation to normal separable coordinates ([11])). A glance to the formulae for cases D and E in [10] tells us immediately that they are contained as a particular case in our forms (3.5) and (3.8) respectively (provided some normalization is performed on coordinates x^1 and x^2). Equations (3.5) and (3.8) give all the separable metrics (in normal separable coordinates) of class $n-2$ and index different from 0. It seems that they contain cases which have not been considered in [10].

4. SEPARABILITY STRUCTURES OF CLASS $n-3$

First, let us consider a Stäckel matrix of order 3:

$$\mathcal{U} = \| \overset{b}{u}_a \| = \left\| \begin{matrix} \overset{1}{u}_1 & \overset{2}{u}_1 & \overset{3}{u}_1 \\ \overset{1}{u}_2 & \overset{2}{u}_2 & \overset{3}{u}_2 \\ \overset{1}{u}_3 & \overset{2}{u}_3 & \overset{3}{u}_3 \end{matrix} \right\| \tag{4.1}$$

([11]) The reader may look at the metric components g^{ij} given in [4] for the case of general separable coordinates and compare them with (1.6) and (3.2) above, together with equations (2.3) of [10].

together with its inverse

$$(4.2) \qquad \mathcal{U}^{-1} = \| \underset{b}{u}^a \| = \begin{Vmatrix} \underset{1}{u}^1 & \underset{1}{u}^2 & \underset{1}{u}^3 \\ \underset{2}{u}^1 & \underset{2}{u}^2 & \underset{2}{u}^3 \\ \underset{3}{u}^1 & \underset{3}{u}^2 & \underset{3}{u}^3 \end{Vmatrix}$$

and let us discuss a canonical representation of $\mathcal{U}^{-1}$. The following result has been proved in [*5*], § 2 ([12]):

LEMMA 1. — *For a* (3×3) *Stäckel matrix* $\mathcal{U}$ *defined on an open subset* $\mathrm{U} \subseteq \mathbb{R}^3$ *one of the following alternatives holds: i) there exists an open subset* $\mathrm{U}' \subseteq \mathrm{U}$ *in which at least one of the first two columns of* $\mathcal{U}$ *contains only nowhere vanishing functions; ii) there exists an open subset* $\mathrm{U}'' \subseteq \mathrm{U}$ *in which both the first and second columns of* $\mathcal{U}$ *contain one and only one possibly vanishing function, not belonging to the same row.*

It is now useful introduce the following concept:

DEFINITION. — *Two Stäckel matrices of order m,* $\mathcal{U}$ *and* $\mathcal{U}'$*, are said to be* $\mathcal{S}$*-equivalent if they have the same m-th row apart from permutations of columns.*

It is clear that the following operations on a Stäckel matrix yield an $\mathcal{S}$-equivalent one: (*a*) arbitrary permutations of rows of $\mathcal{U}$; (*b*) replacement of one column among the first $m - 1$ columns by a linear combination of them (see [*13*]). We remark that $\mathcal{S}$-equivalence is in fact an equivalence relation in the set of all Stäckel matrices of order m (for any $m > 1$). The relations with the separability structure theory are the following. As we remarked in Section 1, the Stäckel matrix $\mathcal{U}$ entering a given separability structure through the general form (1.6) is not uniquely determined. However, equations (1.6) tell us that the m-th row $(\underset{m}{u}^a)$ of $\mathcal{U}^{-1}$ is fixed by the choice of the normal separable coordinates in the separability structure (only rescaling of coordinates are allowed). Therefore, any Stäckel matrix $\mathcal{U}'$ which is $\mathcal{S}$-equivalent to $\mathcal{U}$ spans the same m-dimensional space of K-tensors, characterizing the separability structure. In fact, it turns out that such a choice gives the same metric components g^{ij} and a new basis for the linear subspace of the remaining $m - 1$ K-tensors (cfr. equations (1.11)). We also remark that operation (*a*) above destroys, in general, the preferred order of the second class coordinates (x^a). Therefore its use can produce a modification of the matrix form (1.7). On the contrary, operation (*b*) does not affect the convention we have choosen for labeling the second class coordinates.

(12) This lemma has been proved in [*5*] as a part of $\mathcal{S}_{n-3}$ structures theory (see Prop. 1, [*5*]), but it turns out to be simply a property of matrices satisfying conditions (1.9) and $(1.9)_3$, hence, in particular, a property of Stäckel matrices. Similar remark holds also for lemmas below.

The following holds:

LEMMA 2. — *Let $\mathcal{U}$ be a Stäckel matrix of order 3. There exists always an $\mathcal{S}$-equivalent Stäckel matrix $\mathcal{U}'$ in which the first column contains only functions which are nowhere vanishing.*

This Lemma has been implicitly proved in [*5*], § 2, by rather long direct calculations in a different perspective. However, there is a straightforward proof by using the operation (*b*) above (see the general discussion given in [*13*]).

In order to give canonical forms for an $\mathcal{S}_{n-3;d}$ structure, according to Lemma 2, it is sufficient to assume that the Stäckel matrix entering (1.6) has the following form:

$$\mathcal{U} = \begin{Vmatrix} \overset{1}{u}_1 \neq 0 & \overset{2}{u}_1 & \overset{3}{u}_1 \\ \overset{1}{u}_2 \neq 0 & \overset{2}{u}_2 & \overset{3}{u}_2 \\ \overset{1}{u}_3 \neq 0 & \overset{2}{u}_3 & \overset{3}{u}_3 \end{Vmatrix} \tag{4.3}$$

As already described in [*5*], we take:

$$\Psi_a = -\frac{1}{\overset{1}{u}_a}, \qquad \mu_a = \frac{\overset{2}{u}_a}{\overset{1}{u}_a}, \qquad \nu_a = \frac{\overset{3}{u}_a}{\overset{1}{u}_a}. \tag{4.4}$$

($a = 1, 2, 3$). With these positions the elements of the inverse of matrix (4.3) assume the form:

$$\begin{cases} \underset{1}{u}^a = -\dfrac{\Psi_a}{\varphi}(\mu_{a+1}\nu_{a+2} - \mu_{a+2}\nu_{a+1}), \\ \underset{2}{u}^a = -\dfrac{\Psi_a}{\varphi}(\nu_{a+1} - \nu_{a+2}), \\ \underset{3}{u}^a = \dfrac{\Psi_a}{\varphi}(\mu_{a+1} - \mu_{a+2}), \end{cases} \tag{4.5}$$

where: indices are taken modulo 3, φ is the (non vanishing) determinant:

$$\varphi = \det \begin{Vmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Vmatrix} \tag{4.6}$$

and Ψ_a, μ_a and ν_a are functions of the coordinate x^a only.

In a manifold of dimension $n > 3$ a separability structure of class $n - 3$ may have index 0, 1, 2 or 3, provided the limitation (1.5) holds. Hereafter we discuss these four cases in detail.

i) $d = 0$. This is a case already investigated in [*5*]. There we have shown that the metric tensor components (in normal separable coordinates) admit the following form (which can be derived directly from (1.6) and $(4.5)_1$):

$$(4.7)\quad \mathcal{S}_{n-3;0}\qquad \boxed{\begin{aligned} & g^{aa} = \frac{\Psi_a}{\varphi}(\mu_{a+1} - \mu_{a+2}) \qquad (a = 1, 2, 3), \\ & g^{ai} = 0 \quad (i \neq a), \\ & g^{\alpha\beta} = \frac{1}{\varphi}\sum_a \Psi_a^{\alpha\beta}(\mu_{a+1} - \mu_{a+2}) \qquad (\alpha, \beta = 4, \dots, n), \end{aligned}}$$

where $\Psi_a^{\alpha\beta}$ are functions of x^a only. By (1.11) and $(4.5)_{2,3}$, we find the following forms for the components of the two non-trivial K-tensors:

$$(4.8)\quad \mathcal{S}_{n-3;0}\qquad \boxed{\begin{aligned} & \mathrm{K}_1^{aa} = -\frac{\Psi_a}{\varphi}(\mu_{a+1}\nu_{a+2} - \mu_{a+2}\nu_{a+1}) \\ & \mathrm{K}^{ai} = 0 \quad (i \neq a), \\ & \mathrm{K}_1^{\alpha\beta} = -\frac{1}{\varphi}\sum_a \Psi_a^{\alpha\beta}(\mu_{a+1}\nu_{a+2} - \mu_{a+2}\nu_{a+1}); \\ & \mathrm{K}_2^{aa} = -\frac{\Psi_a}{\varphi}(\nu_{a+1} - \nu_{a+2}) \\ & \mathrm{K}_1^{ai} = 0 \quad (i \neq a), \\ & \mathrm{K}_2^{\alpha\beta} = -\frac{1}{\varphi}\sum_a \Psi_a^{\alpha\beta}(\nu_{a+1} - \nu_{a+2}). \end{aligned}}$$

In this case the non trivial separated equations are the following ones:

$$(4.9)\quad \mathcal{S}_{n-3;0}\qquad \boxed{\Psi_a(\partial_a \mathrm{S})^2 + c_\alpha c_\beta \Psi_a^{\alpha\beta} + c_2\mu_a + e\nu_a + c_1 = 0 \quad (a = 1, 2, 3).}$$

Before discussing the other three cases ($d \neq 0$), we first remark that (in analogy with the case $r = n - 2$) components (α, β) of the metric and K-tensors mantain the same canonical forms (as in $(4.7)_{\mathrm{III}}$, $(4.8)_{\mathrm{III}}$ and $(4.8)_{\mathrm{VI}}$). Therefore, in giving the subsequent canonical forms for $d \neq 0$, we shall write the other components only.

ii) $d = 1$. In this case the metric components in normal separable coordinates take the form:

$$(4.10)\quad \mathcal{S}_{n-3;1}\qquad \boxed{\begin{aligned} & g^{\tilde{a}\tilde{a}} = \frac{\Psi_{\tilde{a}}}{\varphi}(\mu_{\tilde{a}+1} - \mu_{\tilde{a}+2}) \qquad (\tilde{a} = 1, 2), \\ & g^{\tilde{a}i} = 0 \qquad (i \neq \tilde{a}), \qquad g^{33} = 0, \\ & g^{3\alpha} = \frac{\Psi_3^\alpha}{\varphi}(\mu_1 - \mu_2), \qquad g^{\alpha\beta} = \dots \end{aligned}}$$

where $\alpha, \beta = 4, \ldots, n$. For the corresponding K-tensors we have:

(4.11) $\mathscr{S}_{n-3;1}$

$$\begin{aligned}
&K_1^{\tilde{a}\tilde{a}} = -\frac{\Psi_{\tilde{a}}}{\varphi}(\mu_{\tilde{a}+1}\nu_{\tilde{a}+2} - \mu_{\tilde{a}+2}\nu_{\tilde{a}+1}) \qquad (\tilde{a} = 1, 2),\\
&K_1^{\tilde{a}i} = 0 \quad (i \neq \tilde{a}), \qquad K_1^{33} = 0,\\
&K^{3\alpha} = \frac{\Psi_3^\alpha}{\varphi}(\mu_1\nu_2 - \mu_2\nu_1), \qquad K_1^{\alpha\beta} = \ldots;\\
&K_2^{\tilde{a}\tilde{a}} = -\frac{\Psi_{\tilde{a}}}{\varphi}(\nu_{\tilde{a}+1} - \nu_{\tilde{a}+2}) \qquad (\tilde{a} = 1, 2),\\
&K_2^{\tilde{a}i} = 0 \quad (i \neq \tilde{a}), \qquad K_2^{33} = 0,\\
&K_2^{3\alpha} = \frac{\Psi_3^\alpha}{\varphi}(\nu_2 - \nu_1), \qquad K_2^{\alpha\beta} = \ldots
\end{aligned}$$

The non trivial separated equations are the following three ones:

(4.12) $\mathscr{S}_{n-3;1}$

$$\begin{aligned}
&\Psi_{\tilde{a}}(\partial_{\tilde{a}}S)^2 + c_\alpha c_\beta \Psi_{\tilde{a}}^{\alpha\beta} + c_2\mu_{\tilde{a}} + e\nu_{\tilde{a}} + c_1 = 0 \quad (\tilde{a} = 1, 2),\\
&2c_\alpha \Psi_3^\alpha \partial_3 S + c_\alpha c_\beta \Psi_3^{\alpha\beta} + c_2\mu_3 + e\nu_3 + c_1 = 0.
\end{aligned}$$

iii) $d = 2$. When the index is 2 we have the following canonical forms:

(4.13) $\mathscr{S}_{n-3;2}$

$$\begin{aligned}
&g^{11} = \frac{\Psi_1}{\varphi}(\mu_1 - \mu_3), \qquad g^{1i} = 0 \quad (i \neq 1),\\
&g^{\bar{a}\alpha} = \frac{\Psi_{\bar{a}}^\alpha}{\varphi}(\mu_{\bar{a}+1} - \mu_{\bar{a}+2}) \qquad (\bar{a} = 2, 3),\\
&g^{\bar{a}b} = 0 \quad (b = 1, 2, 3), \qquad g^{\alpha\beta} = \ldots
\end{aligned}$$

(4.14) $\mathscr{S}_{n-3;2}$

$$\begin{aligned}
&K_1^{11} = -\frac{\Psi_1}{\varphi}(\mu_2\nu_3 - \mu_3\nu_2), \qquad K_1^{1i} = 0 \quad (i \neq 1),\\
&K^{\bar{a}\alpha} = = \frac{\Psi_{\bar{a}}^\alpha}{\varphi}(\mu_{\bar{a}+1}\nu_{\bar{a}+2} - \mu_{\bar{a}+2}\nu_{\bar{a}+1}) \qquad (\bar{a} = 1, 2),\\
&K_1^{\bar{a}b} = 0 \quad (b = 1, 2, 3), \qquad K_1^{\alpha\beta} = \ldots;\\
&K_2^{11} = -\frac{\Psi_1}{\varphi}(\nu_2 - \nu_3), \qquad K_2^{1i} = 0 \quad (i \neq 0),\\
&K_2^{\bar{a}\alpha} = -\frac{\Psi_{\bar{a}}^\alpha}{\varphi}(\nu_{\bar{a}+1} - \nu_{\bar{a}+2}), \qquad (\bar{a} = 2, 3),\\
&K_2^{\bar{a}b} = 0 \quad (b = 1, 2, 3), \qquad K_2^{\alpha\beta} = \ldots;
\end{aligned}$$

where, as before, $\alpha, \beta = 4, \ldots, n$. The non trivial separated equations are:

(4.15) $\mathscr{S}_{n-3;2}$

$$\boxed{\begin{array}{l} \Psi_1(\partial_1 S)^2 + c_\alpha c_\beta \Psi_1^{\alpha\beta} + c_2\mu_1 + ev_1 + c_1 = 0\,, \\ 2c_\alpha \Psi_{\bar{a}}^{\alpha} \partial_{\bar{a}} S + c_\alpha c_\beta \Psi_{\bar{a}}^{\alpha\beta} + c_2\mu_{\bar{a}} + ev_{\bar{a}} + c_1 = 0 \quad (\bar{a} = 2, 3). \end{array}}$$

iv) $d = 3$. In this last case we have:

(4.16) $\mathscr{S}_{n-3;3}$

$$\boxed{\begin{array}{l} g^{ab} = 0 \qquad (a, b = 1, 2, 3)\,, \qquad g^{\alpha\beta} = \ldots\,, \\ g^{a\alpha} = \dfrac{\Psi_a^\alpha}{\varphi}(\mu_{a+1} - \mu_{a+2})\,; \end{array}}$$

(4.17) $\mathscr{S}_{n-3;3}$

$$\boxed{\begin{array}{l} K_1^{ab} = 0 \qquad (a, b = 1, 2, 3)\,, \qquad K_1^{\alpha\beta} = \ldots\,, \\ K_1^{a\alpha} = \dfrac{\Psi_a^\alpha}{\varphi}(\mu_{a+1}v_{a+2} - \mu_{a+2}v_{a+1})\,; \\ K_2^{ab} = 0 \qquad (a, b = 1, 2, 3)\,, \qquad K_2^{\alpha\beta} = \ldots\,, \\ K_2^{a\alpha} = \dfrac{\Psi_a^\alpha}{\varphi}(v_{a+1} - v_{a+2})\,; \end{array}}$$

while the non trivial separated equations are $(a = 1, 2, 3)$:

(4.18) $\mathscr{S}_{n-3;3}$

$$\boxed{2c_\alpha \Psi_a^\alpha \partial_a S + c_\alpha c_\beta \Psi_a^{\alpha\beta} + c_2\mu_a + ev_a + c_1 = 0}$$

5. SEPARABILITY STRUCTURES OF CLASS $n - 4$

As we already announced in the Introduction, in this section we shall give the canonical forms of the metric and of the remaining three K-tensors for separability structures of class $r = n - 4$ and index $d = 0$. These forms will be given, as usual, in terms of normal separable coordinates. They will be obtained from a canonical representation of the generic Stäckel matrix of order 4. From this representation, by following the scheme presented in the Introduction and described in the preceding Sections 3 and 4, the reader can easily compute the canonical forms for indices $d = 1, 2, 3, 4$ whenever required. We do not list these forms here, for obvious reasons of space. Furthermore, we stress again that for applications in General Relativity only the case $d = 0$ is interesting, since $d \neq 0$ is not allowed in a Lorentzian 4-dimensional space time with a separability structure of class 0. Similar remarks apply to cases with $r \geqslant n - 5$ (see [*13*]).

In the analysis of a Stäckel matrix of order 4 we shall follow a procedure perfectly analogous to that used in Section 4. A Stäckel matrix of order 4 is given in the following form:

$$(5.1) \qquad \mathcal{U} = \| \overset{b}{u}_a \|$$

(where b is the index of column and a the index of row), together with the inverse matrix:

$$(5.2) \qquad \mathcal{U}^{-1} = \| \underset{b}{u}^a \|$$

(where b is the index of row and a the index of column). Each $\overset{b}{u}_a$ is a function of x^a only. We denote by $\bar{\mathcal{U}}$ the (4×3) matrix obtained by deleting the last column of $\mathcal{U}$. We have the following lemma (see [*13*]):

Lemma 3. — *Let* $\mathcal{U}$ *be a* (4×4) *Stäckel matrix defined in an open set* $U \subseteq R^4$. *In each point* $x \in U$ *at most six elements of* $\bar{\mathcal{U}}$ *may vanish together. In each row and in each column of* $\bar{\mathcal{U}}$ *we cannot have more than two functions vanishing together.*

Proof. — As in [*5*], the proof may be given by relying on the relations which give the elements $\underset{a}{u}^b$ of $\mathcal{U}^{-1}$ in terms of the elements $\overset{b}{u}_a$ of $\mathcal{U}$, taking into account that $\underset{4}{u}^a \neq 0$ everywhere in U (see footnote ([12])). We have obviously:

$$(5.3) \quad \underset{4}{u}^1 = \frac{1}{\Delta}[\overset{1}{u}_2(\overset{2}{u}_3\overset{3}{u}_4 - \overset{2}{u}_4\overset{3}{u}_3) + \overset{1}{u}_3(\overset{2}{u}_4\overset{3}{u}_2 - \overset{2}{u}_2\overset{3}{u}_4) + \overset{1}{u}_4(\overset{2}{u}_2\overset{3}{u}_3 - \overset{2}{u}_3\overset{3}{u}_2)],$$

$$(5.4) \quad \underset{4}{u}^2 = \frac{-1}{\Delta}[\overset{1}{u}_1(\overset{2}{u}_3\overset{3}{u}_4 - \overset{2}{u}_4\overset{3}{u}_3) + \overset{1}{u}_3(\overset{2}{u}_4\overset{3}{u}_1 - \overset{2}{u}_1\overset{3}{u}_4) + \overset{1}{u}_4(\overset{2}{u}_1\overset{3}{u}_3 - \overset{2}{u}_3\overset{3}{u}_1)],$$

(and two similar relations for $\underset{4}{u}^3$ and $\underset{4}{u}^4$), where Δ denotes the determinant of $\mathcal{U}$. Let us suppose that two functions on the same row (e. g. $\overset{1}{u}_4$ and $\overset{2}{u}_4$) vanish together at a point $x \in U$. From (5.3) and $\underset{4}{u}^1 \neq 0$ it follows that $\overset{3}{u}_4$ cannot vanish at x. This proves that on each row we cannot find three zeroes. Analogously, the similar result for columns is proven by taking for example $\overset{1}{u}_3 = \overset{1}{u}_4 = \overset{2}{u}_4 = 0$ at x. In this case, (5.3) and (5.4) imply that $\overset{1}{u}_2(x) \neq 0$, $\overset{2}{u}_3(x) \neq 0$, $\overset{1}{u}_1(x) \neq 0$. Finally, we observe that Stäckel matrices of the form:

$$\left\| \begin{matrix} \overset{1}{u}_1 \neq 0 & 0 & 0 & \overset{4}{u}_1 \\ \overset{1}{u}_2 \neq 0 & \overset{2}{u}_2 \neq 0 & 0 & \overset{4}{u}_2 \\ 0 & \overset{2}{u}_3 \neq 0 & \overset{3}{u}_3 \neq 0 & \overset{4}{u}_3 \\ 0 & 0 & \overset{3}{u}_4 \neq 0 & \overset{4}{u}_4 \end{matrix} \right\|$$

are certainly allowed, where $\overset{1}{u}_1, \ldots$ are nowhere vanishing functions. (Q. E. D.)

The analogous of Lemma 2 also holds. In fact we have:

Lemma 4. — *Let* $\mathcal{U}$ *be a* (4×4) *Stäckel matrix defined in* $U \subseteq \mathbb{R}^4$. *There* *exists an $\mathcal{S}$-equivalent Stäckel matrix $\mathcal{U}'$ in an open $U' \subseteq U$ in which the first column contains only nowhere vanishing functions.*

For the proof it is sufficient to apply the general argument of [13], by suitably combining the first three columns of $\mathcal{U}$ (i. e. the columns of $\bar{\mathcal{U}}$). Therefore, it is not restrictive to suppose that the Stäckel matrix $\mathcal{U}$ entering (1.6) is such that the elements $\overset{1}{u}_a$ are nowhere vanishing. In analogy with (4.4) we can take now:

$$(5.5)\qquad \Psi_a = -\frac{1}{\overset{1}{u}_a}, \quad \lambda_a = \frac{\overset{2}{u}_a}{\overset{1}{u}_a}, \quad \mu_a = \frac{\overset{3}{u}_a}{\overset{1}{u}_a}, \quad \nu_a = \frac{\overset{4}{u}_a}{\overset{1}{u}_a} \qquad (a = 1, 2, 3, 4).$$

With this choice the determinant Δ of $\mathcal{U}$ takes the form:

$$(5.6)\qquad \Delta = \frac{\varphi}{\Psi_1\Psi_2\Psi_3\Psi_4}$$

where φ is the following non-vanishing determinant:

$$(5.7)\qquad \varphi = \det\begin{Vmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 \end{Vmatrix} = \varepsilon^{abcd} 1_a \lambda_b \mu_c \nu_d$$

where ε_{abcd} if the Levi-Civita symbol and $1_a = 1$ $(a = 1, 2, 3, 4)$. Then, we can represent the elements $\underset{b}{u}^a$ of the inverse Stäckel matrix $\mathcal{U}^{-1}$ as follows:

$$(5.8)\qquad \begin{cases} \underset{1}{u}^a = \dfrac{\Psi_a}{\varphi}\, \varepsilon^{abcd} \lambda_b \mu_c \nu_d\,, \\ \underset{2}{u}^a = \dfrac{\Psi_a}{\varphi}\, \varepsilon^{abcd} 1_b \mu_c \nu_d\,, \\ \underset{3}{u}^a = \dfrac{\Psi_a}{\varphi}\, \varepsilon^{abcd} 1_b \nu_c \lambda_d\,, \\ \underset{4}{u}^a = \dfrac{\Psi_a}{\varphi}\, \varepsilon^{abcd} 1_b \lambda_c \mu_d\,, \end{cases}$$

We conclude that the metric tensor components in normal separable coordinates of an $\mathcal{S}_{n-4;0}$ structure have the following canonical form:

$$(5.9)\qquad \mathcal{S}_{n-4;0} \qquad \boxed{\begin{aligned} & g^{aa} = \frac{\Psi_a}{\varphi}\, \varepsilon^{abcd} 1_b \lambda_c \mu_d\,, \\ & g^{ai} = 0 \qquad (i \neq a) \qquad (a = 1, 2, 3, 4)\,, \\ & g^{\alpha\beta} = \frac{1}{\varphi}\, \varepsilon^{abcd} 1_b \lambda_c \mu_d \underset{a}{\Psi}^{\alpha\beta} \qquad (\alpha, \beta = 5, \dots, n)\,, \end{aligned}}$$

where Ψ_a, λ_a, μ_a, ν_a, $\Psi_a^{\alpha\beta}$ depend on x^a only. The corresponding three K-tensors are represented as follows:

$$(5.10) \qquad \mathcal{S}_{n-4;0} \qquad \boxed{\begin{aligned} K_1^{aa} &= \frac{\Psi_a}{\varphi}\,\varepsilon^{abcd}\lambda_b\mu_c\nu_d\,, \\ K_1^{ai} &= 0 \qquad (i \neq a)\,, \\ K_1^{\alpha\beta} &= \frac{1}{\varphi}\,\varepsilon^{abcd}\Psi_a^{\alpha\beta}\lambda_b\mu_c\nu_d\,; \\ K_2^{aa} &= \frac{\Psi_a}{\varphi}\,\varepsilon^{abcd}1_b\mu_c\nu_d\,, \\ K_2^{ai} &= 0 \qquad (i \neq a)\,, \\ K_2^{\alpha\beta} &= \frac{1}{\varphi}\,\varepsilon^{abcd}1_b\mu_c\nu_d\Psi_a^{\alpha\beta}\,; \\ K_3^{aa} &= \frac{\Psi_a}{\varphi}\,\varepsilon^{abcd}1_b\nu_c\lambda_d\,, \\ K_3^{ai} &= 0 \qquad (i \neq a)\,, \\ K_3^{\alpha\beta} &= \frac{1}{\varphi}\,\varepsilon^{abcd}1_b\nu_c\lambda_d\Psi_a^{\alpha\beta}\,. \end{aligned}}$$

Finally, the four non-trivial separated equations are:

$$(5.11) \qquad \mathcal{S}_{n-4;0} \qquad \boxed{\Psi_a(\partial_a S)^2 + c_\alpha c_\beta \Psi_a^{\alpha\beta} + e\nu_a + c_3\mu_a + c_2\lambda_a + c_1 = 0\,.}$$

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