ANNALES DE L'I. H. P., SECTION A

J. GINIBRE
G. VELO

The classical field limit of non-relativistic bosons. II. Asymptotic expansions for general potentials

Annales de l'I. H. P., section A, tome 33, nº 4 (1980), p. 363-394
<http://www.numdam.org/item?id=AIHPA_1980__33_4_363_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXXIII, n° 4, 1980, p. 363-394. *Physique théorique.*

The classical field limit of non-relativistic bosons II. Asymptotic expansions for general potentials

by

J. GINIBRE

Laboratoire de Physique Théorique et Hautes Énergies (*),
Université de Paris-Sud, 91405 Orsay, France

and

G. VELO

Istituto di Fisica A. Righi, Università di Bologna,
and I. N. F. N., Sezione di Bologna, Italy

ABSTRACT. — We continue our study of the classical field limit as $\hbar \to 0$ of non relativistic many-boson systems with twobody interaction in the neighborhood of a fixed $\hbar$-independent solution of the classical evolution equation. The unitary group describing the evolution of the quantum system, after multiplication from both sides by suitable $\hbar$-dependent Weyl operators, has an expansion in power series of $\hbar^{1/2}$. In a previous paper, we proved that this series is strongly Borel summable for bounded interaction potentials and small time intervals. In this paper we prove that, for more singular potentials, this series, although probably not Borel summable, is still strongly asymptotic. We also briefly discuss the case of the S-matrix.

1. INTRODUCTION

This is the second paper where we study power series expansions in $\hbar^{1/2}$ for non relativistic many-boson systems with two-body interaction in

(*) Laboratoire associé au Centre National de la Recherche Scientifique.

a neighborhood of a fixed $\hbar$-independent solution of the corresponding classical field equation, namely the Hartree equation. We refer to the introduction of a previous paper on the same subject [*4*] for a detailed description of the problem; here we give only a brief summary.

We consider the unitary group $U(t - s)$ (see (2.6)), describing the evolution of the quantum system. After multiplication on both sides by suitable $\hbar$-dependent Weyl operators (see (2.7)), which somehow subtract the classical motion, one obtains a modified unitary group $W(t, s)$ (see (2.5)) which has an expansion in power series of $\hbar^{1/2}$, the zeroth order term of which is a unitary group describing the fluctuations of the quantum system around the classical motion. In [*4*] we proved that, for bounded two-body interaction potentials and small time intervals, this series is strongly Borel summable in $z = \hbar^{1/2}$ near $z = 0$. In the present paper we investigate the same series for more singular potentials. For the class of potentials considered here we obtain the same analyticity domain of $W(t, s)$ with respect to z as in [*4*]; however the bounds on the coefficients and on the remainder of the series increase faster with the order, thereby destroying Borel summability. Nevertheless we prove that the series is strongly asymptotic for small time intervals by obtaining explicit estimates on the general term and on the remainder, uniform in z for z small in the previous analyticity domain. For physical values of z (i. e. z real and positive) the same series is strongly asymptotic for arbitrary time intervals. Furthermore under additional assumptions including a strong form of asymptotic completeness of the quantum system and a suitable time decay of the classical solution, the expansion of the S-matrix is also strongly asymptotic.

This paper relies heavily on [*4*] and is not supposed to be readable by itself. In particular we shall use freely the notation and results of [*4*]. Proofs will be shortened or even omitted whenever similar to those of [*4*].

This paper is organized as follows. In Section 2 we derive formally the expansion of $W(t, s)$ in a form suitable for the subsequent estimates of Sections 4 and 5. Section 3 contains the operator theoretic results needed to justify this expansion. Section 4 contains the estimates for the general term of the expansion and Section 5 the estimates for the remainder of the series. Finally in Section 6 we briefly discuss the possible uniformity in time of these estimates and its implications for the asymptotic expansion of the S-matrix.

From a logical point of view, the formal expansion, the operator theoretic considerations and the estimates should be done simultaneously since each step of any of them relies on the previous steps of all three. They have been artificially separated to make the exposition simpler. The inter-connection between the three of them is briefly described at the end of Section 3.

2. FORMAL EXPANSIONS

In this section we recall the definitions of the various groups or semi-groups that arise in the problem and use them to put the general term and the remainder of the expansion of $W(t, s)$ as a power series of $z = \hbar^{1/2}$ in a form suitable to perform the estimates. We proceed at a formal level. The operator theoretic justification relies partly on the arguments given in Section 3 and partly on the estimates of Sections 4 and 5. It will be sketched at the end of Section 3.

We consider a non relativistic many-boson system in n-dimensional space. The total hamiltonian of the system is

$$H = \hbar H_0 + \hbar^2 H_4 \tag{2.1}$$

where

$$H_0 = \frac{1}{2}\int \nabla a^*(x).\nabla a(x)dx, \tag{2.2}$$

$$H_4 = \frac{1}{2}\int V(x-y)a^*(x)a^*(y)a(y)a(x)dxdy, \tag{2.3}$$

V is a real even function in $\mathbb{R}^n$ and $a(x)$ and $a^*(x)$ satisfy the usual representation of the CCR in the boson Fock space $\mathscr{H}$. The dependence of H on $\hbar$ has been chosen in such a way as to yield the classical field limit as $\hbar$ tends to zero (see [2] for a discussion of this point). The evolution equation for the limiting classical field φ is the Hartree equation

$$i\dot{\varphi} = -\frac{1}{2}\Delta\varphi + \varphi(V * |\varphi|^2). \tag{2.4}$$

The evolution operator $W(t, s)$ is defined by

$$W(t,s) = C(z, -\varphi(t))U(t-s)C(z,\varphi(s)) \times \exp\left\{-iz^{-2}\int_s^t d\tau H_4(\varphi(\tau))\right\}, \tag{2.5}$$

where

$$U(t-s) = \exp\{-i(t-s)(H_0 + z^2H_4)\}, \tag{2.6}$$

$$C(z,\varphi) = \exp\{z^{-1}(a^*(\varphi) - a(\bar{\varphi}))\}, \tag{2.7}$$

and $H_4(\varphi)$ is obtained by substituting φ for a and $\bar{\varphi}$ for a^* in (2.3). We shall consider $W(t, s)$ also for complex z.

The operator $W(t, s)$ satisfies formally the differential equation

$$i\frac{d}{dt}W(t,s) = (H_2(t) + zH_3(t) + z^2H_4)W(t,s), \tag{2.8}$$

where $H_2(t)$ and $H_3(t)$ are defined by (1.34)-(1.43) of [4]. This differential equation directly leads to an expansion in power series of z for $W(t, s)$, the zeroth order term of which is the unitary group $U_2(t, s)$ defined formally by

$$\begin{cases} i\dfrac{d}{dt}U_2(t, s) = H_2(t)U_2(t, s), \\ U_2(s, s) = \mathbb{1}. \end{cases} \tag{2.9}$$

It is convenient to perform the expansion in two steps using at an intermediate stage the family of operators $U_4(t, s)$ formally defined by

$$\begin{cases} i\dfrac{d}{dt}U_4(t, s) = (H_2(t) + z^2H_4)U_4(t, s), \\ U_4(s, s) = \mathbb{1}. \end{cases} \tag{2.10}$$

One of the steps consists in expanding $U_4(t, s)$ as a power series in z. In order to make the formulas more concise and transparent we use the short-hand notation

$$U_0^\# \prod_{1 \leqslant i \leqslant k} (H_{l_i}U_i^\#) \tag{2.11}$$

to denote the integral

$$\int_{\Delta_k} d\{\tau\}\, U_0^\#(t, \tau_1) \prod_{1 \leqslant i \leqslant k} (H_{l_i}U_i^\#(\tau_i, \tau_{i+1})) \tag{2.12}$$

where $U_i^\#$ are chosen among U_2, U_4, U and W and the integration domain of the variables $\{\tau\} = \{\tau_1, \ldots, \tau_k\}$ is the set

$$\Delta_k = \{\tau_1, \ldots, \tau_k : t \equiv \tau_0 \geqslant \tau_1 \geqslant \ldots \geqslant \tau_k \geqslant \tau_{k+1} \equiv s\}. \tag{2.13}$$

With this notation the power series expansion of $U_4(t, s)$ can be written as

$$U_4(t, s) = \sum_{k \geqslant 0} z^{2k}U_{4,k}(t, s), \tag{2.14}$$

where

$$U_{4,k}(t, s) = (-i)^k(U_2H_4)^kU_2, \tag{2.15}$$

and the remainder of the expansion, defined by

$$Q_{4,k}(t, s) = U_4(t, s) - \sum_{0 \leqslant j < k} z^{2j}U_{4,j}(t, s), \tag{2.16}$$

can be written as

$$Q_{4,k}(t, s) = (-iz^2)^k(U_2H_4)^kU_4. \tag{2.17}$$

We also expand $W(t, s)$ as a power series of z:

$$W(t, s) = \sum_{p \geqslant 0} z^pW_p(t, s), \tag{2.18}$$

and define the remainder of the series by

$$R_p(t, s) = W(t, s) - \sum_{0 \leqslant j < p} z^j W_j(t, s). \tag{2.19}$$

The following lemma, which is essentially a restatement of Lemma 6.1 of [4], gives an explicit expression for $W_p(t, s)$ and $R_p(t, s)$.

LEMMA 2.1. — $W_p(t, s)$ can be represented as

$$W_p(t, s) = \sum_{j+2k=p} (-i)^j \sum_{\{k\}} \left\{ \prod_{0 \leqslant i < j} (U_{4,k_i} H_3) \right\} U_{4,k_j}, \tag{2.20}$$

where the sum over $\{ k \} = \{ k_0, k_1, \ldots, k_j \}$ runs over the integers k_i with $k_i \geqslant 0$ for $0 \leqslant i \leqslant j$ and

$$\sum_{0 \leqslant i \leqslant j} k_i = k. \tag{2.21}$$

$R_p(t, s)$ can be represented as

$$R_p(t, s) = \sum_{0 \leqslant j < p} R_{p,j}(t, s) + R'_p(t, s), \tag{2.22}$$

where

$$R_{p,j}(t, s) = \sum_{0 \leqslant r \leqslant j} \sum_{\{k\}} \left\{ \prod_{0 \leqslant i < r} z^{2k_i} U_{4,k_i}(-izH_3) \right\} \times Q_{4,k_r}(-izH_3U_4)^{j-r}, \tag{2.23}$$

$$R'_p(t, s) = (-izU_4H_3)^p W, \tag{2.24}$$

the sum over $\{ k \} = \{ k_0, \ldots, k_r \}$ runs over the integers k_i with $k_i \geqslant 0$ for $0 \leqslant i \leqslant r - 1$, $k_r \geqslant 1$ and

$$\sum_{0 \leqslant i \leqslant r} k_i = k, \tag{2.25}$$

where k is defined by

$$2k = \begin{cases} p - j & \text{if} \quad p - j \text{ is even}, \\ p - j + 1 & \text{if} \quad p - j \text{ is odd}. \end{cases} \tag{2.26}$$

In the case of bounded potentials that was considered in [4], Lemma 2.1 was a convenient starting point to perform the necessary estimates, because the operators H_3 and H_4 were bounded by powers of N. This will no longer be the case for H_4 with the singular potentials that we want to consider here. We shall nevertheless keep H_3 bounded by a power of N by taking the classical solution φ sufficiently regular, so that H_3 can be treated as in [4]. However, an additional trick is needed to control the local singularities of H_4. For this purpose we first consider the simpler problem of the expansion of $U(t-s)$ (see (2.6)) as a power series of z^2 (which is now simply the coupling constant):

$$U(t-s) = \sum_{l \geqslant 0} z^{2l} U_{(l)}(t-s), \tag{2.27}$$

where

$$U_{(l)}(t-s) = (-i)^l (U_0 H_4)^l U_0 \tag{2.28}$$

and

$$U_0(t-s) = \exp(-i(t-s)H_0). \tag{2.29}$$

The remainder, defined by

$$Q_l(t-s) = U(t-s) - \sum_{0 \leqslant j < l} z^{2j} U_{(j)}(t-s), \tag{2.30}$$

can be written as

$$Q_l(t-s) = (-iz^2)^l (U_0 H_4)^l U. \tag{2.31}$$

As is well known [6], and will be recalled later (see Lemma 4.2 below), the local singularities of H_4 in (2.28), if not too wild, can be controlled by the smoothing effect of the free evolution $U_0(t)$. The same turns out to be true in the present case. In order to see it, it is convenient to work in a new representation defined by

$$\tilde{U}^{\#}(t, s) = U_0(t)^* U^{\#}(t, s) U_0(s), \tag{2.32}$$

where $U^{\#}$ can represent U_2, U_4, $U_{4,k}$, $Q_{4,k}$, U, $U_{(l)}$, Q_l, W, W_p, R_p, R'_p or $R_{p,j}$, and by

$$\tilde{H}_k(t) = U_0(t)^* H_k(t) U_0(t). \tag{2.33}$$

for $2 \leqslant k \leqslant 4$. With these definitions, the relations (2.14)-(2.20), (2.22)-(2.24), (2.27) and (2.30) still hold with all operators replaced by the corresponding ones with tilda. For brevity, the relation obtained from relation $(2.\alpha)$ through this replacement will be referred to as $(2.\tilde{\alpha})$. The relations (2.28) and (2.31) become respectively

$$\tilde{U}_{(l)}(t, s) = (-i)^l \tilde{H}_4{}^l \tag{2.34}$$

and

$$\tilde{Q}_l(t, s) = (-iz^2)^l \tilde{H}_4{}^l \tilde{U} \tag{2.35}$$

since obviously the definition (2.32) yields $\tilde{U}_0(t, s) = \mathbb{1}$.

The singularities of H_4 in $\tilde{U}_{(l)}(t, s)$ and $\tilde{Q}_l(t, s)$ will be smoothed out by the presence of U_0. In order to control them in $\tilde{U}_{4,k}(t, s)$ and $\tilde{Q}_{4,k}(t, s)$, it is therefore sufficient to express these quantities in terms of $\tilde{U}_{(l)}(t, s)$ and $\tilde{Q}_l(t, s)$. This will be done through a partial expansion, as described by the following lemma.

LEMMA 2.2. — For all $k \geqslant 1$, $\tilde{\mathrm{U}}_{4,k}(t, s)$ and $\tilde{\mathrm{Q}}_{4,k}(t, s)$ can be expressed as follows

$$\tilde{\mathrm{U}}_{4,k}(t, s) = \sum_{0 \leqslant r \leqslant k} \sum_{\{l\}} \int_{\Delta_r} d\{\tau\} \left\{ \prod_{0 \leqslant i < r} \tilde{\mathrm{Y}}_{l_i}(\tau_i, \tau_{i+1}) \right\} \times \tilde{\mathrm{U}}_2(\tau_r, s) \tilde{\mathrm{U}}_{(l_r)}(\tau_r, s), \quad (2.36)$$

where

$$\tilde{\mathrm{Y}}_l(\tau, \tau') = i\tilde{\mathrm{U}}_2(\tau, \tau')[\tilde{\mathrm{H}}_2(\tau') - \mathrm{H}_0, \tilde{\mathrm{U}}_{(l)}(\tau, \tau')], \quad (2.37)$$

the integration over $\{\tau\} = \{\tau_1, \ldots, \tau_r\}$ is performed in the domain Δ_r defined by (2.13), and the sum over $\{l\} = \{l_0, \ldots, l_r\}$ runs over the integers $l_i \geqslant 1$ $(0 \leqslant i < r)$, $l_r \geqslant 0$, such that

$$\sum_{0 \leqslant i \leqslant r} l_i = k. \quad (2.38)$$

$$\tilde{\mathrm{Q}}_{4,k}(t, s) = \sum_{0 \leqslant r \leqslant k} \sum_{\{l\}}{}' \left\{ \int_{\Delta_r} d\{\tau\} \left\{ \prod_{0 \leqslant i < r} z^{2l_i} \tilde{\mathrm{Y}}_{l_i}(\tau_i, \tau_{i+1}) \right\} \right.$$
$$\times \tilde{\mathrm{U}}_2(\tau_r, s) \tilde{\mathrm{Q}}_{l_r}(\tau_r, s) + \int_{\Delta_{r+1}} d\{\tau\} \left\{ \prod_{0 \leqslant i < r} z^{2l_i} \tilde{\mathrm{Y}}_{l_i}(\tau_i, \tau_{i+1}) \right\}$$
$$\left. \times \tilde{\mathrm{Z}}_{l_r}(\tau_r, \tau_{r+1}) \tilde{\mathrm{U}}_4(\tau_{r+1}, s) \right\}, \quad (2.39)$$

where $\tilde{\mathrm{Y}}_l(\tau, \tau')$ is defined by (2.37), the integration over $\{\tau\}$ has the same meaning as in (2.36), with however $r + 1$ variables in the second integral, the sum over $\{l\} = \{l_0, \ldots, l_r\}$ runs over the integers $l_i \geqslant 1$ $(0 \leqslant i \leqslant r)$ satisfying (2.38), and

$$\tilde{\mathrm{Z}}_l(\tau, \tau') = i\tilde{\mathrm{U}}_2(\tau, \tau')[\tilde{\mathrm{H}}_2(\tau') - \mathrm{H}_0, \tilde{\mathrm{Q}}_l(\tau, \tau')]. \quad (2.40)$$

Proof. — The idea of the proof, as well as of the lemma itself, is to push the factors $\tilde{\mathrm{U}}_2(\tau, \tau')$ that occur in $\tilde{\mathrm{U}}_{4,k}(t, s)$ and $\tilde{\mathrm{Q}}_{4,k}(t, s)$ to the left by commuting them with the factors $\tilde{\mathrm{H}}_4(\tau)$. This is done by repeated application of formulas similar to

$$\tilde{\mathrm{U}}_2(t, \tau)\tilde{\mathrm{H}}_4(\tau)\tilde{\mathrm{U}}_4(\tau, s) = \tilde{\mathrm{U}}_2(t, s)\tilde{\mathrm{H}}_4(\tau)\tilde{\mathrm{U}}(\tau, s)$$
$$+ i \int_s^\tau d\tau' \tilde{\mathrm{U}}_2(t, \tau')[\tilde{\mathrm{H}}_2(\tau') - \mathrm{H}_0, \tilde{\mathrm{H}}_4(\tau)\tilde{\mathrm{U}}(\tau, \tau')]\tilde{\mathrm{U}}_4(\tau', s), \quad (2.41)$$

which by integration over τ $(s \leqslant \tau \leqslant t)$ yields (2.39) for $k = 1$. We now begin the proof proper. We first prove (2.39). Let $\{\tau\} = \{\tau_1, \ldots, \tau_k\}$.

We define an operator valued function $J(\{\tau\},\tau)$ of $k+1$ time variables by

$$J(\{\tau\},\tau) = (-iz^2)^k \tilde{U}_2(t,\tau)\left\{\prod_{1\leqslant i\leqslant l} \tilde{H}_4(\tau_i)\right\}\tilde{U}_2(\tau,\tau_{l+1}) \times \left\{\prod_{l<j<k} \tilde{H}_4(\tau_j)\tilde{U}_2(\tau_j,\tau_{j+1})\right\}\tilde{H}_4(\tau_k)\tilde{U}_4(\tau_k,s) \quad (2.42)$$

for $\tau \in [\tau_{l+1},\tau_l]$, $1 \leqslant l \leqslant k$, and

$$J(\{\tau\},\tau) = (-iz^2)^k \tilde{U}_2(t,\tau)\left\{\prod_{1\leqslant i\leqslant k} \tilde{H}_4(\tau_i)\right\}\tilde{U}(\tau_k,\tau)\tilde{U}_4(\tau,s) \quad (2.43)$$

for $\tau \in [s,\tau_k]$.

Clearly, $J(\{\tau\},\tau)$ is continuous in τ (at least formally). We next define, for $1 \leqslant l \leqslant k$,

$$\tilde{Q}_k^l(t,s) = \int_{\Delta_k} d\{\tau\}\, J(\{\tau\},\tau_l). \quad (2.44)$$

It follows immediately from the definitions (2.42) and (2.43) that

$$\tilde{Q}_k^1(t,s) = \tilde{Q}_{4,k}(t,s). \quad (2.45)$$

On the other hand, it follows from (2.44) that for $1 \leqslant l < k$,

$$\tilde{Q}_k^l(t,s) - \tilde{Q}_k^{l+1}(t,s) = \int_{\Delta_k} d\{\tau\}\int_{\tau_{l+1}}^{\tau_l} d\tau \frac{d}{d\tau} J(\{\tau\},\tau) = z^{2l}\int_s^t d\tau \tilde{Y}_l(t,\tau)\tilde{Q}_{4,k-l}(\tau,s), \quad (2.46)$$

where $\tilde{Y}_l$ is defined by (2.37).

Similarly

$$\tilde{Q}_k^k(t,s) - \tilde{U}_2(t,s)\tilde{Q}_k(t,s) = \int_{\Delta_k} d\{\tau\}\int_s^{\tau_k} d\tau \frac{d}{d\tau} J(\{\tau\},\tau) = \int_s^t d\tau \tilde{Z}_k(t,\tau)\tilde{U}_4(\tau,s). \quad (2.47)$$

Adding together (2.46) for $1 \leqslant l < k$ and (2.47), we obtain the recursion relation

$$\tilde{Q}_{4,k}(t,s) = \tilde{U}_2(t,s)\tilde{Q}_k(t,s) + \int_s^t d\tau \tilde{Z}_k(t,\tau)\tilde{U}_4(\tau,s) + \sum_{1\leqslant l<k} z^{2l}\int_s^t d\tau \tilde{Y}_l(t,\tau)\tilde{Q}_{4,k-l}(\tau,s), \quad (2.48)$$

from which (2.39) follows by iteration.

The proof of (2.36) is similar to that of (2.39), but simpler. Alternatively, (2.36) can be obtained by replacing $\tilde{Z}_l$ by $\tilde{Y}_l$, $\tilde{Q}_l$ by $\tilde{U}_{(l)}$ and $\tilde{U}_4$ by $\tilde{U}_2$ (in 2.39). The second sum thereby obtained from (2.39) yields the terms with $l_r = 0$ in (2.36). Q. E. D.

3. DEFINITION AND PROPERTIES OF THE EVOLUTION SEMI GROUPS

In this section we give a precise definition of the various groups and semi groups formally defined in Section 2 and study their properties. In particular we prove the analyticity properties of $W(t, s)$ mentioned in the introduction, the integral relations on which the expansions of the previous section are based and various estimates that will be used later to bound the general term and the remainder of the series of $W(t, s)$.

For this purpose we introduce some notation. The basic space of the theory if the boson Fock space

$$\mathcal{H} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N \tag{3.1}$$

where $\mathcal{H}_N$ is the space of totally symmetric square integrable functions of N variables in $\mathbb{R}^n$. The scalar product in $\mathcal{H}$ is denoted by

$$\langle \Phi, \Psi \rangle = \int dX \bar{\Phi}(X)\Psi(X), \tag{3.2}$$

where $X = (x_1, \ldots, x_N)$ and

$$\int dX = \sum_{N=0}^{\infty} (N!)^{-1} \int dx_1, \ldots, dx_N. \tag{3.3}$$

We denote by $\| \, . \, \|$ the norm in $\mathcal{H}$, by $|||\, A \,|||$ the norm of a bounded operator A in $\mathcal{H}$, by $\mathcal{D}(A)$ the domain of an unbounded operator A, by $Q(A)(\equiv \mathcal{D}(A^{1/2}))$ the form domain of a positive operator, and by $Q(A)^*$ the dual of $Q(A)$ in the sense of the scalar product of $\mathcal{H}$. The creation and annihilation operators are defined, for any $\alpha \in L^2(\equiv L^2(\mathbb{R}^n))$, by

$$\begin{cases} (a(\bar{\alpha})\Psi)(X) = \displaystyle\int dx \bar{\alpha}(x)\Psi(X, x), \\ (a^*(\alpha))\Psi)(X) = \displaystyle\sum_{1 \leqslant i \leqslant N} \alpha(x_i)\Psi(X \setminus x_i). \end{cases} \tag{3.4}$$

In (3.2-4), we follow the convention of Friedrichs [*1*]. The particle number operator N is defined in the usual way.

We first consider the family of operators $U(t)$ formally defined by (2.6), (2.2) and (2.3). We take $t \geqslant 0$ and we assume the potential V to satisfy the following stability condition :

Stability of the potential: there exists a constant $B \geqslant 0$ such that

$$H_4 + BN \geqslant 0 . \tag{3.5}$$

For any $\eta \geqslant 0$, we denote by $E(\eta)$ the region of the complex plane:

$$E(\eta) = \{ z \in \mathbb{C} : 0 \geqslant \operatorname{Im} z^2 \geqslant -\eta \}, \tag{3.6}$$

and for any $\eta > 0$, we denote by $E_0(\eta)$ the interior of $E(\eta)$:

$$E_0(\eta) = \{ z \in \mathbb{C} : 0 > \operatorname{Im} z^2 > -\eta \}. \tag{3.7}$$

It will be also convenient to introduce a cut-off in the potential V as follows: for any $\kappa > 0$ we split V as

$$V = V_\kappa + V'_\kappa \tag{3.8}$$

with

$$V_\kappa(x) = \begin{cases} V(x) & \text{if} \quad |V(x)| \leqslant \kappa, \\ 0 & \text{if} \quad |V(x)| > \kappa. \end{cases} \tag{3.9}$$

If $V \in L^\infty + L^{p_2}$, $1 \leqslant p_2 < \infty$, then there exists $\bar{\kappa} > 0$ such that for all $\kappa \geqslant \bar{\kappa}$ and all p_0, $1 \leqslant p_0 \leqslant p_2$, $V'_\kappa \in L^{p_0}$ and

$$\| V'_\kappa \|_{p_0} \leqslant \kappa^{1 - p_2/p_0} \| V'_\kappa \|_{p_2}^{p_2/p_0}, \tag{3.10}$$

where $\| \, . \, \|_p$, $1 \leqslant p \leqslant \infty$, denotes the norm in $L^p \equiv L^p(\mathbb{R}^n)$. We define also $\bar{V}_\kappa = \operatorname{Min}(V, \kappa)$, $V''_\kappa = V - \bar{V}_\kappa$, and H_4^κ by

$$H_4^\kappa = \frac{1}{2} \int dx dy \bar{V}_\kappa(x - y) a^*(x) a^*(y) a(y) a(x). \tag{3.11}$$

We now state the relevant properties of $U(t)$ and of its approximants $U^\kappa(t)$ generated by $H_0 + z^2 H_4^\kappa$ (1).

PROPOSITION 3.1. — Let V be stable and $V \in L^\infty + L^{p_2}$ with $p_2 \geqslant n/2$, $p_2 \geqslant 1$ for $n = 1$, $p_2 > 1$ for $n = 2$. Then

(1) For any $\eta \geqslant 0$, for any $z \in E(\eta)$, the operator $(i(H_0 + z^2 H_4) + B\eta N)$ defined as a sum of quadratic forms, is m-accretive, commutes with N, and generates a contraction semi group $U(t) \exp(-B\eta N t)$ strongly continuous in t for $t \geqslant 0$ and in z for $z \in E(\eta)$. If $\eta > 0$, this semi group is analytic in z for $z \in E_0(\eta)$.

(2) Under the same assumptions as in part (1) the operator

$$(i(H_0 + z^2 H_4^\kappa) + B\eta N)$$

generates a contraction semi group $U^\kappa(t) \exp(-B\eta N t)$ which satisfies the same properties as $U(t) \exp(-B\eta N t)$. Furthermore $U^\kappa(t) \exp(-B\eta N t)$ converges strongly to $U(t) \exp(-B\eta N t)$ as $\kappa \to \infty$, uniformly for t in any compact subset of $\mathbb{R}^+$.

(1) See note added in proof.

Proof. — (1) Let $z \in E(\eta)$. For $\operatorname{Re} z \neq 0$, the form defined by

$$(i(H_0 + z^2 H_4) + B\eta N)$$

is strictly m-sectorial and defines an m-accretive and strictly m-sectorial operator ([7], p. 282). For $\operatorname{Re} z = 0$, we use the following perturbation argument: let v be the operator of multiplication by V in L^2. Then, under the assumptions made on V, for any $\varepsilon > 0$, there exists a $\bar{b}_\varepsilon \geqslant 0$ such that the following inequality holds in the sense of quadratic forms on $Q(-\Delta)$:

$$|v| \leqslant -\varepsilon\Delta + \bar{b}_\varepsilon . \tag{3.12}$$

From this it follows that in $\mathcal{H}_N$,

$$H_4 \leqslant \sum_{i<j} \left(\frac{\varepsilon}{4}(k_i - k_j)^2 + \bar{b}_\varepsilon \right),$$

where k_i is the momentum of the i-th particle,

$$\begin{aligned} &\leqslant \varepsilon \left\{ \sum_{i<j} \frac{1}{4}(k_i - k_j)^2 + \frac{1}{4}\left(\sum_i k_i\right)^2 \right\} + \frac{1}{2}\bar{b}_\varepsilon N(N-1) \\ &\leqslant \frac{1}{2} N(\varepsilon H_0 + \bar{b}_\varepsilon N) . \end{aligned} \tag{3.13}$$

Therefore by a standard perturbation argument the operator

$$H_0 + z^2 H_4 \equiv H_0 - |z|^2 H_4$$

can be defined as a sum of quadratic forms in each $\mathcal{H}_N$ and then in $\mathcal{H}$ as a direct sum of its restrictions to the $\mathcal{H}_N$'s. In both cases by the Hille-Yosida theorem the operator $(i(H_0 + z^2 H_4) + B\eta N)$ generates a strongly continuous contraction semi group.

Strong continuity and analyticity in z of $U(t)\exp(-B\eta N t)$ follow from the same properties in each $\mathcal{H}_N$ since $U(t)\exp(-B\eta N t)$ is the strong limit as $\nu \to \infty$ of its restriction to the subspace with $N \leqslant \nu$, uniformly with respect to $z \in E(\eta)$. For each N these properties follow from the same properties of the resolvent operator.

(2) The first statement is a special case of part (1). Strong convergence of the semi group follows from the corresponding properties in each $\mathcal{H}_N$ for fixed N. This follows, through strong resolvent convergence, from the convergence of H_4^κ to H_4 as an operator from $Q(H_0) \cap \mathcal{H}_N$ to $Q(H_0)^* \cap \mathcal{H}_N$.

Remark 3.1. — It is clear from the proof that part (1) of Proposition 3.1 still holds under the weaker assumptions V stable and $V \in L^1_{\text{loc}}$ for $z \in E(\eta)$, $\operatorname{Re} z \neq 0$.

We now turn to the study of the operator $W(t, s)$ formally defined by (2.5). The function φ appearing in (2.5) will be a solution of the following integral form of the Hartree equation (2.4)

$$\varphi(t) = u_0(t - t_0)\varphi_0 - i \int_{t_0}^{t} d\tau u_0(t - \tau) \{ \varphi(\tau)(V * |\varphi(\tau)|^2) \} \tag{3.14}$$

where $u_0(.)$ is the free evolution operator defined by

$$u_0(t) = \exp\left(\frac{it}{2}\Delta\right). \tag{3.15}$$

The relevant spaces where to solve the equation (3.14) and some information on the solutions are contained in Proposition 2.1 of [*4*] (for more details see [*3*]). The main properties of $W(t, s)$, of $\tilde{W}(t, s)$ (which is defined by (2.32)), and of their approximants $W^\kappa(t, s)$, defined by

$$W^\kappa(t, s) = C(z, -\varphi(t))U^\kappa(t-s)C(z, \varphi(s)) \\ + \exp\left\{-iz^{-2}\int_s^t d\tau H_4(\varphi(\tau))\right\}, \tag{3.16}$$

and $\tilde{W}^\kappa(t, s)$, also defined by (2.32), are expressed by the following proposition.

PROPOSITION 3.2. — Let V be stable and $V \in L^\infty + L^{p_2}$ with $p_2 \geqslant n/2$, $p_2 \geqslant 1$ for $n = 1$, $p_2 > 1$ for $n = 2$. Let $\varphi \in \mathscr{C}(\mathbb{R}, L^2 \cap L^q)$ with $1/p_2 + 1/q = 1$, φ a solution of the classical equation (3.14). Let $\eta \geqslant 0$, $z \in E(\eta)$ with $z \neq 0$, let $\alpha, \beta \in \mathbb{R}$ with $\beta > \alpha$. Then

(1) The operator $\exp(\alpha N)W(t, s)\exp(-\beta N)$ is bounded and strongly continuous with respect to s, t and z for $t \geqslant s$, $B\eta(t-s) < \beta - \alpha$. If in addition $\eta > 0$, this operator is analytic in z for $z \in E_0(\eta)$.

The same properties hold for the operators $\exp(\alpha N)\tilde{W}(t, s)\exp(-\beta N)$, $\exp(\alpha N)W^\kappa(t, s)\exp(-\beta N)$ and $\exp(\alpha N)\tilde{W}^\kappa(t, s)\exp(-\beta N)$.

(2) The operators $\exp(\alpha N)W^\kappa(t, s)\exp(-\beta N)$ and

$$\exp(\alpha N)\tilde{W}^\kappa(t, s)\exp(-\beta N)$$

converge strongly to $\exp(\alpha N)W(t, s)\exp(-\beta N)$ and

$$\exp(\alpha N)\tilde{W}(t, s)\exp(-\beta N)$$

respectively, as $\kappa \to \infty$, uniformly for t, $s\,(t \geqslant s)$ in compact subsets of the region $B\eta(t-s) < \beta - \alpha$.

(3) The operator $\exp(\alpha N)\tilde{W}^\kappa(t, s)\exp(-\beta N)$ is strongly differentiable with respect to t and s for $t \geqslant s$ and $B\eta(t-s) < \beta - \alpha$, with derivative given by

$$i\frac{d}{dt}\exp(\alpha N)\tilde{W}^\kappa(t, s)\exp(-\beta N) \\ = \exp(\alpha N)\{\tilde{H}_2^\kappa(t) - H_0 + z\tilde{H}_3^\kappa(t) + z^2\tilde{H}_4^\kappa(t) \\ - z^{-2}(H_4(\varphi(t)) - H_4^\kappa(\varphi(t))) - z^{-1}a^*(u_0(-t)(\varphi(t)(V_\kappa''*|\varphi(t)|^2))) \\ - z^{-1}a(\overline{u_0(-t)(\varphi(t)(V_\kappa''*|\varphi(t)|^2))})\}\,\tilde{W}^\kappa(t, s)\exp(-\beta N), \tag{3.17}$$

where $\tilde{H}_2^\kappa$ and $\tilde{H}_3^\kappa$ are obtained in the same way as in (2.33) from H_2^κ and H_3^κ

which are themselves obtained by replacing V by $\bar{V}_\kappa$ in the definitions (1.34)-(1.43) of [*4*].

Proof. — (1) The proof is identical with that of Proposition 4.1, part (1), of [*4*].

(2) Follows immediately from Proposition 3.1, part (2).

(3) The proof is almost identical with that of Proposition 4.1, part (2), of [*4*]. One first obtains

$$
\begin{aligned}
& i\frac{d}{dt}\exp(\alpha N)\tilde{W}^\kappa(t,s)\exp(-\beta N) \\
& \quad = \exp\left\{-iz^{-2}\int_s^t d\tau H_4(\varphi(\tau))\right\}\exp(\alpha N)C(z,-\tilde{\varphi}(t))\{z^2\tilde{H}_4^\kappa(t) \\
& \qquad + z^{-2}H_4(\varphi(t)) - iz^{-1}(a^*(\dot{\tilde{\varphi}}(t)) - a(\dot{\tilde{\varphi}}(t))) \\
& \qquad - z^{-2}\,\mathrm{Im}\langle\tilde{\varphi}(t),\dot{\tilde{\varphi}}(t)\rangle\}\tilde{U}^\kappa(t,s)C(z,\tilde{\varphi}(s))\exp(-\beta N),
\end{aligned}
\tag{3.18}
$$

where

$$\tilde{\varphi}(t) = u_0(-t)\varphi(t).$$

$\tilde{\varphi}$ satisfies (4.28) of [*4*] and still belongs to $\mathscr{C}^1(\mathbb{R}, L^2)$ because of the coupled assumptions on V and φ. The result (3.17) now follows from (3.18) and from (4.28), (4.29) and (4.32) of [*4*].

We now turn to the study of the semigroup $U_4(t,s)$ formally defined by (2.10) and in particular of its relation with $W(t,s)$. The operator $U_4(t,s)$ will be constructed as the limit of suitably regularized operators and its main properties will be derived from the corresponding properties of the approximants.

It will be convenient to introduce the scale of Hilbert spaces $\mathscr{H}^\delta$ for $\delta\in\mathbb{R}$ as follows: for $\delta \geqslant 0$, $\mathscr{H}^\delta = \mathscr{D}(N^{\delta/2})$, which is a Hilbert space with norm

$$\|\Phi\|_\delta = \|f(\delta,N)\Phi\| \qquad (\delta\geqslant 0), \tag{3.19}$$

where

$$f(\delta,N) = \{\Gamma(N+\delta+1)/\Gamma(N+1)\}^{1/2}. \tag{3.20}$$

For $\delta\leqslant 0$, $\mathscr{H}^\delta$ is the completion of $\mathscr{H}$ in the norm

$$\|\Phi\|_\delta = \|f(|\delta|,N)^{-1}\Phi\| \qquad (\delta\leqslant 0). \tag{3.21}$$

The space of bounded operators in $\mathscr{H}^\delta$ will be denoted by $\mathscr{B}(\delta)$.

We define formally the approximant $\tilde{U}_4^\kappa(t,s)$ to $\tilde{U}_4(t,s)$ as the solution of the system

$$
\left\{
\begin{aligned}
& i\frac{d}{dt}\tilde{U}_4^\kappa(t,s) = \tilde{A}_4^\kappa(t)\tilde{U}_4^\kappa(t,s), \\
& \tilde{U}_4^\kappa(s,s) = \lambda,
\end{aligned}
\right.
\tag{3.22}
$$

with infinitesimal generator

$$\tilde{A}_4^\kappa(t) = \tilde{H}_2(t) - H_0 + z^2\tilde{H}_4^\kappa(t). \qquad (3.23)$$

Note that the cut off κ appears in $\tilde{H}_4$, but not in $\tilde{H}_2$. Since $\overline{V}_\kappa \in L^\infty$, $\tilde{U}_4^\kappa(t, s)$ satisfies the properties stated for $\tilde{U}_4(t, s)$ in Section 5 of [*4*] up to, and including, Corollary 5.3. In particular $\tilde{U}_4^\kappa(t, s)$ satisfies the equation

$$\tilde{U}_4^\kappa(t, s) = \tilde{U}_2(t, s) - iz^2 \int_s^t d\tau \tilde{U}_2(t, \tau)\tilde{H}_4^\kappa(\tau)\tilde{U}_4^\kappa(\tau, s) \qquad (3.24)$$

in a suitable sense (see Corollary 5.3 of [*4*]).

We next take the limit $\kappa \to \infty$ in $\tilde{U}_4^\kappa(t, s)$.

PROPOSITION 3.3. — Let V be stable and $V \in L^\infty + L^{p_2}$ with $p_2 \geqslant 2$, $p_2 > n/2$. Let $\varphi \in \mathscr{C}(\mathbb{R}, L^2 \cap L^q)$ with $1/p_2 + 2/q = 1$. Let $\eta \geqslant 0$, let $[a, b]$ be an interval of $\mathbb{R}$ and let $\alpha_4(.) \in \mathscr{C}^1([a, b], \mathbb{R}^+)$ satisfy the equation (5.4) of [*4*] for all $\tau \in [a, b]$. Let $z \in E(\eta)$ and let $s, t \in [a, b], t \geqslant s$. Then

(1) For all $\delta \geqslant 0$, the following limit exists in the strong topology of $\mathscr{B}(\delta)$:

$$\begin{aligned} \operatorname*{s-lim}_{\kappa\to\infty} &\exp(\alpha_4(t)(N + 1))\tilde{U}_4^\kappa(t, s) \exp(-\alpha_4(s)(N + 1)) \\ &\equiv \exp(\alpha_4(t)(N + 1))\tilde{U}_4(t, s) \exp(-\alpha_4(s)(N + 1)), \qquad (3.25)\end{aligned}$$

and the convergence is uniform in s, t in any compact subset of $[a, b]$. The operator on the R. H. S. of (3.25) is strongly continuous with respect to s, t in $\mathscr{B}(\delta)$ and satisfies the estimate (5.6) of [*4*].

(2) Let in addition $\varphi \in \mathscr{C}(\mathbb{R}, L^2 \cap L^q)$ with $1/p_2 + 1/q = 1/2$, φ a solution of the classical equation (3.14). Then, for any α, $\beta \in \mathbb{R}$ such that

$$0 \leqslant \alpha \leqslant \alpha_4(t) \leqslant \alpha_4(s) < \beta,$$

the following identity holds

$$\begin{aligned}\exp(\alpha N)\tilde{W}(t, s) \exp(-\beta N) &= \exp(\alpha N)\tilde{U}_4(t, s) \exp(-\beta N) \\ &- iz \int_s^t d\tau \exp(\alpha N)\tilde{U}_4(t, \tau)\tilde{H}_3(\tau)\tilde{W}(\tau, s) \exp(-\beta N). \qquad (3.26)\end{aligned}$$

Proof. — (1) In order to prove the existence of the limit (3.25) we introduce a particle number cut off as follows. Let $\sigma_1 \in \mathscr{C}^1(\mathbb{R}^+, \mathbb{R})$ be positive and decreasing, $\sigma_1(s) = 1$ if $s \leqslant 1$, $\sigma_1(s) = 0$ if $s \geqslant 2$. We denote by σ_ν the operator $\sigma_1(N/\nu)$ in $\mathscr{H}$. We introduce a double approximant $\tilde{U}_4^{\kappa\nu}(t, s)$ formally generated by

$$\tilde{A}_4^{\kappa\nu}(t) = \sigma_\nu(\tilde{H}_2(t) - H_0)\sigma_\nu + \tilde{H}_4^\kappa$$

and defined more precisely by the series (see (2.11), (2.12))

$$\begin{aligned}&\exp(\alpha N)\tilde{U}_4^{\kappa\nu}(t, s) \exp(-\beta N) \\ &= \sum_{j\geqslant 0} (-i)^j \exp(\alpha N)(\tilde{U}^\kappa\sigma_\nu(\tilde{H}_2 - H_0)\sigma_\nu)^j\tilde{U}^\kappa \exp(-\beta N), \qquad (3.27)\end{aligned}$$

which is norm convergent for $B\eta(t-s) < \beta - \alpha$. One sees easily that $\tilde{U}_4^{\kappa\nu}(t, s)$ satisfies the estimate (5.6) of [*4*] uniformly in κ and ν. By proposition 3.1, part (2), one can take in (3.27) the strong limit as $\kappa \to \infty$ for fixed ν. On the other hand, by the same argument as in the proof of Proposition 5.1 of [*4*] and of Proposition 4.1 of [*2*], based on the formula

$$\tilde{U}_4^{\kappa\nu}(t, s) - \tilde{U}_4^{\kappa\mu}(t, s)$$
$$= -i\int_s^t d\tau \tilde{U}_4^{\kappa\mu}(t, \tau)\{\sigma_\nu(\tilde{H}_2(\tau) - H_0)\sigma_\nu - \sigma_\mu(\tilde{H}_2(\tau) - H_0)\sigma_\mu\}\tilde{U}_4^{\kappa\nu}(\tau, s),$$

one can take the limit

$$\underset{\nu\to\infty}{s\text{-lim}}\, f(\delta, N)\exp(\alpha_4(t)(N+1))\tilde{U}_4^{\kappa\nu}(t, s)\Phi$$

uniformly in κ for $\Phi \in \mathcal{D}(\exp(\beta N))$, $\beta > \alpha_4(s)$.

Part (1) follows from these two limiting properties and from the estimate (5.6) of [*4*].

(2) Part (2) follows by integrating the identity

$$i\frac{d}{d\tau}\{\exp(\alpha N)\tilde{U}_4^{\kappa}(t, \tau)\tilde{W}^{\kappa}(\tau, s)\exp(-\beta N)\}$$
$$= \exp(\alpha N)\tilde{U}_4^{\kappa}(t, \tau)\{\tilde{H}_2^{\kappa}(\tau) - \tilde{H}_2(\tau) + z\tilde{H}_3^{\kappa}(\tau) - z^{-2}(H_4(\varphi(\tau)) - H_4^{\kappa}(\varphi(\tau)))$$
$$- z^{-1}a^*(u_0(-\tau)(\varphi(\tau)(V_\kappa''*|\varphi(\tau)|^2)))$$
$$- z^{-1}a(\overline{u_0(-\tau)(\varphi(\tau)(V_\kappa''*|\varphi(\tau)|^2))})\}\tilde{W}^{\kappa}(\tau, s)\exp(-\beta N)$$

and then taking the limit $\kappa \to \infty$, using Proposition 3.2, part (2), and part (1) of this proposition. Q. E. D.

We conclude this section with a brief sketch of the operator theoretic justification of Lemmas 2.1 and 2.2. This is based on Propositions 3.2 and 3.3, on (3.24) and on the estimates of Sections 4 and 5 below. First, all the formulas in Lemmas 2.1 and 2.2, as well as (3.24), are supposed to hold as relations between bounded operators when sandwiched between $\exp(\alpha N)$ and $\exp(-\beta N)$ with $0 \leqslant \alpha \leqslant \alpha_4(t) \leqslant \alpha_4(s) < \beta$ for some $\alpha_4(.)$ satisfying the equation (5.4) of [*4*]. The same remark applies to all statements and formulas in the discussion below. The first step consists in proving Lemma 2.2 with a cut off κ in $\tilde{H}_4$, $\tilde{U}$, $\tilde{U}_{(l)}$, $\tilde{Q}_l$, $\tilde{U}_4$, but, of course, not in $\tilde{H}_2$ and in $\tilde{U}_2$. With this cut off the proof is exactly that given for Lemma 2.2. One then proves the estimates of Sections 4 and 5 with the cut off κ: all the estimates are uniform in κ. The next step consists in taking the limit $\kappa \to \infty$. From the estimates of Lemma 4.2 and the fact that $\tilde{U}_{(l)}$ is multilinear in the potential V, it follows that the approximant $\tilde{U}_{(l)}^{\kappa}$ converges to $\tilde{U}_{(l)}$ strongly (actually in norm). On the other hand $\tilde{U}_\kappa$ converges strongly to $\tilde{U}$ by Proposition 3.1, part (2), so that also $\tilde{Q}_l^{\kappa}$ converges strongly to $\tilde{Q}_l$ (this also follows from the estimates of Lemma 5.2). One then uses these convergences together with that of $\tilde{U}_4^{\kappa}$ to $\tilde{U}_4$ given by

Proposition 3.3, part (1), and the uniform estimates of Lemmas 4.3, 4.4, 5.3 and 5.4 to take the limit $\kappa \to \infty$ in (2.36) and (2.39). One concludes that $\tilde{U}^{\kappa}_{4,k}$ and $\tilde{Q}^{\kappa}_{4,k}$ converge strongly to $\tilde{U}_{4,k}$ and $\tilde{Q}_{4,k}$ respectively, that the estimates of Lemmas 4.4 and 5.4 still hold for the limits, and that $\tilde{U}_4$ has an asymptotic expansion in z

$$\tilde{U}_4(t, s) = \sum_{0 \leqslant j < k} z^{2j}\tilde{U}_{4,j}(t, s) + \tilde{Q}_{4,k}(t, s), \tag{3.28}$$

with coefficients that are bounded operators. One finally combines (3.28) with an iterated form of (3.26) to derive Lemma 2.1, the proof being now purely combinatorial.

4. ESTIMATE OF THE GENERAL TERM

In this section we estimate the general term $W_p(t, s)$ of the expansion of $W(t, s)$ as a power series in $z = \hbar^{1/2}$ given by Eq. (2.20). As in [*4*], we estimate $W_p(t, s)$ in the strong sense, namely, we estimate $\| W_p(t, s)\Phi \|$ for Φ in the dense set $\mathscr{D}(\exp(\varepsilon N))$ for some $\varepsilon > 0$. By Lemma 2.1, the operators $W_p(t, s)$ are expressed in terms of the operators $U_{4,k}(t, s)$ which, by (2.32) and Lemma 2.2, are themselves expressed in terms of the operators $U_{(l)}(t - s)$ and $\tilde{Y}_l(t, s)$. Therefore, we estimate successively $U_{(l)}(t - s)$, $\tilde{Y}_l(t, s)$, $U_{4,k}(t, s)$ and finally $W_p(t, s)$. All the estimates of this and the next section apply equally well to the operators with and without $\tilde{}$, and we shall state and prove them for whichever is more convenient, without further explanation.

The estimates on $U_{(l)}(t - s)$ will be expressed in terms of the function $\mu(t)$ defined by

$$\mu(t) = \operatorname*{Max}_{1 \leqslant i \leqslant 3} \mu^{(i)}(t) \tag{4.1}$$

where

$$\mu^{(1)}(t) = \operatorname*{Sup}_{\Phi, \|\Phi\| = 1} \left| \int_0^t d\tau \, \| \, | V_{ij} |^{1/2} U_0(\tau)\Phi \|^2 \right|,$$

$$\mu^{(2)}(t) = \operatorname*{Sup}_{\lambda \in \mathbb{R}} \left\| \int_0^t d\tau e^{-i\tau\lambda} \, | V_{ij} |^{1/2} U_0(\tau) \, | V_{ij} |^{1/2} \right\|,$$

$$\mu^{(3)}(t) = \operatorname*{Sup}_{\lambda \in \mathbb{R}} \left\| \int_0^t d\tau e^{-i\tau\lambda} \, | V_{ij} |^{1/2} U_0(\tau) \, | V_{jl} |^{1/2} \right\|.$$

The vector Φ in the expression for $\mu^{(1)}$ and the operator norms in the expressions for $\mu^{(2)}$ and $\mu^{(3)}$ are taken in the N-particle space $\mathscr{H}_N$ for any $N \geqslant 3$ (the result is independent of N), and i, j, l label three different particles (the result is also independent of the choice of i, j, l). Standard arguments [*5, 6*] show that $\mu(t)$ is finite for a large class of potentials. For the convenience of the reader we prove an explicit bound in the following lemma.

LEMMA 4.1. — Let $V \in L^\infty + L^{p_2}$ with $p_2 \geqslant 1$, $p_2 > n/2$. Then for any $\kappa \geqslant \bar{\kappa}$ (see after (3.9)),

$$\mu(t) \leqslant \bar{\mu}(t) \equiv 2(\kappa \mid t \mid + C_{p_2} \mid t \mid^{\varepsilon_2} \| V'_\kappa \|_{p_2}), \tag{4.2}$$

where V'_κ is defined by (3.8, 3.9), $\varepsilon_2 = 1 - n/(2p_2)$ and C_{p_2} is a constant depending only on p_2.

Proof. — We first estimate $\mu^{(1)}$. Obviously it suffices to take $t > 0$ and $\Phi \in \mathscr{H}_2$. We drop the subscripts i, j $(= 1, 2)$. Let f be a real positive even function of positive type in $L^1 \cap L^\infty$ with $f(s) \geqslant 1$ for $|s| \leqslant 1$, and let $f_t(\tau) = f(\tau/t)$. Then, for any $\kappa > 0$,

$$\mu^{(1)}(t) \leqslant \kappa t + \sup_{\Phi, \|\Phi\| = 1} g_\Phi(t/2), \tag{4.3}$$

where

$$g_\Phi(t) = \int_{-t}^{t} d\tau \, \| \, |V'_\kappa|^{1/2} U_0(\tau)\Phi \|^2 \leqslant \int d\tau f_t(\tau)^2 \, \| \, |V'_\kappa|^{1/2} U_0(\tau)\Phi \|^2$$
$$= (2\pi)^{-1} \int d\lambda \, \| \, |V'_\kappa|^{1/2} \hat{f}_t(\lambda + H_0)\Phi \|^2$$

by the vector valued version of Plancherel's theorem,

$$\leqslant (2\pi)^{-1} \int d\lambda \, ||| \, |V'_\kappa|^{1/2} \hat{f}_t(\lambda + H_0) \, |V'_\kappa|^{1/2} \, ||| \langle \Phi, \hat{f}_t(\lambda + H_0)\Phi \rangle$$

where we have used the positivity of $\hat{f}$,

$$\leqslant f(0) \sup_\lambda ||| \, |V'_\kappa|^{1/2} \hat{f}_t(\lambda + H_0) \, |V'_\kappa|^{1/2} \, |||$$
$$\leqslant f(0) \int d\tau f_t(\tau) \, ||| \, |V'_\kappa|^{1/2} U_0(\tau) \, |V'_\kappa|^{1/2} \, ||| \, .$$

The last norm is bounded [6] by

$$(4\pi \mid \tau \mid)^{\varepsilon_2 - 1} \| V'_\kappa \|_{p_2}$$

for any $\kappa \geqslant \bar{\kappa}$, so that

$$\sup_\Phi g_\Phi(t) \leqslant f(0) \int d\tau f_t(\tau) (4\pi \mid \tau \mid)^{\varepsilon_2 - 1} \| V'_\kappa \|_{p_2} = C_{p_2} (2t)^{\varepsilon_2} \| V'_\kappa \|_{p_2} \tag{4.4}$$

with

$$C_{p_2} = \frac{1}{2} f(0) \int d\tau (2\pi \mid \tau \mid)^{\varepsilon_2 - 1} f(\tau) \, .$$

Together with (4.3), this proves the estimate (4.2) for $\mu^{(1)}$ (improved by a factor 2). A possible suitable choice for f is

$$f(\tau) = \exp(\varepsilon_2(1 - \tau^2)/4) \, .$$

We next estimate $\mu^{(3)}$ (the estimate of $\mu^{(2)}$ will be omitted, since it is similar and yields a smaller bound). Let $\kappa \geqslant \bar{\kappa}$. We substitute the decomposition $V = V_\kappa + V'_\kappa$ into $\mu^{(3)}(t)$, expand, and estimate the various terms. The diagonal term with V'_κ is bounded [5, 6] by

$$(2\pi)^{\varepsilon_2 - 1}\varepsilon_2^{-1}t^{\varepsilon_2} \| V'_\kappa \|_{p_2} .$$

The non diagonal terms are bounded by

$$2\kappa^{1/2} \operatorname*{Sup}_{\lambda \in \mathbb{R}} \left|\left|\left| \int_0^t d\tau e^{-i\tau\lambda} \, | V'_{\kappa ij} |^{1/2} U_0(\tau) \right|\right|\right|$$
$$\leqslant 2(\kappa t)^{1/2} \left\{ \operatorname*{Sup}_{\Phi, \|\Phi\| = 1} \int_0^t d\tau \, \| \, | V'_{\kappa ij} |^{1/2} U_0(\tau)\Phi \|^2 \right\}^{1/2}$$

by Schwarz's inequality,

$$\leqslant 2(\kappa t)^{1/2} \{ C_{p_2} t^{\varepsilon_2} \| V'_\kappa \|_{p_2} \}^{1/2}$$

by the argument used above to estimate $g_\Phi(t)$ (see (4.4)). Therefore

$$\mu^{(3)}(t) \leqslant \kappa t + (2\pi)^{\varepsilon_2 - 1}\varepsilon_2^{-1}t^{\varepsilon_2} \| V'_\kappa \|_{p_2} + 2(\kappa t)^{1/2} \{ C_{p_2} t^{\varepsilon_2} \| V'_\kappa \|_{p_2} \}^{1/2} ,$$

from which (4.2) for $\mu^{(3)}$ follows immediately since $(2\pi)^{\varepsilon_2 - 1}\varepsilon_2^{-1} \leqslant C_{p_2}$.

Q. E. D.

We now estimate $U_{(l)}(t - s)$ defined by (2.28).

LEMMA 4.2. — Let $V \in L^\infty + L^{p_2}$, with $p_2 \geqslant 1$, $p_2 > n/2$, let $\delta \geqslant 0$. Then, for any integer $l \geqslant 0$, $U_{(l)}(t - s)$ satisfies the estimate

$$||| f(\delta, N) U_{(l)}(t - s) f(\delta + 4l, N)^{-1} ||| \leqslant 2^{-l} \mu(t - s)^l , \tag{4.5}$$

where $f(\delta, N)$ is defined by (3.20).

Proof. — Without any loss in generality, we take $s = 0$ and $t > 0$. It is sufficient to prove the estimate in each N-particle subspace $\mathscr{H}_N$. Since the number of perturbation graphs of $U_{(l)}$ in $\mathscr{H}_N$ is 2^{-l} $[N(N-1)]^l$, it is sufficient to prove that the contribution of each graph is bounded by $\mu(t)$. A general perturbation graph is defined by a sequence of pairs of particles $\alpha = \{ \alpha_i \}$, $1 \leqslant i \leqslant l$, and contributes a term

$$\int_{\Delta_l} d\{ \tau \} \prod_{1 \leqslant i \leqslant l} (U_0(\tau_{i-1} - \tau_i) V_{\alpha_i}) U_0(\tau_l) \equiv U_{(\alpha)}(t) . \tag{4.6}$$

Let Φ and $\Psi \in \mathscr{H}_N$; then

$$\langle \Phi, U_{(\alpha)}(t)\Psi \rangle = f(t) \tag{4.7}$$

where

$$f(\tau) = \langle \Phi, F_{\alpha_1} * G_{\alpha_1\alpha_2} * \ldots * G_{\alpha_{l-1}\alpha_l} * F'_{\alpha_l}\Psi \rangle (\tau) , \tag{4.8}$$

$*$ denotes the convolution, and the operator valued functions F_α, F'_α and $G_{\alpha\beta}$ are defined by

$$\begin{aligned} F_\alpha(\tau) &= \theta(t-\tau)\theta(\tau)U_0(\tau)V_\alpha \,|\, V_\alpha \,|^{-1/2}, \\ F'_\alpha(\tau) &= \theta(t-\tau)\theta(\tau) \,|\, V_\alpha \,|^{1/2} U_0(\tau), \\ G_{\alpha\beta}(\tau) &= \theta(t-\tau)\theta(\tau) \,|\, V_\alpha \,|^{1/2} U_0(\tau) V_\beta \,|\, V_\beta \,|^{-1/2}. \end{aligned}$$

These functions depend on t as a parameter and the convolution product in (4.8) is taken for the value t of the variable. We estimate

$$\begin{aligned} |\, f(t) \,| &\leqslant \|\, f \,\|_\infty \leqslant \|\, \hat{f} \,\|_1 \\ &= (2\pi)^{-1} \int d\lambda \left| \left\langle \Phi, \hat{F}_{\alpha_1}(\lambda) \left\{ \prod_{1 \leqslant i < l} \hat{G}_{\alpha_i \alpha_{i+1}}(\lambda) \right\} \hat{F}'_{\alpha_l}(\lambda)\Psi \right\rangle \right| \\ &\leqslant (2\pi)^{-1} \left\{ \int d\lambda \,\|\, \hat{F}^*_{\alpha_1}(\lambda)\Phi \,\|^2 \right\}^{1/2} \left\{ \int d\lambda \,\|\, \hat{F}'_{\alpha_l}(\lambda)\Psi \,\|^2 \right\}^{1/2} \\ &\quad \times \prod_{1 \leqslant i < l} \left\{ \operatorname*{Sup}_\lambda |||\, \hat{G}_{\alpha_i\alpha_{i+1}}(\lambda) \,||| \right\} \\ &= \left\{ \int_0^t d\tau \,\|\, |\, V_{\alpha_1} \,|^{1/2} U_0(-\tau)\Phi \,\|^2 \right\}^{1/2} \left\{ \int_0^t d\tau \,\|\, |\, V_{\alpha_l} \,|^{1/2} U_0(\tau)\Psi \,\|^2 \right\}^{1/2} \\ &\quad \times \prod_{1 \leqslant i < l} \left\{ \operatorname*{Sup}_\lambda |||\, \hat{G}_{\alpha_i\alpha_{i+1}}(\lambda) \,||| \right\} \end{aligned} \tag{4.9}$$

by Plancherel's theorem. Now the first and last factors in the last member of (4.9) are bounded by $(\mu^{(1)}(t))^{1/2}$. If the two pairs $\alpha_i = \alpha$ and $\alpha_{i+1} = \beta$ overlap, then

$$\operatorname*{Sup}_\lambda |||\, \hat{G}_{\alpha\beta}(\lambda) \,||| = \begin{cases} \mu^{(2)}(t) & \text{if} \quad \alpha = \beta, \\ \mu^{(3)}(t) & \text{if} \quad \alpha \neq \beta. \end{cases}$$

If the pairs α and β are disjoint, we estimate for any Φ and Ψ, $\|\Phi\| \doteq \|\Psi\| = 1$,

$$\begin{aligned} |\langle \Phi, \hat{G}_{\alpha\beta}(\lambda)\Psi \rangle| &= \left| \int_0^t d\tau e^{-i\lambda\tau} \langle \Phi, |\, V_\alpha \,|^{1/2} U_0(\tau) V_\beta \,|\, V_\beta \,|^{-1/2} \Psi \rangle \right| \\ &\leqslant \left\{ \int_0^t d\tau \,\|\, |\, V_\alpha \,|^{1/2} U_0(\tau)\Psi \,\|^2 \right\}^{1/2} \left\{ \int_0^t d\tau \,\|\, |\, V_\beta \,|^{1/2} U_0(-\tau)\Phi \,\|^2 \right\}^{1/2} \\ &\leqslant \mu^{(1)}(t) \end{aligned}$$

by Schwarz's inequality.

Collecting all the estimates we obtain (4.5). Q. E. D.

We next estimate the commutator that occurs in $\tilde{Y}_l(t, s)$.

LEMMA 4.3. — Let $V \in L^\infty + L^{p_2}$ with $p_2 \geqslant 2$, $p_2 > n/2$, let

$\varphi \in \mathscr{C}(\mathbb{R}, L^2 \cap L^q)$ with $1/p_2 + 2/q = 1$, and let $\delta \geqslant 0$. Then for $t \geqslant s$ and for any integer $l \geqslant 0$,

$$||| f(\delta, N)[\tilde{H}_2(s) - H_0, \tilde{U}_{(l)}(t, s)] f(\delta + 4l, N)^{-1} ||| \leqslant 4l\, 2^{-l}\mu(t-s)^l(\| g(s) \|_\infty + 2c(s)), \quad (4.10)$$

where $g(s)$ is the function of x defined by (1.38) of [4] with φ taken at time s, and $c(.)$ is defined by (2.19) of [4].

Proof. — We write the commutator in the L. H. S. of (4.10) as

$$\begin{aligned}[\tilde{H}_2(s) - H_0, \tilde{U}_{(l)}(t, s)] &= (\tilde{G}(s) + \tilde{K}(s)) \angle \tilde{U}_{(l)}(t, s) - \tilde{U}_{(l)}(t, s) \angle (\tilde{G}(s) + \tilde{K}(s)) \\ &\quad + \tilde{L}(s) \angle \tilde{U}_{(l)}(t, s) - \tilde{U}_{(l)}(t, s) \angle \tilde{L}(s)^*, \end{aligned} \quad (4.11)$$

where $\tilde{G}(s)$, $\tilde{K}(s)$, $\tilde{L}(s)$ and $\tilde{L}(s)^*$ are defined by formulas similar to (2.33) in terms of $G(s)$, $K(s)$, $L(s)$ and $L(s)^*$ as given by (1.35-1.40) of [4] with φ taken at time s, and $\angle$ denotes the connected product: this means that for each graph in the perturbation expansion of $\tilde{U}_{(l)}(t, s)$, only those terms in $\tilde{G}(s)$ and $\tilde{K}(s)$ (resp. $\tilde{L}(s)$, resp. $\tilde{L}(s)^*$) are retained that act on (resp. annihilate, resp. create) particles that interact in the given graph [1]. We estimate the various terms in the R. H. S. of (4.11) successively. $\tilde{G}(s)$ and $\tilde{K}(s)$ are sums of one particle operators, and the norms of their one particle components are bounded by $\| g(s) \|_\infty$ and $c(s)$ respectively (see Lemma 2.2 of [4], especially (2.16) and (2.17)). Since there are at most $2l$ particles interacting in $\tilde{U}_{(l)}(t, s)$, we obtain

$$||| f(\delta, N)[\tilde{G}(s) + \tilde{K}(s), \tilde{U}_{(l)}(t, s)] f(\delta + 4l, N)^{-1} ||| \leqslant 4l(\| g(s) \|_\infty + c(s)) 2^{-l}\mu(t-s)^l, \quad (4.12)$$

where we have used the estimate of each individual perturbation term of $\tilde{U}_{(l)}(t, s)$ obtained in Lemma 4.2.

We now turn to the terms containing $\tilde{L}(s)$ and $\tilde{L}(s)^*$. Commuting $f(\delta, N)$ with $\tilde{L}(s)$ or $\tilde{L}(s)^*$, we see that the term with $\tilde{L}(s)^*$ has the larger contribution. We therefore consider only this term. Let Φ, $\Psi \in \mathscr{H}$, $\| \Phi \| = \| \Psi \| = 1$, and let Φ_N, Ψ_N, be the N-particle components of Φ, Ψ. Then

$$\begin{aligned}&\langle \Phi, f(\delta, N)\tilde{U}_{(l)}(t, s) \angle \tilde{L}(s)^* f(\delta + 4l, N)^{-1}\Psi \rangle \\ &\qquad = \sum_N f(\delta, N) f(\delta + 4l, N-2)^{-1} \langle \Phi_N, \tilde{U}_{(l)}(t, s) \angle \tilde{L}(s)^*\Psi_{N-2} \rangle .\end{aligned}$$

We consider separately the contribution of the various perturbation graphs with l interactions that occur in $\tilde{U}_{(l)}(t, s)$. Let m be the number of interacting particles in a given graph, so that $2 \leqslant m \leqslant \mathrm{Min}(2l, N)$. $\tilde{L}(s)^*$ is a sum of two particle creating terms, of which only those connected with the given graph survive. The number of terms with one contraction is

$m(\mathrm{N}-m)$ and the number of terms with two contractions is $\frac{1}{2}m(m-1)$. The total number of terms is therefore less than

$$m(\mathrm{N}-m)+\frac{1}{2}m(m-1)\leqslant m\left(\mathrm{N}-\frac{m}{2}\right)\leqslant 2l(\mathrm{N}-1),$$

so that

$$\begin{aligned}\langle \Phi_{\mathrm{N}}, \tilde{\mathrm{U}}_{(l)}(t,s) \angle \tilde{\mathrm{L}}(s)^*\Psi_{\mathrm{N}-2}\rangle &\leqslant 2^{-l}\mu(t-s)^l \\ &\times (\mathrm{N}(\mathrm{N}-1))^l 2l(\mathrm{N}-1)\,\|\Phi_{\mathrm{N}}\|\,(\mathrm{N}(\mathrm{N}-1))^{-1/2}\,\|l(s)\|_2\,\|\Psi_{\mathrm{N}-2}\|,\end{aligned}$$

where $l(s)\equiv l(s;x,y)$ is defined by (1.40) of [4], $\|l(s)\|_2$ denotes its norm in $\mathrm{L}^2(dx\,dy)$, and the factor $(\mathrm{N}(\mathrm{N}-1))^{-1/2}$ accounts for the transition from the norm in $\mathscr{H}_{\mathrm{N}}$ to the norm in $\mathscr{H}_{\mathrm{N}-2}$ (see (3.3)). Therefore

$$\begin{aligned}&|\langle \Phi, f(\delta,\mathrm{N})\tilde{\mathrm{U}}_{(l)}(t,s) \angle \tilde{\mathrm{L}}(s)^* f(\delta+4l,\mathrm{N})^{-1}\Psi\rangle| \\ &\quad\leqslant 2l\,2^{-l}\mu(t-s)^l\,\|l(s)\|_2\sum_{\mathrm{N}}\|\Phi_{\mathrm{N}}\|\,\|\Psi_{\mathrm{N}-2}\| \\ &\quad\quad\times\{(\mathrm{N}(\mathrm{N}-1))^{l-1/2}(\mathrm{N}-1)f(\delta,\mathrm{N})f(\delta+4l,\mathrm{N}-2)^{-1}\} \\ &\quad\leqslant 2l\,2^{-l}\mu(t-s)^l\,\|l(s)\|_2,\end{aligned}$$

by Schwarz's inequality applied to the sum over N, since the last combination of N-dependent factors is easily seen to be smaller than one for all $\delta\geqslant 0$.

The lemma follows by collecting the previous estimates and noting that $\|l(s)\|_2=c(s)$. Q. E. D.

The next step consists in estimating $\mathrm{U}_{4,k}(t,s)$ or equivalently $\tilde{\mathrm{U}}_{4,k}(t,s)$ as given by (2.32) and Lemma 2.2.

LEMMA 4.4. — Under the same assumptions as in Lemma 4.3, for $t\geqslant s$ and for any integer $k\geqslant 0$, $\mathrm{U}_{4,k}(t,s)$ satisfies the estimate

$$\begin{aligned}&|||\, f(\delta,\mathrm{N})\mathrm{U}_{4,k}(t,s)f(\delta+4k,\mathrm{N})^{-1}\,||| \\ &\quad\leqslant \bar{\mu}(t-s)^k\exp\left\{(\delta+4k)\int_s^t d\tau c(\tau)+8\int_s^t d\tau(\,\|g(\tau)\|_\infty+2c(\tau))\right\},\end{aligned}\tag{4.13}$$

where $\bar{\mu}$ is defined in (4.2).

Proof. — We estimate $\tilde{\mathrm{U}}_{4,k}(t,s)$ in the form given by (2.36):

$$\begin{aligned}&|||\, f(\delta,\mathrm{N})\tilde{\mathrm{U}}_{4,k}(t,s)f(\delta+4k,\mathrm{N})^{-1}\,||| \\ &\quad\leqslant\sum_{0\leqslant r\leqslant k}\sum_{\{l\}}\int_{\Delta_r}d\{\tau\}\left\{\prod_{0\leqslant i<r}(\,|||\, f(\delta_i,\mathrm{N})\tilde{\mathrm{U}}_2(\tau_i,\tau_{i+1})f(\delta_i,\mathrm{N})^{-1}\,|||\right. \\ &\quad\quad\left.|||\, f(\delta_i,\mathrm{N})[\tilde{\mathrm{H}}_2(\tau_{i+1})-\mathrm{H}_0,\tilde{\mathrm{U}}_{l_i}(\tau_i,\tau_{i+1})]f(\delta_{i+1},\mathrm{N})^{-1}\,|||\,)\right\} \\ &\quad\quad\times|||\, f(\delta_r,\mathrm{N})\tilde{\mathrm{U}}_2(\tau_r,s)f(\delta_r,\mathrm{N})^{-1}\,||| \\ &\quad\quad\times|||\, f(\delta_r,\mathrm{N})\tilde{\mathrm{U}}_{(l_r)}(\tau_r,s)f(\delta_{r+1},\mathrm{N})^{-1}\,|||,\end{aligned}\tag{4.14}$$

where

$$\delta_0 = \delta, \delta_i = \delta + 4(l_0 + \dots + l_{i-1}), \delta_{r+1} = \delta + 4k,$$

and the remaining notation is the same as in (2.36). We estimate the norms containing $\tilde{U}_2(\tau, \tau')$ by Proposition 2.2 of [*4*] (see especially (2.24) of [*4*]) and the other norms by Lemmas 4.3 and 4.4 above:

$$\dots \leqslant \exp\left\{ (\delta + 4k) \int_s^t d\tau c(\tau) \right\} \sum_{0 \leqslant r \leqslant k} \sum_{\{l\}} \int_{\Delta_r} d\{\tau\} \prod_{1 \leqslant i \leqslant r} \{ 4l_{i-1}(\| g(\tau_i) \|_\infty + 2c(\tau_i)) \} \prod_{0 \leqslant i \leqslant r} 2^{-l_i} \mu(\tau_i - \tau_{i+1})$$

where we have replaced δ_i by $\delta_{r+1} = \delta + 4k$ in the exponent resulting from the application of (2.24) of [*4*],

$$\dots \leqslant 2^{-k} \bar{\mu}(t - s)^k \exp\left\{ (\delta + 4k) \int_s^t d\tau c(\tau) \right\} \times \sum_{0 \leqslant r \leqslant k} \frac{1}{r!} \left\{ 4 \int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\}^r \sum_{\{l\}} \prod_{0 \leqslant i < r} l_i , \quad (4.15)$$

where we have used the fact that $\bar{\mu}$ is an increasing function of t. Now

$$\sum_{\{l\}} \prod_{0 \leqslant i < r} l_i = \text{coefficient of } y^k \text{ in } y^r(1 - y)^{-(2r+1)} = \binom{k + r}{k - r} \leqslant 2^{k+r}.$$

The result now follows by extending the sum over r in (4.15) up to infinity.

Q. E. D.

We are now ready to estimate the general term $W_p(t, s)$ of the expansion of $W(t, s)$ as a power series in z, using the representation given in Lemma 2.1.

PROPOSITION 4.1. — Let $V \in L^\infty + L^{p_2}$ with $p_2 \geqslant 2$, $p_2 > n/2$, and $\varphi \in \mathscr{C}(\mathbb{R}, L^2 \cap L^q)$ with $1/p_2 + 1/q = 1/2$. Let $\varepsilon > 0$ and $\Phi \in \mathscr{D}(\exp(\varepsilon N))$. Then, for $t \geqslant s$ and for all $p \geqslant 1$, $W_p(t, s)\Phi$ satisfies the following estimate:

$$\| W_p(t, s)\Phi \| \leqslant p(p!)^{1/2} 6^p \{ C \operatorname{Max}(3(Cb_3)^2, pC\bar{b}_4) \}^{p/2} \times \exp\left\{ 8 \int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\} \| \exp(\varepsilon(N + 1))\Phi \|, \quad (4.16)$$

where

$$C = (1 - e^{-2\varepsilon})^{-1} \exp\left\{ 2 \int_s^t d\tau c(\tau) \right\}, \quad (4.17)$$

$c(.)$ is defined by (2.19) of [*4*],

$$b_3 = 2\int_s^t d\tau c_1(\tau)\,, \tag{4.18}$$

$$\bar{b}_4 = \frac{1}{2}\bar{\mu}(t-s)\,, \tag{4.19}$$

and $c_1(.)$ is defined by (3.8) of [*4*].

Proof. — Using the representation (2.20) we estimate $W_p(t, s)$ as follows:

$$\begin{aligned} &\| W_p(t,s)\Phi \| \\ &\leqslant \sum_{j+2k=p} \sum_{\{k\}} \int_{\Delta_j} d\{\tau\} \prod_{1 \leqslant i \leqslant j} ||| f(\delta_i - 3, N)H_3(\tau_i) f(\delta_i, N)^{-1} ||| \\ &\prod_{0 \leqslant i \leqslant j} ||| f(\delta_i, N)U_{4,k_i}(\tau_i, \tau_{i+1}) f(\delta_i + 4k_i, N)^{-1} ||| \; ||| f(3j+4k, N)\Phi ||| \,, \end{aligned} \tag{4.20}$$

where $\delta_0 = 0$ and $\delta_i = 3i + 4(k_0 + \ldots + k_{i-1})$ for $1 \leqslant i \leqslant j$. The norms containing $H_3(\tau_i)$ are bounded by $2c_1(\tau_i)$, by using Lemma 3.2 of [*4*], while the norms containing $U_{4,k}$ are estimated by Lemma 4.4 above. We obtain

$$\begin{aligned} \| W_p(t,s)\Phi \| &\leqslant \exp\left\{ 8\int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\} \\ &\sum_{j+2k=p} \binom{j+k}{j} \exp\left\{ (3j+4k)\int_s^t d\tau c(\tau) \right\} \bar{\mu}(t-s)^k \\ &\int_{\Delta_j} d\{\tau\} \prod_{1 \leqslant i \leqslant j} (2c_1(\tau_i)) \, \| f(3j+4k, N)\Phi \| \,, \end{aligned} \tag{4.21}$$

where we have replaced $\delta_i + 4k_i$ by $\delta_j + 4k_j = 3j + 4k$ in the exponent resulting from the application of (4.13) and we have used the fact that $\bar{\mu}$ is an increasing function of t. The factor $\binom{j+k}{j}$ counts the number of terms in the sum over $\{ k \}$ for fixed j and k, since

$$\sum_{\{k\}} 1 = \text{coefficient of } y^k \text{ in } (1-y)^{-(j+1)} = \binom{j+k}{j} \leqslant 2^{j+k}\,.$$

We estimate the last norm in (2.41) by Lemma 3.1 of [*4*], and obtain

$$\begin{aligned} \| W_p(t,s)\Phi \| &\leqslant \exp\left\{ 8\int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\} C^{p/2} \\ &\times \| \exp(\varepsilon(N+1))\Phi \| \sum_{j+2k=p} ((2p+j)!)^{1/2}(j!)^{-1}(2Cb_3)^j(4C\bar{b}_4)^k \,, \end{aligned} \tag{4.22}$$

with C, b_3 and $\bar{b}_4$ defined by (4.17), (4.18), (4.19) respectively. Using the binomial inequality $(2p + j)! \leqslant (2p)!j!(1 + \alpha)^{2p}[(1 + \alpha)/\alpha]^j$ for $\alpha > 0$, we estimate the last sum as

$$\sum_{j+2k=p} \dots \leqslant ((2p)!)^{1/2}(4C\bar{b}_4)^{p/2} \times \operatorname*{Inf}_{\alpha>0} (1 + \alpha)^p \sum_j (j!)^{-1/2}\left(\frac{1 + \alpha}{\alpha} \cdot Cb_3^2\bar{b}_4^{-1}\right)^{j/2}. \quad (4.23)$$

The number of terms in the sum over j is $1 + [p/2] \leqslant p$ for $p \geqslant 1$. We estimate the sum as p times the maximum term, which we arrange to be the term $j = p$ by a suitable choice of α: for this purpose we define α by

$$\begin{aligned} \alpha &= 1/2 && \text{if} \quad p \leqslant 3Cb_3^2\bar{b}_4^{-1}, \\ \left(\frac{1 + \alpha}{\alpha}\right)Cb_3^2\bar{b}_4^{-1} &= p && \text{if} \quad p \geqslant 3Cb_3^2\bar{b}_4^{-1}. \end{aligned}$$

We bound the factor $(1 + \alpha)^p$ in (4.23) by $(3/2)^p$ and obtain

$$\sum_{j+2k=p} \dots \leqslant ((2p)!)^{1/2}p(p!)^{-1/2}3^p\{\operatorname{Max}(3(Cb_3)^2, pC\bar{b}_4)\}^{p/2}. \quad (4.24)$$

Eq. (4.16) now follows from (4.22), (4.24) and the binomial inequality $(2p)! \leqslant 2^{2p}(p!)^2$. Q. E. D.

The estimate (4.16) should be compared with the similar estimate (3.17) of [*4*] corresponding to the case of bounded potentials. While the term coming from H_3 in (4.16) still contributes a factor of the order $(p!)^{1/2}$ $(\text{Const})^p$ as previously, the term coming from H_4 now has an additional factor $p^{p/2}$ and is therefore estimated only as $p!\,(\text{Const})^p$. The occurrence of this additional factor $p^{p/2}$ can be traced back to Lemma 4.2 where the factor $(l!)^{-1}$ expected from the time ordering in (4.6) actually does not appear in the final estimate because of the local singularities of the potential (a more careful estimate should have a factor $(l!)^{-\varepsilon_2}$ in (4.5) with

$$\varepsilon_2 = 1 - n/(2p_2)).$$

As a consequence, with the analyticity properties available from Proposition 3.2, part (1), which is the same as for bounded potentials (see Proposition 4.1, part (1), of [*4*]), Borel summability is lost in the present case.

5. ESTIMATE OF THE REMAINDER

In this section we estimate the remainder $R_p(t, s)$ of the expansion of $W(t, s)$ as a power series in $z = \hbar^{1/2}$, given by (2.19). More precisely we estimate $R_p(t, s)\Phi$ for any $\Phi \in \mathscr{D}(\exp(\beta N))$ with β sufficiently large and

$(t - s)$ sufficiently small. The remainder $R_p(t, s)$ can be represented as in (2.22), with $R_{p,j}(t, s)$ and $R'_p(t, s)$ given by (2.23) and (2.24) respectively. Now, for V stable, $V \in L^\infty + L^{p_2}$ for $p_2 \geqslant 2$, $p_2 > n/2$, and $\varphi \in \mathscr{C}(\mathbb{R}, L^2 \cap L^q)$ with $1/p_2 + 1/q = 1/2$, φ a solution of the classical equation (3.14), $H_3(t)$ still satisfies the estimates of Lemma 3.2 of [*4*] and in particular (3.10) of [*4*]. Furthermore $U_4(t, s)$ satisfies the estimate (5.25) of [*4*] and $W(t, s)$ satisfies the estimates of Proposition 4.2 of [*4*], especially (4.38) of [*4*]. As a consequence, $R'_p(t, s)$ is estimated exactly as in [*4*] and therefore satisfies the estimate (6.21) of [*4*]. We therefore concentrate on the terms $R_{p,j}(t, s)$.

In order to estimate $R_{p,j}(t, s)$ using the representation (2.23), we need the estimate of $U_{4,k}(t, s)$ given by Lemma 4.4 and a similar estimate for $Q_{4,k}(t, s)$. The latter will be obtained by the use of (2.39) from the estimates contained in Lemmas 4.2 and 4.3 and similar estimates on $Q_{(l)}(t - s)$ and $\tilde{Z}_l(t, s)$ which we now set out to derive.

We first need a preliminary estimate on $U(t)$.

LEMMA 5.1. — Let V be stable, $V \in L^\infty + L^{p_2}$, with $p_2 > 1$, $p_2 > n/2$. Let $\eta \geqslant 0$, $z \in E(\eta)$ and $\varepsilon > 0$. Let $t \geqslant 0$ and define $\mu_1(t)$ by

$$\mu_1(t) = \operatorname*{Sup}_{\Phi, \|\Phi\| = 1} \int_0^t d\tau \, \| N \, | V_{ij} |^{1/2} U(\tau) \exp(- B\eta\tau N - \varepsilon N)\Phi \|^2 . \quad (5.1)$$

Then $\mu_1(t)$ is finite and satisfies for all $z \in E(\eta)$, $|z| \leqslant 1$,

$$\mu_1(t) \leqslant \bar{\mu}_1(t) = \operatorname*{Sup}_{N \geqslant 2} N^2 \exp(- 2\varepsilon N)(1 + 5\kappa_N t + (N\kappa_N t)^2) , \quad (5.2)$$

where

$$\kappa_N \leqslant \operatorname{Max}(\bar{\kappa}, Dt^{(p_0 - n/2)/(p_2 - p_0)} N^{2p_0/(p_2 - p_0)}) \quad (5.3)$$

for some p_0 such that $\operatorname{Max}(1, n/2) < p_0 < p_2$ and some $\bar{\kappa}$ depending only on V, and with D a constant depending only on V, p_2 and p_0.

Proof. — Since $U(.)$ commutes with N, it is sufficient to estimate

$$h(t) = \int_0^t d\tau \, \| \, | V_{12} |^{1/2} U(\tau) \exp(- B\eta\tau N)\Phi \|^2$$

for $\Phi \in \mathscr{H}_N$, $\| \Phi \| = 1$, $N \geqslant 2$.

Let $\kappa \geqslant \bar{\kappa}$, where $\bar{\kappa}$ is defined after (3.9). We split V as $V_\kappa + V'_\kappa$ according to (3.8). Then

$$h(t) \leqslant \kappa t + h'_\kappa(t) ,$$

where $h'_\kappa(t)$ is obtained from $h(t)$ by replacing V by V'_κ. We estimate $h'_\kappa(t)$ by the use of the integral equation

$$\begin{aligned} | V'_{\kappa 12} |^{1/2} U(\tau)\Psi &= | V'_{\kappa 12} |^{1/2} U_0(\tau)\Psi \\ &\quad - iz^2 \sum_{i<j} \int_0^\tau d\tau' \, | V'_{\kappa 12} |^{1/2} U_0(\tau - \tau')(V_{\kappa ij} + V'_{\kappa ij})U(\tau')\Psi , \quad (5.4) \end{aligned}$$

where $\Psi = \exp(-B\eta\tau N)\Phi$. Taking the norms of the various terms in $L^2([0,t], d\tau)$, we obtain, by the same method as in the proof of Lemmas 4.1 and 4.2

$$\begin{aligned} h'_\kappa(t)^{1/2} \leqslant & \left\{ \int_0^t d\tau \, \| \, | V'_{\kappa 12} |^{1/2} U_0(\tau) \Psi \|^2 \right\}^{1/2} \\ & + (\kappa t)^{1/2} \sum_{i<j} \operatorname{Sup}_{\lambda \in \mathbb{R}} \left| \! \left| \! \left| \int_0^t d\tau \, | V'_{\kappa 12} |^{1/2} U_0(\tau) \, | V_{\kappa ij} |^{1/2} e^{-i\tau\lambda} \right| \! \right| \! \right| \\ & + h'_\kappa(t)^{1/2} \sum_{i<j} \operatorname{Sup}_{\lambda \in \mathbb{R}} \left| \! \left| \! \left| \int_0^t d\tau \, | V'_{\kappa 12} |^{1/2} U_0(\tau) \, | V'_{\kappa ij} |^{1/2} e^{-i\tau\lambda} \right| \! \right| \! \right| \\ & \leqslant a_\kappa^{1/2} + \frac{1}{2} N(N-1)\kappa t a_\kappa^{1/2} + \frac{1}{2} N(N-1) a_\kappa h'_\kappa(t), \end{aligned}$$

where (see (4.2)) $a_\kappa = C_{p_0} t^{1-n/(2p_0)} \| V'_\kappa \|_{p_0}$.

We now choose $\kappa = \kappa_N$ sufficiently large, depending on N, such that $a_{\kappa_N} = N^{-2}$ and obtain

$$h'_{\kappa_N}(t)^{1/2} \leqslant 2N^{-1} + N\kappa t$$

and therefore

$$h(t) \leqslant \kappa_N t + (2N^{-1} + N\kappa_N t)^2 \leqslant 1 + 5\kappa_N t + (N\kappa_N t)^2 .$$

This implies the bound (5.2) for μ_1 under the condition $a_{\kappa_N} = N^{-2}$. In order to prove that $\bar{\mu}_1(t)$ is finite, it is sufficient to show that this condition implies that κ_N increases no faster than a power of N and more precisely that κ_N satisfies (5.3). Now from (3.10), we obtain, for $\kappa_N \geqslant \bar{\kappa}$,

$$C_{p_0} t^{1-n/(2p_0)} \| V'_{\bar{\kappa}} \|_{p_2}^{p_2/p_0} \kappa_N^{-(p_2-p_0)/p_0} \geqslant N^{-2} ,$$

from which (5.3) follows immediately. Q. E. D.

Remark 5.1. — Strictly speaking, the integral relation (5.4) can be proved only for sufficiently regular potentials, for instance under the additional assumption that $V \in L^\infty$. Therefore one should first introduce a cut off λ, prove the estimate (5.2) with V replaced by V_λ both in U and at the places where it occurs explicitly in the proof above, and then take the limit $\lambda \to \infty$ in the final result. We omit the details.

We next estimate $Q_l(t-s)$ as defined by (2.30).

LEMMA 5.2. — Let V be stable and $V \in L^\infty + L^{p_2}$ with $p_2 > 1$, $p_2 > n/2$. Let $\eta \geqslant 0$, $z \in E(\eta)$, $t \geqslant s$ and $\varepsilon > 0$. Let $\delta \geqslant 0$. Then, for any integer $l \geqslant 1$, $Q_l(t-s)$ satisfies the estimate

$$\begin{aligned} & ||| f(\delta, N) N Q_l(t-s) f(\delta + 4l, N)^{-1} \exp(-B\eta(t-s)N - \varepsilon N) ||| \\ & \qquad \leqslant 2^{-l} |z|^{2l} \mu(t-s)^{l-1/2} \mu_1(t-s)^{1/2} . \quad (5.5) \end{aligned}$$

Proof. — The proof is almost identical with that of Lemma 4.2, the only difference being that here one should take

$$F'_\alpha(\tau) = \theta(t - \tau)\theta(\tau)\,\mathrm{N}\,|\,\mathrm{V}_\alpha\,|^{1/2}\mathrm{U}(\tau)\exp(-\mathrm{B}\eta\tau\mathrm{N} - \varepsilon\mathrm{N})$$

and estimate this quantity by the use of Lemma 5.1. Q. E. D.

We next estimate the commutator that occurs in $\tilde{Z}_l(t, s)$.

LEMMA 5.3. — Let V be stable, $V \in L^\infty + L^{p_2}$ with $p_2 \geqslant 2$, $p_2 > n/2$. Let $\varphi \in \mathscr{C}(\mathbb{R}, L^2 \cap L^q)$ with $1/p_2 + 2/q = 1$. Let $\eta \geqslant 0$, $z \in E(\eta)$, $t \geqslant s$ and $\varepsilon > 0$. Let $\delta \geqslant 0$. Then, for any integer $l \geqslant 1$,

$$\begin{aligned}|||\, f(\delta, \mathrm{N})\,[\tilde{\mathrm{H}}_2(s) - \mathrm{H}_0, \tilde{\mathrm{Q}}_l(t, s)\,]\,f(\delta + 4l, \mathrm{N})^{-1}\exp(-\mathrm{B}\eta(t-s)\mathrm{N} - \varepsilon\mathrm{N})\,|||\\ \leqslant 2^{-l}\,|\,z\,|^{2l}\mu(t-s)^{l-1/2}\mu_1(t-s)^{1/2}\\ \times \{\, 2\,\|\, g(s)\,\|_\infty + c(s)(3 + \cosh(2\mathrm{B}\eta(t-s) + 2\varepsilon))\,\}, \quad (5.6)\end{aligned}$$

where $g(.)$ and $c(.)$ are the same functions are in Lemma 4.3.

Proof. — We define $\theta = \mathrm{B}\eta(t - s) + \varepsilon$ and estimate successively the contributions of $\tilde{\mathrm{G}}$, $\tilde{\mathrm{K}}$, $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{L}}^*$ to the L. H. S. of (5.6). Since $\tilde{\mathrm{G}}$ and $\tilde{\mathrm{K}}$ commute with N and are sums of one particle operators, the norms of which are bounded by $\|\, g(s)\,\|_\infty$ and $c(s)$ respectively, we obtain

$$\begin{aligned}&|||\, f(\delta, \mathrm{N})\,[\tilde{\mathrm{G}}(s) + \tilde{\mathrm{K}}(s), \tilde{\mathrm{Q}}_l(t, s)\,]\,f(\delta + 4l, \mathrm{N})^{-1}\exp(-\theta\mathrm{N})\,|||\\ &\leqslant 2(\,\|\, g(s)\,\|_\infty + c(s))\,|||\, f(\delta, \mathrm{N})\mathrm{N}\tilde{\mathrm{Q}}_l(t, s)f(\delta + 4l, \mathrm{N})^{-1}\exp(-\theta\mathrm{N})\,|||\,. \quad (5.7)\end{aligned}$$

The last norm on the R. H. S. of (5.7) is then estimated by Lemma 5.2. In order to estimate the terms with $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{L}}^*$ we commute these operators away from $\tilde{\mathrm{Q}}_l$ to the left or to the right and then use (2.18) of [*4*]. We obtain

$$\begin{aligned}&|||\, f(\delta, \mathrm{N})\,[\tilde{\mathrm{L}}(s) + \tilde{\mathrm{L}}(s)^*, \tilde{\mathrm{Q}}_l(t, s)\,]\,f(\delta + 4l, \mathrm{N})^{-1}\exp(-\theta\mathrm{N})\,|||\\ &\leqslant \frac{1}{2}c(s)\,\{\, |||\,(\mathrm{N}(\mathrm{N}-1))^{1/2}f(\delta, \mathrm{N}-2)\tilde{\mathrm{Q}}_l(t, s)f(\delta + 4l, \mathrm{N})^{-1}\exp(-\theta\mathrm{N})\,|||\\ &\quad + |||\,((\mathrm{N}+1)(\mathrm{N}+2))^{1/2}f(\delta, \mathrm{N}+2)\tilde{\mathrm{Q}}_l(t, s)f(\delta + 4l, \mathrm{N})^{-1}\exp(-\theta\mathrm{N})\,|||\\ &\quad + e^{-2\theta}\,|||\, f(\delta, \mathrm{N})\tilde{\mathrm{Q}}_l(t, s)f(\delta+4l, \mathrm{N}+2)^{-1}((\mathrm{N}+1)(\mathrm{N}+2))^{1/2}\exp(-\theta\mathrm{N})\,|||\\ &\quad + e^{2\theta}\,|||\, f(\delta, \mathrm{N})\tilde{\mathrm{Q}}_l(t, s)f(\delta + 4l, \mathrm{N}-2)^{-1}(\mathrm{N}(\mathrm{N}-1))^{1/2}\exp(-\theta\mathrm{N})\,|||\,\}\\ &\leqslant c(s)(1 + \cosh 2\theta)\,|||\, f(\delta, \mathrm{N})\tilde{\mathrm{Q}}_l(t, s)f(\delta + 4l - 2, \mathrm{N})^{-1}\exp(-\theta\mathrm{N})\,|||\,, \quad (5.8)\end{aligned}$$

where we have kept the largest of the four preceding norms (for $l \geqslant 1$). The last norm in (5.8) is estimated by the same method as in Lemma 5.2 and also satisfies (5.5). Collecting all the estimates, we obtain (5.6). Q. E. D.

The next step consists in estimating $\mathrm{Q}_{4,k}(t, s)$ or equivalently $\tilde{\mathrm{Q}}_{4,k}(t, s)$ as given by (2.32) and Lemma 2.2.

LEMMA 5.4. — Let the assumptions of Lemma 5.3 be satisfied. Let s, t, $t \geqslant s$, be such that the equation (5.4) of [*4*] with boundary condition

$\alpha_4(t) = \varepsilon$ has a solution $\alpha_4(.) \in \mathscr{C}^1([s, t], \mathbb{R}^+)$. Let $\delta \geqslant 0$. Then, for any integer $k \geqslant 1$, $Q_{4,k}(t, s)$ satisfies the estimate

$$\begin{aligned} &||| f(\delta, N)Q_{4,k}(t, s)f(\delta + 4k, N)^{-1} \exp(-\alpha_4(s)(N + 1)) ||| \\ &\quad \leqslant \frac{1}{2} | z |^{2k} \bar{\mu}(t - s)^{k - 1/2} \mu_1(t - s)^{1/2} \\ &\quad\quad \times \exp\left\{ (\delta + 4k) \int_s^t d\tau c(\tau) \exp(2\alpha_4(\tau)) + 8 \int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\} \\ &\quad\quad \times \left\{ 1 + \int_s^t d\tau(2 \| g(\tau) \|_\infty + (3 + \cosh(2\alpha_4(\tau)))c(\tau)) \right\}. \end{aligned} \tag{5.9}$$

Proof. — We first note that, because of (5.4) of [*4*], $\alpha_4(.)$ satisfies the inequality

$$\alpha_4(\tau) \geqslant \varepsilon + B\eta(t - \tau) \tag{5.10}$$

for all $\tau \in [s, t]$. By (2.39), $\tilde{Q}_{4,k}(t, s)$ is given as the sum of two terms which we denote by $\tilde{Q}^{I}_{4,k}(t, s)$ and $\tilde{Q}^{II}_{4,k}(t, s)$ and which we estimate separately. Exactly as in the proof of Lemma 4.4 (see (4.15)),

$$\begin{aligned} &||| f(\delta, N)\tilde{Q}^{I}_{4,k}(t, s)f(\delta + 4k, N)^{-1} \exp(-B\eta(t - s)N - \varepsilon N) ||| \\ &\quad \leqslant | z |^{2k} 2^{-k} \bar{\mu}(t - s)^{k - 1/2} \mu_1(t - s)^{1/2} \exp\left\{ (\delta + 4k) \int_s^t d\tau c(\tau) \right\} \\ &\quad\quad \times \sum_{0 \leqslant r \leqslant k} \frac{1}{r!} \left\{ 4 \int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\}^r \sum_{\{l\}}' \prod_{0 \leqslant i < r} l_i . \end{aligned} \tag{5.11}$$

We then estimate the sum over $\{ l \}$ as

$$\begin{aligned} \sum_{\{l\}}' \prod_{0 \leqslant i < r} l_i &= \text{coefficient of } y^k \text{ in } y^{r+1}(1 - y)^{-(2r+1)} \\ &= \binom{k + r - 1}{k - r - 1} \leqslant 2^{k+r-1} , \end{aligned} \tag{5.12}$$

and extend the sum over r up to infinity:

$$\begin{aligned} \ldots &\leqslant \frac{1}{2} | z |^{2k} \bar{\mu}(t - s)^{k - 1/2} \mu_1(t - s)^{1/2} \\ &\quad \times \exp\left\{ (\delta + 4k) \int_s^t d\tau c(\tau) + 8 \int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\}. \end{aligned} \tag{5.13}$$

We now estimate $\tilde{Q}^{II}_{4,k}(t, s)$ exactly in the same way and obtain an estimate similar to (4.14) with however the last norm replaced by

$$\begin{aligned} &||| f(\delta_r, N)\tilde{Z}_{l_r}(\tau_r, \tau_{r+1})f(\delta + 4k, N)^{-1} \exp(-\alpha_4(\tau_{r+1})(N + 1)) ||| \\ &\quad \times ||| f(\delta + 4k, N) \exp(\alpha_4(\tau_{r+1})(N + 1))\tilde{U}_4(\tau_{r+1}, s) \exp(-\alpha_4(s)(N + 1)) \\ &\quad \times f(\delta + 4k, N)^{-1} ||| . \end{aligned}$$

We estimate these two norms by Lemma 5.3 and by (5.6) of [4] respectively and the other terms as in the proof of Lemma 4.4. Thus

$$\begin{aligned}
&||| f(\delta, \mathrm{N})\tilde{\mathrm{Q}}^{\mathrm{II}}_{4,k}(t, s)f(\delta + 4k, \mathrm{N})^{-1} \exp(-\alpha_4(s)(\mathrm{N} + 1)) ||| \\
&\quad \leqslant |z|^{2k}2^{-k}\bar{\mu}(t-s)^{k-1/2}\mu_1(t-s)^{1/2} \exp\left\{ (\delta + 4k) \int_s^t d\tau c(\tau) \exp(2\alpha_4(\tau)) \right\} \\
&\quad \times \sum_{0 \leqslant r \leqslant k} \int_{\Delta_{r+1}} d\{\tau\} \prod_{1 \leqslant i \leqslant r} (4(\| g(\tau_i) \|_\infty + 2c(\tau_i))) \\
&\quad \times \{ 2 \| g(\tau_{r+1}) \|_\infty + (3 + \cosh(2\alpha_4(\tau_{r+1})))c(\tau_{r+1}) \} \sum_{\{l\}}{}' \prod_{0 \leqslant i < r} l_i, \qquad (5.14)
\end{aligned}$$

where we have used (5.10). We estimate the integral over τ_{r+1} by extending the integration to the whole interval $[s, t]$, perform the integral over $(\tau_1, \ldots, \tau_r)$ as before, and estimate the sum over $\{ l \}$ by (5.12). Then

$$\begin{aligned}
\ldots &\leqslant \frac{1}{2} |z|^{2k}\bar{\mu}(t-s)^{k-1/2}\mu_1(t-s)^{1/2} \\
&\quad \times \exp\left\{ (\delta + 4k) \int_s^t d\tau c(\tau) \exp(2\alpha_4(\tau)) + 8 \int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\} \\
&\quad \times \int_s^t d\tau(2 \| g(\tau) \|_\infty + (3 + \cosh(2\alpha_4(\tau)))c(\tau)). \qquad (5.15)
\end{aligned}$$

Collecting (5.13) and (5.15) yields (5.9). Q. E. D.

We are now ready to estimate the contribution of the terms $\mathrm{R}_{p,j}(t, s)$ to the remainder $\mathrm{R}_p(t, s)$ of the expansion of $\mathrm{W}(t, s)$ as a power series in z, using the representation given in Lemma 2.1.

PROPOSITION 5.1. — Let V be stable, $\mathrm{V} \in \mathrm{L}^\infty + \mathrm{L}^{p_2}$ with $p_2 \geqslant 2$, $p_2 > n/2$. Let $\varphi \in \mathscr{C}(\mathbb{R}, \mathrm{L}^2 \cap \mathrm{L}^q)$ with $1/p_2 + 1/q = 1/2$. Let $\eta \geqslant 0$, $z \in \mathrm{E}(\eta)$, and $\varepsilon > 0$. Let $s, t \in \mathbb{R}$, $t \geqslant s$, be such that the equation (5.4) of [4] with boundary condition $\alpha_4(t) = \varepsilon$ has a solution $\alpha_4(.) \in \mathscr{C}^1([s, t], \mathbb{R}^+)$, and let

$$\Phi \in \mathscr{D}(\exp((\alpha_4(s) + \varepsilon)\mathrm{N}).$$

Then the contribution to $\mathrm{R}_p(t, s)$ of the terms $\mathrm{R}_{p,j}(t, s)$, namely of the first sum in the R. H. S. of (2.22), satisfies the estimate

$$\begin{aligned}
&\sum_{0 \leqslant j < p} \| \mathrm{R}_{p,j}(t, s)\Phi \| \\
&\quad \leqslant \frac{1}{2} \mu_1(t-s)^{1/2}\bar{\mu}(t-s)^{-1/2} \exp\left\{ 8 \int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\} \\
&\quad \left\{ 1 + \int_s^t d\tau(2 \| g(\tau) \|_\infty + (3 + \cosh 2\alpha_4(\tau))c(\tau)) \right\} \\
&\quad \times p(p!)^{1/2}6^p |z|^p \{ \mathrm{C}' \operatorname{Max}(3(\mathrm{C}'b'_3)^2, p\mathrm{C}'\bar{b}_4) \}^{p/2} \\
&\quad \times \{ 1 + 6|z|((p+1)e\mathrm{C}' \operatorname{Max}(3(\mathrm{C}'b'_3)^2, (p+1)\mathrm{C}'\bar{b}_4))^{1/2} \} \\
&\quad \times \| \exp((\alpha_4(s) + \varepsilon)(\mathrm{N} + 1))\Phi \|, \qquad (5.16)
\end{aligned}$$

where C' and b_3' are defined by (6.24) and (6.25) of [*4*], namely

$$C' = (1 - e^{-2\varepsilon})^{-1} \exp\left\{ 2 \int_s^t d\tau c(\tau) \exp(2\alpha_4(\tau)) \right\}, \tag{5.17}$$

$$b_3' = 2 \int_s^t d\tau c_1(\tau) \cosh(\alpha_4(\tau)), \tag{5.18}$$

and $\bar{b}_4 = \frac{1}{2} \bar{\mu}(t - s)$ (see (4.19)).

Proof. — We use the representation (2.23) of Lemma 2.1 and estimate the various factors as in the proof of Proposition 4.1 (see especially (4.20), (4.21)), with however the operators $U_4(.,.)$ now sandwiched between additional factors $\exp(\pm \alpha_4(.)(N + 1))$. Using the estimate (5.25) of [*4*] (see Proposition 3.3, part (1)), the estimates (4.13) of Lemma 4.4 and (5.9) of Lemma 5.4, and the estimate (6.27) of [*4*], we obtain

$$\begin{aligned}
\sum_{0 \leqslant j < p} \| R_{p,j}(t, s)\Phi \| \leqslant \frac{1}{2} \sum_{0 \leqslant j < p} & | z |^{j+2k} \bar{\mu}(t - s)^{k - 1/2} \mu_1(t - s)^{1/2} \\
& \times \exp\left\{ (3j + 4k) \int_s^t d\tau c(\tau) \exp(2\alpha_4(\tau)) + 8 \int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\} \\
& \times \left\{ 1 + \int_s^t d\tau (2 \| g(\tau) \|_\infty + (3 + \cosh(2\alpha_4(\tau)))c(\tau) \right\} \\
& \times (j!)^{-1} \left\{ \int_s^t d\tau 2c_1(\tau) \cosh(\alpha_4(\tau)) \right\}^j \binom{j + k}{j} \| f(3j + 4k, N)\Phi \| .
\end{aligned}$$

We estimate the last norm by the use of Lemma 3.1 of [*4*]:

$$\begin{aligned}
\ldots \leqslant \frac{1}{2} \mu_1(t - s)^{1/2} \bar{\mu}(t - s)^{-1/2} & \exp\left\{ 8 \int_s^t d\tau(\| g(\tau) \|_\infty + 2c(\tau)) \right\} \\
& \times \left\{ 1 + \int_s^t d\tau (2 \| g(\tau) \|_\infty + (3 + \cosh(2\alpha_4(\tau)))c(\tau)) \right\} \\
& \times \sum_{0 \leqslant j < p} | z |^{j+2k} C'^{(j+2k)/2} ((3j + 4k)!)^{1/2} (j!)^{-1} (2C'b_3')^j (4C'\bar{b}_4)^k \\
& \times \| \exp((\alpha_4(s) + \varepsilon)(N + 1))\Phi \| ,
\end{aligned}$$

with C' and b_3' defined by (5.17) and (5.18) respectively. The last sum over j is of the same type as that in (4.22), with however $j + 2k$ now taking the values p or $p + 1$. We estimate it exactly as before (see (4.23), (4.24)), thereby obtaining (5.16). Q. E. D.

The estimate (5.16) should be compared with the similar estimate (6.22) of [*4*] corresponding to the case of bounded potentials. As in the case of the general term (4.16) and for the same reason, the estimate (5.16) contains an additional factor $p^{p/2}$ for singular potentials (see the remark at the end of Section 4).

We are now in a position to state our main result on the expansion of $W(t, s)$ as a power series in z.

THEOREM 5.1. — Let V be stable, $V \in L^\infty + L^{p_2}$, with $p_2 \geqslant 2$, $p_2 > n/2$. Let $\varphi \in \mathscr{C}(\mathbb{R}, L^2 \cap L^q)$ with $1/p_2 + 1/q = 1/2$, φ a solution of the classical equation (3.14). Let $\eta > 0$, let $\beta > 0$ and $\Phi \in \mathscr{D}(\exp(\beta N))$. Then for all $s \in \mathbb{R}$, there exists $\theta > 0$ such that, if $t \in [s, s + \theta]$, $W(t, s)\Phi$ is analytic in z for $z \in E_0(\eta)$ and has an asymptotic expansion at $z = 0$ which satisfies an estimate of the form

$$\| R_p(t, s)\Phi \| \leqslant C_0 p!(C_1 \mid z \mid)^p \| \exp(\beta N)\Phi \| \tag{5.19}$$

for all $z \in E(\eta)$, where $E_0(\eta)$ and $E(\eta)$ are defined by (3.6), (3.7).

The constants θ, C_0 and C_1 depend on V, η, β, s and φ. For fixed V, η, β and φ, they can be taken uniform in s on the compact subsets of $\mathbb{R}$. For fixed η and β, they can be taken uniform in V, s and φ if

$$V = V_1 + V_2 , \quad V_1 \in L^\infty , \quad V_2 \in L^{p_2}$$

with $\| V_1 \|_\infty$, $\| V_2 \|_{p_2}$ and the stability constant B uniformly bounded, and if $\varphi(\tau)$ is bounded in L^2 and in L^q uniformly in φ and τ.

Proof. — The theorem follows from the analyticity properties of $W(t, s)$ given in Proposition 3.2, part (1), and from the estimates of the remainder $R_p(t, s)$ contained in Proposition 5.1 for the terms $R_{p,j}(t, s)$ and in Proposition 6.1 of [*4*] for the term $R'_p(t, s)$ (see the introduction of this section).
Q. E. D.

6. UNIFORMITY IN TIME AND ASYMPTOTIC EXPANSION OF THE S-MATRIX

The results stated so far in this paper and in [*4*] are restricted to small time intervals. The reason is that in general the solutions of (5.4) and (4.36) of [*4*] blow up in a finite time. However in the special case of real positive z, namely for physical values of $\hbar$, these equations have solutions for arbitrary time intervals and therefore Theorem 6.1 of [*4*] and Theorem 5.1 hold without restrictions on $t - s$. Even in this situation, the estimates on the general term and on the remainder of the expansion of $\tilde{W}(t, s)$ in general depend on t and s. Under stronger assumptions, these estimates can be made uniform in t and s, thereby yielding an asymptotic expansion in $\hbar^{1/2}$ for $\tilde{W}(+\infty, -\infty)$. This quantity is related to the S-matrix, as explained in [*2*] (see especially (1.56) of [*2*]).

THEOREM 6.1. — Let the assumptions of Theorem 5.1 be satisfied. In addition let $z \in \mathbb{R}^+$, $n \geqslant 3$, $V \in L^{p_1}$ for some p_1, $1 \leqslant p_1 < n/2$. Assume that there exists a $\rho > 0$ such that

$$\operatorname*{Sup}_{0 \leqslant z \leqslant 1} \ \operatorname*{Sup}_{\Phi, \|\Phi\| = 1} \int_{-\infty}^{\infty} d\tau \, \| \mid V_{12} \mid^{1/2} U(\tau) N^{-\rho} \Phi \| < \infty . \tag{6.1}$$

Let the functions $\| g(.) \|_\infty$, $c(.)$ and $c_1(.)$ defined by (1.38), (2.19) and (3.8) of [4] be in $L^1(\mathbb{R})$. Then $\tilde{W}(+\infty, -\infty)$ has an asymptotic expansion at $z = 0$ which satisfies an estimate of the form (5.19) for all z with $0 \leqslant z \leqslant 1$.

The proof of this theorem is a simple check that the estimates of Sections 4 and 5 are uniform in time under the assumptions made. In particular the assumption $V \in L^{p_1} \cap L^{p_2}$ implies uniform boundedness of μ (defined by (4.1)) as a function of t [5, 6]. Similarly (6.1) implies uniform boundedness of μ_1 (defined by (5.1)) as a function of t. (This condition is a strong form of the asymptotic completeness for the N-body problem). The integrability conditions on $\| g(.) \|_\infty$, $c(.)$ and $c_1(.)$ mean that the classical solution φ decays suitably for large time (see Proposition 2.2 of [2], and for more details [3]). Under suitable assumptions on V, the integrability of $\| g(.) \|_\infty$ and of $c(.)$ can be proved for a large class of solutions, while the integrability of $c_1(.)$ lies beyond the reach of the methods used in [3].

ACKNOWLEDGMENTS

One of us (G. V.) is grateful to K. Chadan for the kind hospitality at the Laboratoire de Physique Théorique in Orsay, where part of this work was done.

Note added in proof. — In all the statements made in the rest of this section and involving H_4^κ, namely in Proposition 3.1, part (2), in Proposition 3.2 and in Proposition 3.3, we assume that $\bar{V}_\kappa$ is stable for sufficiently large κ, we assume that κ is large accordingly, and we choose a stability constant B that ensures the stability condition (3.5) both for V and $\bar{V}_\kappa$. Stability of $\bar{V}_\kappa$ for large κ holds in particular if V is the sum of a bounded stable potential and of a positive potential. In the general case, one can use a different, more complicated cut off procedure, similar to that used in [2], to define a different set of approximate operators U_κ, $U_{4\kappa}$ and W_κ, for which all the results of Section 3 hold.

The content of this remark has no incidence on the estimates and results of the rest of the paper (Sections 4 to 6).

REFERENCES

[1] K. O. FRIEDRICHS, Perturbation of spectra in Hilbert space. Providence, R. I.: *Am. Math. Soc.*, 1965.

[2] J. GINIBRE, G. VELO, *Comm. Math. Phys.*, t. **66**, 1979, p. 37-76.

[3] J. GINIBRE, G. VELO, *Math. Z.*, t. **170**, 1980, p. 109-136.

[4] J. GINIBRE, G. VELO, The classical field limit of non-relativistic bosons. I. Borel summability for bounded potentials, *Ann. Phys. N. Y., in press.*

[5] R. J. IORIO Jr., M. O'CARROLL, *Comm. Math. Phys.*, t. **27**, 1972, p. 137-145.

[6] T. KATO, *Math. Ann.*, t. **162**, 1966, p. 258-279.

[7] M. REED, B. SIMON, Methods of Modern Mathematical Physics, Vol. **I**, *Functional Analysis*, New York, Academic Press, 1972.

(Manuscrit reçu le 20 juin 1980)