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Resonances in an abstract analytic scattering theory

by

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ABSTRACT. — An abstract analytic scattering theory is constructed by adding one analyticity assumption in the Kato-Kuroda scattering theory. Resonances are defined and are identified with poles of the S-matrix. Applications are given to Schrödinger operators (with and without Stark effect) in $L^2(\mathbb{R}^n)$.

1. INTRODUCTION

An abstract analytic scattering theory is constructed for a pair of self-adjoint operators (H_2, H_1) (in a Hilbert space H) satisfying the assumptions of the Kato-Kuroda abstract theory of scattering in the form given by Kuroda in [*9*] [*10*] by adding one analyticity assumption. The results are localized with respect to the spectral parameter. A meromorphic continuation of the S-matrix is constructed. For an explicitly characterized dense set of vector f, g a meromorphic continuation of $((H_2 - \zeta)^{-1}f, g)$ is constructed from $\{ \operatorname{Im} \zeta > 0 \}$ into a subset of the lower half plane. The set of poles of this continuation for all allowable f, g is called the set of resonances. The main abstract result is that this set of resonances is equal to the set of poles of the continued S-matrix. Hence resonances have an intrinsic meaning for the pair (H_2, H_1).

An important example by Howland [*7*] shows that it is not possible to give a satisfactory definition of a resonance depending only on the structure of a single operator in an abstract Hilbert space. We show that for a pair of operators it is possible to define resonances with an intrinsic meaning. For further discussion on this point, see [*14*].

The analytic continuations are obtained using the local distortion technique (spectral deformation technique) in a spectral representation for H_1. This technique has previously been applied to various concrete problems (see for instance [*2*] [*3*] [*11*] [*15*]). Our approach is based on the following result (see [*4*]): let H be a selfadjoint operator in a Hilbert space H with spectral family $E(\lambda)$. Let $f, g \in H$. $((H - \zeta)^{-1}f, g)$ can be continued analytically across an interval $I \subset \mathbb{R}$ from above and below if and only if $\lambda \mapsto (E(\lambda)f, g)$ is (real) analytic in I.

In order to apply the abstract results one must be able to find the spectral representation of H_1 explicitly. We give two examples. The first is concerned with the pair $(-\Delta, -\Delta + V)$ in $L^2(\mathbb{R}^n)$. Our technique requires an exponentially decaying V and simplifies and generalises to $L^2(\mathbb{R}^n)$ the result for the case $n = 3$ given in [*2*] [*8*]. Our second example is concerned with the Stark effect Hamiltonians $(-\Delta + \vec{E}\cdot\vec{x}, -\Delta + \vec{E}\cdot\vec{x} + V)$ in $L^2(\mathbb{R}^n)$. Here one needs a potential decaying exponentially in the negative $\vec{E}$-direction. We use some results on the same problem obtained in [*16*] [*17*]. We consider a larger class of potentials than in [*17*]. Also our proof on the resonances differs from the one given in [*17*]. Further applications will be given elsewhere.

2. NOTATION AND ASSUMPTIONS

In this section we give our assumptions. We follow the notation of [*9*] except as noted below and refer to [*9*] for several results we need.

Consider two selfadjoint operators H_1, H_2 on a Hilbert space H. $\rho(H_j)$ is the resolvent set and $R_j(\zeta)$ the resolvent of H_j, $j = 1, 2$. H_1 and H_2 are formally related as $H_2 = H_1 + B^*A = H_1 + A^*B$; note that we omit the operator C from [*9*]. We can always reintroduce C by replacing B by CB or A by C^*A.

ASSUMPTION 2.1. — A and B are closed operators from H to another Hilbert space K with $D(A) \supset D(H_1)$ and $D(B) \supset D(H_1)$.

ASSUMPTION 2.2. — We assume that $BR_1(\zeta)A^*$ is closable and its closure $[BR_1(\zeta)A^*]^a \in B(K)$ for one (or equivalently all) $\zeta \in \rho(H_1)$.

For $\zeta \in \rho(H_1)$ let $Q_1(\zeta) = [BR_1(\zeta)A^*]^a$ and $G_1(\zeta) = 1 + Q_1(\zeta)$.

ASSUMPTION 2.3. — For every $\zeta \in \rho(H_1) \cap \rho(H_2)$ $G_1(\zeta)^{-1} \in B(K)$. Furthermore $R_2(\zeta) = R_1(\zeta) - [R_1(\zeta)A^*]^a G_1(\zeta)^{-1} BR_1(\zeta)$ holds for every $\zeta \in \rho(H_1) \cap \rho(H_2)$.

The results [*9*; Proposition 2.6, 2.7] are now available.

As mentioned in the introduction our results are localized with respect to the spectral parameter. Let $I \subset \mathbb{R}$ be an open (non-empty) interval, and let E_j denote the spectral measure associated with H_j, $j = 1, 2$.

ASSUMPTION 2.4. — There exists a Hilbert space C and a unitary operator F from $E_1(I)H$ onto $L^2(I; C)$ such that for every Borel set $I' \subset I$ one has $FE_1(I')F^{-1} = \chi_{I'}\cdot$, where $\chi_{I'}\cdot$ stands for multiplication by the characteristic function of I'.

ASSUMPTION 2.5. — There exist $B(K, C)$-valued functions $T(\lambda, A)$ and $T(\lambda, B)$, $\lambda \in I$, such that

i) there exists an open connected set $\Omega \subset \mathbb{C}$ with $\Omega \cap \mathbb{R} = I$ and $\{ \bar{z} \mid z \in \Omega \} = \Omega$ such that $T(\lambda, A)$ and $T(\lambda, B)$ can be extended to Ω as analytic functions with values in $B(K, C)$;

ii) there exist dense subsets $D \subset D(A^*)$ and $D' \subset D(B^*)$ such that for any $u \in D$ and $v \in D'$ one has

$$T(\lambda\,;\,A)u = (FE_1(I)A^*u)(\lambda) \qquad \text{for a. e.} \quad \lambda \in I$$
$$T(\lambda\,;\,B)v = (FE_1(I)B^*v)(\lambda) \qquad \text{for a. e.} \quad \lambda \in I$$

ASSUMPTION 2.6. — For one (or equivalently all) $\zeta \in \rho(H_1)$ either $AR_1(\zeta) \in B_\infty(H, K)$ or $BR_1(\zeta) \in B_\infty(H, K)$. Here $B_\infty(H, K)$ denotes the compact operators from H to K.

ASSUMPTION 2.7. — The subspace generated by $\{ E_j(I')A^*u \mid u \in D(A^*),$ $I' \subset I$ a Borel set $\}$ is dense in $E_j(I)H$, $j = 1, 2$.

REMARK 2.8. — These assumptions are identical with the assumptions 2.1-2.4 and 3.2-3.5 in [9] except that 3.5 *i)* has been strengthened by requiring $T(\lambda, A)$ and $T(\lambda, B)$ be real analytic instead of locally Hölder continuous on I. For further comments, see [9, Remark 3.6].

These assumptions imply that we have all the results of [9, § 3, § 4, Theorem 6.3] at our disposal. Discreteness of the singular spectrum of H_2 in I is shown below and is an easy consequence of the analyticity assumptions.

3. MEROMORPHIC CONTINUATION OF THE S-MATRIX

We begin by constructing analytic continuations of some operator- and vector-valued functions. Let us use the notation $\pi^\pm = \{ \zeta \in \mathbb{C} \mid \pm \operatorname{Im} \zeta > 0 \}$, and $\Omega^\pm = \pi^\pm \cap \Omega$. Let $Q_{1\pm}(\zeta)$ be the restriction of $Q_1(\zeta)$ to $\pi^\pm$.

PROPOSITION 3.1. — As an analytic function in π^+ with values in $B(K)$ $Q_{1+}(\zeta)$ has an analytic continuation to $\pi^+ \cup \Omega^- \cup I$, denoted

$\tilde{Q}_{1+}(\zeta)$. Similarly, $Q_{1-}(\zeta)$ has an analytic continuation, denoted $\tilde{Q}_{1-}(\zeta)$, from π^- to $\pi^- \cup \Omega^+ \cup I$. We have for $\zeta \in \Omega$ the relation

$$\tilde{Q}_{1+}(\zeta) - \tilde{Q}_{1-}(\zeta) = 2\pi i T(\bar{\zeta}\,;\,B)^* T(\zeta\,;\,A) \tag{3.1}$$

Proof. — Let us first consider $Q_{1+}(\zeta)$. For $u \in D$, $v \in D'$ we have (with $I^c = \mathbb{R} \setminus I$)

$$\begin{aligned}(Q_1(\zeta)u, v) &= (R_1(\zeta)A^*u, B^*v) \\ &= (R_1(\zeta)E_1(I^c)A^*u, B^*v) + (R_1(\zeta)E_1(I)A^*u, E_1(I)B^*v) \\ &= ([BR_1(\zeta)E_1(I^c)A^*]^a u, v) + (R_1(\zeta)E_1(I)A^*u, E_1(I)B^*v)\end{aligned}$$

It is easy to see that $\zeta \mapsto [BR_1(\zeta)E_1(I^c)A^*]^a$ is analytic in $\pi^+ \cup I \cup \pi^-$ with values in $B(K)$. For the second term we use the spectral representation. Assume $\zeta \in \pi^+$.

$$\begin{aligned}(R_1(\zeta)E_1(I)A^*u, E_1(I)B^*v) &= (FR_1(\zeta)E_1(I)A^*u, FE_1(I)B^*v) \\ &= \int_I (\lambda - \zeta)^{-1}(T(\lambda\,;\,A)u, T(\lambda\,;\,B)v)d\lambda\,.\end{aligned}$$

Let Γ be a piecewise C^1-curve as indicated on Figure 1.

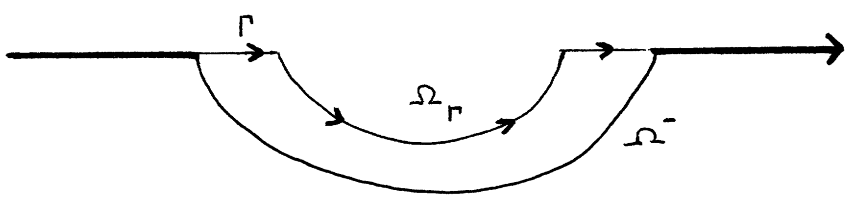

FIG. 1.

Let Ω_Γ be the domain enclosed by Γ and the real axis. Assumption 2.5 and Cauchy's theorem imply that we have

$$(R_1(\zeta)E_1(I)A^*u, E_1(I)B^*v) = \int_\Gamma (\lambda - \zeta)^{-1}(T(\lambda\,;\,A)u, T(\bar{\lambda}\,;\,B)v)d\lambda\,.$$

Thus the left hand side can be continued analytically from π^+ into Ω_Γ. We get a continuation as an operator-valued function by observing that due to Assumption 2.5 *i)* for every compact set $K \subset \pi^+ \cup \Omega_\Gamma$ there exists a constant $c_K > 0$ such that

$$\left| \int_\Gamma (\lambda - \zeta)^{-1}(T(\lambda\,;\,A)u, T(\bar{\lambda}\,;\,B)v)d\lambda \right| \leqslant c_K \,\| u \|\, \| v \|$$

for all $u \in D$, $v \in D'$, $\zeta \in K$. By varying Γ the result follows, using uniqueness of analytic continuation.

If we fix $\zeta_0 \in \Omega^-$ and assume Γ chosen such that $\zeta_0 \in \Omega_\Gamma$, we obtain from Cauchy's theorem the following expression

$$\begin{aligned}(\tilde{Q}_{1+}(\zeta_0)u, v) &= (R_1(\zeta_0)E_1(I^c)A^*u, B^*v) \\ &\quad + \int_\Gamma (\lambda - \zeta_0)^{-1}(T(\lambda ; A)u, T(\bar{\lambda} ; B)v)d\lambda \\ &= (R_1(\zeta_0)E_1(I^c)A^*u, B^*v) + \int_I (\lambda - \zeta_0)^{-1}(T(\lambda ; A)u, T(\lambda ; B)v)d\lambda \\ &\quad + 2\pi i(T(\zeta_0 ; A)u, T(\bar{\zeta}_0 ; B)v) \\ &= (Q_{-1}(\zeta_0)u, v) + (2\pi i T(\bar{\zeta}_0 ; B)^*T(\zeta_0 ; A)u, v) .\end{aligned}$$

Hence we have proved for $\zeta \in \Omega^-$

$$\tilde{Q}_{1+}(\zeta) - \tilde{Q}_{1-}(\zeta) = 2\pi i T(\bar{\zeta} ; B)^*T(\zeta ; A) ,$$

and uniqueness of analytic continuation implies that for every $\zeta \in \Omega$ we have (3.1). We use the notation $\tilde{G}_{1\pm}(\zeta) = 1 + \tilde{Q}_{1\pm}(\zeta)$.

PROPOSITION 3.2. — There exist discrete sets $e_\pm \subset I$ with the end points of I as the only possible points of accumulation and discrete sets $r_\pm \subset \Omega^\mp$ with $\partial\Omega^\mp \setminus I$ as the only possible points of accumulation such that $\tilde{G}_{1\pm}(\zeta)$ are invertible for $\zeta \in (\pi^\pm \cup \Omega^\mp \cup I) \setminus (e_\pm \cup r_\pm)$ and $\tilde{G}_{1\pm}(\zeta)^{-1}$ are meromorphic in $\pi^\pm \cup \Omega^\mp \cup I$ with poles in $e_\pm \cup r_\pm$.

REMARK 3.3. — We can avoid accumulation at $\partial\Omega^\pm \setminus I$ by considering a domain slightly smaller than Ω, so these accumulation points are not very important.

Proof. — The result is an immediate consequence of Proposition 3.1 and the analytic Fredholm theorem.

The next step is to construct analytic continuations of some matrix elements of $R_1(\zeta)$ and $R_2(\zeta)$.

DEFINITION 3.4. — We denote by $\mathscr{R}_I$ the set of function $f \in L^2(I, C)$ such that $f : I \to C$ has an analytic continuation to Ω with values in C.

REMARK 3.5. — $\mathscr{R}_I$ is dense in $L^2(I, C)$.

PROPOSITION 3.6. — For $f, g \in F^{-1}\mathscr{R}_I$, $(R_1(\zeta)f, g)$ has an analytic continuation from $\pi^\pm$ into $\pi^\pm \cup \Omega^\mp \cup I$.

Proof. — Choose Γ as in Figure 1. For $\zeta \in \pi^+$ we have, using Cauchy's theorem and $Ff, Fg \in \mathscr{R}_I$

$$\begin{aligned}(R_1(\zeta)f, g) &= (R_1(\zeta)E_1(I^c)f, g) + \int_I (\lambda - \zeta)^{-1}(Ff(\lambda), Fg(\lambda))d\lambda \\ &= (R_1(\zeta)E_1(I^c)f, g) + \int_\Gamma (\lambda - \zeta)^{-1}(Ff(\lambda), Fg(\bar{\lambda}))d\lambda .\end{aligned}$$

Hence we have an analytic continuation of $(R_1(\zeta)f, g)$ into Ω_Γ. The result now follows by varying Γ.

REMARK 3.7. — $\mathcal{R}_I$ is the largest set of vectors for which we can obtain analytic continuation of $(R_1(\zeta)f, g)$, cf. the result from [*4*] mentioned in the introduction.

PROPOSITION 3.8. — *For* $f, g \in F^{-1}\mathcal{R}_I$ $(R_2(\zeta)f, g)$ *has a meromorphic continuation from* $\pi^\pm$ *to* $\pi^\pm \cup \Omega^\mp \cup I$ *with poles contained in* $e_\pm \cup r_\pm$.

Proof. — Let $f = F^{-1}u$, $g = F^{-1}v$. Assumption 2.3 implies that

$$(R_2(\zeta)f, g) = (R_1(\zeta)f, g) - (G_1(\zeta)^{-1}BR_1(\zeta)f, AR_1(\bar{\zeta})g) \qquad (3.2)$$

Note that $f \in E_1(I)H$. For $w \in D'$ we have

$$(BR_1(\zeta)f, w) = (R_1(\zeta)f, E_1(I)B^*w) = \int_I (\lambda - \zeta)^{-1}(u(\lambda), T(\lambda\,; B)w)d\lambda$$

We can extend this to an arbitrary $w \in K$ such that

$$(BR_1(\zeta)f, w) = \int_I (\lambda - \zeta)^{-1}(T(\lambda\,; B)^*u(\lambda), w)d\lambda\,.$$

Again by deforming the integration contour we get that $\zeta \mapsto BR_1(\zeta)f$ has an analytic continuation to $\pi^+ \cup \Omega^- \cup I$ with values in K. A similar result holds in the case and for $AR_1(\zeta)g$. The result now follows from (3.1), the above result, and Proposition 3.2, 3.6.

The sets e_+ and e_- are discrete and can be identified with the point spectrum of H_2 in I. Due to the analyticity assumption the proof is elementary compared to the general case [*9*; § 5], which also requires additional assumptions.

THEOREM 3.9. — $e = e_+ = e_- = I \cap \sigma_p(H_2)$. *The points in* e *are simple poles of* $\tilde{G}_{1+}(\zeta)^{-1}$ *and* $\tilde{G}_{1-}(\zeta)^{-1}$.

Proof. — The proof is similar to the proof of [*2*; Lemma 4.6]. Let $\lambda_0 \in I$ and $P = E_2(\{\lambda_0\})$. It is well known that

$$P = \underset{\zeta \to \lambda_0}{s\text{-lim}}\,(\lambda_0 - \zeta)R_2(\zeta)\,,$$

where ζ approaches λ_0 non-tangentially.

Assume that $\lambda_0 \in \sigma_p(H_2)$. Then $P \neq 0$ and we can find $f, g \in F^{-1}\mathcal{R}_I$ such that $(Pf, g) \neq 0$. Now

$$(Pf, g) = \lim_{\substack{\zeta \to \lambda_0 \\ \mathrm{Im}\,\zeta > 0}} (\lambda_0 - \zeta)(R_2(\zeta)f, g)\,,$$

so the continuation of $(R_2(\zeta), f, g)$ has a simple pole at λ_0. It follows from the proof of Proposition 3.8 that $\tilde{G}_{1+}(\zeta)^{-1}$ has a pole at λ_0. Thus $\sigma_p(H_2) \cap I \subset e_+$.

Assume that $\lambda_0 \notin \sigma_p(H_2)$. For $\operatorname{Im} \zeta > 0$ we have $G_{2+}(\zeta) = G_{1+}(\zeta)^{-1}$, $G_2(\zeta) = 1 - [BR_2(\zeta)A^*]^a$ (see [9; Lemma 2.13]). Let $f \in D(A^*)$, $g \in D(B^*)$ be arbitrary. From $(G_{1+}(\zeta)^{-1}f, g) = (f, g) - (R_2(\zeta)A^*f, B^*g)$ we see that

$$\lim_{\substack{\zeta \to \lambda_0 \\ \operatorname{Im} \zeta > 0}} (\lambda_0 - \zeta)(G_{1+}(\zeta)^{-1}f, g) = 0 .$$

The density of $D(A^*)$ and $D(B^*)$ in K now implies that $\lambda_0 \notin e_+$.

Now note that the equation

$$(\tilde{G}_{1+}(\zeta)^{-1}f, g) = (f, g) - (R_2(\zeta)A^*f, B^*g) \tag{3.3}$$

implies that the poles of $\tilde{G}_{1+}(\zeta)^{-1}$ at e_+ are simple. The proof in the-case is similar.

Let us now give a characterization of r_+, which is based on continuation of $(R_2(\zeta)f, g)$ for suitable f, g. Previously (Proposition 3.8) we used $f, g \in F^{-1}\mathcal{R}_I$. The characterization is easily obtained using $R(A^*)$ and $R(B^*)$ instead. Let us note that for $I' \subset \subset I$, a relatively compact subinterval, we have $FE_1(I')R(A^*) \subset \mathcal{R}_I$, and $FE_1(I')R(B^*) \subset \mathcal{R}_{I'}$, because for $u \in D(A^*)$ we have $(FE_1(I')A^*u)(\lambda) = T(\lambda ; A)u$, $\lambda \in I'$, and $T(\lambda ; A)$ is analytic in λ (cf. [9; Proposition 3.7]), and similarly for B^*. We have assumed $E_1(I')R(A^*)$ dense in $E_1(I)H$ (when we vary I' also), but we have no similar assumption for B^*. Therefore we find it more convenient to use $R(A^*)$ and $R(B^*)$ directly. Let $f = A^*u$, $g = B^*v$, $u \in D(A^*)$, $v \in D(B^*)$. We now use (3.3)

$$(R_2(\zeta)f, g) = (u, v) - (\tilde{G}_{1+}(\zeta)^{-1}u, v) ,$$

which directly gives the continuation of $(R_2(\zeta)f, g)$. The density of $D(A^*)$ and $D(B^*)$ now implies the result:

$$r_+ = \{ \zeta_0 \in \Omega^- \mid \text{there exist } f \in R(A^*), g \in R(B^*) \text{ such that the continuation of } (R_2(\zeta)f, g) \text{ from } \pi^+ \text{ into } \Omega^- \text{ has a pole at } \zeta_0 \} . \tag{3.4}$$

We state the basic results in scattering theory given in [9; Theorem 3.11-3.13; Theorem 6.3] and our analyticity result for the S-matrix in the following two theorems. Let $\tilde{G}_{2\pm}(\zeta) = \tilde{G}_{1\pm}(\zeta)^{-1}$ whenever the inverse exists.

THEOREM 3.10. — *a*) Let Assumption 2.1-2.7 be satisfied. Then there exists a uniquely determined operator $F_\pm$ from $E_2(I \setminus e)H$ onto $L^2(I ; C)$ such that for every Borel set $I' \subset I \setminus e$ and every $u \in D(A^*)$ one has

$$(F_\pm E_2(I')A^*u)(\lambda) = \chi_{I'}(\lambda)T(\lambda ; A)\tilde{G}_{2\pm}(\lambda)u \qquad \text{a. e. in } I .$$

Furthermore, $F_\pm$ satisfies $F_\pm E_2(I')F_\pm^{-1} = \chi_{I'} \cdot$ for every Borel set $I' \subset I \setminus e$.

b) Let $W_\pm = W_\pm(H_2, H_1 ; I) = F_\pm^* F$. Then $W_\pm$ is a unitary operator from $E_1(I)H$ onto $E_2(I \setminus e)H$ and satisfies the intertwinning relation

$H_2W_\pm = W_\pm H_1$ on $E_1(I)H$. The operator $S = S(H_2, H_1 ; I) = W_+^* W_-$ is a unitary operator on $E_1(I)H$ which commutes with H_1.

c) Let ϕ be a real-valued Borel measurable function on I such that

$$\int_0^\infty \left| \int_I f(\lambda) \exp(-it\phi(\lambda) - i\xi\lambda)d\lambda \right|^2 d\xi \to 0 \qquad \text{as} \quad t \to \infty$$

for any $f \in L^2(I)$. Then for any $u \in E_1(I)H$

$$\lim_{t \to \pm\infty} e^{it\phi(H_2)} e^{-it\phi(H_1)} u = W_\pm u .$$

THEOREM 3.11. — For any $\lambda \in I \setminus e$ let

$$S(\lambda) = 1 - 2\pi i T(\lambda ; A)\tilde{G}_{2+}(\lambda)T(\bar{\lambda} ; B)^* \tag{3.5}$$

For any $f \in L^2(I ; C)$ we have

$$(FSF^*f)(\lambda) = S(\lambda)f(\lambda) \qquad \text{a. e. in I}$$
$$(FS^{-1}F^*f)(\lambda) = S(\lambda)^{-1}f(\lambda) \qquad \text{a. e. in I}$$

where $S(\lambda)^{-1}$ satisfies

$$S(\lambda)^{-1} = 1 + 2\pi i T(\lambda ; A)\tilde{G}_{2-}(\lambda)T(\bar{\lambda} ; B)^* \tag{3.6}$$

$S(\lambda)$ is a unitary operator in C and can be extended to a meromorphic function (also denoted $S(\lambda)$) in Ω with poles at most in r_+. The continuation is given by (3.5) for $\zeta \in \Omega \setminus (e \cup r_+)$.

Proof of Theorem 3.11. — The first part is identical with [*9*; Theorem 6.3]. The existence of the meromorphic continuation of $S(\lambda)$ from $I \setminus e$ into $\Omega \setminus (e \cup r_+)$ is an immediate consequence of Assumption 2.5 and Proposition 3.2. It is easy to see that unitary of $S(\lambda)$ for $\lambda \in I \setminus e$ implies that the poles at e are removable singularities. We omit the details.

We now come to the main result showing that the poles of the meromorphically continued matrix elements of $R_2(\zeta)$ are intrinsic, because they agree with the poles of the meromorphically continued S-matrix. We call r_+ the set of *resonances.*

THEOREM 3.12. — The resonances r_+ and the poles of $S(\lambda)$ in Ω agree. Furthermore, for $\zeta_0 \in r_+$ $\ker(S(\zeta_0)^{-1})$ is isomorphic to $\ker(\tilde{G}_{1+}(\zeta_0))$ via

$$-2\pi i T(\zeta_0 ; A) : \ker(\tilde{G}_{1+}(\zeta_0)) \to \ker(S(\zeta_0)^{-1})$$

which has the inverse

$$G_{2-}(\zeta_0)T(\bar{\zeta}_0 ; B)^* : \ker(S(\zeta_0)^{-1}) \to \ker(\tilde{G}_{1+}(\zeta_0)) .$$

REMARK 3.13. — *i)* The proof given below is an adaptation of the proof of the same result in [*8*] in the special case considered in [*2*].

ii) Notice the difference in the behaviour of $S(\lambda)$ and $\tilde{G}_{1+}(\lambda)$ at points in e. As noted in Theorem 3.10 $S(\lambda)$ exists and is unitary at points $\lambda_0 \in e$, whereas $\tilde{G}_{1+}(\zeta)$ has a simple pole at $\zeta = \lambda_0$ (see Theorem 3.9). This shows up in the above operators as follows. For $f \in \ker(\tilde{G}_{1+}(\lambda_0))$ we have $T(\lambda_0 ; A)f = 0$ (at least under some additional assumptions, see [*9*; § 5]), whereas the other operator obviously does not exist.

For the proof of Theorem 3.12 we need the following lemma.

LEMMA 3.14. — $\zeta_0 \in \Omega^-$ is a pole of $S(\zeta)$ if and only if $\ker(S(\zeta_0)^{-1}) \neq 0$.

Proof. — See [12; Lemma 6.2] and [*8*; Lemma 3].

Proof of Theorem 3.12. — Assume $\zeta_0 \in \Omega^-$. We divide the proof into four steps.

1° $-2\pi i T(\zeta_0 ; A)$ maps $\ker(\tilde{G}_{1+}(\zeta_0))$ into $\ker(S(\zeta_0)^{-1})$.

Let $\psi \in \ker(\tilde{G}_{1+}(\zeta_0))$ and define $f = -2\pi i T(\zeta_0 ; A)\psi$. We must show that $S(\zeta_0)^{-1}f = 0$. This is done by using (3.6), $G_{2-}(\zeta_0)(1 + Q_{1-}(\zeta_0)) = 1$, and (3.1) as follows

$$\begin{aligned}
S(\zeta_0)^{-1}f &= -2\pi i(T(\zeta_0 ; A)\psi + 2\pi i T(\zeta_0 ; A)G_{2-}(\zeta_0)T(\bar{\zeta}_0 ; B)^*T(\zeta_0 ; A)\psi) \\
&= -2\pi i T(\zeta_0 ; A)[1 + 2\pi i G_{2-}(\zeta_0)T(\bar{\zeta}_0 ; B)^*T(\zeta_0 ; A)]\psi \\
&= -2\pi i T(\zeta_0 ; A)G_{2-}(\zeta_0)[1 + Q_{1-}(\zeta_0) + 2\pi i T(\bar{\zeta}_0 ; B)^*T(\zeta_0 ; A)]\psi \\
&= -2\pi i T(\zeta_0 ; A)G_{2-}(\zeta_0)[1 + \tilde{Q}_{1+}(\zeta_0)]\psi \\
&= 0 .
\end{aligned}$$

2° $G_{2-}(\zeta_0)T(\bar{\zeta}_0 ; B)^*$ maps $\ker(S(\zeta_0)^{-1})$ into $\ker(\tilde{G}_{1+}(\zeta_0))$.

Let $f \in \ker(S(\zeta_0)^{-1})$, and define $\psi = G_{2-}(\zeta_0)T(\bar{\zeta}_0 ; B)^*f$. The result follows from the computation given below.

$$\begin{aligned}
(1 + \tilde{Q}_{1+}(\zeta_0))\psi &= (1 + \tilde{Q}_{1+}(\zeta_0))G_{2-}(\zeta_0)T(\bar{\zeta}_0 ; B)^*f \\
&= (1 + Q_{1-}(\zeta_0) + 2\pi i T(\bar{\zeta}_0 ; B)^*T(\zeta_0 ; A))G_{2-}(\zeta_0)T(\zeta_0 ; B)^*f \\
&= T(\zeta_0 ; B)^*[1 + 2\pi i T(\zeta_0 ; A)G_{2-}(\zeta_0)T(\bar{\zeta}_0 ; B)^*]f \\
&= T(\bar{\zeta}_0 ; B)^*S(\zeta_0)^{-1}f \\
&= 0 .
\end{aligned}$$

3° $$(-2\pi i T(\zeta_0 ; A)) \circ (G_{2-}(\zeta_0)T(\bar{\zeta}_0 ; B)^*) = 1_{\ker(S(\zeta_0)^{-1})} .$$

This is a trivial consequence of (3.6).

4° $$(G_{2-}(\zeta_0)T(\bar{\zeta}_0 ; B)^*) \circ (-2\pi i T(\zeta_0 ; A) = 1_{\ker(\tilde{G}_{1+}(\zeta_0))} .$$

For $\psi \in \ker(\tilde{G}_{1+}(\zeta_0))$ we have from (3.1):

$$\begin{aligned}
G_{2-}(\zeta_0)T(\bar{\zeta}_0 ; B)^*(-2\pi i T(\zeta_0 ; A))\psi &= G_{2-}(\zeta_0)(-2\pi i T(\bar{\zeta}_0 ; B)^*T(\zeta_0 ; A))\psi \\
&= G_{2-}(\zeta_0)(Q_{1-}(\zeta_0) - \tilde{Q}_{1+}(\zeta_0))\psi \\
&= G_{2-}(\zeta_0)(1 + Q_{1-}(\zeta_0))\psi = \psi .
\end{aligned}$$

4. DEPENDENCE OF RESONANCES ON (H_2, H_1)

Let H_2 be a selfadjoint operator on a Hilbert space H. If H_2 is the Hamiltonian for a quantum mechanical system, the choice of H_1 in the decomposition $H_2 = H_1 + V_{21}$ corresponds to decomposing H_2 into the free Hamiltonian H_1 and the interaction V_{21}. Under the assumptions in § 2 on (H_2, H_1) we can associate resonances to the pair (H_2, H_1). It is relevant to ask how our choice of H_1 affects the resonances. We shall give a partial answer to this question.

Consider three selfadjoint operator H_0, H_1, H_2 on a Hilbert space H satisfying the following relations:

$$H_2 = H_1 + V_{21}, \qquad H_1 = H_0 + V_{10},$$
$$H_2 = H_0 + V_{20}, \qquad V_{20} = V_{21} + V_{10},$$

where V_{21}, V_{10}, V_{20} are closed symmetric relatively compact operators with $D(V_{21}) = D(V_{10}) = D(V_{20}) \supset D(H_j)$, $j = 0, 1, 2$.

Define

$$Q_{21}(\zeta) = V_{21}(H_1 - \zeta)^{-1}, \qquad G_{21}(\zeta) = 1 + Q_{21}(\zeta),$$
$$Q_{20}(\zeta) = V_{20}(H_0 - \zeta)^{-1}, \qquad G_{20}(\zeta) = 1 + Q_{20}(\zeta),$$
$$Q_{10}(\zeta) = V_{10}(H_0 - \zeta)^{-1}, \qquad G_{10}(\zeta) = 1 + Q_{10}(\zeta).$$

Instead of the factorization technique used previously we make the following assumption. There exists a Hilbert space X, $X \hookrightarrow H$ dense and continuously embedded such that $Q_{20}(\zeta) \in B(X)$ and such that for an interval $I \subset \mathbb{R}$ $Q_{20}(\zeta)$ has an analytic continuation $\tilde{Q}_{20}(\zeta) \in B(X)$ from $\{\zeta \mid \text{Im}\,\zeta > 0\}$ into a domain $\Omega^- \subset \{\zeta \mid \text{Im}\,\zeta < 0\}$ across I. Similar assumptions are made for $Q_{21}(\zeta)$ and $Q_{10}(\zeta)$. Furthermore, all $\tilde{Q}$-operator are assumed to be compact operators in $B(X)$.

In a factorization scheme of the type $H_2 = H_1 + A^*B$ with $B = CA$ X can be chosen to be $R(A^*)$ with the norm of $f \in R(A^*)$ given by

$$\inf \{ \| \mu \| \mid \mu \in D(A^*), A^*\mu = f \}.$$

The resonances for the pair (H_2, H_1) are obtained as those $\zeta_0 \in \Omega^-$ for which $\tilde{G}_{21}(\zeta_0)$ is not invertible, see Proposition 3.2. Note that we must also have the assumptions of Section 2 to have proved that this definition of a resonance is satisfactory. The set of resonances for the pair (H_2, H_1) is denoted $r(H_2, H_1)$.

The equation $G_{20}(\zeta) = G_{21}(\zeta)G_{10}(\zeta)$ and its continuation

$$\tilde{G}_{20}(\zeta) = \tilde{G}_{21}(\zeta)\tilde{G}_{10}(\zeta), \qquad \zeta \in \Omega^-,$$

on $B(X)$ then imply the result.

If $r(H_1, H_0) = \emptyset$, then $r(H_2, H_0) = r(H_2, H_1)$.

Hence for operators H_0, H_1, H_2 satisfying all our assumptions the resonances for (H_2, H_0) agree with the resonances for (H_2, H_1), provided the pair (H_1, H_0) has no resonances. In the quantum mechanical framework this is a reasonable result.

In Section 5 we give an example, where all the above assumptions are satisfied, see Remark 5.5 *iii*).

5. APPLICATIONS I

As our first example we consider Schrödinger operators in $L^2(\mathbb{R}^n)$. We take $H_1 = -\Delta$ and $H_2 = -\Delta + V$, where V is exponentially decaying.

Let $H = K = L^2(\mathbb{R}^n)$ and $H_1 = -\Delta$ with $D(H_1) = H^2(\mathbb{R}^n)$, the usual Sobolev space. Let $p \in C^\infty(\mathbb{R}^n)$ be a real function with the properties $p(x) \geq 0$, $x \in \mathbb{R}^n$ and $p(x) = |x|$ for $|x| \geq 1$. Let $H_2 = H_1 + V$, where V is a closed, symmetric, H_1-compact operator such that there exist a constant $a > 0$ and a compact operator U from $H^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with $V = e^{-ap(x)}Ue^{-ap(x)}$. Here $e^{-ap(x)}$ denotes multiplication by the function $e^{-ap(x)}$. Note that $e^{-ap(x)}$ maps $H^2(\mathbb{R}^n)$ into $H^2(\mathbb{R}^n)$ boundedly.

Let $A = e^{-ap(x)}$ with $D(A) = H$ and $B = Ue^{-ap(x)}$ with $D(B) = H^2(\mathbb{R}^n)$.

LEMMA 5.1. — Assumptions 2.1, 2.2 and 2.3 are satisfied.

Proof. — Obvious.

The spectral representation for H_1 is given by the Fourier transform followed by a change of variables. Let $\mathscr{F}$ be the Fourier transform

$$(\mathscr{F}f)(\xi) = (2\pi)^{-n/2}\int f(x)e^{-ix\cdot\xi}dx\,. \tag{5.1}$$

In Assumption 2.4 we let $I = (0, \infty)$. Then $E_1(I)$ is the identity on H. Let $C = L^2(S^{n-1})$.

$$F : H \to L^2(I : C)$$

is defined by

$$(Ff)(\lambda)(\omega) = 2^{-1/2}\lambda^{(n-2)/4}(\mathscr{F}f)(\lambda^{1/2}\omega)\,, \qquad \lambda \in I\,, \quad \omega \in S^{n-1}\,.$$

LEMMA 5.2. — Assumptions 2.4, 2.6 and 2.7 are satisfied.

Proof. — Assumption 2.4 is easily verified for F defined above. Assumption 2.6 is satisfied by B. $A^* = A = e^{-ap(x)}$ and $D(A^*) = K$, so assumption 2.7 is satisfied.

LEMMA 5.3. — Assumption 2.5 is satisfied.

Proof. — $A^* = e^{-ap(x)}$ with $D(A^*) = K$. Let us take $D = C_0^\infty(\mathbb{R}^n)$. For $u \in D$ consider

$$(FE_1(I)A^*u)(\lambda)(\omega) = 2^{-1/2}\lambda^{(n-2)/4}\mathscr{F}(e^{-ap(x)}u)(\lambda^{1/2}\omega)$$

and define

$$(\mathrm{T}(\lambda\,;\,\mathrm{A})u)(\omega) = 2^{-1/2}\lambda^{(n-2)/4}\mathscr{F}(e^{-ap(x)}u)(\lambda^{1/2}\omega)\,.$$

$\mathscr{F}(e^{-ap(x)}u) = (2\pi)^{n/2}\mathscr{F}(e^{-ap(x)})*\mathscr{F}(u)$. This implies that

$$\lambda \mapsto \mathscr{F}(e^{-ap(x)}u)(\lambda^{1/2}\cdot)$$

extends to an analytic function in

$$\Omega = \{\, \zeta = x + iy \in \mathbb{C} \setminus (-\infty, 0] \mid y^2 \leqslant 4a^2(x + a^2) \,\}$$

with values in C, and also

$$\| \mathscr{F}(e^{-ap(x)}u)(\lambda^{1/2}\cdot) \|_C \leqslant c \| u \|$$

with c independent of λ for λ in a compact subset of Ω. Note that $\lambda^{1/2}$ and $\lambda^{1/4}$ are well defined on Ω.

Write B as follows

$$\mathrm{B} = \mathrm{U}(1 - \Delta)^{-1}(1 - \Delta)e^{-ap(x)}\,.$$

$\mathrm{W} = \mathrm{U}(1 - \Delta)^{-1}$ is a bounded operator on $\mathrm{L}^2(\mathbb{R}^n)$ $\mathrm{B}^* = e^{-ap(x)}(1 - \Delta)\mathrm{W}^*$.

We choose $D' = \mathrm{C}_0^\infty(\mathbb{R}^n)$ and define $\mathrm{T}(\lambda\,;\,\mathrm{B})$ by the expression

$$(\mathrm{FE}_1(\mathrm{I})\mathrm{B}^*u)(\lambda)\,, \qquad u \in D'\,,$$

as above. We have

$$\begin{aligned}(\mathrm{FE}_1(\mathrm{I})\mathrm{B}^*u)(\lambda)(\omega) &= 2^{-1/2}\lambda^{(n-2)/4}\mathscr{F}(e^{-ap(x)}(1 - \Delta)\mathrm{W}^*u)(\lambda^{1/2}\omega) \\ &= 2^{-1/2}\lambda^{(n-2)/4}(2\pi)^{n/2}\mathscr{F}(e^{-ap(x)}) * \mathscr{F}((1 - \Delta)\mathrm{W}^*u)(\lambda^{1/2}\omega) \\ &= 2^{-1/2}\lambda^{(n-2/4}(2\pi)^{n/2}(1 + \lambda)\mathscr{F}(e^{-ap(x)}) * \mathscr{F}(\mathrm{W}^*u)(\lambda^{1/2}\omega)\,.\end{aligned}$$

Now W^* is a bounded operator on $\mathrm{L}^2(\mathbb{R}^n)$, so the rest of the proof follows as above. Let us briefly state the result:

THEOREM 5.4. — Theorems 3.10, 3.11, 3.12 are true for the pair $(\mathrm{H}_2, \mathrm{H}_1)$ given above.

REMARK 5.5. — *i*) The main results are the existence of a meromorphic continuation of the S-matrix and the characterization of the resonances. The continuation of the S-matrix was given in [*2*] for $n = 3$ using explicit kernels for various operators. In [*2*] the potential is a form perturbation of H_1. The above proof can be extended to cover this case. The approach in [*2*] was to give a direct proof of the unitary of the S-matrix. The connection with the scattering operator was not proved. The results on the poles of $\mathrm{S}(\lambda)$ was given in [*8*] under the same assumptions as in [*2*].

ii) For $n > 2$ continuation of the S-matrix and characterization of the poles of the S-matrix have been given in [*13*] for $-\Delta + q(x)$ in an exterior domain with $q(x)$ uniformly Hölder continuous with compact support.

iii) The results of Section 4 apply to the class of potentials considered above with $X = \{\, f \mid e^{a|x|}f \in \mathrm{L}^2(\mathbb{R}^n) \,\}$ with the norm $\| f \|_X = \| e^{a|x|}f \|_{\mathrm{L}^2}$.

The main result needed in the above application is the construction of an explicit spectral representation for $-\Delta$ using the Fourier transform followed by a change of variables. Similar results can be obtained for other H_1, which are constant coefficient (pseudo)-differential operators and for which we can find the spectral representation explicitly and continue $T(\lambda ; A)$, $T(\lambda ; B)$; e. g.

$$H_1 = (-\Delta + m^2)^{1/2}, \qquad m > 0;$$
$$H_1 = (-\Delta)^k, \qquad k = 2, 3, \ldots .$$

We then obtain the results for $H_2 = H_1 + V$ with V exponentially decaying as above.

In the following section we give an application to Stark effect Hamiltonians.

6. APPLICATION II

As another application we consider Stark effect Hamiltonians. The results given below are essentially given in [*16*] [*17*], except that we have a different proof of the connection between poles and resonances, and also that we allow general non-local potentials. We will refer to [*16*] [*17*] for some results needed below. In the following we use the same letters H_1, H_2, etc. as in Section 5 to denote different operators.

We consider $H_1 = -\Delta + \varepsilon x_1$, $\varepsilon > 0$ fixed, and $H_2 = -\Delta + \varepsilon x_1 + V$. We have the following assumption on V. Let $\chi \in C^\infty(\mathbb{R})$ satisfy $0 \leqslant \chi(x_1) \leqslant 1$, $\chi(x_1) = 1$ for $x_1 > -1$, $\chi(x_1) = 0$ for $x < -2$. We assume that there exists a constant $a > 0$ such that

$$V = (e^{ax_1}\chi(-x_1) + \chi(x_1))U(e^{ax_1}\chi(-x_1) + \chi(x_1)),$$

where U is a closed, symmetric, H_1-compact operator.

To verify the assumptions in Section 2 we take $H = K = L^2(\mathbb{R}^n)$, $A = e^{ax_1}\chi(-x_1) + \chi(x_1)$ (multiplication operator) with $D(A) = H$, and $B = U(e^{ax_1}\chi(-x_1) + \chi(x_1))$ with $D(B) = D(H_1)$. (One can verify that $e^{ax_1}\chi(-x_1) + \chi(x_1)$ maps $D(H_1)$ into $D(H_1)$.)

LEMMA 6.1. — Assumptions 2.1, 2.2, 2.3 are satisfied.

Proof. — Obvious.

We now describe the spectral representation for H_1. Let $\mathscr{F}$ be the Fourier transform in $L^2(\mathbb{R}^n)$ (see (5.1)). Let

$$G(p) = \frac{1}{3} p_1^3 + p_1(p_2^2 + \ldots + p_n^2), \qquad p \in \mathbb{R}^n .$$

Let $I = \mathbb{R}$, $C = L^2(\mathbb{R}^{n-1})$ and define $F : E_1(I)H = L^2(\mathbb{R}^n) \to L^2(I ; C)$ by

$$(Ff)(x_1)(x') = (2\pi\varepsilon)^{-\frac{n}{2}} \int \exp(i(x \cdot p - G(p))/\varepsilon)(\mathscr{F}f)(p)dp ,$$

where we write $x = (x_1, x')$, $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$.

LEMMA 6.2. — Assumptions 2.4, 2.6 and 2.7 are satisfied.

Proof. — It is well known that F defined above gives a spectral representation for H_1, see [*1*] [*5*] [*16*] [*17*]. Assumption 2.6 is satisfied by B. $A^* = A = e^{ax_1}\chi(-x_1) + \chi(-x_1)$ and $D(A^*) = K$, so Assumption 2.7 is satisfied.

LEMMA 6.3. — Assumption 2.5 is satisfied.

Proof. — We can use $\Omega = \mathbb{C}$. We have $I = \mathbb{R}$, so $E_1(I)$ is the identity. We let $D = D' = C_0^\infty(\mathbb{R}^n)$ and define for $u \in D$

$$(T(\lambda ; A)u)(x') = (FE_1(I)A^*u)(\lambda ; x') .$$

The required analyticity properties of $T(\lambda ; A)$ are proved in [*17*; Lemma 1.1].

For B we proceed as follows

$$\begin{aligned} B &= U(e^{ax_1}\chi(-x_1) + \chi(x_1)) \\ &= U(H_1 + i)^{-1}(H_1 + i)(e^{ax_1}\chi(-x_1) + \chi(x_1)) . \end{aligned}$$

$W = U(H_1 + i)^{-1}$ is a bounded operator on $L^2(\mathbb{R}^n)$. We have

$$\begin{aligned} B^* &= (e^{ax_1}\chi(-x_1) + \chi(x_1))(H_1 - i)W^* \\ &= (H_1 - i)(e^{ax_1}\chi(-x_1) + \chi(x_1))X \end{aligned}$$

where

$$X = (e^{ax_1}\chi(-x_1) + \chi(x_1))^{-1}(H_1 - i)^{-1}(e^{ax_1}\chi(-x_1) + \chi(x_1))(H_1 - i)W^*$$

can be shown to define a bounded operator on $L^2(\mathbb{R}^n)$. The result for $T(\lambda ; B)$ defined by $(T(\lambda ; B)u)(x') = (FE_1(I)B^*u)(\lambda, x')$ for $u \in D'$ now follows from [*17*, Lemma 1.1] as above. Thus we have proved:

THEOREM 6.4. — Theorem 3.10, 3.11, 3.12 are true for (H_2, H_1) given above.

REMARK 6.5. — *i*) The conditions on V used in [*17*] were the following: V is a realvalued function satisfying the following conditions. There exist $a > 0$ and realvalued functions V_1, V_2 such that

a) $V(x) = (e^{ax_1}\chi(-x_1) + \chi(x_1))(V_1(x) + V_2(x))$,

b) $V_1(x) \in L^\infty(\mathbb{R}^n)$, $\lim_{|x| \to \infty} V_1(x) = 0$,

c) $(1 + |x|)^\gamma V_2 \in L^q(\mathbb{R}^n)$ with $q > \frac{n}{2}$ and $\gamma > 0$.

It follows from the results in [*16*] that such a V will satisfy our assumptions.

ii) There are several recent papers discussing Stark effect Hamiltonians. See [*1*] [*5*] [*6*] [*16*] [*17*] and the references given there.

iii) If V is dilation-analytic and satisfies our assumption, the resonances defined in the dilation-analytic theory [*5*] agree with the poles of the continued S-matrix. This can be seen from (3.4) and results in [*5*].

iv) At remark similar to *iii*) holds for V translation-analytic (see [*1*]) and satisfying our assumptions.

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