ANNALES DE L'I. H. P., SECTION A

P. T. LEUNG

On stellar collapse : continual or oscillatory ? A short comment

Annales de l'I. H. P., section A, tome 33, n° 2 (1980), p. 205-208
<http://www.numdam.org/item?id=AIHPA_1980__33_2_205_0>

Ann. Inst. Henri Poincaré Section A :
Vol. XXXIII, n° 2, 1980, p. 205-208. *Physique théorique.*

On Stellar Collapse: continual or oscillatory ? A short comment

by

P. T. LEUNG
Department of Physics, Suny at Buffalo, NY 14260, USA

ABSTRACT. — We comment on a previously published paper on the oscillatory dynamics of stellar collapse and conclude that the Schwarzschild interior solution applied to the « inflection points » can never give rise to a « turning back » motion, in spite of the fact that the geodesic equation really does not always describe an attractive gravitational acceleration.

RÉSUMÉ. — Nous faisons un commentaire sur un article publié antérieurement sur les dynamiques oscillatoires de l'effondrement astral, et nous concluons que la solution intérieure Schwarzschild, appliquée aux « points d'inflexion », ne peut jamais produire une motion rétrograde, en dépit du fait que l'équation géodésique ne décrit pas toujours une accélération gravitationnelle attractive.

A recent publication ([1]) claims that the non-positive-definite affine connection field ($\Gamma^{\rho}_{\mu\nu}$) appears in the geodesic equation allows the possibility of « gravitational expansion » for a previously collapsing star really shows the existence of « inflection points » at which the contraction stops and expansion starts to occur, thus predicting that the stellar motion is oscillatory rather than a continuous collapse into the black hole limit. This short note seeks to clarify some of the misleading arguments and concludes that if Schwarzschild interior solution is really applicable to such « inflection points », then there can never exist such points and the stellar collapse is still in general a continual contraction process.

I

The geodesic equation for a « test particle » moving under the influence of the gravitational field of a distribution of matter reads:

$$\ddot{X}^\rho = - \Gamma^\rho_{\mu\nu}\dot{X}^\mu\dot{X}^\nu \tag{1}$$

where

$$\Gamma^\rho_{\mu\nu}\left(\equiv \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu})\right)$$

is to be determined from Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = (- 8\pi G/c^2)T_{\mu\nu} \tag{2}$$

by assuming a model ($T_{\mu\nu}$) for the distribution of matter. The indefiniteness of sign of $\Gamma^\rho_{\mu\nu}$ indeed leaves with the possibility of « gravitational repulsion » acted on the test particle. That is, we cannot exclude that there might exist a model $T_{\mu\nu}$ which when put in (2) gives solutions for $g_{\mu\nu}$ which in turn gives rise to a negative $\Gamma^\rho_{\mu\nu}$ so that $\ddot{X}^\rho > 0$ in (1) which means a gravitational repulsion (we have specify that $\ddot{X}^\rho < 0$ corresponds to the ordinary gravitational attraction). But such case is quite rare (see for example, ref. 2, sect. 4.2, 4.3). In the following, we want to ask precisely the question: does the Schwarzschild interior solution from (2) predict any turning point due to gravitational repulsion for a test particle moving inside the collapsing star or for the whole star itself? We shall see that in the first case, it is very improbable for most of the stars and in the second case it is impossible for the collapsing star to « bounce back ». The *necessary* condition that the acceleration $\ddot{X}^\rho$ in equation (1) must change sign at the « inflection point » (relative to the initial sign of the « inward motion ») will be applied throughout our discussion.

II

The Schwarzschild interior solution for a spherical symmetric star as given in ref. [*1*] (see for example, ref. [*3*]) is as follows:

$$\left.\begin{aligned} g_{00} &= \left\{ \frac{3}{2}\left[1 - \left(\frac{r_0}{R_0}\right)^2\right]^{1/2} - \frac{1}{2}\left[1 - \left(\frac{r}{R_0}\right)^2\right]^{1/2} \right\}^2 \\ g_{11} &= - \left[1 - \left(\frac{r}{R_0}\right)^2\right]^{-1} = 1/g^{11} \\ g_{22} &= - r^2, \qquad g_{33} = - r^2 \sin^2 \theta, \qquad g_{\mu\neq\nu} = 0 \end{aligned}\right\} \tag{3}$$

where $r < r_0$, $R_0^2 = 3c^2/8\pi G\rho$,

$$\rho = \frac{M}{V_0}\left[1 - 0.3\left(\frac{r_s}{r_0}\right) + \left(\frac{r_s^2}{r_0^2}\right)\right]$$

is the assumed constant mass density of the star; r_s and r_0 being the Schwarzschild and gravitational radius of the star respectively. This model gives rise to the pertinent radial equation of motion to be:

$$\ddot{X}^1 = -\Gamma^1_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = -\frac{1}{2}g^{11}[\partial_1 g_{11}(\dot{r})^2 - \partial_1 g_{00}(\dot{X}^0)^2] \tag{4}$$

as quoted from ref. [1]. Again we follow ref. [1] by assuming that at the « inflection points » (i. e. $r = r_0$ where collapse starts and $r = R$, where the outward motion starts), $\dot{r} = 0$ and $\dot{X}^0 \approx c$; then (4) reduses to:

$$\ddot{X}^1 = \frac{1}{2}c^2 g^{11}\partial_1 g_{00}, \qquad \text{at} \quad r = r_0, R \tag{5}$$

Hence, from (3), (5) becomes:

$$\ddot{X}^1(r) = \frac{c^2 r}{2R_0^2}\sqrt{1-\left(\frac{r}{R_0}\right)^2}\left[\sqrt{1-\left(\frac{r}{R_0}\right)^2} - 3\sqrt{1-\left(\frac{r_0}{R_0}\right)^2}\right], \qquad \text{at} \quad r = r_0, R \tag{6}$$

Now we divide into two cases. First we examine the inward radial motion of *a* test particle inside the star starting at $r = r_0$ to the turning point $r = R$, assuming such point exists. By assuming the magnitudes of $\ddot{X}^1$ are equal at these points, the result found in reference [1] is $R = \frac{1}{2}r_0$. To have this to be an acceptable answer, we must check the *necessary* condition mentioned in section I. Since from (6), $\ddot{X}^1(r = r_0) < 0$, so we require necessarily, $\ddot{X}^1(r = R) > 0$. Now, if $R = \frac{1}{2}r_0$ is an acceptable solution, (6) implies that:

$$\sqrt{1-\left(\frac{r_0/2}{R_0}\right)^2} - 3\sqrt{1-\left(\frac{r_0}{R_0}\right)^2} > 0$$

which simplifies to:

$$\frac{r_0}{R_0} > \sqrt{\frac{32}{35}}(\approx 0.96) \tag{7}$$

Combining with the condition $r_0 < R_0$ for the validity of Schwarzschild interior solution (see ref. [3]), we conclude that for such « turning point » exists, the gravitational radius (r_0) of the star has to satisfy the following inequality:

$$0.96\, R_0 < r_0 < R_0 \tag{8}$$

(i. e. $r_0 \approx R_0$) which is not likely for most of the stars since one can show that $\left(\frac{r_s}{r_0}\right) = \left(\frac{r_0}{R_0}\right)^2$ and in general $r_0 \gg r_s$ for most stars. Hence we can conclude that for particles moving inside a star under the influence of the

Schwarzschild interior field, it is improbable for it to « bounce back » due to possible gravitational repulsion.

However, the above consideration has nothing to do with a collapsing star. To treat this case (if we assume the *whole* collapsing star will bounce back after it contracts from $r = r_0$ to $r = \bar{R}$ and Schwarzschild interior solution applies well to the star at these points), we must apply the Schwarzschild interior model *twice* with the constants R_0, ρ, V_0 being redefined at $r = \bar{R}$. Let these be symbolically denoted by $\bar{R}_0$, $\bar{\rho}$, $\bar{V}_0$ (e. g. $\bar{R}_0^2 = 3c^2/8\pi G\bar{\rho}$).

Then from equation (6), we have:

$$\ddot{X}^1(r = r_0) = \frac{c^2 r_0}{2R_0^2}\sqrt{1 - \left(\frac{r_0}{R_0}\right)^2}\left[\sqrt{1 - \left(\frac{r_0}{R_0}\right)^2} - 3\sqrt{1 - \left(\frac{r_0}{R_0}\right)^2}\right] < 0$$

as the collapse starts and at $r = \bar{R}$, we have similar expression:

$$\ddot{X}^1(r = \bar{R}) = \frac{c^2 \bar{R}}{2\bar{R}_0^2}\sqrt{1 - \left(\frac{\bar{R}}{\bar{R}_0}\right)^2}\left[\sqrt{1 - \left(\frac{\bar{R}}{\bar{R}_0}\right)^2} - 3\sqrt{1 - \left(\frac{\bar{R}}{R_0}\right)^2}\right] < 0$$

Hence our necessary condition mentioned in section I is violated and therefore $r = \bar{R}$ as a turning point can never exist.

III

Thus we conclude that if Schwarzschild interior solution is to be applicable at the « inflection points » of a collapsing star, there can never exist a turning point for the inward stellar contraction and the black hole limit is reachable through such continual collapse in contrary to the conclusion of ref. [*1*]. However, this does not exclude the possibility that some other more adequate stellar model might exist which when applied to the turning points really shows an oscillatory dynamics of the stellar collapse. Indeed, if we take equations (1) and (2) as a complete description of gravitational phenomena, there is no guarantee for the « always-attractive » property of gravitational force ([2]), although they give the correct Newtonian limit for weak field. Hence, it might be an interesting problem to investigate under what conditions for the $T_{\mu\nu}$ in (2) will give rise to a « repulsive motion » in equation (1).

REFERENCES

[*1*] M. SACHS, *Ann. Inst. Henri Poincaré*, Section A : *Physique théorique*. Vol. **XXVIII**, n° 4, 1978, p. 399-405.

[*2*] R. K. SACHS and H. WU, *General Relativity for Mathematicians*, by Springer-Verlag, New York, Inc., 1977.

[*3*] R. ADLER, M. BAZIN and M. SCHIFFER, *Introduction to General Relativity*, by McGraw-Hill, Inc., Sec. 9.6, 1965.

(Manuscrit reçu le 22 avril 1980)