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Section A :
Physique théorique.

On the mass spectrum of Higgs particles

by

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ABSTRACT. — The mass spectrum of massive gauge particles which arise from a spontaneous symmetry breakdown of a gauge theory is investigated. This spontaneous symmetry breakdown is assumed to relate to a model with Lagrangian function

$$L = L_1(A_\mu, \phi) + V(\phi)$$

where $L_1(A_\mu, \phi)$ depends on a certain type of gauge field A_μ as well as on some Higgs field ϕ. $V(\phi)$ is an effective potential depending on ϕ.

It is shown that, in the use of R. Thom's catastrophe theory, there is a mass matrix $M = (M_{ik})$ which can be derived from $V(\phi)$. This mass-matrix describes the mass spectrum of gauge particles which acquire a mass through the Higgs mechanism.

I. INTRODUCTION

The phenomenon of spontaneous symmetry breaking can be apprehended in the case of a ferromagnet idealized as an array of independently orientable dipoles. Due to the interaction of neighboring dipoles, the free energy $F = F(\vec{M})$, $\vec{M}$ denotes the magnetization, takes its minimum values when all dipoles are oriented in the same direction. That is, below a certain critical temperature T_c a magnetization $\vec{M}$ appears which is regarded as a magnetic order parameter. Otherwise stated: There is a spontaneous symmetry breakdown of the initial rotational symmetry O(3), revealed in

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the magnetization $\vec{M}$. This symmetry breaking masks the initially given rotational symmetry (where no particular direction was preferred). Conversely, an ordered state becomes unstable if the temperature of the system is raised, i. e. $T > T_c$. The magnetization vanishes and rotational symmetry is restored.

A prototype for the spontaneous symmetry-breaking mechanism is the Landau-Ginzburg theory of superconductivity. Let

$$(1)\quad F(\phi, T) = F_n + \alpha(T)\,|\phi|^2 + \frac{\beta}{2}|\phi|^4 + \frac{1}{2m}\left|\left(-i\hbar\nabla - \frac{2e}{c}\vec{A}\right)\phi\right|^2 + \frac{H^2}{8\pi}$$

be the free energy density (n denotes « normal »). The complex Cooper-pair field ϕ is treated as an order parameter. Below a critical temperature $T < T_c$, F has a minimum, i. e. the equilibrium value $|\phi_e|^2$ is determined by minimizing F

$$\left|\frac{\partial F}{\partial |\phi|^2} = 0 = \alpha(T) + \beta(T)\,|\phi_e|^2\right|$$

and thus

$$(2)\quad |\phi_e|^2 = -\frac{\alpha(T)}{\beta(T)} > 0 \qquad \left\{\begin{array}{l}\text{ground state value of } \phi \text{ in considering} \\ F = F_n + \alpha(T)\,|\phi|^2 + \dfrac{\beta(T)}{2}|\phi|^4\end{array}\right\}$$

A model for the relativistic counterpart of the Landau-Ginzburg theory, the Higgs-model, a vortex-line model for hadrons, is available in terms of the following Lagrangian density

$$(1')\qquad L(x) = -1/4\, F_{\mu\nu}(x)F_{\mu\nu}(x) + 1/2(\nabla_\mu\phi)^2 + V(\phi),$$

where

$$(3)\qquad V(\phi) = +\frac{\mu^2}{2}|\phi|^2 - \frac{\lambda}{4}|\phi|^4$$

The expressions for $F_{\mu\nu}$ and $\nabla_\mu\phi$ are

$$(4)\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + eA_\mu \wedge A_\nu$$

$$(5)\qquad \nabla_\mu\phi = \partial_\mu\phi + eA_\mu \wedge \phi$$

Given the Higgs-model, the aim of this paper is the study of the mass spectrum of Higgs particles related to spontaneous symmetry breakdown.

II. SPONTANEOUS SYMMETRY BREAKING AND MASS SPECTRUM

Consider a classical Lagrangian field theory with Lagrangian L and consider the solutions of the eq. of motion which are derived from the action principle $\delta \int_{c_4} L d^4 x = 0$, $c_4 \subset \mathbb{R}^4$. Two classes of solutions of these equations are of relevance:

(*i*) *Constant solutions:* To be identified with the single or multiple vacuum state ϕ_0, and

(*ii*) *Static solutions.*

A classification of solutions refers to certain homotopy groups $\pi_k(M_0)$, $k = 1, 2 \ldots$ [*1*], where M_0 constitutes the vacuum manifold associated with L. M_0 labels the different possible vacuum states. M_0 is determined as follows. Let $\phi(x)$ be a Higgs field transforming under a representation $g \mapsto \varphi(g)$, $g \in G$ of the gauge group G. ϕ lies in M_0, the manifold which minimizes the self-interaction $V(\phi)$ of ϕ. Assume G to act transitively on M_0. Then for any fixed $\phi_0 \in M_0$

(6) $$M_0 = \{ \varphi(g)\phi_0 : g \in G \}.$$

(7) $$G_0 = \{ g \in G \mid \varphi(g)\phi = \phi \}$$

is the isotropy subgroup of G at $\phi \in M_0$ and M_0 is a homogeneous space of G [*2*], i. e.

(8) $$M_0 = G/G_0 = \{ gG_0 ; g \in G \}$$

DÉFINITION 1. — A gauge symmetry G associated with a Lagrange field-theoretical model is said to be spontaneously broken iff there exists a vacuum manifold

(9) $$M_0 = \{ \varphi(g)\phi_0 \mid g \in G \} = G/G_0 \qquad G_0 \neq G$$

for a given vacuum state ϕ_0.

The meaning of eq. (8) is that the internal gauge symmetry group G sweeps the whole vacuum manifold M_0.

The following cases are to be distinguished :

(9 *a*) $G = G_0$: the vacuum ϕ_0 is unique; the gauge symmetry G is an exact symmetry.

(9 *b*) $\{e\} \subset G_0 \subset G$: there is a partly spontenaeously symmetry breaking of G.

(9 *c*) $G_0 = \{e\}$: the gauge symmetry G is completely broken.

II.*a*. Study of the Lagrange model (1′)-(4′).

The Lagrangian (1′) is invariant under the gauge group $G = SO(3)$. A^i_μ($\mu = 0, 1, 2, 3$ = Lorentz index, $i = 1, 2, 3$) and ϕ_i denote the Yang-Mills field and Higgs field triplet respectively with respect to this SO(3)-symmetry. The vacuum solutions are

(10) $$A^i_\mu = 0$$ and (11) $$\phi_i = n_i \sqrt{+\frac{\mu^2}{\lambda}}\,; \qquad \| n_i \| = 1$$

where (11) is a covariant constant solution, $\nabla_\mu \phi_i = 0$, of the type (*i*). On account of the field eq.

$$\nabla_\mu \nabla_\mu \phi_i = +\mu^2 \phi_i - \lambda \phi_i^2 \phi_i , \tag{12}$$

the solution (11) minimizes the potential energy $V(\phi)$ (eq. (3), i. e.

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_i = \phi_i^0} = 0 = +\mu^2 \phi_i - \lambda \phi_i^2 \phi_i \tag{13}$$

hence

$$\phi_i^{02} = +\frac{\mu^2}{\lambda} > 0 . \tag{14}$$

The minimum condition (14), similar as eq. (2) corresponding to the Landau-Ginzburg theory, determines the vacuum-manifold M_0 (eq. (9)) of field configurations that minimize the energy V:

$$M_0 = S^2 = \left\{ \phi_i \mid \phi_i^2 = +\frac{\mu^2}{\lambda} \right\} \tag{15}$$

Otherwise stated: the symmetry breakdown $G = SO(3) \to G_0 = SO(2)$ yields the vacuum manifold labelling the different possible vacuum states:

$$M_0 = SO(3)/SO(2) = S^2 \tag{16}$$

Hereby, the natural C^∞-action

$$SO(3) \times S^2 \to S^2 ; \quad (g, \phi) \mapsto \varphi(g)\phi \qquad \varphi(g) \in SO(3) \tag{17}$$

$$\| \varphi(g)\phi \| = \| \phi \| = \sqrt{+\frac{\mu^2}{\lambda}}$$

is transitive [*2*]. The classification of solutions to the field eq. is given in terms of $\pi_2(S^2) = \pi_2(SO(3)/SO(2)) = \mathbb{Z}$ [*1*].

II.*b*. Study of the mass spectrum of Higgs-particles

Spontaneously broken gauge symmetries imply that a number of previously massless gauge particles acquire a mass. This phenomenon is known as Higgs mechanism. The aim of this section is to show, that there is a mass matrix which accounts for the study of the mass spectrum of particles associated with a Higgs field ϕ (« Higgs-particles »).

Let

$$L = L_1(A_\mu, \phi) + V(\phi) \tag{1''}$$

be a Lagrangian corresponding to a gauge theory with spontaneous symmetry breakdown, where $L_1(A_\mu, \phi)$ depends on a certain class of gauge fields A_μ as well as on some Higgs field ϕ. $V(\phi)$ is the effective potential given by

(3′) $$V(\phi) = -1/2\frac{\mu^2}{2}\phi_i^2 - \lambda/4(\phi_i^2)^2\,; \qquad \mu^2 < 0,\ \lambda > 0$$

and which characterizes spontaneous symmetry breakdown. The minimum condition requires:

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_i=\phi_i^0} = 0$$

(eq. (13)) and

(18) $$\left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}\right|_{\phi=\phi^0} \geqslant 0 \qquad (i, k = 1, \dots, \dim G)$$

Expansion of $V(\phi)$ in a Taylor series in $(\phi - \phi_0)$ yields

(19) $V(\phi) = V(\phi_0) + 1/2\, M_{ik}\xi^i\xi^k + 0(\xi)\,;\qquad \xi = \phi - \phi_0$ i. e.

(20) $V(\phi) = 1/2(\phi - \phi_0)^T M(\phi - \phi_0)$ $\Big\{$ in matrix form. ($V(\phi_0)$ has been omitted).

Spontaneous symmetry breaking referring to

(1‴) $$L = 1/2(\partial_\mu\phi_i)(\partial_\mu\phi_i) - V(\phi)$$

(*without* Yang-Mills fields) gives massless (*Goldstone*) *bosons* by

THEOREM 1. — *Let* $V(\phi)$ *be invariant under the symmetry Lie group* G ($\dim G = n$) *and let*

(21) $$M_{ik} := \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}\right|_{\phi=\phi_0} \qquad i, k = 1, \dots, \dim G$$

where $\phi_0 \in M_0$ *is a ground state which minimizes* $V(\phi)$. *If* $M := (M_{ik})$, *is regarded as a mass matrix associated with* $V(\phi)$, *it enjoys the following properties:*

(*a*) $M = (M_{ik})$ *has* $(n - n_1)$*-times the eigenvalue* 0 ($n_1 = \dim G_0$ *is the dimension of the isotropy subgroup* G_0 *of* G at ϕ_0).

(*b*) *The remaining* n_1 *eigenvalues of* M_{ik} *are positive and equal the inertial masses* m_i^2.

Proof. — Invariance of V under the gauge group G implies:

$$dV(\phi)h = V(\phi + h) - V(\phi) = V(\phi) - V(\phi) = \sum_{i=1}^{n}\frac{\partial V}{\partial\phi_i}(\phi)h_i = 0$$

By virtue of $A_{ik} = A_{ik}(\alpha_1, \dots, \alpha_n) \in \varphi(G)$ and the transformation law

(23) $$\bar{\phi}_i = A_{ik}\phi_k$$

the increments h_i are related to the infinitesimal generators of G,

(24) $$\mathrm{I}^j_{ik} = \left.\frac{\partial \mathrm{A}_{ik}}{\partial \alpha_j}\right|_{\alpha_j=0} \in \mathscr{I}(\mathrm{G}), \qquad \begin{matrix} j = 1, \ldots, n \\ i = 1, \ldots, n \end{matrix}$$

i. e. the elements of the Lie algebra $\mathscr{I}(\mathrm{G})$ of the group G by

$$i \,.\, \mathrm{I}^j_{ik}\alpha_j\phi_k = h_i \qquad j = 1, 2, \ldots, n = \dim \mathrm{G} \quad (i = \sqrt{-1}), \quad \lambda_j : \text{small}$$

In fact:

$$\bar{\phi}_i = \mathrm{A}_{ik}\phi_k = (\delta_{ik} + i\mathrm{I}^j_{ik}\alpha_j)\phi_k = \delta_{ik}\phi_k + i\mathrm{I}^j_{ik}\alpha_j\phi_k = \phi_i + h_i .$$

Hence

$$\frac{\partial \mathrm{V}}{\partial \phi_i}(\phi)h_i = i\frac{\partial \mathrm{V}}{\partial \phi_i}(\phi)\mathrm{I}^j_{ik}\phi_k\alpha_j = 0$$

and thus

(25) $$\frac{\partial \mathrm{V}}{\partial \phi_i}(\phi)\mathrm{I}^j_{ik}\phi_k = 0 \qquad j = 1, \ldots, n$$

Differentiation of (25) yields

(26) $$\frac{\partial^2 \mathrm{V}}{\partial \phi_i \partial \phi_k}\mathrm{I}^j_{il}\phi_l + \frac{\partial \mathrm{V}}{\partial \phi_i}\mathrm{I}^j_{ik} = 0$$

which entails, by (13) for $\phi = \phi_0$

(27) $$\mathrm{M}_{il}\mathrm{I}^j_{ik}\phi^0_k = 0 \qquad j = 1, \ldots, n_1, \ldots, n$$

The defining property of the isotropy subgroup G_0 ($\dim \mathrm{G}_0 = n_1$) of G, $\varphi(g)\phi_0 = \phi_0$ (cf. eq. (7)) amounts to

$$\mathrm{A}_{ik}\phi^0_k = (\delta_{ik} + i\mathrm{I}^j_{ik}\lambda_j)\phi^0_k = \phi^0_i$$

which yields

(28) $$\phi^0_i + i\mathrm{I}^j_{ik}\phi^0_k\alpha_j = \phi^0_i \Rightarrow \mathrm{I}^j_{ik}\phi^0_k = 0, \qquad \begin{matrix} j = 1, \ldots, n_1 \\ i = 1, \ldots, n \end{matrix}$$

The n_1 eqs. (28) are compatible with n_1 of the eqs. (27). The n_1 eigenvalues λ_i of M_{ik} are positive, i. e. the $\mathrm{M}_{ik} = \left.\frac{\partial^2 \mathrm{V}}{\partial \phi_i \partial \phi_k}\right|_{\phi_0} \geqslant 0$ are non-negative for a minimum of $\mathrm{V}(\phi)$. The remaining $(n - n_1)$ eqs. then entail the eigenvalue $m^2 = 0$ to hold. ■

Remark. — If

(29) $$\mathrm{H}\psi = \mathrm{M}\psi \qquad (\mathrm{H} \in \mathrm{L}(\mathscr{H}, \mathscr{H}) \text{ is}$$

the total Hamiltonian acting on observables states in Hilbert space $\mathscr{H}$), then diagonalization of the mass matrix M means, that there is a unitary transformation $U \in L(\mathscr{H}, \mathscr{H})$, such that

$$\bar{\psi} = U\psi . \tag{30}$$

We transfer (30) into (29) and multiply from the left with U^{-1}, so that

$$H\bar{\psi} = HU\psi = MU\psi \Rightarrow U^{-1}HU\psi = U^{-1}MU\psi = \bar{M}\psi .$$

Since eq. (28) can be written as (28′) $I\phi_0 = 0$ (I annihilates the vacuum ϕ_0), so $H(I\phi_0) = M(I\phi_0) = 0$. But also $U(I\phi_0) = 0$. Hence

$$U^{-1}MU(I\phi_0) = \bar{M}(I\phi_0) = 0 .$$

Therefore, if eqs. (27) hold for M, they are also true of the diagonal matrix $\bar{M}$.

Example for theorem 1. — Let

$$L = L_1 + V(\phi^+\phi), \qquad V = +\frac{\mu^2}{2}\phi^+\phi - \frac{\lambda}{4}(\phi^+\phi)^2 \tag{31}$$

be a Lagrangian with full invariance group $G = SU(2) \times U(1)$, $\dim G = 4$, i. e. the Higgs-field is supposed to be

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix} \qquad \phi_i \in \mathbb{R}, \quad i = 1, 2, 3, 4$$

Hence

$$\phi^+\phi = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2\phi_3 - i\phi_4)\frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix} = \frac{1}{2}\sum_{i=1}^{4}(\phi_i)^2$$

and $\phi_0^+\phi_0 = +\dfrac{\mu^2}{\lambda} > 0$ whenever $V(\phi_0^+\phi_0) =$ minimum.

To study the mass spectrum of the particles associated with the Higgs-field ϕ one determines the transformations of G which leave ϕ_0 invariant. This is equivalent to finding those infinitesimal transformations

$$I \in \mathscr{I}(SU(2) \times U(1))$$

which satisfy $I\phi_0 = 0$ (eq. (28′)), i. e. $\varphi(g)\phi_0 = \phi_0$ or $e^{i\alpha I}\phi_0 = \phi_0$ $\forall\alpha$. Out of the 4 generators $I_k \in \mathscr{I}(SU(2) \times U(1))$ only one annihilates the vacuum state ϕ_0. Three give rise to massless Goldstone bosons. To the $\mu^2 > 0$ case, however, corresponds one massive scalar boson with mass $m = \sqrt{+\mu^2}$.

In order to study the Higgs-mechanism in terms of Thom's catastrophe theory we consider a potential which is of the same topological type as the potentials (3) or (3′), i. e. $V : \mathbb{R}^2 \to \mathbb{R}$,

$$V = V(\xi, m) := \frac{\lambda}{4}\xi^4 - \frac{1}{2}(m - m_i)^2\xi^2 + km_i^n\xi ; \qquad m_i, k \in \mathbb{R}, \ n \in \mathbb{N} . \tag{3''}$$

where $m \in \mathbb{R}$ is assumed to be variable and $\xi = \sqrt{\phi_i^2}$.

Consider now the graph of $V : \mathbb{R}^2 \to \mathbb{R}$ which is the manifold

$$(32)\quad M^2 := \left\{ (\xi, m, V(\xi, m)) \mid V(\xi, m) = \frac{\lambda}{4}\xi^4 - \frac{1}{2}(m - m_i)^2\xi^2 + km^n\xi \right\}$$

Otherwise stated, M^2 is the image of the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$(33)\qquad f(\xi, m) = (\xi, m, V(\xi, m)).$$

Given the submanifold (M^2, f) of $\mathbb{R}^3$ in terms of (32)-(33) we consider next the spherical (or Gauss) map [*4*]

$$(34)\qquad n : M^2 \to S^2_\phi \subset \mathbb{R}^3$$

which is a $\mathbb{R}^3$-vector-valued map on M^2. That is, for $p \in M^2$,

$$n(p) \in \mathbb{R}^3 = T_{f(p)}(\mathbb{R}^3)$$

(tangent space of $\mathbb{R}^3$ at $f(p)$). The vector $n(p)$ being an element of S^2 is a unit vector, i. e. $\| n(p) \| = 1$. On account of the obvious identifications

$$df(T_p(M^2)) = T_p(M^2) = T_{n(p)}(S^2)$$

($df(T_p(M^2))$ and $T_{n(p)}(S^2)$ are the same subspaces of $\mathbb{R}^3$), the linear transformation

$$(35)\qquad dn(p) \in L(T_p(M^2), T_{n(p)}(S^2)) = L(T_p(M^2), T_p(M^2))$$

is then the second fundamental form of M^2 at p [*2*], [*4*], that is

$$(36)\qquad II(p)(X, Y) = \langle dn(p)X, Y \rangle \qquad X, Y \in T_p(M^2)$$

M^2 is endowed with the Riemannian structure induced from f, i. e.

$$(37)\qquad \langle X, Y \rangle_p = \langle df(X), df(Y) \rangle_{f(p)}$$

By virtue of the property of self-adjointness:

$$(38)\qquad \langle dn(p)X, Y \rangle = \langle X, dn(p)Y \rangle, \qquad X, Y \in T_p(M^2)$$

then $a_{ij} = a_{ji}\ \forall i, j$; $(a_{ij}) = (dn(p)_{ij})$. Let $(\xi, m) \equiv (x^1, x^2)$:

THEOREM 2. — *The matrix representation of the differential* (35) i. e.

$$(35')\qquad dn(p) : T_p(M^2) \to T_{n(p)}(S^2), \qquad p \in M^2,$$

relative to the basis $\{ e_1, e_2 \} = \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\}$ *in* $T_p(M^2)$ *is given in terms of a potential* $V : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$(39)\qquad (a_{ij}) = (dn(p)_{ij}) = \begin{pmatrix} -\dfrac{\partial^2 V}{\partial x_1^2} & -\dfrac{\partial^2 V}{\partial x_1 \partial x_2} \\ -\dfrac{\partial^2 V}{\partial x_2 \partial x_1} & -\dfrac{\partial^2 V}{\partial x_2^2} \end{pmatrix}; \qquad a_{ij} = a_{ji}$$

Proof. — Let $\varphi : \mathrm{U} \subset \mathbb{R}^2 \to \mathrm{M}^2$ be a homeomorphism, i. e. (U, φ) be a local chart of M^2. The map (33) is then given as

$$f(\varphi(x^1, x^2)) = (x^1, x^2, \mathrm{V}(x^1, x^2)) \qquad (x^1, x^2) \in \mathrm{U}\,. \tag{33'}$$

Relative to two bases $\{ e_1, e_2 \}$ in $\mathbb{R}^2$ and $\{ e'_1, e'_2, e'_3 \}$ in $\mathbb{R}^3$, the vector-valued map (33) has then the matrix representation

$$\left(\frac{\partial f^j}{\partial x^i}\right) = \begin{pmatrix} \dfrac{\partial f^1}{\partial x^1}(x) & \dfrac{\partial f^1}{\partial x^2}(x) \\ \dfrac{\partial f^2}{\partial x^1}(x) & \dfrac{\partial f^2}{\partial x^2}(x) \\ \dfrac{\partial f^3}{\partial x^1}(x) & \dfrac{\partial f^3}{\partial x^2}(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{\partial \mathrm{V}}{\partial x^1}(x) & \dfrac{\partial \mathrm{V}}{\partial x^2}(x) \end{pmatrix}; \qquad x \in \mathrm{U} \tag{40}$$

Thus, the spherical map (34) is characterized in terms of the normalized cross product of the two vectors

$$\frac{\partial f}{\partial x^1}(x) = \left(1, 0, \frac{\partial \mathrm{V}}{\partial x^1}\right) \qquad \text{and} \qquad \frac{\partial f}{\partial x^2}(x) = \left(0, 1, \frac{\partial \mathrm{V}}{\partial x^2}\right)$$

such that

$$n(x^1, x^2, \mathrm{V}(x^1, x^2)) \tag{41}$$

$$\frac{\left(1, 0, \dfrac{\partial \mathrm{V}}{\partial x^1}\right) \wedge \left(0, 1, \dfrac{\partial \mathrm{V}}{\partial x^2}\right)}{\sqrt{\left(\dfrac{\partial \mathrm{V}}{\partial x^1}\right)^2 + \left(\dfrac{\partial \mathrm{V}}{\partial x^2}\right)^2 + 1}} = - \frac{\dfrac{\partial \mathrm{V}}{\partial x^1} e'_1 - \dfrac{\partial \mathrm{V}}{\partial x^2} e'_2 + e'_3}{\sqrt{\left(\dfrac{\partial \mathrm{V}}{\partial x^1}\right)^2 + \left(\dfrac{\partial \mathrm{V}}{\partial x^2}\right)^2 + 1}}$$

This implies, on account of the rule $\mathrm{X}(f) = df(\mathrm{X}), \mathrm{X} \in \mathrm{T}_p(\mathrm{M}), f \in \mathrm{F}^0(\mathrm{M})$ [2]:

$$dn(p)\frac{\partial}{\partial x^i}(f) \equiv n_* \frac{\partial}{\partial x^i}(f) = n_* f_*\left(\frac{\partial}{\partial x^i}\right) = (n \circ f)_*\left(\frac{\partial}{\partial x^i}\right) = d(n \circ f)\frac{\partial}{\partial x^i}$$

$$= \frac{\partial}{\partial x^i}(n \circ f)\bigg|_{n(p)} = \left(\frac{\partial(n^1 \circ f)}{\partial x^i}, \frac{\partial(n^2 \circ f)}{\partial x^i}, \frac{\partial(n^3 \circ f)}{\partial x^i}\right)\bigg|_{n(p)}; \; i \in \{ 1, 2 \}$$

hence

$$a_i^j = \frac{\partial(n^j \circ f)}{\partial x^i} = \frac{\partial}{\partial x^i}\left[- \frac{1}{\mathrm{F}}\left(\frac{\partial \mathrm{V}}{\partial x^1}, \frac{\partial \mathrm{V}}{\partial x^2}, 1\right)\right]; \tag{42}$$

$$\mathrm{F} := \sqrt{\left(\frac{\partial \mathrm{V}}{\partial x^1}\right)^2 + \left(\frac{\partial \mathrm{V}}{\partial x^2}\right)^2 + 1}$$

Thus, since $\varphi(0) = p$ and $\dfrac{\partial \mathrm{V}}{\partial x^1}(0) = \dfrac{\partial \mathrm{V}}{\partial x^2}(0) = 0$ and $\mathrm{F}(0) = 1$, one finds

$$a_{11} = \frac{\partial}{\partial x^1}\left(- \frac{1}{\mathrm{F}}\frac{\partial \mathrm{V}}{\partial x^1}\right)\bigg|_0 = \frac{1}{\mathrm{F}^2}\frac{\partial \mathrm{F}}{\partial x^1}\frac{\partial \mathrm{V}}{\partial x^1}\bigg|_0 - \frac{1}{\mathrm{F}}\frac{\partial^2 \mathrm{V}}{\partial x_1^2}\bigg|_0 = - \frac{\partial^2 \mathrm{V}}{\partial x_1^2}\bigg|_0$$

and similarly

$$a_{12} = a_{21} = -\left.\frac{\partial^2 V}{\partial x_1 \partial x_2}\right|_0 ; \qquad a_{22} = -\left.\frac{\partial^2 V}{\partial x_2^2}\right|_0 \qquad \blacksquare$$

Remark. — The matrix elements a_{ij} relative to the basis $\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\}$ can also be obtained in the use of formula (36):

$$\mathrm{II}(p)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle dn(p)\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \qquad i, j = 1, 2$$

An alternate version of Goldstone's theorem (Theorem 1 of sect. II *b)* is now available in terms of the following.

COROLLARY 3. — *Let* $V : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$V := 1/4\, \xi^4 - 1/2(m - m_i)^2 \xi^2 + \mathscr{X} m_i^n \xi ; \qquad n \in \mathbb{N},\ \mathscr{X} \in \mathbb{R}$$

fixed, m variable and denote by $(x^1, x^2) = (\xi, m)$. *Then the matrix* (39) *can assume one of the following forms*

$$M_{kl} = \begin{pmatrix} 3m_i^2 & 0 \\ 0 & m_i^2 \end{pmatrix} \qquad \text{whenever } \mathscr{X} = 1 \tag{43}$$

or

$$M_{kl}^0 = \begin{pmatrix} m_{kl}^2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{whenever } \mathscr{X} = 0 \tag{43'}$$

For $\mathscr{X} \in \mathbb{R} - \{0\}, \mathscr{X} \neq 1$ M_{kl} *is of the form* (43).

Proof. — Minima of $V : \mathbb{R}^2 \to \mathbb{R}$,

$$V(\xi, m) = 1/4\, \xi^4 - 1/2(m - m_i)^2 \xi^2 + \mathscr{X} m_i^3 \xi$$

require

$$\operatorname{grad} V = \left(\frac{\partial V}{\partial \xi}, \frac{\partial V}{\partial m}\right) = 0,$$

i. e. one must solve the system of equations

$$\frac{\partial V}{\partial \xi} = 0, \quad \frac{\partial V}{\partial m} = 0.$$

Let (ξ, m) be any solution of this system. If

$$\frac{\partial^2 V}{\partial \xi^2}(\xi, m) > 0 \qquad \text{and} \qquad \left.\left[\frac{\partial^2 V}{\partial \xi^2}\frac{\partial^2 V}{\partial m^2} - \left(\frac{\partial^2 V}{\partial \xi \partial m}\right)^2\right]\right|_{(\xi,m)} > 0$$

then (ξ, m) is a minimum. We analyze two cases.

(*i*) $\mathscr{X} = 0$, then

$$\frac{\partial V}{\partial \xi} = 0 = \xi^3 - (m - m_i)^2\xi \Rightarrow \underset{m - m_i = m_0 \neq 0}{\xi = 0} \quad \text{or} \quad \underset{m = m_i}{\xi = \pm (m - m_i)}$$

$$\frac{\partial V}{\partial m} = 0 = (m - m_i)\xi^2$$

Thus one obtains the solutions:

$$(\xi, m) = (0, m_k)\,; \quad (\xi, m) = (\pm (m - m_i), m_i)$$

A minimum requires:

$$\frac{\partial^2 V}{\partial \xi^2}(0, m_k) > 0 \quad \text{or} \quad \frac{\partial^2 V}{\partial \xi^2}(\pm (m - m_i), m_i) > 0$$

(*i*1) $\xi = \pm (m - m_i) \Rightarrow \frac{\partial^2 V}{\partial \xi^2}(\pm (m - m_i), m_i)$

$$= 3(m - m_i)^2 - (m - m_i)^2 = 2(m - m_i)^2 \,|_{m=m_i} = 0$$

$$\therefore \frac{\partial^2 V}{\partial \xi^2}(\pm (m - m_i), m_i) = \frac{\partial^2 V}{\partial m^2}(\pm (m - m_i), m_i) = \frac{\partial^2 V}{\partial \xi \partial m}(\pm (m - m_i), m_i) = 0$$

$$\Rightarrow M_{ik}(\pm (m - m_i), m_i) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

No information is obtained and further investigation is necessary.

(*i*2) $\xi = 0, \ m_k = m_0 + m_i$

$$\frac{\partial^2 V}{\partial \xi^2}(0, m_k) = (m_k - m_i)^2 = m_{ik}^2 > 0$$

$$\frac{\partial^2 V}{\partial m^2}(0, m_k) = \frac{\partial^2 V}{\partial m \partial \xi}(0, m_k) = \frac{\partial^2 V}{\partial \xi \partial m} = 0 = 2(m - m_i)\xi \Big|_{(0,m_k)}$$

Thus

$$M_{ik} = \begin{pmatrix} m_{ik}^2 & 0 \\ 0 & 0 \end{pmatrix}; \quad \left[\frac{\partial^2 V}{\partial \xi^2}\frac{\partial^2 V}{\partial m^2} - \left(\frac{\partial^2 V}{\partial \xi \partial m}\right)^2\right]_{(0,m_k)} = 0$$

hence further (geometric) investigation is required

(*ii*) $\mathscr{X} = 1\,(n = 3)$; one finds $\frac{\partial^2 V}{\partial m^2}(m_i, m_i) = m_i^2\,;\ \frac{\partial^2 V}{\partial \xi^2}(m_i, m_i) = 3m_i^2$

$$\frac{\partial^2 V}{\partial m \partial \xi}(m_i, m_i) = \frac{\partial^2 V}{\partial \xi \partial m}(m_i, m_i) = 2(m - m_i)\xi \Big|_{(m_i,m_i)} = 0$$

$$\left[\frac{\partial^2 V \partial^2 V}{\partial \xi^2 \, \partial m^2} - \left(\frac{\partial^2 V}{\partial \xi \partial m}\right)^2\right]_{(m_i,m_i)} = 3m_i^4 > 0 \quad \blacksquare$$

III. CONCLUSION

The way how the Higgs-mechanism associates massive particles with gauge fields (those corresponding to broken group generators) can be exhibited in two different ways:

(1) Consider the minimum condition (14), sect. II *a* for $V(\phi)$, i. e.

$$\phi_i^0 = \sqrt{+\frac{\mu^2}{\lambda}};$$

There is a transformation law

$$(44) \qquad \phi(x) = U(x)\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \phi_i^0 + \eta(x) \end{pmatrix}; \qquad U(x) = e^{i\alpha_i(x)I_i}, \; i \in \{1, \ldots (n - n_1)\}$$

where $\eta(x)$ constitutes a field which measures the amount by which ϕ_i differs from the minimum value ϕ_i^0. Substituting (44) into the Lagrangian (1′) and formula (5) of sect. I yields:

$$(45) \quad L = 1/2(\partial_\mu\eta)(\partial_\mu\eta) + \mu^2\eta - \frac{g^2\phi_i^0}{2}(I_iA_\mu^i)(I_iA_\mu^i) - 1/4\, F_{\mu\nu}F_{\mu\nu} + \text{higher order terms}.$$

The following can then been read off from (45):

(*j*) Spontaneous symmetry breakdown of gauge theories does not lead to massless (Goldstone) bosons.

(*jj*) The η-field has become massive, i. e. $m = \sqrt{+2\mu^2}$.

(*jjj*) There are now $(n - n_1)$ massive vector bosons (46) $m_i = g\phi_i^0$. They correspond to the spontaneously broken symmetry generators. I. e. the matrix M_{ik} (eq. (21)) of theorem 1 accounts for $(n - n_1)$ Goldstone bosons. There is one Goldstone boson for each group generator $I \in \mathscr{I}(G)$ such that $I\phi_0 \neq 0$ (= broken symmetry generators). These previously massless gauge mesons acquire now the masses (46). They are associated with those gauge fields which correspond to the broken group generators $I : I\phi_0 \neq 0$.

(*jv*) The remaining n_1 vector bosons are those corresponding to the exact symmetry (isotropy subgroup G_0). They are massless.

(2) In terms of R. Thom's catastrophe theory [3] the Higgs mechanism is governed by a potential $V(m, \xi)$ (eq. (3″)). The mass matrix $M_{ik} = \begin{pmatrix} 3m_i^2 & 0 \\ 0 & m_i^2 \end{pmatrix}$

(eq. (43)), is derived from V, and contrary to the matrix (21), accounts directly for the massive particles, whose number is $n - n_1 = 2$, since $n = \dim \mathrm{SO}(3) = 3$ and $n_1 = \dim \mathrm{SO}(2) = 1$. This situation reflects the case of the Lagrangian (45) and (*j*)-(*jv*). The matrix (43′) is discarded for physical and geometrical reasons.

The potential $V(\xi, m)$ when compared with a potential of the same topological type, that is

$$(46) \qquad U(\xi, u_1, u_2) = 1/4\, \xi^4 + u_1 \frac{\xi^2}{2} + u_2 \xi\,; \qquad u_i \in \mathbb{R}\ i = 1, 2$$

admits the following interpretations:

— There is an information of « mass degeneracy » stored in the singularity $U_0(\xi) = V_0 = 1/4\, \xi^4$. The corresponding mass-matrix (39), i. e.

$$(39') \qquad M_{ik} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has diagonal elements $a_{11} = m_i^2 = 0$ $a_{22} = m_k^2 = 0$. These $(n - n_1)$ vanishing masses are regarded as corresponding to Goldstone-particles.

— The universal unfolding family [*3*], which corresponds to the given potential (3″), i. e.

$$(46') \quad U = U_0(\xi) + u_1 U_1(\xi) + u_2 U_2(\xi)\,; \qquad U_1(\xi) = \frac{1}{2}\xi^2\,,\ U_2(\xi) = \xi$$

decodes the information of mass degeneracy stored in $U_0(\xi)$.

So we conclude that Thom's catastrophe theory is susceptible to describe the Higgs effect.

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