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Section A :
Physique théorique.

Presymplectic Lagrangian systems II: the second-order equation problem

by

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ABSTRACT. — The « second-order equation problem » for degenerate Lagrangian systems is solved. Using techniques of global presymplectic geometry we find that, typically, solutions of « consistent » Lagrangian equations of motion are not globally second-order equations. For a broad class of Lagrangians, which we term « admissible », we characterize as well as prove the existence of certain submanifolds of velocity phasespace along which the Lagrange equations are second-order. Furthermore, we provide an explicit construction of both these submanifolds and their associated second-order equation solutions of the Lagrange equations.

I. INTRODUCTION

In previous papers [*1-4*], we have investigated the global presymplectic geometry of Lagrangian systems. A constraint algorithm was developed which enables us to define and solve « consistent » Lagrangian equations of motion in the degenerate case. In the present communication, we examine an important global aspect of the solutions of these equations, the « second-order equation problem ».

The consistent Lagrange equations that follow from the constraint algorithm are typically a set of coupled *first-order* differential equations—a feature characteristic of theories which are described mathematically by presymplectic geometries. Variational as well as physical considerations demand, however, that the Lagrange equations be a set of coupled *second-order* differential equations [*5*]. Geometrically, this reflects to some extent the fact that the fundamental mathematical object in the Lagrangian formulation—velocity phasespace—is a tangent bundle.

It is therefore important to find the conditions under which the first-order Lagrange equations that follow from the constraint algorithm are equivalent to a set of second-order differential equations. This question was first raised by Künzle [*6*], who discussed degenerate homogeneous Lagrangian systems. He developed an algorithm which in principle constructs a submanifold of velocity phasespace such that all solutions of the Lagrange equations are second-order when restricted to this submanifold. Unfortunately, Künzle was unable to show that his algorithm does in fact terminate, except under very restrictive conditions.

Here, in contrast to Künzle, Lagrangian systems are considered from the point of view of the inhomogeneous formalism. This has the virtue of greatly simplifying the problem, for now a second-order equation can be described by a single vectorfield rather than by a two-dimensional distribution.

The second-order equation problem is nontrivial, for there exist Lagrangians which give rise to consistent equations of motion which are *nowhere* second-order [*7*]. Typically, it appears that the Lagrange equations will be second-order globally iff the Lagrangian is regular (cf. § III). Furthermore, specific results can be obtained only when the Lagrangian is « admissible », in which case it will presently be shown that there exist certain submanifolds of velocity phasespace along which the Lagrange equations are second-order. This « Second-Order Equation Theorem », the proof of which contains an explicit construction of both the above mentioned submanifolds and their associated second-order equation solutions, constitutes the main result of this paper.

Section II provides a concise introduction to the almost tangent structure canonically associated to velocity phasespace and its exterior calculus; its application to Lagrangian systems [*8*] is briefly outlined. Also included here is a short summary of the methods by which one defines and solves consistent Lagrangian equations of motion in the degenerate case. In the third section, we discuss the second-order equation requirement and formally state the Second-Order Equation Theorem. The proof of this theorem is in two steps: local existence and uniqueness in § IV and global existence, uniqueness and classification in the last section. In general, we try to keep our notation and terminology consistent with that of reference [*1*].

II. ALMOST TANGENT STRUCTURE AND THE PRESYMPLECTIC GEOMETRY OF LAGRANGIAN DYNAMICS [*1-4*, *9*, *10*, *12*]

A manifold is said to be *symplectic* if it carries a distinguished closed nondegenerate 2-form. If we relax the requirement that this form have maximal rank the manifold is *presymplectic*. We have shown [*1*] that the presymplectic geometry of Lagrangian systems is sufficient in and by itself to define a canonical formalism for Lagrangian dynamics. Consequently, one need not resort to the Hamiltonian formulation in order to « cast the theory into canonical form », or even to quantize the theory. This generality is not illusory, as there exist well-behaved Lagrangian systems whose Hamiltonian counterparts are highly singular or even nonexistent (cf. [*1*]).

We begin by developing just enough of the theory of vector-valued differential forms to enable us to put a presymplectic structure on velocity phasespace. The basis of this approach to Lagrangian mechanics is Klein's concept of « almost tangent geometry » [*9*, *10*]. This formalism possesses numerous technical advantages over the more standard methods relying upon the fiber derivative (cf. [*1*]); furthermore, almost tangent geometry provides a natural framework in which interesting generalizations of Lagrangian dynamics may be developed [*11*]. We first establish some notation.

If Q is a manifold, we denote by TQ both the tangent bundle of Q and the space of all smooth vectorfields on Q. The bundle projection is $\tau_Q : TQ \to Q$. The prolongation of τ_Q to T(TQ) is denoted $T\tau_Q$, and is such that the following diagram commutes:

$$\begin{array}{ccc} T(TQ) & \xrightarrow{T\tau_Q} & TQ \\ {\scriptstyle \tau_{TQ}}\downarrow & & \downarrow{\scriptstyle \tau_Q} \\ TQ & \xrightarrow[\tau_Q]{} & Q \end{array}$$

The *vertical bundle* V(TQ) is the subbundle of T(TQ) defined by

$$V(TQ) := \ker T\tau_Q := \{ Z \in T(TQ) \mid T\tau_Q(Z) = 0 \}.$$

If $(U; \widehat{q}^i)$ is a chart on Q, the chart $(TU; q^i, v^i)$ on TQ defined by

$$q^i(w) = \widehat{q}^i \circ \tau_Q(w) \quad , \quad v^i(w) = \langle w \mid d\widehat{q}^i \rangle$$

for $w \in TQ$ is said to be a *natural bundle chart*. Here, $\langle \mid \rangle$ denotes the natural pairing $TQ \times T^*Q \to \mathbb{R}$. Natural bundle charts of the form $(T(TU); q^i, v^i, \dot{q}^i, \dot{v}^i)$ on T(TQ) are defined similarly.

Let ξ_y denote the vertical lift $T_xQ \to V_y(TQ)$, that is, for $x = \tau_Q(y) = \tau_Q(w)$,

$$\xi_y(w) := \frac{d}{d\lambda}(y + \lambda w)\,|_{\lambda=0}.$$

Using this, one can define a map $J_y : T_y(TQ) \to T_y(TQ)$ by

$$J_y(z) := \xi_y \circ T\tau_Q(z)$$

for all $z \in T_y(TQ)$. We thus obtain a linear endomorphism

$$J : T(TQ) \to T(TQ)$$

such that $J^2 = 0$ and $\ker J = \operatorname{Im} J = V(TQ)$. The vector-valued 1-form J is called the *almost tangent structure* naturally associated to TQ. In a natural bundle chart on T(TQ), the action of J is

$$J(q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, v^i, 0, \dot{q}^i).$$

Define the *adjoint* J^* of J to be the linear endomorphism of the exterior algebra $\Lambda(TQ)$ given by

$$J^*f := f \quad , \quad \langle X \mid J^*\alpha \rangle := \langle JX \mid \alpha \rangle, \tag{2.1}$$

where $f \in C^\infty(TQ)$, $\alpha \in T^*(TQ)$ and $X \in T(TQ)$. J^* is then defined on $\Lambda(TQ)$ by homomorphic extension. Define the *interior product* of J with a p-form β by

$$i_J\beta(X_1, \ldots, X_p) := \sum_{i=1}^{p} \beta(X_1, \ldots, JX_i, \ldots, X_p) \tag{2.2}$$

where $X_1, \ldots, X_p \in T(TQ)$, and set $i_J f := 0$ for any function f. Finally, the *vertical derivative* d_J is

$$d_J := [i_J, d].$$

It is apparent that d and d_J anticommute, and furthermore that $d_J^2 = 0$. Also, from (2.1) and the definition of i_J one has that $d_J f = J^*df$, and consequently $i_J d_J f = 0$.

Physically, Q represents the configuration space of a physical system, while TQ is its velocity phasespace. The almost tangent geometry of TQ, in and by itself, is not enough to define a presymplectic structure. However, if we distinguish a Lagrangian $L : TQ \to \mathbb{R}$, then J determines a preferred presymplectic form

$$\Omega := -\, dd_J L. \tag{2.3}$$

In a natural bundle chart (2.3) becomes simply

$$\Omega = \frac{\partial^2 L}{\partial v^i \partial q^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dq^i \wedge dv^j. \tag{2.4}$$

By construction and (2.2), J is Hamiltonian for Ω, i. e.,

$$i_J\Omega = 0. \tag{2.5}$$

The Lagrangian L is said to be *regular* iff Ω is nondegenerate; otherwise L is *degenerate* or *irregular*. In more familiar terms, (2.4) shows that Ω is nondegenerate iff the velocity Hessian $\left[\dfrac{\partial^2 L}{\partial v^i \partial v^j}\right]$ of L is invertible. The triple (TQ, L, Ω) is said to be a *Lagrangian system*.

In order to geometrize the equations of motion we need to define yet one other object, the *Liouville vectorfield* V, characterized as follows:

$$V(w) := \xi_w(w).$$

Note that V is *vertical*, that is, $JV = 0$. We call the function $A_f : TQ \to \mathbb{R}$ defined by $A_f := V[f]$, for $f \in C^\infty(TQ)$, the *action* of f.

Return now to the Lagrangian system (TQ, L, Ω). If A_L is the action of L, then the *energy* E of L is simply $A_L - L$. The almost tangent structure intertwines E and Ω according to

$$i_J dE = -i_V \Omega, \tag{2.6}$$

and, in these terms, the Lagrangian equations of motion are

$$i_X \Omega = dE. \tag{2.7}$$

In a natural bundle chart, (2.4) gives for (2.7)

$$\left.\begin{aligned} \frac{\partial^2 L}{\partial v^i \partial v^j}(a^j - v^j) &= 0 \\ \frac{\partial^2 L}{\partial v^i \partial v^j} b^i = \frac{\partial^2 L}{\partial v^{[i} \partial q^{j]}} a^i &- \frac{\partial^2 L}{\partial v^i \partial q^j} v_i + \frac{\partial L}{\partial q^j}, \end{aligned}\right] \tag{2.8}$$

where $X = a^i(\partial/\partial q^i) + b^i(\partial/\partial v^i)$.

In the regular case, Ω is symplectic so that equations (2.7) possess a unique solution X whose integral curves are the dynamical trajectories of the system in velocity phasespace. The assumption of regularity is, however, too restrictive (e. g., the Maxwell and Einstein systems). The major implication of the degeneracy of the Lagrangian is that Ω will now be merely *presymplectic* (i. e., Ω is no longer of maximal rank). Consequently, the equations of motion (2.7) as they stand need not be consistent and will not in general possess globally defined solutions (and, typically, even if solutions exist they will not be unique).

We have developed a geometric constraint algorithm [*1-4*, *12*] which gives necessary and sufficient conditions for the solvability of « consistent » Lagrangian equations of motion. Specifically, the algorithm finds whether or not there exists a submanifold P of TQ along which the equations (2.7) hold; if such a submanifold exists, then the algorithm gives a *constructive* method for finding it. This is accomplished as follows: the algorithm generates a sequence of submanifolds

$$\dots \to P_3 \to P_2 \to P_1 := TQ$$

defined by:

$$P_{l+1} := \{ m \in P_l \mid \langle TP_l^{\perp} \mid dE \rangle (m) = 0 \},$$

where

$$TP_l^{\perp} := \{ Z \in T(TQ) \mid \Omega(Z, TP_l) = 0 \}$$

with the obvious notational shorthand. The constraint algorithm must terminate with some *final constraint submanifold* $P \equiv P_K$, $1 \leq K < \infty$. If $P \neq \phi$, then on P one has completely consistent equations of motion of the form

$$(i_X\Omega - dE) \mid P = 0, \tag{2.9}$$

and furthermore one is assured that at least one solution $X \in TP$ of these equations exists. The solutions of (2.9) are not necessarily unique, however, being determined only up to vectofields in $\ker \Omega \cap TP$. The final constraint submanifold P is *maximal* in the sense that if N is any other submanifold along which the equations (2.7) are satisfied (with solutions tangent to N), then $N \subseteq P$. The triple (TQ, Ω, P) is called a *Lagrangian canonical system.*

III. THE SECOND-ORDER EQUATION THEOREM [*2-4*]

We have shown [*5*] that the equations of motion

$$i_X\Omega = dE \tag{3.1}$$

associated to a presymplectic Lagrangian system (TQ, L, Ω) will follow from a variational principle iff the *second-order equation condition*

$$JX = V \tag{3.2}$$

is satisfied. In a natural bundle chart $(TU; q^i, v^i)$ for TQ, (3.2) implies that X must have the form

$$X(q, v) = v^i(\partial/\partial q^i) + b^i(\partial/\partial v^i) \tag{3.3}$$

for some coefficients b^i. In more familiar terms, X will be a *second-order equation* iff

$$T\tau_Q(X) = \tau_{TQ}(X). \tag{3.4}$$

When (TQ, L, Ω) is regular, the unique solution X of (3.1) is automatically a second-order equation. Indeed, applying i_J to (3.1) and utilizing (2.6), one has

$$i_J i_X \Omega = - i_V \Omega.$$

In view of (2.5) and the identity $[i_J, i_X] = - i_{JX}$, this becomes

$$- i_{JX}\Omega = - i_V\Omega,$$

and the nondegeneracy of Ω implies the desired result (3.2). Alternatively, one could derive this result from the first of equations (2.8) and the non-singularity of the matrix $(\partial^2 L/\partial v^i \partial v^j)$.

As a concrete example, consider the kinetic-energy Lagrangian

$$L(m) = 1/2 \langle m, m \rangle,$$

where $m \in TQ$ and $\langle\ ,\ \rangle$ is a Riemannian metric on Q. The solution

$$X = v^i \partial/\partial q^i - \Gamma^i_{jk} v^j v^k \partial/\partial v^i$$

of the Lagrange equations is not only a second-order equation, but is in fact a *spray* (i. e., the coefficients b^i in (3.3) are quadratic in the velocities v^i). The second-order Lagrange equations in this case are the geodesic equations for the metric $\langle\ ,\ \rangle$.

If L is irregular, then (3.2) need not follow. Furthermore, even if consistent Lagrangian equations of motion

$$(i_X \Omega - dE) \mid P = 0 \tag{3.5}$$

can be defined on some final constraint submanifold P, the constraint algorithm is not sufficient to assure that the solutions of (3.5) will satisfy the second-order equation condition.

The problem, then, is to find simultaneous solutions of equations (3.5) and (3.2); generically, such solutions—if they exist—will be defined only along some submanifold of P. Hence, one searches for a submanifold S of P and a smooth vectorfield $X \in TS$ such that $(i_X \Omega - dE) \mid S = 0$ and $(JX - V) \mid S = 0$. Unfortunately, for a completely general canonical system (TQ, Ω, P), S and X need not exist, and even if they do they will not necessarily be unique.

The additional structure required to *ensure* the existence of at least one such submanifold S and smooth solution X is that of « admissibility » [*13*]. A Lagrangian system (TQ, L, Ω) is *admissible* provided the leaf space $\mathscr{L}$ of the foliation $\mathscr{D}$ of TQ generated by the involutive distribution $D := \ker \Omega \cap V(TQ)$ admits a manifold structure such that the canonical projection $\zeta : TQ \to \mathscr{L}$ is a submersion [*14*].

Let P be the final constraint submanifold associated to the Lagrangian system (TQ, L, Ω) by the algorithm, and assume that P is imbedded in TQ. If (TQ, L, Ω) is admissible, then so is the canonical system (TQ, Ω, P) in the following sense:

PROPOSITION 1. — D restricts to an involutive distribution in TP so that $\mathscr{D}_P := \mathscr{D} \mid P$ foliates P. Furthermore, the leaf space $\mathscr{L}_P := P/\mathscr{D}_P$ is a manifold imbedded in $\mathscr{L}$ and the induced projection $\zeta_P : P \to \mathscr{L}_P$ is a submersion.

Proof. — First, show that $D \mid P \subseteq TP$ by inductior on the constraint submanifolds P_l. Of course, $D \subseteq T(TQ)$; suppose that $D \mid P_l \subseteq TP_l$.

The constraint submanifold P_{l+1} is characterized by the vanishing of functions of the form $\phi = i_Z dE$, where $Z \in TP_l^{\perp}$. Thus, a vectorfield $Y \in D$ is tangent to P_{l+1} iff $Y[\phi] \mid P_{l+1} = 0$ for all such ϕ. Now, if $\pounds$ denotes the Lie derivative,

$$Y[\langle Z \mid dE \rangle] = \langle [Y, Z] \mid dE \rangle + \langle Z \mid \pounds_Y dE \rangle. \qquad (3.6)$$

Since $Y \in D$, Y is vertical. Consequently, locally there exists a vectorfield W such that $JW = Y$. Then, by (2.6),

$$\begin{aligned} \pounds_Y dE &= di_{JW} dE = di_W i_J dE \\ &= -\, di_W i_V \Omega = -\, d[\Omega(V, W)]. \end{aligned}$$

But V is vertical, so locally $V = JW'$ for some vectorfield W'. By (2.5),

$$\begin{aligned} \pounds_Y dE &= -\, d[\Omega(JW', W)] \\ &= d[\Omega(W', JW)]. \end{aligned}$$

However, $JW = Y \in D \subseteq \ker \Omega$, and hence the second term in (3.6) vanishes. Letting $W \in TP_l$ be arbitrary, one has along P_l

$$\begin{aligned} \Omega([Y, Z], W) &= -\, \pounds_Y \Omega(Z, W) + \pounds_Y[\Omega(Z, W)] - \Omega(Z, [Y, W]) \\ &= 0 \end{aligned}$$

by the assumptions on Y, Z and W. Consequently, $[Y, Z] \mid P_l \in TP_l^{\perp}$, so that $\langle [Y, Z] \mid dE \rangle \mid P_{l+1} = 0$, which implies that $D \mid P_{l+1} \subseteq TP_{l+1}$.

Thus, $D \mid P$ gives rise to a foliation $\mathscr{D}_P$ of P. The leaf space $\mathscr{L}_P$ of $\mathscr{D}_P$ can be identified with the image of P under the map $\zeta_P : = \zeta \circ j$, where j is the imbedding $P \to TQ$. Consequently, $\mathscr{L}_P$ inherits a submanifold structure from $\mathscr{L}$ such that ζ_P is a submersion and $j : \mathscr{L}_P \to \mathscr{L}$ is an imbedding. ∇

The main result, the *Second-Order Equation Theorem*, can be stated as follows:

THEOREM. — Let (TQ, L, Ω) be an admissible Lagrangian system with final constraint submanifold P imbedded in TQ. Then there exists at least one submanifold S of P and a *unique* (for fixed S) smooth vectorfield $X \in TS$ which simultaneously satisfies

$$(i_X \Omega - dE) \mid S = 0$$

and

$$(JX - V) \mid S = 0.$$

Every such submanifold S is diffeomorphic to $\mathscr{L}_P$. ∇

If (TQ, L, Ω) is regular, then it is trivially admissible, and the theorem reduces to the result derived above that the unique solution of (3.1) is a second-order equation everywhere, i. e., $S = TQ$.

In the degenerate case, it is worth noting that the submanifold S whose existence is guaranteed by the above theorem is *strictly* a submanifold of P. Furthermore, there will usually exist many such submanifolds, the collection of which is parameterized by a certain quotient of the kernel of the presymplectic structure Ω (§ V).

The Second-Order Equation Theorem is simply to be regarded as an *existence* theorem. In certain instances—regardless of whether or not (TQ, L, Ω) is admissible—there may exist submanifolds with the desired properties which are « arbitrarily large », perhaps even comprising all of the final constraint submanifold P itself [*15*]. Thus, no attempt is made to assert that the submanifolds whose existence is assured by the theorem are « maximal » [*16*].

The proof of the theorem will be broken into two major parts. Section IV consists of ultra-local arguments to the effect that there is a unique point in each leaf of $\mathscr{D}_P$ at which every « reasonable » solution X of (3.5) also satisfies (3.2). In § V it is shown that these points actually define a submanifold of P diffeomorphic to $\mathscr{L}_P$, and that there exists a unique second-order equation solution of the Lagrange equations tangent to this submanifold.

IV. LOCAL EXISTENCE AND UNIQUENESS

Let (TQ, Ω, P) be an admissible Lagrangian canonical system, and suppose that $X \in TP$ is a solution of (3.5). The problem is to characterize the subset of P on which X, subject to certain regularity conditions, is algebraically a second-order equation.

Since (TQ, L, Ω) is admissible and $\langle D \mid dE \rangle = 0$, the techniques of reference [*4*] enable one to conclude that there exists a unique presymplectic system $(\mathscr{L}, \tilde{\Omega}, d\tilde{E})$ on the leaf space $\mathscr{L}$ such that $\zeta^*\tilde{\Omega} = \Omega$ and $\zeta^*\tilde{E} = E$. The connection between the dynamics on TQ and that on $\mathscr{L}$ is given by the following version of the Equivalence Theorem [*1*, *17*] :

Lemma. — There exists at least one solution $\tilde{X}$ of the reduced Lagrange equations

$$(i_{\tilde{X}}\tilde{\Omega} - d\tilde{E}) \,|\, \mathscr{L}_P = 0. \tag{4.1}$$

Any vectorfield $X \in T\zeta^{-1}\{\tilde{X}\}$ will then solve (3.5). Conversely, if $X \in TP$ solves (3.5) and $T\zeta(X)$ is well-defined, then $T\zeta(X) \in T\mathscr{L}_P$ solves (4.1). ∇

A vectorfield X is *prolongable* if it projects to $\mathscr{L}$, i. e., $T\zeta(X)$ is well-defined. The above lemma suffices to show that prolongable solutions of (3.5) always exist when (TQ, L, Ω) is admissible: indeed, if $\tilde{X} \in T\mathscr{L}_P$ solves (4.1), then any vectorfield $X \in T\zeta^{-1}\{\tilde{X}\}$ is a prolongable solution of (3.5). Moreover, we say that X is *semi-prolongable* if it is prolongable modulo V(TQ).

For any vectorfield Y, define the *deficiency vectorfield* W_Y of Y to be $W_Y := JY - V$. Evidently, W_Y measures the failure of Y to be a second-order equation. One has the following important facts:

PROPOSITION 2. — *i)* $i_J(i_Y\Omega - dE) = i_{W_Y}\Omega$.
ii) If Y solves the Lagrange equations (3.5), then $W_Y \in D \mid P$.

Proof. — Part *i)* follows immediately from (2.6) and the definition of W_Y. For *ii)*, $W_Y \in (\ker \Omega) \mid P$ by *i)* and the assumption on Y, and W_Y is clearly vertical. Thus, $W_Y \in (\ker \Omega \cap V(TQ)) \mid P$. ∇

The following result solves the ultra-local existence problem for second-order equation solutions of the Lagrange equations:

PROPOSITION 3. — Let (TQ, L, Ω) be admissible, and let $X \in TP$ be a semi-prolongable solution of the equations (3.5). Then there exists a unique point in each leaf of $\mathscr{D}_P$ at which X is a second-order equation [*18*].

Proof. — Let $m \in P$, and denote by $\mathscr{D}_m$ the leaf of $\mathscr{D}_P$ through m. We claim that $n_X := T\tau_Q(X(m))$ is the required point. It is necessary to show that (*a*) n_X does not depend upon the choice of $m \in \mathscr{D}_m$; (*b*) $n_X \in \mathscr{D}_m$; and (*c*) that X is a second-order equation at n_X.

Since X is semi-prolongable, it can be decomposed $X = Y + Z$, where Y is prolongable and Z is vertical. Since D is vertical, there exists a map $\rho : T\mathscr{L} \to TQ$ such that the following diagram commutes

$$\begin{tikzcd} TQ & T(TQ) \arrow[l, "T\tau_Q"'] \arrow[d, "T\zeta"] \\ & T\mathscr{L} \arrow[ul, "\rho"] \end{tikzcd}$$

Consequently, $T\tau_Q(X(m)) = T\tau_Q(Y(m)) = \rho \circ T\zeta(Y(m))$. But $T\zeta(Y(m))$ is insensitive to the choice of $m \in \mathscr{D}_m$, and thus so is $T\tau_Q(X(m))$.

To prove (*b*), consider the vertical integral curves of $-W_X$. By Proposition 2, $-W_X \in D \mid P$ so that these trajectories are contained in the leaves of $\mathscr{D}_P$. In a natural bundle chart, one has

$$V(q, v) = v^i\partial/\partial v^i \quad , \quad JX(q, v) = n^i_X\partial/\partial v^i, \tag{4.2}$$

and so the equation determining the integral curves of $-W_X$ is

$$dv^i(\sigma)/d\sigma = v^i(\sigma) - n^i_X.$$

Note that, by (*a*), the functions n^i_X are constant on $\mathscr{D}_m$. The integral curve starting at $m = (q_0, v_0)$ is thus

$$v^i(\sigma) = n^i_X + e^\sigma(v^i_0 - n^i_X).$$

As $\sigma \to -\infty$, $v^i(\sigma) \to n^i_X$, so that n_X is a limit point of the line $v(\sigma)$. But $v(\sigma) \in \mathscr{D}_m$ for all σ, and $\mathscr{D}_m$ is closed (as ζ_P is continuous); consequently $n_X \in \mathscr{D}_m$.

Finally, X is a second-order equation at n_X, for $W_X(n_X) = 0$. Furthermore, it is clear from (4.2) that X is a second-order equation only at n_X. ∇

Two semi-prolongable solutions X, Y of (3.5) are said to be J-*equivalent* provided $J(X - Y) = 0$. This defines an equivalence relation, and J-equivalence classes of vectorfields X are denoted by [X]. It follows that if X, $Y \in [X]$, then $W_X = W_Y$, so that n_X depends only upon [X] (and will henceforth be denoted $n_{[X]}$).

V. GLOBAL EXISTENCE, CLASSIFICATION AND UNIQUENESS

Let X be a semi-prolongable solution of the Lagrange equations, and let $S_{[X]}$ denote the union of all the points $n_{[X]}$, one for each leaf of $\mathscr{D}_P$. The set $S_{[X]}$ depends only upon [X].

Given $X \in [X]$, one has a C^∞ injection $\alpha_{[X]} : \mathscr{L}_P \to P$ defined by $\alpha_{[X]}(\tilde{m}) = T\tau_Q(X(m))$, where, according to Proposition 3, $\alpha_{[X]}$ does not depend upon the choice of $m \in \zeta_P^{-1}\{\tilde{m}\}$. A calculation in charts suffices to show that $T\alpha_{[X]}$ is non-singular, and since $S_{[X]}$ is the image of $\mathscr{L}_P$ under $\alpha_{[X]}$, $S_{[X]}$ is a submanifold of P diffeomorphic to $\mathscr{L}_P$.

Each J-equivalence class [X] of solutions of the Lagrange equations defines a unique $S_{[X]}$: if $[X] \neq [Y]$, then $S_{[X]} \neq S_{[Y]}$ by Proposition 3. The solutions of (3.5) are unique up to vectorfields in $(\ker \Omega \cap TP)$; thus, the set of all submanifolds $S_{[X]}$ is parameterized by the quotient $(\ker \Omega \cap TP)/V(TQ)$.

For a fixed semi-prolongable solution X of the Lagrange equations, Proposition 3 and the above results guarantee the existence of a submanifold $S_{[X]}$ of P along which X *algebraically* satisfies both (3.1) and (3.2). Unfortunately, X need not be a second-order equation in a *differential* sense, that is, $X \mid S_{[X]} \notin TS_{[X]}$ necessarily. Physically, of course, one is interested in precisely those solutions X for which X is tangent to $S_{[X]}$.

Thus in global terms, one must search for vectorfields $X \in [X]$ such that

$$(i_X\Omega - dE) \mid S_{[X]} = 0 \tag{5.1}$$

and

$$(JX - V) \mid S_{[X]} = 0 \tag{5.2}$$

are simultaneously satisfied, with $X \in TS_{[X]}$.

Now, fix a prolongable solution X of (3.5), and let $\tilde{X} = T\zeta(X)$. By the Lemma of § IV, $\tilde{X}$ satisfies (4.1) and $\tilde{X} \in T\mathscr{L}_P$. The desired result follows from

PROPOSITION 4. — The vectorfield $T\alpha_{[X]}(\tilde{X})$ tangent to $S_{[X]}$ simultaneously satisfies (5.1) and (5.2).

Proof. — Clearly, $T\alpha_{[X]}(\tilde{X}) \in TS_{[X]}$. Now, $\zeta \circ \alpha_{[X]} = id \mid \mathscr{L}_P$, hence

$$\begin{aligned}(i_{T\alpha_{[X]}(\tilde{X})}\Omega) \mid S_{[X]} &= (i_{T\alpha_{[X]}(\tilde{X})}\zeta^*\tilde{\Omega}) \mid S_{[X]} \\ &= [\zeta^*(i_{T\zeta \circ T\alpha_{[X]}(\tilde{X})}\tilde{\Omega})] \mid S_{[X]} \\ &= \zeta^*[(i_{\tilde{X}}\tilde{\Omega}) \mid \mathscr{L}_P] \\ &= \zeta^*[dE \mid \mathscr{L}_P] \\ &= dE \mid S_{[X]}\end{aligned}$$

by the Lemma of the last section. Furthermore, $T\zeta(X) = T\zeta \circ T\alpha_{[X]}(\tilde{X})$, so that $X - T\alpha_{[X]}(\tilde{X}) \in D \mid P$ and hence is vertical. Thus

$$W_{T\alpha_{[X]}(\tilde{X})} = W_X = 0$$

by the definition of $S_{[X]}$. ∇

Not every solution $Y \in TS_{[X]}$ of (5.1) need be a second-order equation on $S_{[X]}$. One has, however, the following uniqueness theorem:

PROPOSITION 5. — There exists a unique vectorfield $Y \in TS_{[X]}$ which simultaneously satisfies (5.1) and (5.2).

Proof. — Suppose that Y and Z were two such vectorfields. Then $W_Y = W_Z = 0$ which implies that $J(Y - Z) = 0$. Furthermore, (5.1) implies that $Y - Z \in \ker \Omega$. Thus, $Y - Z \in D \mid P$ which is impossible as $TS_{[X]} \cap D = \{ 0 \}$ since $\zeta_P \mid S_{[X]} : S_{[X]} \to \mathscr{L}_P$ is a diffeomorphism. ∇

This concludes the proof of the Second-Order Equation Theorem.

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[7] For example, take $L = (1 + y)v_x^2 - zx^2 + y$ on $TQ = T\mathbb{R}^3$.

[8] Throughout this paper, we assume for simplicity that all physical systems under consideration have a finite number of degrees of freedom; however, all of the theory developed in this paper can be applied when this restriction is removed with little or no modification. For details concerning the infinite-dimensional case, see references [*3*], [*4*] and [*12*].

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[13] Elsewhere [*3*] we have developed a technique which will construct such an S—if it exists—for a completely general Lagrangian canonical system. However, the corresponding second-order equation X on S need not be smooth if (TQ, Ω, P) is not admissible.

[14] The requirement of admissibility is slightly weaker than that of almost regularity, cf. [*1*].

[15] This is the case, e. g., in electromagnetism, cf. [*4*].

[16] Nonetheless, by utilizing the technique alluded to in [*13*], it is possible to construct a unique maximal submanifold S′ with the desired properties for any Lagrangian system whatsoever. However, unless the existence of S′ actually follows from the Second-Order Equation Theorem, one is guaranteed neither that S′ will be non-empty nor that the associated second-order equation X on S′ will be smooth.

[17] With regard to the constructions of reference [*1*], one is effectively replacing « almost regular » by « admissible » and (FL(TQ), ω_1, dH_1) by ($\mathscr{L}$, $\tilde{\Omega}$, $d\tilde{E}$).

[18] This proposition has the following useful corollary: if a solution of (3.5) is *globally* a second-order equation (i. e. (3.2) is satisfied on all of P), then it is not semi-prolongable, cf. [*15*].