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# Orthogonal Polynomial Bases for Holomorphically Induced Representations of the General Linear Groups

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ABSTRACT. — We present a global procedure for constructing orthogonal polynomial bases over group elements of the general linear group GL($n$, C). This is done inductively by constructing maps from the polynomial bases of holomorphically induced representations of GL($n$ − 1, C) to irreducible representations of GL($n$, C).

## INTRODUCTION

In the representation theory of compact Lie groups much of the structure of the irreducible representations can be obtained by infinitesimal methods, using the underlying Lie algebras. One is usually able to obtain bases, work out branching rules and compute the reduction of tensor products (when there is no multiplicity) through the use of raising and lowering operators, or what is equivalent, using the structure of Dynkin diagrams. But while these infinitesimal methods have been very fruitful in developing the representation theory of compact Lie groups, a number of problems remain for which infinitesimal methods seem ill-suited to finding a solution. Central among these problems is the multiplicity problem, arising from the decomposition of tensor products of two unitary irreducible representations into a direct sum of irreducible constituents. It is not difficult to compute the multiplicity: what is much harder is to obtain explicit bases for the irreducible representations and find the maps that reduce the tensor products in the presence of multiplicity.

It is our contention that the global representation theory of compact groups formulated via the holomorphic induction procedure by Borel and Weil, when concretely realized by polynomials over group variables [6], offers a much more fruitful possibility for dealing with the multiplicity problem. In a previous paper [4] we have shown that in the decomposition of tensor products of U($n$), the notion of invariant double cosets plays an important role in dealing with the multiplicity problem. Basically the idea is to show that the reducible tensor product space can be mapped into other spaces labeled by the double cosets $B \times B\backslash G \times G/G$, where B is a Borel subgroup of the complexification $G = GL(n, C)$ of U($n$).

But one of the problems standing in the way of actually using these double coset spaces is the construction of convenient bases in G. The goal of this paper is to construct a basis for GL($n$, C) by constructing maps that carry representations of GL($n - 1$, C) into the irreducible representation spaces of GL($n$, C). The interesting feature of these maps is that they are the compositions of three « double coset » maps; it is our contention that the general idea of double coset maps reveals an important structural element in the representation theory of compact groups, a structural element that is not present when infinitesimal methods are used. In particular in this paper we show the interplay between the reduction of tensor products and the decomposition of irreducible representations of a group into irreducible constituents of a subgroup, as related by double coset maps.

A double coset map can be defined via the following setup. Let B be an inducing subgroup with a representation $\pi$, so that a representation for G by right translation is defined on the vector space of polynomials F on G that transform to the left as $F(bg) = \pi(b)F(g)$, $b \in B$, $g \in G$. Then a representation space of a subgroup H of G can be obtained via the double coset map $\Phi_D$ defined as $(\Phi_D F)(h) = F(g_D h)$, for $h \in H$ and $g_D$ a double coset representative of $B\backslash G/H$. The representation of H is induced by a subgroup of H defined by the set $\{ h \in H : g_D h g_D^{-1} \in B \}$. One of the surprising features of these double coset maps is that often one of double cosets generates a representation space of the same (or larger) dimension than the original space.

For the basis problem considered in this paper $G = GL(n, C)$ and B is a Borel subgroup of G while $H = GL(n - 1, C)$. Then a double coset map of the kind given above carries the irreducible representation space of GL($n$, C) into a reducible representation space of GL($n - 1$, C) (called $H^{(m)}$). Which representations of GL($n - 1$, C) occur in $H^{(m)}$ is well known; in fact the multiplicity is either zero or one. We assume that a basis for irreducible representations of GL($n$ − 1, C) is known and proceed to construct the map that carries these irreducible representations into $H^{(m)}$. Again these maps are of the double coset type, arising from the tensor product decomposition of irreducible representations of GL($n$ − 1, C) where the double cosets are of the form $B \times B\backslash H \times H/H \approx S_{n-1}$, in particular the identity double coset map leads to the highest weight representation while another double coset map (denoted $\Phi_{p_1}$) sends functions into $H^{(m)}$.

Thus the double coset maps define an inductive procedure for obtaining bases in GL($n$, C); a basis for GL(2, C) is easy to exhibit and is used to construct a basis for GL(3, C), actually computed as an example at the end of the paper. Our inductive procedure is then completed by showing that if a basis for GL($n$ − 1, C) is known, it can generate a basis for GL($n$, C). Such an inductive procedure has previously been used by Gelfand and Žetlin [2] and Gelfand and Graev [3] to construct bases for GL($n$, C). But they used infinitesimal methods that we feel cloud the underlying structure shown by our global approach. Their method leads them to realize the irreducible representation spaces of G = GL($n$, C) as polynomials over B\G. However to work with an inner product associated with polynomials on B\G (which may involve integration over B\G) is generally quite impractical. For application in physics and chemistry it is important to have an easily computable inner product, for example in computing quantities such as the Clebsch-Gordan coefficients. In contrast the inner product associated with polynomials over G is essentially a « differentiation » inner product (see [5], Eq. 3.23), which makes calculations very straightforward, in fact even suitable for a computer, as we hope to be able to show in future publications. In spite of the differences in approach between our work and that of Gelfand, Žetlin and Graev, it is straightforward to show the relationship between the two methods. This is done via the Gauss decomposition of G into B and B\G and is discussed in connection with the weight vectors that label irreducible representations.

## 1. Results.

Let $n$ be an integer ≥ 2. Set $G_n$ = GL($n$, C) and let $B_n$, $Z_n$, and $D_n$ denote the subgroups of $G_n$ consisting respectively of lower triangular, unipotent upper triangular, and diagonal matrices. If $(m) = (m_1, \ldots, m_n)$ is an $n$-tuple of integers satisfying $m_1 \geq \ldots \geq m_n \geq 0$, $\pi^{(m)}$ will denote a holomorphic character of $B_n$ defined by

$$\pi^{(m)}(b) = b_{11}^{m_1} \ldots b_{nn}^{m_n}, \qquad b \in B_n .$$

If $R^{(m)}$ denotes the representation of $G_m$ obtained by right translation on the linear space of all holomorphic functions on $G_n$ which transform covariantly with respect to $\pi^{(m)}$, then it is well known [6] that $R^{(m)}$ is irreducible with signature $(m_1, \ldots, m_n)$. Actually, by analytic continuation we may assume that the representation space of $R^{(m)}$ consists of all polynomial functions F on $C^{n \times n}$ which satisfy the covariant condition

$$F(by) = \pi^{(m)}(b)F(y), \qquad b \in B_n, \quad y \in C^{n \times n}.$$

Let $V^{(m)}$ denote such a representation space.

The group $G_{n-1}$ can be diffeomorphically identified with a closed subgroup of $G_n$ via the mapping $g \to \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$, $g \in G_{n-1}$. Thus, whenever there is no possible confusion we shall not distinguish an element $g \in G_{n-1}$ with its image $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ in $G_n$. In the next lemma, elements of $G_n$ will be partitioned in blocks as

$$\begin{array}{c} \\ 1 \\ n-1 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & n-1 & 1 \end{array} \\ \left(\begin{array}{c|c|c} \phantom{x} & \phantom{xxxx} & \phantom{x} \\ \hline & & \\ & & \\ \hline & & \end{array}\right) \end{array}; \tag{1}$$

moreover, if $I_{n-2}$ denotes the identity matrix of $G_{n-2}$ choose the double coset representative as

$$y_D = \begin{pmatrix} 1 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

LEMMA 1.1. — *Almost every element $y$ of $G_n$ can be uniquely represented as $y = b\{y\} y_D g\{y\}$, where $b\{y\}$ is in $B_n$,*

$$g\{y\} = \begin{pmatrix} g_{11}\{y\} & g_{1J}\{y\} & 0 \\ g_{I1}\{y\} & z\{y\} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad z\{y\} \in Z_{n-2}.$$

*Proof.* — Let $y = \begin{pmatrix} y_{11} & y_{1J} & y_{1n} \\ y_{I1} & y_{IJ} & y_{In} \\ y_{n1} & y_{nJ} & y_{nn} \end{pmatrix}$. Assume $y_{1n} \neq 0$ and the matrix $(y_{IJ} - y_{1n}^{-1} y_{In} y_{1J})$ has a Gauss decomposition [7]; then a straightforward computation shows that $y$ can be decomposed as $b\{y\} y_D g\{y\}$. The uniqueness follows from the uniqueness in the Gauss decomposition. It is noteworthy to remark that $b_{22}\{y\} = b(y_{IJ} - y_{1n}^{-1} y_{In} y_{1J})$ and $z\{y\} = z(y_{IJ} - y_{1n}^{-1} y_{In} y_{1J})$, where $b(\ )$ and $z(\ )$ occur in the Gauss decomposition of $y_{IJ} - y_{1n}^{-1} y_{In} y_{1J}$. Finally, the conclusion that the complement of the subset of all elements in $G_n$ having such a decomposition is a submanifold of lower dimension follows from the fact that the set $B_{n-2}Z_{n-2}$ is open and dense in $G_{n-2}$ [7].

Set $B_D = \{ g \in G_{n-1} : y_D g y_D^{-1} \in B_n \}$ ; then every element of $B_D$ has the block form (1) as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\beta$ in $B_{n-2}$. Thus we may identify $B_D$ with $B_{n-2}$ and it follows that $\beta y_D = y_D \beta$ for all $\beta$ in $B_D$. Set

$$H^{(m)} = \{ f : G_{n-1} \to C : f \text{ holomorphic}, \quad f(\beta g) = \pi^{(m)}(\beta) f(g), \quad \beta \in B_D, \quad g \in G_{n-1} \} \tag{2}$$

and define a « double coset » map $\Phi_D : V^{(m)} \to H^{(m)}$ by

$$(\Phi_D F)(g) = F(y_D g), \qquad F \in V^{(m)}, \qquad g \in G_{n-1}.$$

Then it follows from Lemma 1.1 and the definition of $B_D$ that $\Phi_D$ is a $G_{n-1}$-module isomorphism of $V^{(m)}$ into $H^{(m)}$. By the branching theorem [1, p. 172], the representation $R_D^{(m)}$ of $G_{n-1}$ which is obtained by right translation on $\Phi_D(V^{(m)})$ is decomposed into a direct sum of multiplicity free irreducible subrepresentations with signatures $(k) = (k_1, \ldots, k_{n-1})$. The $k_i$ in the $(n-1)$-tuple $(k)$ are allowed to run over all integers such that $m_1 \geq k_1 \geq m_2 \geq \ldots \geq k_{n-1} \geq m_n$. Next, we shall proceed to give a concrete realization of this decomposition assuming inductively the existence of polynomial bases for irreducible holomorphic representations of $G_{n-1}$. By mapping back this decomposition into $V^{(m)}$ we shall get a polynomial basis for $V^{(m)}$.

Fix an $n$-tuple $(m)$, set $k_{i,n-1} = k_i$, $1 \leq i \leq n-1$, and consider the following tableau

$$[m] = \begin{bmatrix} m_1 & & m_2 \ \ldots \ m_n \\ & k_{1,n-1} & \ldots \ k_{n-1,n-1} \\ & \cdot & \cdot \\ & \cdot & \cdot \\ & \cdot & \cdot \\ & k_{12}k_{22} & \\ & k_{11} & \end{bmatrix} \tag{3}$$

where each integer $k_{ij}$ in row $j$ is subject to the constraint

$$k_{i,j+1} \geq k_{ij} \geq k_{i+1,j+1} .$$

Now fix an $(n-1)$-tuple $(k) = (k_1, \ldots, k_{n-1})$ and consider the irreducible holomorphic representation $R^{(k)} = R_{n-1}^{(k)}$ of $G_{n-1}$ in the space of all $\pi^{(k)}$-covariant polynomial functions $V^{(k)} = V_{n-1}^{(k)}$. Let $\{ h_{[l]}^{(k)} \}$ denote the canonical basis of $V^{(k)}$. By canonical we mean the basis of $V^{(k)}$ obtained by our inductive procedure, the canonical basis for $V^{(k_1,k_2)}$ of $G_2$ being

$$h_l^{(k_1,k_2)}(g) = g_{11}^{l-k_2} g_{12}^{k_1-l} \mid g \mid^{k_2}$$

where $l$ varies between $k_1$ and $k_2$, and $|g|$ stands for the determinant of $g$. Thus in the canonical basis $\{ h^{(k)}_{[l]} \}$, $[l]$ ranges over all patterns

$$\begin{bmatrix} k_{1,n-1} & \dots & k_{n-1,n-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & k_{11} & \end{bmatrix}$$

in which the top row is fixed.

We now wish to find a map $\Omega^{(k)}$ from $V^{(k)}$ into $H^{(m)}$. To obtain this map we let $(k')$ and $(k'')$ be two $(n-1)$-tuples defined respectively by

$$\begin{cases} k'_i = \sum_{j=i}^{n-2} k_j - \sum_{j=i+1}^{n-1} m_j, & 1 \leq i \leq n-2; \quad k'_{n-1} = 0 \\ k''_i = \sum_{j=i+1}^{n-1} m_j - \sum_{j=i+1}^{n-2} k_j, & 1 \leq i \leq n-3; \quad k''_{n-2} = m_{n-1}, \ k''_{n-1} = k_{n-1}; \end{cases} \tag{4}$$

then it is easy to verify that $(k')$ and $(k'')$ are dominant and $(k')+(k'')=(k)$. Consider the linear spaces $V^{(k')}$ and $V^{(k'')}$ endowed with their respective canonical bases. We shall inject $V^{(k)}$ isomorphically onto the highest weight submodule of the tensor product $V^{(k')} \otimes V^{(k'')}$ via the mapping

$$(\Psi_e h^{(k)}_{[l]})(g', g'') = \sum_{[l']+[l'']=[l]} C^{(k)(k')(k'')}_{[l][l'][l'']} h^{(k')}_{[l']}(g') h^{(k'')}_{[l'']}(g'') \tag{5}$$

where in Eq. (5) $g'$, $g''$ belong to $G_{n-1}$, and the summation $[l'] + [l'']$ means pattern addition; that is, the addition of an element in $[l']$ with the corresponding element in $[l'']$. The mapping $\Psi_e$ extends obviously by linearity to all elements of $V^{(k)}$. At this point let us remark that such a map $\Psi_e$ was used extensively in [*4*] to obtain the reduction of tensor product representations of $SU(n)$. In order to make $\Psi_e$ intertwine the coefficients $C^{(k)(k')(k'')}_{[l][l'][l'']}$ are required to satisfy the following relation

$$C^{(k)(k')(k'')}_{[l][l'][l'']} D^{(k)}_{[l][\bar{l}]}(g) = \sum_{[\bar{l}']+[\bar{l}'']=[\bar{l}]} C^{(k)(k')(k'')}_{[\bar{l}][\bar{l}'][\bar{l}'']} D^{(k')}_{[l'][\bar{l}']}(g) D^{(k'')}_{[l''][\bar{l}'']}(g) \tag{6}$$

where the D-functions are just the matrix entries defined by

$$R^{(k)}(g) h^{(k)}_{[l]} = \sum_{[\bar{l}]} D^{(k)}_{[\bar{l}][l]}(g) h^{(k)}_{[\bar{l}]} . \tag{7}$$

Eq. (6) is just a Clebsch-Gordan relation and the coefficients $C^{(k)(k')(k'')}_{[l][l'][l'']}$ are the (non-normalized) Clebsch-Gordan coefficients of the highest weight in the tensor product $(k') \otimes (k'')$. For the actual computation of these coefficients see Lemma 3 of Ref. 4.

The map $\Psi_e$ has been introduced in order to obtain the map $\Omega^{(k)}$ that sends $V^{(k)}$ into $H^{(m)}$. To get $\Omega^{(k)}$ we consider the permutation element $p_1$ of the Weyl group $W(G_{n-1}, D_{n-1})$ defined by

$$p_1 = \left[\begin{array}{c|c} 0 & I_{n-2} \\ \hline 1 & 0 \end{array}\right].$$

If we identify $p_1$ with the cyclic permutation $\begin{pmatrix} 1 & 2 & \dots & n-1 \\ 2 & 3 & \dots & 1 \end{pmatrix}$ and define $p_1 \cdot (k'') = (k''_{p_1^{-1}(1)}, \dots, k''_{p_1^{-1}(n-1)})$ then

$$p_1 \cdot (k'') = (k''_{n-1}, k''_2, \dots, k''_{n-2}).$$

Now define a mapping $\Phi_{p_1} : \Psi_e(V^{(k)}) \to H^{(m)}$ by setting

$$[\Phi_{p_1}(\Psi_e h^{(k)}_{[l]})](g) = \Psi_e h^{(k)}_{[l]}(g, p_1 g) \tag{8}$$

and extend by linearity to all functions in $\Psi_e(V^{(k)})$. The map $\Phi_{p_1}$ is a double coset map defined in [4] for reducing tensor products $(k') \otimes (k'')$; the image space $V^{(k') + p_1 \cdot (k'')} = \Phi_{p_1}(V^{(k')} \otimes V^{(k'')})$ contains functions that transform $(k') + p_1 \cdot (k'')$-covariantly with respect to

$$B_{p_1} = \{ b \in B_{n-1} : p_1 b p_1^{-1} \in B_{n-1} \}.$$

Thus by combining Eq. (5) and (8) we can define a linear map $\Omega^{(k)}$ from $V^{(k)}$ into $H^{(m)}$ by

$$C^{(k)(k')(k'')}_{[l][l'][l'']} = [l', l'']$$

and

$$\Omega^{(k)} h^{(k)}_{[l]}(g) = \sum_{[l'] + [l''] = [l]} [l', l''] h^{(k')}_{[l']}(g) h^{(k'')}_{[l'']}(p_1 g). \tag{9}$$

A straightforward verification shows that $\Omega^{(k)}$ is indeed well defined and maps $V^{(k)}$ as a simple $G_{n-1}$-module isomorphically into $H^{(m)}$. Furthermore, the branching theorem will imply that $\Omega^{(k)}(V^{(k)})$ is contained in $\Phi_D(V^{(m)})$. Finally we want to define a map from $\Phi_D(V^{(m)})$ back into $V^{(m)}$. For each $(n-1)$-tuple $(k)$ in (3) let

$$\Psi^{(k)}_D : \Omega^{(k)}(V^{(k)}) \to V^{(m)}$$

by

$$(\Psi^{(k)}_D f)(y) = \pi^{(m)}(b\{y\}) f(g\{y\}) \tag{10}$$

for all $f$ in $\Omega^{(k)}(V^{(k)})$, $b\{y\}$ and $g\{y\}$ being defined as in Lemma 1.1. The fact that $\Psi^{(k)}_D f$ transforms covariantly with respect to $\pi^{(m)}$ follows from the uniqueness of the decomposition $y = b\{y\} y_D g\{y\}$. It remains to show that $\Psi_D^{(k)}$ is an intertwining operator. For this let us compute $R^{(m)}(g)(\Psi_D^{(k)}f)$ and $\Psi_D^{(k)}(R_D^{(m)}(g)f)$, $g \in G_{n-1}$, and compare

$$[R^{(m)}(g)(\Psi_D^{(k)}f)](y) = \pi^{(m)}(b\{yg\})f(g\{yg\})$$

$$[\Psi_D^{(k)}(R_D^{(m)}(g)f)](y) = \pi^{(m)}(b\{y\})f(g\{y\}g). \tag{11}$$

Now we have

$$yg = b\{y\}y_D g\{y\}g = b\{yg\}y_D g\{yg\},$$

it follows that

$$y_D^{-1}(b\{y\})^{-1}b\{yg\}y_D = g\{y\}g(g\{yg\})^{-1}.$$

Thus, by the definition of $B_D$, $(b\{y\})^{-1}b\{yg\})$ belongs to $B_D$ and

$$y_D^{-1}(b\{y\})^{-1}b\{yg\}g_D = (b\{y\})^{-1}b\{yg\}.$$

Therefore,

$$g\{yg\} = (b\{yg\})^{-1}b\{y\}g\{y\}g$$

and

$$\begin{aligned}\pi^{(m)}(b\{yg\})f(g\{yg\}) &= \pi^{(m)}(b\{yg\})\pi^{(m)}((b\{yg\})^{-1}b\{y\})f(g\{y\}g)\\ &= \pi^{(m)}(b\{y\})f(g\{y\}g).\end{aligned}$$

Finally,

$$\begin{aligned}[\Phi_D(\Psi_D^{(k)}f)](g) &= (\Psi_D^{(k)}f)(y_D g)\\ &= \pi^{(m)}(b\{y_D g\})f(g\{y_D g\})\end{aligned}$$

but a simple computation shows that if $g = \begin{bmatrix} g_{11} & g_{1J} & 0 \\ g_{I1} & g_{IJ} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$$\{b(g_{IJ}), z(g_{IJ})\}$$

is a Gauss decomposition of $g_{IJ}$ then

$$b\{y_D g\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b(g_{IJ}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$g\{y_D g\} = \begin{pmatrix} g_{11} & g_{1J} & 0 \\ (b(g_{IJ}))^{-1}g_{I1} & z(g_{IJ}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\pi^{(m)}(b\{y_D g\})f(g\{y_D g\}) = f(g).$$

Thus, $\Phi_D \Psi_D^{(k)}$ is the identity map on $\Omega^{(k)}(V^{(k)})$ (*). Therefore by the branching theorem the $G_{n-1}$-module $V^{(m)}$ is decomposed into irreducible submodules as

$$V^{(m)} = \sum_{(k)} \oplus \Psi_D^{(k)} \Omega^{(k)}(V^{(k)});$$

that we actually have a direct sum follows from a well-known fact concerning semi-simple G-modules.

Consider again the tableau (3) and set

$$\mu_i = \sum_{j=1}^{i} k_{ji}, \quad 1 \le i \le n-1, \quad \text{and} \quad \mu_n = \sum_{j=1}^{n} m_j \tag{12}$$

$$F_{[l]}^{(k)} = \Psi_D^{(k)} \Omega^{(k)}(h_{[l]}^{(k)}).$$

LEMMA 1.2. — *For a fixed n-tuple* $(m) = (m_1, \ldots, m_n)$ *and for* $[m]$ *ranging over all tableaus* (3) *the functions* $F_{[l]}^{(k)}$ *form an orthogonal basis for* $V^{(m)}$ *considered as a Hilbert space equipped with the inner product defined by* Eq. 3.23 [5]. *Moreover, each* $F_{[l]}^{(k)}$ *is a weight vector for the representation* $R^{(m)}$ *with weight* $(\mu_1, \mu_2 - \mu_1, \mu_3 - \mu_2, \ldots, \mu_n - \mu_{n-1})$; *that is,*

$$R^{(m)}(d) F_{[l]}^{(k)} = d_{11}^{\mu_1} d_{22}^{\mu_2 - \mu_1} \ldots d_{nn}^{\mu_n - \mu_{n-1}} F_{[l]}^{(k)}$$

*for all d in* $D_n$.

*Proof.* — By our inductive procedure we assume that $h_{[l]}^{(k)}$ is a weight vector with weight $(\mu_1, \mu_2 - \mu_1, \ldots, \mu_{n-1} - \mu_{n-2})$. Obviously for $n=2$ the $h_l^{(k_1,k_2)}$ are weight vectors with weight $(l, k_1 + k_2 - l)$. Consider again

(*) Logically one should first show that $\Psi_D^{(k)} f$ is in $V^{(m)}$ if $f \in \Omega^{(k)} V^{(k)}$. But since $\Psi_D^{(k)}$ is an intertwining operator it suffices to show that $\Psi_D^{(k)} \Omega^{(k)} h_{[l_M]}^{(k)}$ is in $V^{(m)}$. A straightforward computation shows that

$$\Psi_D^{(k)} \Omega^{(k)} h_{[l_M]}^{(k)}(y) = y_{1n}^{m_1 - k_1} \Delta_p^{k_1 - m_2}(y) \ldots \Delta_{n-1}^{k_{n-1} - m_n}(y) \Delta_n^{m_n}(y) \, | p_1 |^{k_{n-1}} \Delta_2^{m_2 - k_2}(y\tilde{p}_1) \ldots \Delta_{n-1}^{m_{n-1} - k_{n-1}}(y\tilde{p}_1) \quad \text{where } \tilde{p}_1$$

is an $n \times m$ matrix represented in block form as $\begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}$. Therefore if $m_1 - k_1 \ge 0$ it follows that $\Psi_D^{(k)} f$ is in $V^{(m)}$.

the block matrix partition as in Lemma 1.1, and for $y \in G_n$, $d \in D_n$ write $y = b\{y\}y_D g\{y\}$ with

$$b\{y\} = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{I1} & b_{IJ} & 0 \\ b_{n1} & b_{nJ} & b_{nn} \end{bmatrix}, \qquad b_{IJ} \in B_{n-2};$$

$$g\{y\} = \begin{bmatrix} g_{11} & g_{1J} & 0 \\ g_{I1} & z & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad z \in Z_{n-2};$$

$$d = \begin{bmatrix} d_{11} & & \\ & d_{IJ} & \\ & & d_{nn} \end{bmatrix}, \qquad d_{IJ} \in D_{n-2}.$$

Write $yd = b\{yd\}y_D g\{yd\}$; then a simple computation shows that

$$b\{yd\} = \begin{bmatrix} b_{11}d_{nn} & 0 & 0 \\ b_{I1}d_{nn} & b_{IJ}d_{IJ} & 0 \\ b_{n1}d_{nn} & b_{nJ}d_{IJ} & b_{nn}d_{nn} \end{bmatrix}$$

and

$$g\{yd\} = \begin{bmatrix} g_{11}d_{11}d_{nn}^{-1} & g_{1J}d_{IJ}d_{nn}^{-1} & 0 \\ d_{IJ}^{-1}g_{I1}d_{11} & d_{IJ}^{-1}zd_{IJ} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} F_{[l]}^{(k)}(yd) &= [\Psi_D^{(k)}\Omega^{(k)}(h_{[l]}^{(k)})](yd) \\ &= \pi^{(m)}(b\{yd\})[\Omega^{(k)}(h_{[l]}^{(k)})](g\{yd\}) \\ &= d_{nn}^{m_1+m_n}d_{22}^{m_2} \dots d_{n-1,n-1}^{m_{n-1}}\pi^{(m)}(b\{y\}) \\ &\quad \times [\Omega^{(k)}(h_{[l]}^{(k)})](g\{yd\}). \end{aligned}$$

Now

$$g\{yd\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & d_{IJ}^{-1}d_{nn} & 0 \\ 0 & 0 & 1 \end{bmatrix} g\{y\} \begin{bmatrix} d_{11}d_{nn}^{-1} & 0 & 1 \\ 0 & d_{IJ}d_{nn}^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\Omega^{(k)}$ intertwines $R^{(k)}$ with $R_D^{(m)} \mid \Omega^{(k)}(V^{(k)})$ it follows that $\Omega^{(k)}h_{[l]}^{(k)}$ is also a weight vector with weight $(\mu_1, \mu_2 - \mu_1, \dots, \mu_{n-1} - \mu_{n-2})$. Thus

$$(\Omega^{(k)}h_{[l]}^{(k)})(g\{yd\}) = d_{11}^{\mu_1}d_{22}^{\mu_2-\mu_1} \dots d_{n-1,n-1}^{\mu_{n-1}-\mu_1}d_{nn}^{-m_{n-1}} \times (\Omega^{(k)}h_{[l]}^{(k)})(\delta g\{y\})$$

where

$$\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & d_{IJ}^{-1}d_{nn} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\Omega^{(k)}h^{(k)}_{[l]}$ is in $H^{(m)}$ and $\delta$ belongs to $B_D$ we have

$$\Omega^{(k)}h^{(k)}_{[l]}(\delta g\{y\}) = \pi^{(m)}(\delta)\Omega^{(k)}h^{(k)}_{[l]}(g\{y\}).$$

Now $\pi^{(m)}(\delta) = d_{22}^{-m_2} \dots d_{n-1,n-1}^{-m_{n-1}} d_{nn}^{m_2+\dots+m_{n-1}}$; thus by comparing all these equalities we indeed get

$$[\Psi_D^{(k)}\Omega^{(k)}(h^{(k)}_{[l]})](yd) = d_{11}^{\mu_1}d_{22}^{\mu_2-\mu_1} \dots d_{nn}^{\mu_n-\mu_{n-1}}[\Psi_D^{(k)}\Omega^{(k)}(h^{(k)}_{[l]})](y) \tag{13}$$

for all $d$ in $D_n$ and $y$ in $G_n$. It remains to show that the $F^{(k)}_{[l]}$ form an orthogonal basis for $V^{(m)}$. For this purpose, we observe that if we restrict the representation $R^{(m)}$ to the compact subgroup $U(n)$ of $G_n$ then it is unitary (cf. [5], Eq. (3.23)); using the invariance of the orthogonal complement for unitary representations and the uniqueness of decomposition into multiplicity free irreducible representations (of $U(n-1)$) we conclude that the direct sum $V^{(m)} = \sum_{(k)} \oplus \Psi_D^{(k)}\Omega^{(k)}(V^{(k)})$ is in fact orthogonal. By our inductive hypothesis the previous argument is applicable to each $V^{(k)}$ (and hence each $\Psi_D^{(k)}\Omega^{(k)}(V^{(k)})$) and thus by iterating this procedure to the chain $U(n)$, $U(n-1)$, ..., $U(1)$ we obtain the desired result.

REMARK 1.3. — To make the connection with the Gelfand-Žetlin-Graev basis [2, 3] we only need to consider the Gauss decomposition for $y=b(y)z(y)$, $y\in G_n$, and for all $F^{(k)}_{[l]}$ write $F^{(k)}_{[l]}(y)=\pi^{(m)}(b(y))F^{(k)}_{[l]}(z(y))$. Up to multiple constants the $F^{(k)}_{[l]}(z(y))$ are exactly the Gelfand-Žetlin basis functions as given in [7, Chap. X].

The main result of this article can be summarized as follows.

THEOREM 1.4. — *Let $(m)$ be an n-tuple satisfying $m_1 \geq \dots \geq m_n \geq 0$ and let $\pi^{(m)}: B_n \to C^*$ be a holomorphic character. Let $R^{(m)}$ denote the irreducible representation of $G_n$ which transform covariantly with respect to $\pi^{(m)}$. Then a basis for $V^{(m)}$ can be obtained from orthogonal bases $h^{(k)}_{[l]}\in V^{(k)}$ of $G_{n-1}$ by defining a map from $V^{(k)}$ into $V^{(m)}$ as the composition of the following « double coset » maps:*

*i) $\Psi_e: V^{(k)} \to V^{(k')\otimes(k'')}$, where the $(n-1)$-tuple $(k)$ satisfies*

$$m_1 \geq k_1 \geq m_2 \geq k_2 \geq \dots \geq k_{n-1} \geq m_n$$

*and $(k')$, $(k'')$ are dominant weights of $G_{n-1}$ given in terms of $(k)$ and $(m)$ such that $(k') + (k'') = (k)$. $\Psi_e$ is the inverse of the double coset map $\Phi_e$ defined by $(\Phi_e F)(g) = F(g, g)$, $g \in G_{n-1}$, $F \in V^{(k')\otimes(k'')}$.*

*ii) $\Phi_{p_1}: \Psi_e(V^{(k)}) \to H^{(m)}$, where $(\Phi_{p_1}\Psi_e f)(g) = f(g, p_1 g)$, $g\in G_{n-1}$, $f \in V^{(k)}$, and $p_1$ is the double coset representative of*

$$B_{n-1} \times B_{n-1}\backslash G_{n-1} \times G_{n-1}/G_{n-1} \simeq S_{n-1}$$

*identified with the permutation* $\begin{pmatrix} 1 & 2 & \dots & n-1 \\ 2 & 3 & \dots & 1 \end{pmatrix}$.

*iii)* $\Psi_D : \Phi_{p_1}\Psi_e(V^{(k)}) \to V^{(m)}$, *where* $\Psi_D$ *is the « inverse » of the double coset map* $(\Phi_D F)(g) = F(y_D g)$ *with* $g \in G_{n-1}$, $F \in V^{(m)}$ *and* $y_D$ *a double coset representative of* $B_n \backslash G_n / G_{n-1}$.

*Then* $\Psi_D \Phi_{p_1} \Psi_e(h^{(k)}_{[l]})$ *form an orthogonal basis in* $V^{(m)}$ *with respect to the inner product* [5, Eq. (3.23)], *and furthermore they are weight vectors of* $R^{(m)}$.

## 2. An example: $G_3$.

Consider the tableau

$$\begin{bmatrix} m_1 & & m_2 & & m_3 \\ & k_1 & & k_2 & \\ & & l & & \end{bmatrix}$$

and fix $(k) = (k_1, k_2)$. The canonical basis for $V^{(k)}$ in $G_2$ is given by

$$h_l^{(k_1,k_2)}(g) = g_{11}^{l-k_2} g_{12}^{k_1-l} \mid g \mid^{k_2},$$

$k_1 \geq l \geq k_2$. For this case the permutation matrix $p_1$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus $(k'_1, k'_2) = (k_1 - m_2, 0)$ and $(k''_1, k''_2) = (m_2, k_2)$. A simple computation shows that

$$[l', l''] = \begin{pmatrix} k_1 - k_2 \\ (l' + l'') - k_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1 - m_2 \\ l' \end{pmatrix} \begin{pmatrix} m_2 - k_2 \\ l'' - k_2 \end{pmatrix} \tag{14}$$

for all $l'$ and $l''$ such that $k_1 - m_2 \geq l' \geq 0$ and $m_2 \geq l'' \geq k_2$. Therefore,

$$\Omega^{(k)} h_l^{(k)}(g) = \sum_{l'+l''=l} [l', l''] g_{11}^{l'} g_{12}^{k_1-m_2-l'} g_{21}^{l''-k_2} g_{22}^{m_2-l''} \mid g \mid^{k_2} (-1)^{k_2}$$

for all $g$ in $G_2$.

Now for $y \in G_3$ if we let $\Delta^{i_1 i_2}_{j_1 j_2}(y)$ denote the minor of $y$ formed from the rows $i_1$, $i_2$ and columns $j_1$, $j_2$ (the principal minors being denoted simply by $\Delta_1(y)$, $\Delta_2(y)$ and $\mid y \mid$) then the decomposition $y = g\{y\} y_D g\{y\}$ corresponds to

$$y_D = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b_{11}\{y\} = y_{13}, \qquad b_{22}\{y\} = -\Delta^{12}_{23}(y) y_{13}^{-1}$$

$$b_{33}\{y\} = \mid y \mid \Delta_2^{-1}(y), \qquad g_{11}\{y\} = y_{11} y_{13}^{-1},$$
$$g_{12}\{y\} = y_{12} y_{13}^{-1}, \qquad g_{21}\{y\} = \Delta^{12}_{13}(y)(\Delta^{12}_{23}(y))^{-1},$$

$z\{y\} = 1$, and $\mid g\{y\} \mid = -\Delta_2(y)(\Delta^{12}_{23}(y))^{-1}$. The remaining entries of the matrix $b\{y\}$ are not needed for our purpose. From this decomposition it follows that

$$\begin{aligned}\Psi_{D}^{(k)}\Omega^{(k)}h_{l}^{(k)}(y) &= \pi^{(m)}(b\{y\})\Omega^{(k)}h_{l}^{(k)}(g\{y\})\\ &= y_{13}^{m_1}(-\Delta_{23}^{12}(y)y_{13}^{-1})^{m_2}(|y|\Delta_2^{-1}(y))^{m_3}\sum_{l'+l''=l}[l', l'']\\ &\quad\times(y_{11}y_{13}^{-1})^{l'}(y_{12}y_{13}^{-1})^{k_1-m_2-l'}(-1)^{k_2}\\ &\quad\times[\Delta_{13}^{12}(y)(\Delta_{23}^{12}(y))^{-1}]^{l''-k_2}[-\Delta_2(y)(\Delta_{23}^{12}(y))^{-1}]^{k_2}\\ &= y_{13}^{m_1-k_1}\Delta_2^{k_2-m_3}(y)|y|^{m_3}\sum_{l'+l''=l}(-1)^{m_2}[l', l'']\\ &\quad\times y_{11}^{l'}y_{12}^{k_1-m_2-l'}(\Delta_{13}^{12}(y))^{l''-k_2}(\Delta_{23}^{12}(y))^{m_2-l''}\end{aligned}\tag{15}$$

where the coefficients $[l', l'']$ are given by Eq. (14) for all $l'$, $l''$ subject to the condition $l' + l'' = l$, $k_1 - m_2 \geq l' \geq 0$, and $m_2 \geq l'' \geq k_2$. Thus if we let $F_l^{(k)}$ denote the function defined by Eq. (15) and allow $k_1$, $k_2$, and $l$ to vary within all patterns

$$\begin{pmatrix} m_1 & & m_2 & & m_3 \\ & k_1 & & k_2 & \\ & & l & & \end{pmatrix}$$

we obtain the desired canonical basis for $G_3$.

To make the connection with the Gelfand-Žetlin basis for $G_3$ as given in [7, Chap. X], we first write down the Gauss decomposition for a general element $y$ in $G_3$:

$$\begin{aligned}&y = b(y)z(y) \qquad \text{with} \qquad b(y)\in B_3 \qquad \text{and} \qquad z(y)\in Z_3;\\ &b_{11}(y) = y_{11}, \qquad b_{22}(y) = \Delta_2(y)y_{11}^{-1}, \qquad b_{33}(y) = \Delta_2^{-1}(y)|y|\\ &z_{12}(y) = y_{12}y_{11}^{-1}, \qquad z_{13}(y) = y_{13}y_{11}^{-1}, \qquad z_{23}(y) = \Delta_{12}^{12}(y)\Delta_2^{-1}(y)\\ &\hat{z}_{13}(y) = z_{12}(y)z_{23}(y) - z_{13}(y) = \Delta_{23}^{12}(y)\Delta_2^{-1}(y).\end{aligned}$$

Now simply by rewriting $F_l^{(k)}(y)$ as $F_l^{(k)}(y) = \pi^{(m)}(b(y))F_l^{(k)}(z(y))$ we obtain

$$\begin{aligned}F_l^{(k)}(y) &= y_{11}^{m_1-m_2}(\Delta_2(y))^{m_2-m_3}\sum_{l'+l''=l}(-1)^{m_2}[l', l'']\\ &\quad\times y_{11}^{l'+m_2-m_1}y_{12}^{k_1-m_2-l'}y_{13}^{m_1-k_1}\\ &\quad\times(\Delta_{13}^{12}(y))^{l''-k_2}(\Delta_{23}^{12}(y))^{m_2-l''}(\Delta_2(y))^{k_2-m_2}\\ &= \pi^{(m)}(b(y))\sum_{l'+l''=l}(-1)^{m_2}[l', l'']\\ &\quad\times(z_{12}(y))^{k_1-m_2-l'}(z_{13}(y))^{m_1-k_1}\\ &\quad\times(z_{23}(y))^{l''-k_2}(\hat{z}_{13}(y))^{m_2-l''}.\end{aligned}\tag{16}$$

Thus the basis vector $F_l^{(k)}$ corresponds to the Gelfand-Žetlin basis vector

$$F_l^{(k)}(z) = \sum_{l'+l''=l} (-1)^{m_2}[l', l'']$$
$$\times z_{12}^{k_1-m_2-l'} z_{13}^{m_1-k_1} z_{23}^{l''-k_2} (\hat{z}_{13})^{m_2-l''} \qquad 0 \le l' \le k_1 - m_2 ; \quad k_2 \le l'' \le m_2$$

for all $z$ in $Z_3$.

## CONCLUSION

We have shown how to construct orthogonal bases for irreducible holomorphic representations of GL($n$, C) if bases for GL($n$ − 1, C) are given, using a global rather than infinitesimal approach. This procedure can be abstractly generalized in the following fashion. Let G be a complex group whose irreducible representations are holomorphically induced by a subgroup B. Let H be a « sufficiently large » subgroup of G so that B\G/H is finite. Assume that irreducible representations of H are concretely realized on vector spaces with given orthogonal bases. Then the double cosets B\G/H define maps that carry an irreducible representation of G into a reducible representation of H. The map from irreducible representations of H back to irreducible representations of G is given from an analysis of the tensor product structure of H. Formulated in this way the reduction of an irreducible representation of G into irreducible representations of H in general involves multiplicity; we conjecture that this multiplicity can be dealt with using the general double coset structure. We plan in future publications to deal with bases of SO($n$ − 1, C) contained in SO($n$, C) and with SO($n$ C) contained in GL($n$, C) using the above ideas.

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