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Discreteness of the ground state for non-relativistic bosons in one and two dimensions

by

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ABSTRACT. — A multiparticle analog of the theorem that attractive potentials bind in one and two dimensions leads to stability of « negative ions » and strict monotonicity of the ground state energy for neutral boson « plasma » in the number of pairs.

It is well known that attractive potentials bind in one and two dimensions but not necessarily in three. We shall describe a multiparticle analog of this theorem.

Consider the N particles Schrödinger Hamiltonian in one or two dimensions obeying Bose or Maxwell statistics:

$$H_N = \sum_{i=1}^{N} p_i^2 - Z \sum_{i=1}^{N} V_i(x_i) + \sum_{1 \leq i < j \leq N} V_{ij}(x_i - x_j) \qquad (1)$$

$$V_i(\infty) = V_{ij}(\infty) = 0, \qquad \forall i, j$$

A sufficient condition for H_N (possibly with center of mass removed) to have a discrete ground state is:

THEOREM 1. — *a*) Suppose:

$$\int V_i(x) d^n x \geq \int V_{kj}(x) d^n x \geq 0, \quad n = 1, 2, \quad \forall i, j, k$$

$$V_i(\infty) = V_{jk}(\infty) = 0, \qquad V_i \not\equiv 0, \quad \forall i, j, k$$

then H_N has a discrete ground state for all $N \leqslant Z + 1$.

b) With the same assumptions as above but

$$\int V_{jk}(x)d^nx = 0, \qquad \forall j, k$$

H_N has a discrete ground state for all N ($Z \geqslant 0$).

Remarks. — *a*) If the integrals of V_i, V_{jk} diverge the inequalities should be interpreted suitably (e. g. over finite balls).

b) The result may be improved in special cases, e. g. $N=2, n=1, V(x)=\delta(x)$, $Z>2/3$ (rather than $Z \geqslant 1$) is sufficient.

The proof of the theorem is elementary nevertheless its physical content is interesting. With $V(x) = V_i(x) = V_{jk}(x)$, $\forall i, j, k$, H_N describes a nucleous of charge Z and N mutually repelling « electrons ». Part (*a*) of the theorem guarantees the stability of once negatively charged anions and *a-fortiori* the stability of atoms and cations. Thus, there are no ideal gases for bosons in one or two dimensions. It is also interesting that details of the potential V are irrelevant. This should be contrasted with the deep result of Žislin [*1*] that for *Coulomb* potentials atoms (and *a-fortiori* cations) but (in general) not anions are stable in three dimensions (Žislin's result holds irrespective of statistics). The proof relies on special properties of the Coulomb force. The result given here holds also for short range potentials.

The absence of restriction on N in (*b*) of the theorem reflects the fact that the interparticle interaction is not truly repulsive.

Deeper facts on the bound states of the N-body Schrodinger Hamiltonian can be found in the review of Simon [*2*]; for newer results see [*4*].

The proof of the theorem is an application of a sharp version of the binding in low dimension phenomenon due to Simon [*3*]:

LEMMA (Simon). — Let

$$\int W(x)d^nx \leq 0, \qquad n = 1, 2$$

with W(*x*) vanishing at infinity and not identically zero ($-\infty$ allowed) then $p^2 + W(x)$ has at least one bound state in $(-\infty, 0)$.

Remark. — For precise conditions on admissible local singularities of W(*x*) see the original work [*3*].

For reasons that are irrelevant in the present context it is assumed in [*3*] that W(*x*) has a sufficiently fast fall off at infinity.

Proof. — By Simon's theory of weak coupling $p^2 + \lambda W(x)$ has a ground state in $(-\infty, 0)$ which is monotonically decreasing function of λ for λ positive near zero. By the convexity of the ground state the lemma then holds for $\lambda = 1$.

To prove the theorem note first that it holds for $N = 1$. We shall use induction on N and first assume Boltzman statistics. We shall bootstrap the result to Bose statistics by a standard argument.

Let φ_{N-1} be the ground state of H_{N-1} with energy ε_{N-1}. H_N has a discrete ground state if for a suitable ψ, $(\psi, H_N\psi) < \varepsilon_{N-1}$. Let

$$W_N(x) = \sum_{j=1}^{N-1} \int |\varphi_{N-1}(x_1, \ldots, x_{N-1})|^2 V_{jN}(x_j - x) d^n x_1 \ldots d^n x_{N-1}$$

and

$$h = p^2 - ZV_N(x) + W_N(x)$$

h has a discrete ground state $f_N(x)$ by the lemma since:

$$-Z \int V_N(x) d^n x + \int W_N(x) d^n x \leq \frac{1}{N-1} \sum_{j=1}^{N-1} (-Z+N-1) \int V_{jN}(x) d^n x \leq 0$$

Choosing $\psi(x_1, \ldots, x_N) = \varphi(x_1, \ldots, x_{N-1}) f_N(x_N)$ proves the theorem for Boltzman statistics.

Suppose now that H_N is invariant under particle permutations. By Perron-Frobenius theorem $\varphi_N > 0$ and is hence symmetric under permutations. Thus the theorem holds also for Bose statistics. For translation invariant hamiltonians the proof goes *mutatis mutandis.*

The theorem has obvious analogs for, say, mutually attracting bosons in suitable external potentials. A more interesting case is that of a neutral « plasma »: 2N bosons of charges $e_i = \pm 1$ such that like charges repel;

$$\tilde{H}_N = \sum_{i=1}^{2N} p^2 + \sum_{j \neq 1} e_i e_j V(x_i - x_j), \qquad \int V(x) d^n x \geq 0 \tag{2}$$

$$V(\infty) = 0, \qquad V \not\equiv 0$$

Theorem 2. — The ground state of $\tilde{H}_N$ in one and two dimensions is strictly monotonically decreasing function of N.

Remark. — *a*) Theorem 2 should be compared with a result of Dyson [*5*] that bosons without hard cores and long range interactions lead to unstable three dimensional matter $\left(\frac{\varepsilon_N}{N} \text{ diverges to } -\infty \text{ with } N\right)$ and no monotonicity claimed. Real matter (i. e. fermions) clusters so theorem 2 fails.

b) Fermions do not benefit from the binding phenomenon in low dimensions (at least if spin is neglected). Counter examples to either theorem 1 or theorem 2 are easily constracted [*6*].

c) The obstacle to extend the method to three dimensions is that

$$I \equiv \int V(x) d^n x$$

cannot provide a sufficient condition for binding. Transformations that keep I fixed but spread the potential over a larger part of space decrease the number of bound states. In particular dilations of V that keep I fixed have no bound states (Use Birman-Schwinger bound. See also [*4*]). On the other hand it is easy to give a sufficient condition for binding in high dimensions involving moments of the potential. This procedure can be used to obtain results of Dyson type (upper and lower bounds on the ground state) but not « local » results described here. For probabilistic characterisation of binding potentials in three dimensions see [*4*].

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