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Quantum Scattering by External Metrics and Yang-Mills Potentials

by

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RÉSUMÉ. — Nous étudions la théorie des collisions dans le cas d'Hamiltoniens donnés comme opérateurs de Laplace-Bochner sur \mathbb{R}^n associés à une métrique et à un potentiel extérieur du type Yang-Mills. On montre que les opérateurs de Møller et la matrice S existent et sont unitaires pour une certaine classe de métriques asymptotiquement plates et de potentiels s'annulant à l'infini. On montre que la matrice S est invariante sous l'effet de transformations de coordonnées et de jauge qui tendent vers l'identité à l'infini. Les méthodes connues pour assurer l'absence de spectre discret sont étendues au cas présent. On présente enfin les résultats d'un calcul à l'ordinateur qui indique l'existence d'une métrique partout régulière et asymptotiquement plate conduisant à un état lié.

ABSTRACT. — We consider the scattering theory for Hamiltonians given as Laplace-Bochner operators on \mathbb{R}^n associated with a metric and an external Yang-Mills potential. The Møller operators and the S -matrix are shown to exist and to be unitary for a certain class of asymptotically flat metrics and Yang-Mills potentials vanishing at infinity. The S -matrix is shown to be invariant under those coordinate and gauge transformations, which tend to the identity at infinity. Known methods to ensure the absence of a discrete spectrum are extended to the present case. A computer calculation is presented indicating the existence of an everywhere regular and asymptotically flat metric leading to a bound state.

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§ 1. INTRODUCTION

The theory of one particle quantum scattering by a potential W has reached a very satisfactory status and progress is continuously being made in the multiparticle case (see e. g. [33] for a detailed treatment and extensive reference).

In this note we will discuss the situation, where scattering occurs through a C^∞ metric $g = \{g_{jk}\}_{1 \leq j, k \leq n}$ and a C^∞ Yang-Mills potential $\underline{A} = \{A_j\}_{1 \leq j \leq n}$ on \mathbb{R}^n . The Hamiltonian will thus be of the form

$$0 \leq \hat{H} = \hat{H}(g, \underline{A}) = -\frac{1}{\sqrt{g}} \sum_{j,k=1}^n (\partial_j + iA_j) \sqrt{g} g^{jk} (\partial_k + iA_k) \quad (1.1)$$

acting on $L^2(\mathbb{R}^n, \sqrt{g} d^n x) \otimes V$. We have set $g = \det g_{jk} > 0$ and V is the finite dimensional Hilbert space on which the A_j act, i. e. the A_j are C^∞ functions on \mathbb{R}^n with values in the hermitean matrices on V . Also $x = (x^1, x^2, \dots, x^n)$ denotes cartesian coordinates with $\partial_j = \frac{\partial}{\partial x^j}$. Our interest in \hat{H} results from the fact that in (euclidean) quantum field theories including gravity or Yang-Mills fields coupled to scalar fields, the resolvent of \hat{H} enters as the covariance of a Gaussian process.

Doing scattering theory means we will compare \hat{H} with the free Hamiltonian $H_0 = -\Delta = -\sum_{j=1}^n (\partial_j)^2$, the ordinary Laplacian, viewed as an operator on $L^2(\mathbb{R}^n, d^n x) \otimes V$. In particular we will discuss the relative Møller operators $\Omega^\pm(\hat{H}, H_0)$. Now \hat{H} and H_0 act in different Hilbert spaces, so Ω^\pm depend on the choice of the identification operator (see e. g. [26]). We will make the following choice

$$J : L^2(\mathbb{R}^n, \sqrt{g} d^n x) \otimes V \rightarrow L^2(\mathbb{R}^n, d^n x) \otimes V$$

given by

$$(Jf) = g^{\frac{1}{4}} f \quad (1.2)$$

J is of course a unitary operator and instead of \hat{H} we may therefore consider the operator H on $L^2(\mathbb{R}^n, d^n x) \otimes V$ given by

$$H = H_{g, \underline{A}} = J\hat{H}(g, \underline{A})J^{-1} = -g^{-\frac{1}{4}} \sum_{j,k=1}^n (\partial_j + iA_j) \sqrt{g} g^{jk} (\partial_k + iA_k) g^{-\frac{1}{4}} \quad (1.3)$$

If they exist, the Møller operators $\Omega^\pm(H, H_0)$ are then given by

$$\Omega^\pm(H, H_0) = \mathcal{S}\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} \tag{1.4}$$

Of course Ω^\pm are only expected to exist whenever H and H_0 are asymptotically equal, which means $g_{jk} - \delta_{jk}$ and A_j have to tend to zero at infinity. Hence we will be concerned with this situation. Since we will assume regularity everywhere and decay properties stronger than $r^{-1} = |x|^{-1}$, Schwarzschild like (Riemannian) metrics as well as instanton like forms of the A 's will be outside the scope of our work as far as Ω^\pm and the continuous spectrum is concerned. We expect, however, that the instanton solution in the singular form given by 'tHooft [20], [21] should be treatable using arguments of Kupsch and Sandhas [28] (see also Simon [40]), since the behaviour at infinity is good enough. Since in our present situation we will obtain a completely satisfactory scattering theory, one may look at the inverse scattering problem. To what extent are the g_{jk} and A_j determined by the phase shifts? In particular, what are necessary and (or) sufficient conditions on the phase shifts to result from metrics and Yang-Mills potentials? Since gauge equivalent A 's give the same phase shifts as well as metrics related by a coordinate transformation (see below), this could lead to interesting differential geometric problems. We now outline the methods employed and the results obtained. The scattering theory for elliptical differential operators with variable coefficients has been discussed by Kuroda [29], [30] (see also Agmon [1], Beals [4], Hörmander [22] and Schechter [37], [38]). Kuroda obtains results about the absolute continuous spectrum and his conditions ensure the absence of a singular continuous spectrum.

As shown by Simon [40], the recent work of Enss [14] on scattering theory may also be applied to elliptic differential operators with variable coefficients. The two conditions of Enss then too exclude a singular continuous spectrum.

In our present case the conditions of Enss and Kuroda are essentially equivalent requiring $g_{jk} - \delta_{jk}$ and A_j together with their derivatives up to second and first order respectively to decrease like $r^{-1-\delta}$ ($\delta > 0$). These conditions are gauge invariant in the restricted sense, where only gauge transformations are allowed, which tend to the identity sufficiently fast at infinity.

Also we will show the gauge invariance of the S-matrix for gauge transformations, which approximate the identity at infinity. The gauge invariance of the S-matrix for a relativistic scalar field is already well known, see e. g. Ruijsenaars [36] and Dimock [13]. We also prove invariance of the S-matrix under coordinate transformations. This again has been proved by Dimock [13] in the context of a relativistic scalar field. In both cases our proof is based on a simple application of Kato's concept of identification operators.

We will present all this in § 2 where we also give a simple proof of essential selfadjointness of \hat{H} and thus of H on $C_0^\infty(\mathbb{R}^n) \otimes V$. We may note in passing that the trace class criterion of Birman [6], de Branges [12] and Kato [25] combined with trace class estimates of Stinespring [41] in the extended form given by Reed and Simon, also give the existence of wave operators. The resulting conditions, however, require decays like $r^{-n-\delta}$ ($\delta > 0$), so they are in general stronger.

In section 3 we will extend the result of Ikebe and Uchiyama [23], who only considered electromagnetic A 's, on the absence of a discrete spectrum to the present case (see also the earlier paper by Jäger [24], who considered the case $A = 0$ and with stronger conditions on the g_{jk}). Apart from differentiability requirements, our conditions will be identical to theirs; i. e. $g_{jk} - \delta_{jk} \rightarrow 0$ at infinity, the first derivatives of g_{jk} tend to zero stronger than r^{-1} and the field strength tensor given by the A 's also tends to zero stronger than r^{-1} .

In the case of potential scattering the corresponding result about the absence of positive eigenvalues is known as the Kato-Agmon-Simon theorem (see e. g. [32] where references are given). Our proof is as close as possible to the spirit of the proof of this theorem. Technically this is achieved by establishing decay properties of the metric tensor in polar coordinates.

In section 4 we finally present arguments based on computer calculations showing that there may exist everywhere regular metrics on \mathbb{R}^3 with deviations from the flat metric going like r^{-1} and such that the corresponding Laplace-Beltrami operator has at least one positive eigenvalue. This indicates that the conditions in § 3 concerning the metric are close to being optimal. The situation for metrics therefore looks analogous to the case for potentials, where W 's such as the von Neumann-Wigner potential [43] (see also e. g. [2], [32]) exist behaving like $0(r^{-1})$ at infinity and having nonnegative eigenvalues. In fact our discussion is modelled on the von Neumann-Wigner ansatz. For A 's with compact support, it follows trivially from known facts, see § 3. The case $n = 1$ is of course trivial since then A may be gauged away. We note that in arbitrary dimensions electromagnetic A 's alone may bind. This may be achieved for example by the so called magnetic bottles (see for example [45]).

§ 2. SELFADJOINTNESS AND EXISTENCE OF WAVE OPERATORS

As already mentioned, we will assume throughout that the g_{jk} and A_j are C^∞ functions. Also the symmetric matrix $\underline{g}(x) = \{g_{jk}(x)\}_{1 \leq j, k \leq n}$ is assumed to be positive definite for all $x \in \mathbb{R}^n$. Then $\underline{g}^{\sharp}(x) = \{g^{\sharp jk}(x)\}_{1 \leq j, k \leq n}$ defined by $g^{\sharp jk}(x)g_{kl}(x) = \delta^j_l$ is also positive definite and in particular $g(x) = \det g_{jk}(x) > 0$ for all $x \in \mathbb{R}^n$.

If the Riemannian space $(\mathbb{R}^n, \underline{g})$ given by $\underline{g} \cdot(x)$ is complete, then the corresponding Laplace-Beltrami operator Δ_g on $L^2(\mathbb{R}^n, \sqrt{g} d^n x)$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^n)$ due to a well known theorem of Gaffney [15] and Roelcke [35]. This even holds for powers of Δ_g , see e. g. Cordes [11] and Chernoff [9]. On compact Riemannian manifolds Bochner-Laplacians are essential selfadjoint on C^∞ sections, one proof [18], [19] being based on Kato's inequality [27]. This inequality also holds in the present case and it should be possible to give a proof of essential selfadjointness of $\hat{H}(g, \underline{A})$ on $C_0^\infty(\mathbb{R}^n) \otimes V$ using this fact whenever (\mathbb{R}^n, g) is complete. We will, however, give a very simple proof of essential selfadjointness, which will cover the situations we are interested in.

PROPOSITION 2.1. — Assume $\underline{g} \cdot(x)$ satisfies the estimate

$$\|\underline{g} \cdot(x)\| \leq C(1 + |x|^2) \tag{2.1}$$

for some $C < \infty$ and all $x \in \mathbb{R}^n$. Here $\|\cdot\|$ denotes the matrix norm and $|\cdot|$ the euclidean metric on \mathbb{R}^n . Then $\hat{H}(g, \underline{A})$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^n) \otimes V$. In particular $\hat{H}(g, \underline{A})$ is the Friedrichs extension obtained from the (closable) nonnegative quadratic form Q on

$$C_0^\infty(\mathbb{R}^n) \otimes V \times C_0^\infty(\mathbb{R}^n) \otimes V$$

given by

$$Q(u, v) = \int \sum_{j,k} \langle (\partial_j + iA_j)u, (\partial_k + iA_k)v \rangle_{\mathcal{V}} g^{jk} \sqrt{g} d^n x \tag{2.2}$$

Here $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ denotes the scalar product in V .

Remark. — It is easy to check that condition (2.1) implies completeness but not *vice versa*.

Proof. — Multiplying \hat{H} by a positive number does not change the consequences, so we may assume $C < 1$. Now assume $\hat{H}(g, \underline{A})$ is not essentially selfadjoint on $C_0^\infty(\mathbb{R}^n) \otimes V$. Then there is

$$0 \neq u \in L^2(\mathbb{R}^n, \sqrt{g} d^n x) \otimes V,$$

such that

$$((\hat{H} + 1)v, u) = 0$$

for all $v \in C_0^\infty(\mathbb{R}^n) \otimes V$. Hence

$$(\hat{H} + 1)u = 0 \tag{2.3}$$

holds in the sence of distributions in $\mathcal{D}'(\mathbb{R}^n) \otimes V$. Since \hat{H} is elliptic, u is C^∞ .

Now let $u_n := u \cdot h_n \in C_0^\infty(\mathbb{R}^n) \otimes V$, where $h_n \in C_0^\infty(\mathbb{R}^n)$ is constructed in the following way: By smoothing out the functions

$$\bar{h}_n(x) = \begin{cases} 1 & \text{for } |x| \leq \sqrt{n} \\ \frac{\sqrt{n} - |x|}{n - \sqrt{n}} + 1 & \text{for } \sqrt{n} \leq |x| \leq n \\ 0 & \text{for } |x| \geq n \end{cases}$$

it is easy to obtain a family $\{h_n\}_{n \in \mathbb{Z}^+ + 1}$, $h_n \in C_0^\infty(\mathbb{R}^n)$, $0 \leq h_n \leq 1$ such that

$$\text{supp } h_n \in \{x \mid |x| \leq n\}$$

$$h_n = 1 \quad \text{for } |x| \leq \sqrt{n} - \frac{1}{2}$$

and

$$\sum_{j=1}^n \left(\frac{\partial}{\partial x_j} h_n(x) \right)^2 \leq \frac{1}{(n - \sqrt{n})^2}$$

In particular we obtain (*)

$$\limsup_{n \rightarrow \infty} \sup_x \left(\frac{\partial}{\partial x_j} h_n(x) \right) g^{jk}(x) \left(\frac{\partial}{\partial x_k} h_n(x) \right) \leq C < 1 \quad (2.4)$$

Now a short calculation gives

$$-\hat{H}u_n = (-\hat{H}(\underline{g}, \underline{A})u) \cdot h_n + u \left(\frac{1}{\sqrt{g}} \partial_j \sqrt{g} g^{jk} \partial_k \right) h_n + 2(\partial_j + iA_j)u g^{jk} \partial_k h_n \quad (2.5)$$

Using relations (2.3) and (2.4), we obtain:

$$\begin{aligned} 0 \geq (u_n, -\hat{H}u_n) &= \int \langle u_n, -\hat{H}u_n \rangle_{\vee}(x) \sqrt{g} d^n x \\ &= \int h_n^2(x) \langle u, -\hat{H}u \rangle_{\vee} \sqrt{g} d^n x \\ &+ \int h_n(x) \langle u, u \rangle_{\vee}(x) \left[\left(\frac{1}{\sqrt{g}} \partial_j \sqrt{g} g^{jk} \partial_k \right) h_n \right](x) \sqrt{g} d^n x \\ &+ 2 \operatorname{Re} \int h_n(x) \langle u, (\partial_j + iA_j)u \rangle_{\vee}(x) g^{jk}(x) (\partial_k h_n(x)) \sqrt{g} d^n x \quad (2.6) \end{aligned}$$

Using the hermiticity of the A_j the last term in rel. (2.6) may be written as

$$\int h_n(x) (\partial_j \langle u, u \rangle_{\vee}(x)) g^{jk}(x) (\partial_k h_n(x)) \sqrt{g} d^n x$$

(*) At places where no confusion may arise, we will use the Einstein summation convention. Also $(\partial_j Y)$ means ∂_j acts on all of Y .

A partial integration on this term and the assumption (2.3) put the estimate (2.6) into the form

$$0 \geq \int \langle u, u \rangle_{\nu} \left(h_n^2(x) - \left(\frac{\partial}{\partial x_j} h_n(x) \right) g^{jk} \left(\frac{\partial}{\partial x_k} h_n(x) \right) \right) \sqrt{g} d^n x$$

Taking the lim inf over n , the Lebesgue dominated convergence theorem and relation (2.4) give

$$0 \geq (1 - C)(u, u)$$

By assumption on C , this is only possible if $u \equiv 0$, contradicting the assumption. This proves the first part of the proposition. The second is trivial.

Next we come to the existence of the wave operators Ω^{\pm} . We will base our discussion on Simon's formulation [40] of Enss' work. Letting

$$U = U(g, \underline{A}) = H_{g, \underline{A}} - H_0$$

we will give sufficient conditions for U, H to be a regular perturbation of H_0 in the sense of [40].

First we write U as

$$\begin{aligned} U &= \frac{1}{4} g^{-1} g^{jk} (\partial_j \partial_k g) + \frac{1}{8} g^{-1} (\partial_j g) g^{jk} (\partial_k g) + \frac{1}{4} g^{-1} (\partial_j g^{jk}) (\partial_k g) \\ &+ A_j g^{jk} A_k - i(\partial_j g^{jk}) A_k - i g^{jk} (\partial_j A_k) - (\partial_j g^{jk}) \partial_k - 2i A_j g^{jk} \partial_k - (g^{jk} - \delta^{jk}) \partial_j \partial_k \\ &= \sum_{|\beta| \leq 2} w^{\beta} D_{\beta} \end{aligned} \tag{2.7}$$

The first conditions for U, H to be a regular perturbation of H_0 is the Enss condition:

For some $N, \mathcal{D}(U) \geq \mathcal{D}(H_0^N)$ and

obeys

$$h(\mathbf{R}) := \| U(H_0^N + 1)^{-1} F(|x| \geq \mathbf{R}) \|$$

$$\int_0^{\infty} h(x) dx < \infty, \quad h(0) < \infty \tag{2.8}$$

Here $F(|x| \geq \mathbf{R})$ denotes multiplication by the characteristic function of $\{x \mid |x| \geq \mathbf{R}\}$.

Now introduce the following norm for measurable functions on \mathbb{R}^n . Let $\Delta_{\alpha}, \alpha \in \mathbb{Z}^n$ be a covering of \mathbb{R}^n by unit cubes with center at α . For a matrix-valued function w let

$$w_{\alpha} = \left[\int_{\Delta_{\alpha}} \|w(x)\|^2 d^n x \right]^{\frac{1}{2}}$$

and set

$$L_u^2 = \{w \mid \sup w_{\alpha} < \infty\}$$

Then Enss' condition is satisfied with $N > 1 + \frac{n}{2}$ if the estimate

$$(1 + |x|)^{1+\delta} w^\beta \in L_u^2 \quad (2.9)$$

holds for some $\delta > 0$ and for all w^β appearing in rel. (2.7).

This is a consequence of Strichartz theorem [42] and the discussion in [40]. An inspection of the terms appearing in rel. (2.7) combined with elementary matrix algebra shows that condition (2.9) is satisfied if the following point-wise bounds hold:

$$\begin{aligned} \sum_{j,k} |g_{jk} - \delta_{jk}| &\leq C(1 + |x|)^{-1-\delta} \\ \sum_{j,k,l} \left| \frac{\partial}{\partial x^l} g_{jk} \right| &\leq C(1 + |x|)^{-1-\delta} \\ \sum_{j,k,l,m} \left| \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^l} g_{jk} \right| &\leq C(1 + |x|)^{-1-\delta} \\ \sum_j \left(\|A_j\| + \sum_k \left\| \frac{\partial}{\partial x^k} A_j \right\| \right) &\leq C(1 + |x|)^{-1-\delta} \end{aligned} \quad (2.10)$$

for some $C > 0$ and $\delta > 0$.

These are conditions of the type also employed by Kuroda [29], [30]. Note in particular that condition (2.1) is fulfilled, if the estimates (2.10) hold. Also we have

$$C_1 |\xi|^2 \leq \sum_{j,k} \xi_j g^{jk}(x) \xi_k \leq C_2 |\xi|^2 \quad (2.11)$$

for all $\xi \in \mathbb{R}^n$ and all $x \in \mathbb{R}^n$ for some $0 < C_1 \leq C_2 < \infty$.

The second condition for (U, H) to be a regular perturbation is that $H_{g,\Delta}$ has to be a selfadjoint extension of $(H_0 + U) \upharpoonright \mathcal{D}(H_0^N)$ which is obviously satisfied due to proposition 2.1.

The third and last condition is that for all a, b in \mathbb{R} there is a positive continuous Q going to infinity at infinity such that

$$Q(H_0)E_{(a,b)}(H)$$

is a bounded operator. Here $E(\cdot)$ denotes the spectral measure of an operator. This is equivalent to H_0 being subordinate to H in the sense of Birman [7].

This last condition is in particular fulfilled if $(H + 1)^{-\frac{1}{2}}(H_0 + 1)^{\frac{1}{2}}$ is

a bounded operator and we will derive sufficient conditions for this to be satisfied. Let $u \in C_0^\infty(\mathbb{R}^n) \otimes V$, then a short calculation gives

$$\begin{aligned} (u, Hu) &= \int \left\langle \left(\partial_j + iA_j \right) \left(\frac{u}{g^{\frac{1}{2}}} \right), \left(\partial_k + iA_k \right) \left(\frac{u}{g^{\frac{1}{2}}} \right) \right\rangle_\vee g^{jk} (g)^{\frac{1}{2}} d^n x \\ &= \int \langle \partial_j u, \partial_k u \rangle_\vee g^{jk} d^n x - \frac{1}{2} \int (\partial_j g) g^{-1} \langle u, \partial_k u \rangle_\vee g^{jk} d^n x \\ &\quad + \frac{1}{16} \int g^{-2} \langle u, u \rangle_\vee (\partial_j g) g^{jk} (\partial_k g) d^n x \\ &\quad + \int g^{jk} \langle A_j u, A_k u \rangle_\vee d^n x \\ &\quad - \frac{i}{4} \int (\langle u, A_k u \rangle_\vee \partial_j g - \langle A_j u, u \rangle_\vee (\partial_k g)) g^{-1} g^{jk} d^n x \\ &\quad + i \int (\langle \partial_j u, A_k u \rangle_\vee - \langle A_j u, \partial_k u \rangle_\vee) g^{jk} d^n x \end{aligned}$$

The fifth term vanishes due to the hermiticity of the A_j and the symmetry of g^{jk} .

Taking Schwarz inequality gives for any $\varepsilon_1, \varepsilon_2 > 0$

$$\begin{aligned} (u, Hu) &\geq \left(1 - \frac{\varepsilon_1}{4} - \frac{\varepsilon_2}{2} \right) \int \langle \partial_j u, \partial_k u \rangle_\vee g^{jk} d^n x \\ &\quad + \left(\frac{1}{16} - \frac{1}{4\varepsilon_1} \right) \int \langle u, u \rangle_\vee (\partial_j \ln g) g^{jk} (\partial_k \ln g) d^n x \\ &\quad + \left(1 - \frac{1}{2\varepsilon_2} \right) \int g^{jk} \langle A_j u, A_k u \rangle_\vee d^n x \end{aligned}$$

Taking $\frac{\varepsilon_1}{4} + \frac{\varepsilon_2}{2} < 1$ we see there is $0 < \bar{c} < \infty$ such that

provided

$$(u, H_{\underline{g}, \Delta} u) \geq \bar{c} (u, (-\Delta - 1)u)$$

$$\underline{g} \cdot (x) \geq c \mathbb{1} \quad (c > 0) \tag{2.11 a}$$

$$(\partial_j \ln g) g^{jk} (\partial_k \ln g) \in L^\infty(\mathbb{R}^n, d^n x) \tag{2.11 b}$$

$$\|A_j g^{jk} A_k\| \in L^\infty(\mathbb{R}^n, d^n x) \tag{2.11 c}$$

Note that (2.11 a) and (2.11 c) imply $\|A_k\| \in L^\infty(\mathbb{R}^n, d^n x)$. In particular these conditions are satisfied provided the bounds (2.10) hold.

Collecting our results and employing Theorem 2.1 in [40] gives

PROPOSITION 2.2. — Assume the metric and the Yang-Mills potential satisfy the bounds (2.10). Then the Møller operators $\Omega^+(H_{\underline{g}, \Delta}, H_0)$ exist.

$H_{\underline{g}, \underline{A}}$ has no singular continuous spectrum. Moreover, $\text{Ran } \Omega^+ = \text{Ran } \Omega^-$ is the absolutely continuous space for $H_{\underline{g}, \underline{A}}$.

We now turn to a discussion of gauge invariance. By a gauge transformation G we will understand a differentiable map

$$x \mapsto G(x)$$

from \mathbb{R}^n into the unitaries of V . This induces a unitary map of $L^2(\mathbb{R}^n, d^n x) \otimes V$ into itself, which we also denote by G . Note that

$$GH_{\underline{g}, \underline{A}}G^{-1} = H_{\underline{g}, \underline{GA}} \tag{2.12}$$

where

$$(GA)_j(x) = G(x)A_j(x)G^{-1}(x) + iG(x)\frac{\partial}{\partial x^j}G^{-1}(x) \tag{2.13}$$

Following Kato, we now consider the following generalized Møller operators (if they exist)

$$\Omega^\pm(H_{\underline{g}, \underline{A}}, H_0; G) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH_{\underline{g}, \underline{A}}} G^{-1} e^{-itH_0}$$

By (2.12) - (2.13)

$$\Omega^\pm(H_{\underline{g}, \underline{A}}, H_0; G) = G^{-1} \Omega^\pm(H_{\underline{g}, \underline{GA}}, H_0) \tag{2.14}$$

We now have the following

PROPOSITION 2.3. — Assume the unitary operators $G(x)$ in V satisfy the following asymptotic condition

$$\lim_{|x| \rightarrow \infty} \|1 - G(x)\| = 0 \tag{2.15}$$

and let \underline{g} and \underline{A} satisfy the bounds (2.10). Then $\Omega^\pm(H_{\underline{g}, \underline{GA}}, H_0)$ exist and are equal to $G\Omega^\pm(H_{\underline{g}, \underline{A}}, H_0)$ such that the S-matrix $S = (\Omega^+)^* \Omega^-$ is gauge invariant:

$$S(\underline{g}, \underline{A}) = S(\underline{g}, \underline{GA}) \tag{2.16}$$

In particular the S-matrix is trivial for the case of a flat metric and a pure gauge, i. e. an \underline{A} of the form $\underline{A} = \underline{GO}$ for some G of the above form.

Proof. — According to theorem 4.2 in [26] it is sufficient to show that

$$s\text{-}\lim_{t \rightarrow \pm\infty} (1 - G^{-1})e^{-itH_0} = 0 \tag{2.17}$$

since this implies $\Omega^\pm(H_{\underline{g}, \underline{A}}, H_0; G) = \Omega^\pm(H_{\underline{g}, \underline{A}}, H_0)$.

To prove (2.17) it is again sufficient to prove

$$\lim_{t \rightarrow \pm\infty} \|(1 - G^{-1})e^{-itH_0} f\| = 0 \tag{2.18}$$

for f in a dense set in $L^2(\mathbb{R}^n, d^n x) \otimes V$. We will choose f 's such that their Fourier transforms \tilde{f} are in $C_0^\infty(\mathbb{R}^n) \otimes V$. For such an $f \neq 0$ and arbi-

bitrary $\varepsilon > 0$, let R be such that $\|(1 - G^{-1}(x))\| < \frac{\varepsilon}{3\|f\|}$ for $|x| \geq R$. Then for all t we have

$$\begin{aligned} \|(1 - G^{-1})e^{-iH_0}f\| &\leq \|(1 - G^{-1}(x))F(|x| \leq R)e^{-iH_0}f\| \\ &\quad + \|(1 - G^{-1}(x))F(|x| \geq R)e^{-iH_0}f\| \\ &\leq 2\|F(|x| \leq R)e^{-iH_0}f\| + \frac{\varepsilon}{3} \end{aligned} \tag{2.19}$$

To estimate the first term define V_t by

$$(V_t f)(x) = (2it)^{-\frac{n}{2}} e^{\frac{ix^2}{4t}} \tilde{f}\left(\frac{x}{2t}\right) \tag{2.20}$$

Then

$$\|e^{-iH_0}f - V_t f\| < \frac{\varepsilon}{6}$$

for some $|t| > t_1(\varepsilon)$ (see e. g. [32], Theorem IX.31). Hence

$$\begin{aligned} \|F(|x| \leq R)e^{-iH_0}f\| &\leq 2\|e^{-iH_0}f - V_t f\| + 2\|F(|x| \leq R)V_t f\| \\ &\leq \frac{\varepsilon}{3} + 2\left(\int_{|x| \leq R} \frac{1}{(2t)^n} \left|\tilde{f}\left(\frac{x}{2t}\right)\right|^2 d^n x\right)^{\frac{1}{2}} \\ &= \frac{\varepsilon}{3} + 2\left(\int_{|y| \leq \frac{R}{2t}} |\tilde{f}(y)|^2 d^n y\right)^{\frac{1}{2}} \end{aligned} \tag{2.21}$$

For given R and f the last term in (2.21) can be made smaller than $\frac{\varepsilon}{3}$ by choosing $|t| > t_2(\varepsilon) > 0$. Inserting estimate (2.21) in (2.18) proves the proposition.

Our next result will be to prove that the S-matrix is also invariant under those coordinate transformations, which tend suitably to the identity at infinity. More precisely let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism of \mathbb{R}^n and denote by T^{-1} its inverse. Consider the following unitary operator in $L^2(\mathbb{R}^n, d^n x) \otimes V$ also denoted by T and given as

$$(Tf)(x) = \left| \frac{\partial T^{-1}}{\partial x} \right|^{\frac{1}{2}}(x) f(T^{-1}(x)) \tag{2.22}$$

where the first factor is the square root of the Jacobian. We have the following relation, which is easy to verify

$$T H_{g, A} T^{-1} = H_{Tg, TA} \tag{2.23}$$

Here

$$\begin{aligned} (Tg)_{ij}(x) &= \sum_{k,l} \frac{\partial y^k}{\partial x^i} g_{kl}(y) \frac{\partial y^l}{\partial x^j} \\ (TA)_j(x) &= \sum_k \frac{\partial y^k}{\partial x^j} A_k(y) \end{aligned} \tag{2.24}$$

where $y = T^{-1}(x)$. In analogy to the previous proposition we have

PROPOSITION 2.4. — Assume T is a diffeomorphism of \mathbb{R}^n which satisfies the following two conditions

$$\lim_{|x| \rightarrow \infty} |x - T(x)| = 0 \quad (2.25 a)$$

$$\lim_{|x| \rightarrow \infty} \left| \frac{\partial T}{\partial x} \right| = 1 \quad (2.25 b)$$

and let \underline{g} and \underline{A} satisfy the bounds (2.10). Then $\Omega^\pm(H_{T\underline{g}, T\underline{A}}, H_0)$ exist and are equal to $T\Omega^\pm(H_{\underline{g}, \underline{A}}, H_0)$ such that the S-matrix is invariant under T , i. e.

$$S(T\underline{g}, T\underline{A}) = S(\underline{g}, \underline{A}) \quad (2.26)$$

In particular, the S-matrix for a Hamiltonian given by a metric \underline{g} being the transform of the flat metric and $\underline{A} = 0$ is trivial.

Proof. — Consider now the following generalized Møller operators

$$\Omega^\pm(H_{\underline{g}, \underline{A}}, H_0; T) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH_{\underline{g}, \underline{A}}} T^{-1} e^{-itH_0}$$

which if they exist, satisfy the identity

$$\Omega^\pm(H_{\underline{g}, \underline{A}}, H_0; T) = T^{-1} \Omega^\pm(H_{T\underline{g}, T\underline{A}}, H_0)$$

which follows from relations (2.24). Again it is sufficient to show that $\Omega^\pm(H_{\underline{g}, \underline{A}}, H_0; T) = \Omega^\pm(H_{\underline{g}, \underline{A}}, H_0)$ and by the previous arguments this will follow if

$$\lim_{t \rightarrow \pm\infty} \|(1 - T)V_t f\| = 0 \quad (2.27)$$

holds for f in a dense set which we will choose as before. Now we write

$$\begin{aligned} \|(1 - T)V_t f\| &\leq \|F(|x| \leq R)(1 - T)V_t f\| \\ &\quad + \|F(|x| \geq R)(1 - T)V_t f\| \\ &\leq \|F(|x| \leq R)V_t f\| + \|TF(|T(x)| \leq R)V_t f\| \\ &\quad + \|F(|x| \geq R)(1 - T)V_t f\| \end{aligned} \quad (2.28)$$

We will again give an $\frac{\varepsilon}{3}$ argument and we start with an estimate of the last term which will fix R . Note that

$$((1 - T)V_t f)(x) = (2it)^{-\frac{n}{2}} \left[e^{\frac{ix^2}{4t}} \tilde{f}\left(\frac{x}{2t}\right) - e^{\frac{i(T^{-1}(x))^2}{4t}} \left| \frac{\partial T^{-1}}{\partial x} \right|^{\frac{1}{2}} \tilde{f}\left(\frac{T^{-1}(x)}{2t}\right) \right] \quad (2.29)$$

Next let \tilde{r} be such that $|p| \leq \tilde{r}$ for $p \in \text{supp } \tilde{f}$ and let $0 < M < \infty$ be such that

$$\sup_p \left(|\tilde{f}(p)| + \sup_t \left| \frac{\partial}{\partial p_i} \tilde{f}(p) \right| \right) + \sup_x |x - T(x)| \leq M$$

Such an M exists, since T^{-1} also satisfies the conditions (2.25). We then have the estimates

$$\left| e^{\frac{ix^2}{4t}} - e^{\frac{i(T^{-1}(x))^2}{4t}} \right| \leq \frac{|x| + |T^{-1}(x)|}{4|t|} |x - T^{-1}(x)|$$

$$\leq \left(\frac{2|x| + M}{4|t|} \right) |x - T^{-1}(x)|$$

and

$$\left| \tilde{f}\left(\frac{T^{-1}(x)}{2t}\right) - \tilde{f}\left(\frac{x}{2t}\right) \right| \leq \frac{|x - T^{-1}(x)|}{2|t|} \cdot M \cdot F(|x| \leq 2t(M + \tilde{r}))$$

Using these estimates in (2.29) combined with a telescoping sum argument gives for $|t| \geq 1$ and some $\tilde{M} < \infty$

$$|(1-T)V_t f(x)|$$

$$\leq (2|t|)^{-\frac{n}{2}} \left[\frac{2|x| + M}{4|t|} |x - T^{-1}(x)| F(|x| \leq 2|t|\tilde{r}) \right.$$

$$+ \left\| \frac{\partial T^{-1}}{\partial x} \right|^{\frac{1}{2}} - 1 \left| F(|x| \leq 2|t|\tilde{r}) \right.$$

$$\left. + \left| \frac{\partial T^{-1}}{\partial x} \right|^{\frac{1}{2}} M \frac{|x - T^{-1}(x)|}{2|t|} F(|x| \leq 2|t|(\tilde{r} + M)) \right]$$

$$\leq (2t)^{-\frac{n}{2}} \tilde{M} \left(\frac{|x|}{2|t|} + 1 \right) \left(\left\| \frac{\partial T^{-1}}{\partial x} \right|^{\frac{1}{2}} - 1 + |x - T^{-1}(x)| \right) F(|x| \leq 2|t|(\tilde{r} + M))$$

(2.30)

Using condition (2.25) in estimate (2.30) shows that for $R = R(\varepsilon)$ sufficiently large the third term in (2.28) may indeed be made smaller than $\frac{\varepsilon}{3}$. For this R the first term in (2.28) can also be made smaller than $\frac{\varepsilon}{3}$ by choosing $|t|$ sufficiently large by previous arguments. The second term may be estimated by

$$\|F(|T(x)| \leq R)V_t f\| \leq \|F(|x| \leq R + M)V_t f\|$$

and this again may be made smaller than $\frac{\varepsilon}{3}$ for $|t|$ sufficiently large by the same arguments. This concludes the proof of proposition 2.4. We note that the results of the last two propositions may be combined and extended to the cases of bundle isomorphisms of the bundle $\mathbb{R}^n \otimes V$ with base space \mathbb{R}^n which approach the identity at infinity. We leave the details to the reader.

§ 3. ABSENCE OF A DISCRETE SPECTRUM

The aim of this section is to exclude nonnegative eigenvalues. To formulate the sufficient conditions for this to hold, define the field strength tensor F_{jk} by

$$F_{jk} = \frac{\partial}{\partial x^j} A_k - \frac{\partial}{\partial x^k} A_j + i[A_j, A_k] \quad (3.1)$$

where $[,]$ denotes the commutator. Furthermore let $O(r^\varepsilon)$ and $o(r^\varepsilon)$ be the usual Landau symbols, which in our context will describe the asymptotic behaviour for $r = |x| \rightarrow \infty$.

THEOREM 3.1. — Assume the metric and the Yang-Mills potential satisfy the following asymptotic estimates for large $|x|$

$$\begin{aligned} \sum_{j,k=1}^n |g_{jk} - \delta_{jk}| = o(1); \quad \sum_{j,k,l} \left| \frac{\partial}{\partial x^l} g_{jk} \right| = o(r^{-1}) \\ \sum_{j,k} \|F_{jk}\| = o(r^{-1}) \end{aligned} \quad (3.2)$$

Then $\hat{H}(g, \underline{A})$ and therefore also $H_{g,\Delta}$ has no discrete spectrum.

Remark. — Note that the first condition in (3.2) implies (2.11) since we assume g_{jk} to define an C^∞ metric tensor on \mathbb{R}^n . Also note that the decay condition on the field strength tensor is gauge invariant. In particular the one instanton field in the form given by Belavin, Polyakov, Schwarz and Tyupkin [5] satisfies these conditions (with the flat metric). The same is true for the N-instanton field of 'tHooft in the regular form given by Griambiagi and Rothe [16] and Sciuto [39].

Proof. — Assume first $E = 0$ is an eigenvalue of $\hat{H}(g, \underline{A})$ and let u be an eigenfunction. Since $u \in \mathcal{D}(\hat{H})$, by the last part of proposition 2.1, u is contained in the form domain of the closure \bar{Q} of Q , which is given by relation (2.2). Thus we must have

$$0 = (u, \hat{H}u) = \bar{Q}(u, u)$$

so in particular

$$(\partial_j + iA_j)u = 0$$

for all j almost everywhere. If we integrate this differential equation for $j = 1$ we obtain

$$u(x^1, \dots, x^n) = \left(T \exp i \int_0^{x^1} A(y, x^2, \dots, x^n) dy \right) u(0, x^2, \dots, x^n)$$

where $T \exp$ is the so called ordered exponential. In particular

$$\langle u(x^1, x^2, \dots, x^n), u(x^1, \dots, x^n) \rangle_\nu = \langle u(o, x^2, \dots, x^n), u(o, x^2, \dots, x^n) \rangle_\nu$$

almost everywhere. Iterating this argument shows that $\langle u(x), u(x) \rangle_\nu$ is constant. This contradicts the square integrability of u unless u is identically zero. Note that no decay property of F_{jk} is needed in this case.

Next consider the case $E > 0$ and assume there is an eigenfunction u with

$$\hat{H}(g, \underline{A})u = Eu$$

If all $g_{ik} - \delta_{ik}$ and A_i had compact support, then

$$-\Delta u(x) = Eu(x)$$

would hold for $|x| \geq R_1$ and $R_1 > \infty$ suitably chosen. By a classical theorem of Rellich [34] this implies

$$u(x) = 0 \quad \text{for } |x| \geq R_1 \tag{3.3}$$

Then by the unique continuation theorem of Aronszajn [3] and Cordes [10], which extends a theorem of Heinz [17] and which may be applied in the present case, u vanishes everywhere.

In the general case, assuming the estimates (3.2) to hold, we will still be able to show (3.3) such that once again the unique continuation theorem may be applied to conclude that u vanishes everywhere. We will use the same strategy as in the proof of the Kato-Agmon-Simon theorem ([32], Th. XIII-58). From now on we will assume $n \geq 2$, since the case $n = 1$ is much simpler. To proceed further, let $y = (y^1, \dots, y^n) = (\phi^1, \dots, \phi^{n-1}, r)$ be the usual polar coordinates such that $r = |x|$ and $\Omega = (\phi^1, \dots, \phi^{n-1})$ parametrizes a point on the unit sphere. Let η_{ij} be the metric tensor in terms of these coordinates, i. e.

$$\eta_{ij} = \sum_{m,n} g_{mn} \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j} \tag{3.4}$$

We will often write η_{rj} for η_{nj} , etc. Furthermore let

$$\bar{\eta} = \eta r^{-2(n-1)} = (\det \eta_{ij}) r^{-2(n-1)}; \quad \eta_{ij} = r^2 \bar{\eta}_{ij};$$

$$\eta^{ij} = r^{-2} \bar{\eta}^{ij} \quad (1 \leq i, j \leq n-1) \tag{3.5}$$

Also let

$$D_j = \frac{\partial}{\partial y^j} + i\tilde{A}_j = \partial_j + i\tilde{A}_j \quad (1 \leq j \leq n)$$

be the covariant derivative in polar coordinates, i. e.

$$\tilde{A}_j = \sum_{i=1}^n \frac{\partial x^i}{\partial y^j} A_i \tag{3.6}$$

In polar coordinates, \hat{H} takes the form

$$-\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4$$

with

$$\begin{aligned} \hat{H}_1 &= \frac{1}{\sqrt{\eta}} D_r \sqrt{\eta} \eta^{rr} D_r \\ \hat{H}_2 &= \frac{1}{\sqrt{\eta}} \sum_{i=1}^{n-1} D_r \sqrt{\eta} \eta^{ri} D_i \\ \hat{H}_3 &= \frac{1}{\sqrt{\eta}} \sum_{i=1}^{n-1} D_i \sqrt{\eta} \eta^{ir} D_r \\ \hat{H}_4 &= \frac{1}{r^2} \frac{1}{\sqrt{\bar{\eta}}} \sum_{i,j=1}^{n-1} D_i \sqrt{\bar{\eta}} \bar{\eta}^{ij} D_j = \frac{1}{r^2} \bar{H}_4 \end{aligned} \quad (3.7)$$

and we consider \hat{H} to act in $L^2\left(\mathbb{R}^+ \times S^{n-1}, r^{n-1} \sqrt{\bar{\eta}} dr \prod_{i=1}^{n-1} d\phi_i\right) \otimes V$.

Given the eigenfunction u , which we may assume to be C^∞ by the ellipticity of \hat{H} , let

$$w = r^{\frac{n-1}{2}} u \quad (3.8)$$

such that

$$w \in L^2\left(\mathbb{R}^+ \times S^{n-1}, \sqrt{\bar{\eta}} dr \prod_{i=1}^{n-1} d\phi_i\right) \otimes V$$

w obeys the equation

$$(\hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4)w + Ew + r^{\frac{n-1}{2}} [\hat{H}_1 + \hat{H}_2 + \hat{H}_3, r^{-\frac{(n-1)}{2}}]w = 0$$

For given r let $\mathcal{L}_r = L^2\left(S^{n-1}, \sqrt{\bar{\eta}} \prod_{i=1}^{n-1} d\phi_i\right) \otimes V$ and let $(\cdot, \cdot)_r$ be the scalar product in \mathcal{L}_r . Consider

$$w, w' = D_r w, w'' = (D_r)^2 w$$

as functions on \mathbb{R}^+ with values in \mathcal{L}_r . Let

$$F(r) = (w', \eta^{rr} w')_r + \frac{1}{r^2} (w, \bar{H}_4 w)_r + E(w, w)_r$$

Due to the assumption $u \in \mathcal{D}(\hat{H})$, it is easy to check that $F(r)$ is integrable in r at infinity. Next, using the hermiticity of A_j , we obtain

$$\begin{aligned} \frac{d}{dr}(rF(r)) &= (w', \eta^{rr}w')_r - \frac{1}{r^2}(w, \bar{H}_4w)_r + E(w, w)_r \\ &\quad + 2r \operatorname{Re} \left(w', \left\{ \eta^{rr}w'' + Ew + \frac{1}{r^2} \bar{H}_4w \right\} \right)_r \\ &\quad + r \left(w', \frac{1}{\sqrt{\bar{\eta}}} (\partial_r \eta^{rr} \sqrt{\bar{\eta}}) w' \right)_r + r E \left(w, \frac{1}{\sqrt{\bar{\eta}}} (\partial_r \sqrt{\bar{\eta}}) w \right)_r \\ &\quad + \frac{1}{r} \left(w, \frac{1}{\sqrt{\bar{\eta}}} (\partial_r \sqrt{\bar{\eta}} \bar{H}_4) w \right)_r \end{aligned}$$

Next

$$\begin{aligned} \hat{H}_1 + r^{\frac{n-1}{2}} [\hat{H}_1, r^{-\frac{(n-1)}{2}}] &= \eta^{rr} (D_r)^2 + \frac{1}{\sqrt{\bar{\eta}}} (\partial_r \sqrt{\bar{\eta}} \eta^{rr}) D_r \\ &\quad - \frac{(n-1)(n-3)}{4r^2} \eta^{rr} - \frac{(n-1)}{2} \frac{1}{r} \frac{1}{\sqrt{\bar{\eta}}} (\partial_r \sqrt{\bar{\eta}} \eta^{rr}) \end{aligned}$$

Thus

$$\begin{aligned} \frac{d}{dr}(rF(r)) &= (w', \eta^{rr}w')_r - \frac{1}{r^2}(w, \bar{H}_4w)_r + E(w, w)_r \\ &\quad - r \left(w', \frac{1}{\sqrt{\bar{\eta}}} (\partial_r \sqrt{\bar{\eta}} \eta^{rr}) w' \right)_r + \frac{1}{2r} (n-1)(n-3) \operatorname{Re} (w', \eta^{rr}w)_r \\ &\quad + (n-1) \operatorname{Re} \left(w', \frac{1}{\sqrt{\bar{\eta}}} (\partial_r \sqrt{\bar{\eta}} \eta^{rr}) w \right)_r + r E \left(w, \frac{1}{\sqrt{\bar{\eta}}} (\partial_r \sqrt{\bar{\eta}}) w \right)_r \\ &\quad - 2r \operatorname{Re} \left(w', r^{\frac{(n-1)}{2}} [\hat{H}_2, r^{-\frac{(n-1)}{2}}] w \right)_r \\ &\quad - 2r \operatorname{Re} \left(w', r^{\frac{n-1}{2}} [\hat{H}_3, r^{-\frac{(n-1)}{2}}] w \right)_r \\ &\quad + \frac{1}{r} \left(w, \frac{1}{\sqrt{\bar{\eta}}} (\partial_r \sqrt{\bar{\eta}} \bar{H}_4) w \right)_r \\ &\quad - 2r \operatorname{Re} (w', \hat{H}_2 w)_r - 2r \operatorname{Re} (w', \hat{H}_3 w)_r \\ &= C_1 + C_2 + C_3 + \sum_{i=1}^9 E_i \end{aligned} \tag{3.9}$$

Now $C_i \geq 0$ and our next aim will be to show that for sufficiently large r , the C_i dominate the E_j . This implies that $rF(r)$ is monotonically non-decreasing for all large r . These estimates are possible due to the next lemma, which we prove in the appendix. To formulate it, let $\overset{\circ}{\eta}_{ij}$ ($1 \leq i, j \leq n$) be the flat metric tensor expressed in polar coordinates. Also let $\overset{\circ}{\eta}_{ij}$

($1 \leq i, j \leq n-1$) be the induced metric on S^{n-1} . With $\overset{\circ}{\eta}^{ij}$, $\overset{\circ}{\eta}^{ij}$ given as usual, we have in addition to relations (3.5)

$$\begin{aligned} \overset{\circ}{\eta}_{rr} = \overset{\circ}{\eta}^{rr} = 1, \quad \overset{\circ}{\eta}_{rj} = \overset{\circ}{\eta}^{rj} = 0, \quad \overset{\circ}{\eta} = \det \overset{\circ}{\eta}_{ij} = \prod_{i=1}^{n-1} \overset{\circ}{\eta}_{ii} \quad (3.10) \\ \overset{\circ}{\eta}_{ij}(r, \Omega) = r^2 \overset{\circ}{\eta}_{ij}(\Omega) = r^2 \delta_{ij} \overset{\circ}{\eta}_{ii}; \\ \overset{\circ}{\eta}^{ij}(r, \Omega) = r^{-2} \overset{\circ}{\eta}^{ij}(\Omega) = r^{-2} \delta^{ij} \overset{\circ}{\eta}^{ii}(\Omega) \quad (1 \leq i, j \leq n-1) \\ \text{(no summation)} \end{aligned}$$

Also let \tilde{F}_{ij} be the field tensor in polar coordinates. An easy calculation shows that

$$\tilde{F}_{ij} = F_{mn} \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j} \quad (3.11)$$

LEMMA 3.2. — If the metric tensor and the Yang-Mills potential satisfy estimates (3.2) in terms of cartesian coordinates, then the following estimates hold uniformly for large r in terms of polar coordinates

$$|\eta_{ij} - \overset{\circ}{\eta}_{ij}| \leq o(1) |\overset{\circ}{\eta}_{ii}|^{\frac{1}{2}} |\overset{\circ}{\eta}_{jj}|^{\frac{1}{2}} \quad (1 \leq i, j \leq n) \quad (\alpha)$$

$$|\eta^{ij} - \overset{\circ}{\eta}^{ij}| \leq o(1) |\overset{\circ}{\eta}^{ii}|^{\frac{1}{2}} |\overset{\circ}{\eta}^{jj}|^{\frac{1}{2}} \quad (1 \leq i, j \leq n) \quad (\beta)$$

$$|\bar{\eta}^{ij} - \overset{\circ}{\eta}^{ij}| \leq o(1) |\overset{\circ}{\eta}^{ii}|^{\frac{1}{2}} |\overset{\circ}{\eta}^{jj}|^{\frac{1}{2}} \quad (1 \leq i, j \leq n-1) \quad (\gamma)$$

$$|\sqrt{\bar{\eta}} - \sqrt{\overset{\circ}{\eta}}| \leq o(1) \sqrt{\overset{\circ}{\eta}} \quad (\delta)$$

$$|\partial_r(\eta_{ij} - \overset{\circ}{\eta}_{ij})| \leq o(r^{-1}) |\overset{\circ}{\eta}_{ii}|^{\frac{1}{2}} |\overset{\circ}{\eta}_{jj}|^{\frac{1}{2}} \quad (1 \leq i, j \leq n) \quad (\varepsilon)$$

$$|\partial_r(\eta^{ij} - \overset{\circ}{\eta}^{ij})| \leq o(r^{-1}) |\overset{\circ}{\eta}^{ii}|^{\frac{1}{2}} |\overset{\circ}{\eta}^{jj}|^{\frac{1}{2}} \quad (1 \leq i, j \leq n) \quad (\zeta)$$

$$|\partial_r(\bar{\eta}^{ij} - \overset{\circ}{\eta}^{ij})| \leq o(r^{-1}) |\overset{\circ}{\eta}^{ii}|^{\frac{1}{2}} |\overset{\circ}{\eta}^{jj}|^{\frac{1}{2}} \quad (1 \leq i, j \leq n) \quad (\eta)$$

$$\left| \frac{1}{\sqrt{\bar{\eta}}} (\partial_r \sqrt{\bar{\eta}}) \right| \leq o(r^{-1}) \quad (\theta)$$

$$\left| \sum_{j=1}^{n-1} \frac{1}{\sqrt{\bar{\eta}}} (\partial_j \sqrt{\bar{\eta}} \eta^{rj}) \right| \leq o(r^{-1}) \quad (i)$$

$$\|\tilde{F}_{rj}\| \leq o(1) \quad (1 \leq j \leq n-1) \quad (\kappa)$$

The proof will be given in the appendix.

We now estimate the different E_i . By (α) $|\eta^{rr} - 1| = o(1)$.

Also by (ζ) $|\partial_r \eta^{rr}| \leq o(r^{-1})$. Combining this with (θ) we have

$$|E_1| \leq o(1)(w', \eta^{rr} w')_r = o(1)C_1$$

Next

$$|E_2| \leq o(r^{-1})[(w', \eta^{rr} w')_r, (w, w)]^{\frac{1}{2}} = o(r^{-1})(C_1 C_3)^{\frac{1}{2}}$$

Similarly

$$|E_3| \leq o(r^{-1})[(w', \eta^{rr} w')_r, (w, w)]^{\frac{1}{2}} = o(r^{-1})(C_1 C_3)^{\frac{1}{2}}$$

and

$$|E_4| \leq o(1)(w, w)_r = o(1)C_3$$

To estimate the remaining terms, let $d\Omega = \sqrt{\frac{\zeta}{\eta}} \prod_{i=1}^{n-1} d\phi_i$ be the standard measure on S^{n-1} and let $(\cdot, \cdot)_0$ denote the scalar product in

$$\mathcal{L}_0 = L^2(S^{n-1}, d\Omega) \otimes V.$$

Also let B be the operator on \mathcal{L}_0 defined by the following quadratic form

$$(v_1, (-B)v_2)_0 = \sum_{j=1}^{n-1} \int \langle D_j v_1, D_j v_2 \rangle_{\vee} \overset{\circ}{\eta}{}^{jj} d\Omega$$

Then by (γ) and (δ) and the basic estimate $2ab \langle a^2 + b^2 \rangle$ we have

$$\begin{aligned} |(w, (-\bar{H}_4)w)_r - (w, (-B)w)_0| &\leq o(1) \int \sum_{i,j=1}^{n-1} \langle D_i w, D_i w \rangle_{\vee}^{\frac{1}{2}} (\overset{\circ}{\eta}{}^{ii})^{\frac{1}{2}} (\overset{\circ}{\eta}{}^{jj})^{\frac{1}{2}} \langle D_j w, D_j w \rangle_{\vee}^{\frac{1}{2}} d\Omega \\ &\leq o(1) \int \sum_{j=1}^{n-1} \langle D_j w, D_j w \rangle_{\vee} \overset{\circ}{\eta}{}^{jj} d\Omega = o(1)(w, (-B)w)_0 \end{aligned}$$

In particular $(w, (-\bar{H}_4)w)_r = 0(1)(w, (-B)w)_0$
 $(w, (-B)w)_0 = 0(1)(w, (-\bar{H}_4)w)_r$ (3.12)

To estimate E_5 , we first note that

$$r^{-\frac{n-1}{2}} [\hat{H}_2, r^{-\frac{(n-1)}{2}}] = -\frac{(n-1)}{2r} \sum_{j=1}^{n-1} \eta^{jr} D_j$$

Since $\overset{\circ}{\eta}{}^{ri} = 0$, estimate (β) therefore gives combined with (3.10)-(3.12)

$$\begin{aligned} |E_5| &\leq o(r^{-1}) [(w', \eta^{rr} w')_r]^{\frac{1}{2}} \sum_{j=1}^{n-1} \| \overset{\circ}{\eta}{}^{jj\frac{1}{2}} D_j w \|_0 \\ &\leq o(1)(C_1 C_2)^{\frac{1}{2}} \end{aligned}$$

Next

$$\begin{aligned} r^{-\frac{n-1}{2}} [\hat{H}_3, r^{-\frac{(n-1)}{2}}] &= -\frac{(n-1)}{2} \frac{1}{r} \frac{1}{\sqrt{\eta}} \sum_{j=1}^{n-1} D_j \sqrt{\eta} \eta^{jr} \\ &= -\frac{(n-1)}{2} \frac{1}{r} \frac{1}{\sqrt{\eta}} \sum_{j=1}^{n-1} (\partial_j \sqrt{\eta} \eta^{jr}) \\ &\quad - \frac{(n-1)}{2} \frac{1}{r} \sum_{j=1}^{n-1} \eta^{jr} D_j \end{aligned}$$

Hence by (3.12) and the arguments used to estimate E_5

$$\begin{aligned} |E_6| &\leq o(r^{-1})(w', \eta^{rr}w')_r^{\frac{1}{2}}((w, (-\bar{H}_4)w)_r^{\frac{1}{2}} + (w, w)_r^{\frac{1}{2}}) \\ &\leq o(1)((C_1C_2)^{\frac{1}{2}} + (C_1C_3)^{\frac{1}{2}}) \end{aligned}$$

By (9), (η) and by now familiar arguments

$$|E_7| \leq o(1)C_2$$

To estimate $E_8 + E_9$, we perform a partial integration in the variable ϕ^i for each i . The boundary terms vanish, since $|\eta^{ir}\sqrt{\eta}| \leq o(1)|\dot{\eta}^{ii}|^{\frac{1}{2}}\sqrt{\eta}$ by lemma 3.2 and since $\dot{\eta}^{ii}\sqrt{\eta}$ vanishes on the boundary of the ϕ^i variable. Using the fact that $[D_r, D_j] = i\tilde{F}_{rj}$ an easy calculation gives

$$\begin{aligned} E_8 + E_9 &= -2r \operatorname{Re} \left(w', \frac{1}{\sqrt{\eta}} \sum_{j=1}^{n-1} (\partial_r \sqrt{\eta} \eta^{rj}) D_j w \right)_r \\ &\quad - 2r \operatorname{Re} \left(w', i \sum_{j=1}^{n-1} \eta^{rj} \tilde{F}_{rj} w \right)_r \end{aligned} \quad (3.13)$$

Estimate (θ) gives $\left| \frac{1}{\sqrt{\eta}} \partial_r \sqrt{\eta} \right| \leq 0(r^{-1})$. Combining this with (3.10), (β)

and (ζ) show that the first term on the right hand side of (3.13) may be estimated by

$$o(r^{-1})(w', \eta^{rr}w')_r^{\frac{1}{2}}(w, (-\bar{H}_4)w)_r^{\frac{1}{2}} = o(1)(C_1C_2)^{\frac{1}{2}}$$

By (κ) the second term in (3.13) is estimated by

$$o(1)(w', \eta^{rr}w')_r^{\frac{1}{2}}(w, w)_r^{\frac{1}{2}} = o(1)(C_1C_3)^{\frac{1}{2}}$$

Altogether this establishes

$$\frac{d}{dr}(rF(r)) \geq 0 \quad \text{for all } r > R_1,$$

R_1 suitably chosen. In particular

$$F(r) \geq \frac{1}{r} r_1 F(r_1) \quad \text{for all } r > r_1 > R_1 \quad (3.14)$$

Our next aim will be to show that if $w(r_0) \neq 0$ for some $r_0 > R_1$, then there is $r_1 > R_1$ such that $F(r_1) > 0$. But then estimate (3.14) contradicts the integrability of $F(r)$ at infinity.

To show that such an r_1 may be found, let

$$\begin{aligned} h_m(r) &= \exp m \int_{R_1}^r x^{-\frac{3}{4}} dx \quad \text{for } r > R_1 \\ &= \exp 4m(r^{\frac{1}{4}} - R_1^{\frac{1}{4}}) \end{aligned}$$

and put

$$w_m(r) = h_m(r) \cdot w(r)$$

Define

$$\begin{aligned} G(m, r) &= (w'_m, \eta^{rr} w'_m)_r + \frac{1}{r^2} (w_m, \bar{H}_4 w_m)_r \\ &+ E \left(1 - \frac{R_1}{r} \right) (w_m, w_m)_r + \frac{m^2}{r^{\frac{3}{2}}} (w_m, \eta^{rr} w_m)_r \end{aligned}$$

Then by arguments similar to those which led to rel. (3.3) a trivial, but somewhat lengthy calculation gives

$$\begin{aligned} \frac{d}{dr} (r^{\frac{7}{4}} G(m, r)) &= \frac{7}{4} r^{\frac{3}{4}} (w'_m, \eta^{rr} w'_m)_r - \frac{1}{4r^{\frac{5}{4}}} (w_m, \bar{H}_4 w_m)_r + \frac{7}{4} E r^{\frac{3}{4}} (w_m, w_m)_r \\ &+ \frac{m^2}{4r^{\frac{3}{4}}} (w_m, \eta^{rr} w_m)_r + 4mr (w'_m, \eta^{rr} w'_m)_r \\ &- r^{\frac{7}{4}} \left(w'_m, \frac{1}{\sqrt{\eta}} (\partial_r \sqrt{\eta} \eta^{rr}) w'_m \right)_r + \frac{1}{2r^{\frac{1}{4}}} (n-1)(n-3) \operatorname{Re} (w', \eta^{rr} w)_r \\ &+ r^{\frac{3}{4}} (n-1) \operatorname{Re} \left(w'_m, \frac{1}{\sqrt{\eta}} (\partial_r \sqrt{\eta} \eta^{rr}) w_m \right)_r + r^{\frac{7}{4}} E \left(w_m, \frac{1}{\sqrt{\eta}} (\partial_r \sqrt{\eta}) w_m \right)_r \\ &- 2r^{\frac{7}{4}} \operatorname{Re} \left(w'_m, r^{\frac{n-1}{2}} [\hat{H}_2, r^{-\frac{(n-1)}{2}}] w_m \right)_r - 2r^{\frac{7}{4}} \operatorname{Re} \left(w'_m, r^{\frac{n-1}{2}} [\hat{H}_3, r^{-\frac{(n-1)}{2}}] w_m \right)_r \\ &+ \frac{1}{r^{\frac{1}{4}}} \left(w_m, \frac{1}{\sqrt{\eta}} (\partial_r \sqrt{\eta} \bar{H}_4) w_m \right)_r - 2r^{\frac{7}{4}} \operatorname{Re} (w'_m, \hat{H}_2 w_m)_r - 2r^{\frac{7}{4}} \operatorname{Re} (w'_m, \hat{H}_3 w_m)_r \\ &- \frac{3}{4} E R_1 r^{-\frac{1}{4}} (w_m, w_m)_r - E R_1 r^{\frac{3}{4}} \left(w_m, \frac{1}{\sqrt{\eta}} (\partial_r \sqrt{\eta}) w_m \right)_r \\ &+ r^{\frac{1}{4}} m^2 \left(w_m, \frac{1}{\sqrt{\eta}} (\partial_r \sqrt{\eta} \eta^{rr}) w_m \right)_r - 2E R_1 r^{\frac{3}{4}} \operatorname{Re} (w'_m, w_m)_r - \frac{3}{2} m \operatorname{Re} (w'_m, \eta^{rr} w_m)_r \\ &+ 2mr \operatorname{Re} \left(w'_m, \frac{1}{\sqrt{\eta}} (\partial_r \sqrt{\eta} \eta^{rr}) w_m \right)_r \\ &:= \sum_{i=1}^5 \bar{C}_i + \sum_{i=1}^{15} \bar{E}_i \end{aligned} \tag{3.15}$$

The expressions \bar{E}_i ($1 \leq i \leq 9$) may be estimated in terms of $\bar{C}_1, \bar{C}_2, \bar{C}_3$ in the same way as the E_i ($1 \leq i \leq 9$) in terms of C_1, C_2, C_3 . In particular this holds uniformly in m . This is also true for the next three trivial estimates

$$\begin{aligned} |\bar{E}_{10}| + |\bar{E}_{11}| &\leq 0(r^{-1})\bar{C}_3 \\ |\bar{E}_{12}| &\leq o(1) \bar{C}_4 \\ |\bar{E}_{13}| &\leq 0(r^{-\frac{1}{8}}) |mr(w'_m, \eta^{rr}w'_m)_r r^{\frac{3}{4}}(w_m, w_m)_r|^{\frac{1}{2}} \\ &= 0(r^{-\frac{1}{8}})(\bar{C}_3\bar{C}_5)^{\frac{1}{2}} \end{aligned}$$

for $m \geq 1$.

Again by Schwarz inequality, we have for all $m \geq 1$

$$\begin{aligned} |\bar{E}_{14}| &\leq 0(r^{-\frac{1}{8}}) [mr(w'_m, \eta^{rr}w'_m)_r]^{\frac{1}{2}} \left[\frac{m^2}{r^{\frac{3}{4}}}(w_m, \eta^{rr}w_m)_r \right]^{\frac{1}{2}} \\ &= 0(r^{-\frac{1}{8}})(\bar{C}_4\bar{C}_5)^{\frac{1}{2}} \end{aligned}$$

and again for $m \geq 1$

$$|\bar{E}_{15}| \leq 0(r^{-\frac{1}{8}})(\bar{C}_4\bar{C}_5)^{\frac{1}{2}}$$

Collecting all these estimates in (3.15), we may assume, by increasing R_1 if necessary $r^{\frac{7}{4}}G(m, r)$ to be monotonically nondecreasing in r for $r > R_1$ and all $m \geq 1$. Suppose now that $w(r_0) \neq 0$ for some $r_0 > R_1$. Writing

$$\begin{aligned} G(m, r) = h_m(r)^2 &\left[\left(w' + \frac{m}{r^{\frac{3}{4}}}w, \eta^{rr} \left(w' + \frac{m}{r^{\frac{3}{4}}}w \right) \right)_r + \frac{1}{r^2}(w, \bar{H}_4w)_r \right. \\ &\left. + E \left(1 - \frac{R_1}{r} \right) (w, w)_r + \frac{m^2}{r^{\frac{3}{2}}}(w, \eta^{rr}w)_r \right] \quad (3.16) \end{aligned}$$

it follows that $G(m, r_0) > 0$ for m sufficiently large.

Hence by the monotonicity proven above, it follows that $G(m, r) > 0$ for $m > M$ and $r > r_0$ provided $w(r_0) \neq 0$. Now given such an r_0 choose first $m_0 > M$ and then $R_2 > r_0$ such that for $r > R_2$

$$-\frac{ER_1}{r} + \frac{2m_0^2}{r^{\frac{3}{2}}} \sup_{\Omega} |\eta^{rr}(r, \Omega)| + \frac{m_0}{r^{\frac{3}{4}}} \sup_{\Omega} \left| \left(\frac{1}{\sqrt{\bar{\eta}}} \partial_r \sqrt{\bar{\eta}} \eta^{rr} \right) (r, \Omega) \right| \leq 0$$

which is possible due to lemma 3.2. Since $(w, \eta^{rr}w)_r$ is integrable at infinity, there is some $r_1 > R_2$ such that

$$\frac{d}{dr} (w, \eta^{rr}w)_r \Big|_{r=r_1} = 2 \operatorname{Re} (w', \eta^{rr}w)_r + \left(w, \left(\frac{1}{\sqrt{\bar{\eta}}} \partial_r \sqrt{\bar{\eta}} \eta^{rr} \right) w \right)_r \Big|_{r=r_1} \leq 0 \quad (3.17)$$

But then rel. (3.16) and (3.17) imply

$$0 < h_{m_0}^{-2}(r_1)G(m_0, r_1) \leq F(r_1)$$

According to our previous arguments this completes the proof of theorem 3.1.

Remark. — Let W be a potential, i. e. a function on \mathbb{R}^n with values in the hermitean matrices of V , such that $\|W(x)\| = o(r^{-1})$. Then theorem 3.1 also holds for $\hat{H}(g, A) + W$. Using the identification operator J (see rel. (1.2)), it easily follows that our discussion also covers the cases discussed by Ikebe and Uchiyama [23].

§ 4. AN ASYMPTOTIC FLAT METRIC GIVING DISCRETE SPECTRUM

In this section we will discuss a computer calculation we carried out which indicates the existence of an asymptotic flat metric with at least one discrete eigenvalue. Take $n = 3$ and consider Robertson-Walker like metrics of the form

$$ds^2 = h(r)^{-4}(dr)^2 + r^2 (\sin^2 \theta(d\phi)^2 + (d\theta)^2)$$

where (r, ϕ, θ) are the polar coordinates. This metric will be asymptotically flat if $h(r) \rightarrow 1$ for $r \rightarrow \infty$. As before let

$$d\Omega = \frac{1}{4\pi} \sin \theta d\theta d\phi$$

Then the Hilbert space for the corresponding Laplace-Beltrami operator Δ_g is $L^2\left(\mathbb{R}^+ \times S^2, \frac{r^2}{h(r)^2} dr d\Omega\right)$. On functions u with no angular dependence (angular momentum zero), Δ_g takes the form

$$\Delta_g u = \frac{h(r)^2}{r^2} \frac{\partial}{\partial r} h(r)^2 r^2 \frac{\partial}{\partial r} u \tag{4.1}$$

We will fix such a function u and then consider the equation

$$-\Delta_g u = u \tag{4.2}$$

as an equation in h . Write $u(r) = (rh(r))^{-1} \chi(r)$. Hence if we choose $\chi \in L^2(\mathbb{R}^+, dr)$ and find $h(r) > 0$ with $h(r) \rightarrow 1$ for $r \rightarrow \infty$ then u will be in $L^2(\mathbb{R}^+ \times S^2, h(r)^{-2} r^2 dr d\Omega)$. Expressed in terms of χ , the differential equation (4.2) takes the form

$$h^4(r) \left(-\frac{\chi''(r)}{\chi(r)} \right) + \frac{h^3(r)}{r} (2h' + rh'') = 1 \tag{4.3}$$

As for the von Neumann-Wigner potential [43] we try the following ansatz (see in particular the discussion in [32], XIII.13):

$$\chi = \sin r \frac{1}{1 + g(r)^2}$$

with

$$g(r) = 2r - \sin 2r = 4 \int_0^r \sin^2 x dx$$

Then

$$-\frac{\chi''(r)}{\chi(r)} = 1 + (1 + g(r)^2)^{-1} (32 \sin r) [g(r)^3 \cos r - 3g(r)^2 \sin^3 r + g(r) \cos r + \sin^3 r]$$

Equation (4.3) is a nonlinear differential equation in $h(r)$ and we look for a solution which is everywhere regular. Since $\left[r \left(-\frac{\chi''}{\chi} \right) \right]_{r=0} = 0$ we must then satisfy the relation

$$h^3(o)h'(o) = 0 \quad (4.4)$$

Therefore the point $r = 0$ is a « singular » point of the differential equation in the sense that $h(o)$ and $h'(o)$ can not be arbitrarily chosen.

On the other hand, any solution with $h'(o) = h''(o) = 0$ satisfies $h(o) = 1$. Also note that $h(r)$ cannot vanish for $r \neq 0$. Hence a real solution h always satisfies the relation $h(r) > 0$ or $h(r) < 0$ for all $r > 0$.

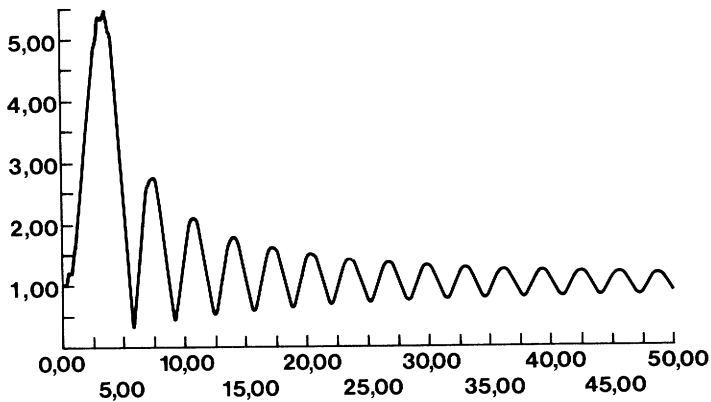


FIG. 1. — $h(r)$.

The computer calculations indicate the existence of a solution with $h(o) = 1$, $h'(o) = h''(o) = 0$ and asymptotic behaviour for which $r(h(r) - 1)$ is bounded. Figure captions 1 and 2 give the plot of $h(r)$ and $r(h(r) - 1)$ versus r for a computation which started at $r_0 = 10^{-6}$ with initial values $h(r_0) = 1$, $h'(r_0) = 0$.

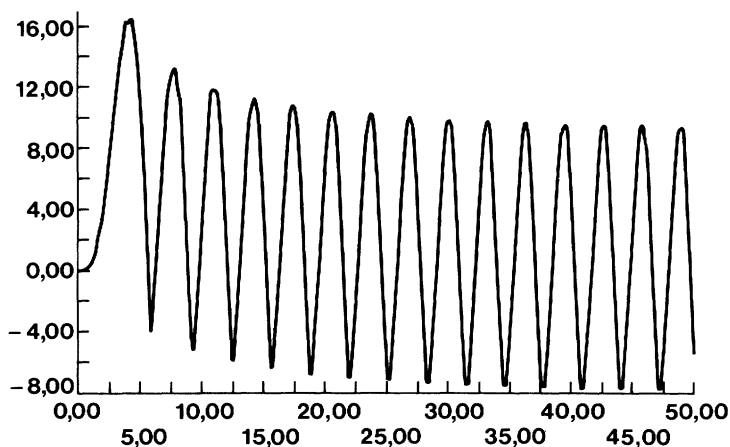


FIG. 2. — $r(h - 1)$.

Moreover, the difficulty with $r = 0$ may be avoided due to the following numerical result. If one starts at an arbitrary $r = r_1 > 0$ with arbitrary $h(r_1) > 0$ and $h'(r_1)$ and integrates out to $r > r_1$, then for the corresponding solution h , $r(h(r) - 1)$ remains bounded. In fact it oscillates. Therefore we may replace the above χ by a function $\bar{\chi}$ which equals the function 1 for $r < \frac{\pi}{4}$ and χ for $r > \frac{\pi}{2}$ and interpolates smoothly and nonvanishingly inbetween. Then the resulting solution h of equation (4.3) with χ being replaced by $\bar{\chi}$ and initial conditions $h(r) = 1$, $h'(r) = 0$ for $r < \frac{\pi}{2}$ has the desired asymptotic behaviour $h(r) = 1 + O(r^{-1})$.

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APPENDIX

In this appendix, we want to prove lemma 3.2. First, with the notation established in § 3, we have

$$\eta_{ij} = \sum_{\mu, \nu=1}^n g_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} \quad (\text{A1})$$

$$\overset{\circ}{\eta}_{ij} = \sum_{\mu, \nu=1}^n \delta_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} = \delta_{ij} \sum_{\mu=1}^n \left(\frac{\partial x^\mu}{\partial y^i} \right)^2$$

Similarly

$$\eta^{ij} = \sum_{\mu, \nu=1}^n g^{\mu\nu} \frac{\partial y^i}{\partial x^\mu} \frac{\partial y^j}{\partial x^\nu} \quad (\text{A2})$$

$$\overset{\circ}{\eta}^{ij} = \sum_{\mu, \nu=1}^n \delta^{\mu\nu} \frac{\partial y^i}{\partial x^\mu} \frac{\partial y^j}{\partial x^\nu} = \delta^{ij} \sum_{\mu=1}^n \left(\frac{\partial y^i}{\partial x^\mu} \right)^2$$

Therefore by Schwarz inequality and the assumption (3.1) on $g_{\mu\nu} - \delta_{\mu\nu}$

$$|\eta_{ij} - \overset{\circ}{\eta}_{ij}| \leq o(1) |\overset{\circ}{\eta}_{ii}|^{\frac{1}{2}} |\overset{\circ}{\eta}_{jj}|^{\frac{1}{2}}$$

which is (α). Since the assumption (3.2) also implies $|g^{\mu\nu} - \delta^{\mu\nu}| \leq o(1)$, A(2) gives

$$|\eta^{ij} - \overset{\circ}{\eta}^{ij}| \leq o(1) |\overset{\circ}{\eta}^{ii}|^{\frac{1}{2}} |\overset{\circ}{\eta}^{jj}|^{\frac{1}{2}}$$

which is (β).

Combined with the relations (3.5) and (3.10), (γ) and (δ) easily follow. Next, by Leibniz rule

$$\begin{aligned} \partial_r(\eta_{ij} - \overset{\circ}{\eta}_{ij}) &= \sum_{\mu, \nu, \tau=1}^n \frac{\partial(g_{\mu\nu} - \delta_{\mu\nu})}{\partial x^\tau} \frac{\partial x^\tau}{\partial r} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} \\ &\quad + \sum_{\mu, \nu} (g_{\mu\nu} - \delta_{\mu\nu}) \left\{ \frac{\partial}{\partial r} \left(\frac{\partial x^\mu}{\partial y^i} \right) \frac{\partial x^\nu}{\partial y^j} + \frac{\partial x^\mu}{\partial y^i} \frac{\partial}{\partial r} \left(\frac{\partial x^\nu}{\partial y^j} \right) \right\} \quad (\text{A3}) \end{aligned}$$

Now

$$\frac{\partial}{\partial r} \frac{\partial x^\mu}{\partial y^i} = \begin{cases} 0 & \text{if } i = n \\ 1 \frac{\partial x^\mu}{\partial y^i} & \text{if } i < n \end{cases}$$

so (A3) easily gives

$$|\partial_r(\eta_{ij} - \overset{\circ}{\eta}_{ij})| \leq o(r^{-1}) |\overset{\circ}{\eta}_{ii}|^{\frac{1}{2}} |\overset{\circ}{\eta}_{jj}|^{\frac{1}{2}}$$

which is (ϵ). Using (A2), (ζ) is proved by similar arguments. Using again the rel. (3.5) and (3.10), (η) then also follows. Next, by a familiar identity concerning the derivative of a determinant

$$\frac{1}{\sqrt{\eta}} \partial_r \sqrt{\eta} = \frac{1}{\sqrt{\overset{\circ}{\eta}}} \partial_r \sqrt{\overset{\circ}{\eta}} - \frac{1}{\sqrt{\overset{\circ}{\eta}}} \partial_r \sqrt{\overset{\circ}{\eta}} = \frac{1}{2} \sum_{i,j=1}^n \eta^{ij} \partial_r \eta_{ij} - \frac{1}{2} \sum_{i,j=1}^n \overset{\circ}{\eta}^{ij} \partial_r \overset{\circ}{\eta}_{ij}$$

Using the already proven estimates (α) - (γ) , this gives (θ) . We turn to a proof of (ι) . Since $\hat{\eta}^{ri} = 0$ ($i < n$), we have

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{1}{\sqrt{\hat{\eta}}} (\partial_r \sqrt{\hat{\eta}} \eta^{ri}) &= \sum_{i=1}^{n-1} \left(\frac{1}{\sqrt{\hat{\eta}}} \partial_r \sqrt{\hat{\eta}} \eta^{ri} - \frac{1}{\sqrt{\hat{\eta}}} (\partial_r \sqrt{\hat{\eta}} \hat{\eta}^{ri}) \right) \\ &= \sum_{i=1}^{n-1} \frac{1}{\sqrt{\hat{\eta}}} (\partial_r \sqrt{\hat{\eta}} \eta^{ri}) - \frac{1}{\sqrt{\hat{\eta}}} (\partial_r \sqrt{\hat{\eta}} \hat{\eta}^{ri}) \\ &\quad - \frac{1}{\sqrt{\hat{\eta}}} (\partial_r \sqrt{\hat{\eta}} \eta^{rr}) + \frac{1}{\sqrt{\hat{\eta}}} (\partial_r \sqrt{\hat{\eta}} \hat{\eta}^{rr}) \end{aligned} \tag{A4}$$

Now the transformation properties of the Christoffel symbols allow us to write the first two terms in (A4) as (see e. g. [44], p. 162-163)

$$\begin{aligned} \sum_{\mu, \nu} (g^{\mu\nu} - \delta^{\mu\nu}) \frac{\partial^2 r}{\partial x^\mu \partial x^\nu} + \sum_{\mu, \nu} \frac{\partial r}{\partial x^\mu} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\nu} \sqrt{g} g^{\mu\nu} \\ = \sum_{\mu, \nu} (g^{\mu\nu} - \delta^{\mu\nu}) \left(\frac{\delta_{\mu\nu}}{r} - \frac{x^\mu x^\nu}{r^3} \right) + \sum_{\mu, \nu} \frac{\partial r}{\partial x^\mu} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\nu} \sqrt{g} g^{\mu\nu} \end{aligned} \tag{A5}$$

Also the last two terms in (A4) may be written as

$$- \frac{\eta^{rr}}{\sqrt{\hat{\eta}}} \partial_r \sqrt{\hat{\eta}} - \partial_r \eta^{rr} - (\eta^{rr} - 1) \frac{1}{\sqrt{\hat{\eta}}} \partial_r \sqrt{\hat{\eta}} \tag{A6}$$

Using (β) , (ζ) , (θ) and assumption (3.2) to estimate (A5) and (A6), this easily leads to the desired estimate (ι) . Finally (σ) follows easily from relation (3.11).

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