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W. O. AMREIN
D. B. PEARSON

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Commutators of Hilbert-Schmidt type and the scattering cross section

by

W. O. AMREIN
Département de Physique Théorique, Université de Genève,
1211 Genève 4, Switzerland

and

D. B. PEARSON (**)
Laboratoire de Physique Théorique et Hautes Énergies (*),
Université de Paris-Sud, 91405 Orsay, France

ABSTRACT. — We present an abstract theory describing pairs of bounded operators in Hilbert space, for which the commutator is Hilbert-Schmidt.
Our results are applied to non-relativistic potential scattering, where we show that, for short range potentials, the cross-section for scattering between disjoint cones is finite and continuous in the energy, implying that cross-sections are finite away from the forward direction. (For potentials decreasing more slowly than $1/r^2$ the *total* cross-section is in general infinite.)

1. INTRODUCTION

Hilbert-Schmidt operators play an important role in Quantum Mechanics in the study of scattering cross-sections, and a number of results may most conveniently be expressed in terms of this class of operators. For example, if $S(\lambda)$ denotes the scattering matrix at energy λ, then the total

(*) Laboratoire associé au Centre National de la Recherche Scientifique.
(**) Permanent Address: Department of Applied Mathematics, University of Hull, Hull (England).

scattering cross-section $\bar{\sigma}(\lambda)$, averaged over all initial directions, is proportional to the Hilbert-Schmidt norm of $R(\lambda) \equiv S(\lambda) - I_0$, I_0 denoting the identity operator in the Hilbert space $\mathscr{H}_0$ describing states at fixed energy λ.

In potential scattering, the Hilbert space is $L^2(\mathbb{R}^3)$ and $S(\lambda)$ acts in $\mathscr{H}_0 \equiv L^2(S^{(2)})$, where $S^{(2)}$ is the unit sphere $S^{(2)} = \{ \underline{\omega} \in \mathbb{R}^3 ; |\underline{\omega}| = 1 \}$. Then λ corresponds to the kinetic energy of a particle, $\underline{\omega}$ to its direction of momentum [*1*]. If $R(\lambda)$ is in the Hilbert-Schmidt class $\mathscr{B}_2(\mathscr{H}_0)$, it is an integral operator in $L^2(S^{(2)})$. Denoting its kernel by $R(\lambda ; \underline{\omega}, \underline{\omega}')$, the scattering amplitude $f(\lambda ; \underline{\omega}' \to \underline{\omega})$ at energy λ from initial direction $\underline{\omega}'$ to final direction $\underline{\omega}$ is just

$$f(\lambda ; \underline{\omega}' \to \underline{\omega}) = -2\pi i \, |\underline{k}|^{-1} R(\lambda ; \underline{\omega}, \underline{\omega}') \tag{1}$$

and the total cross-section is given by

$$\bar{\sigma}(\lambda) = \pi \, |\underline{k}|^{-2} (\| S(\lambda) - I_0 \|_2)^2 . \tag{2}$$

Here, $\| A \|_2$ denotes the Hilbert-Schmidt norm of an operator A, and $|\underline{k}|$ is the length of a wave vector corresponding to energy λ (for non-relativistic kinematics $\lambda = \hbar^2 \underline{k}^2/2m$, i. e. $|\underline{k}| = \hbar^{-1}(2m\lambda)^{1/2}$, whereas more generally one has $\lambda = \phi(|\underline{k}|)$, $|\underline{k}| = \phi^{-1}(\lambda)$, where ϕ is an increasing function and ϕ^{-1} its inverse).

To derive (1) from the definition of the scattering operator in $\mathscr{H}$, one considers an ensemble of independent scattering events. The initial states of the ensemble are obtained from a fixed state f, which is a wave-packet of almost sharp momentum $\underline{p}_0$, by translating f by vectors $\underline{a}$, the values of which are uniformly distributed over a plane orthogonal to $\underline{p}_0$. A proof that (1) holds for almost all λ in some interval Δ is given in [*1*, Section 7-3] under the assumption that

$$\int_\Delta d\lambda \, |\phi^{-1}(\lambda)|^{-2} (\| R(\lambda) \|_2)^2 < \infty. \tag{3}$$

In view of (2), this assumption will not be satisfied for potentials with infinite total cross-section $\bar{\sigma}(\lambda)$. Now $\bar{\sigma}(\lambda)$ is known to be finite if $V(\underline{r})$ decreases at infinity more rapidly than $|\underline{r}|^{-2}$ [*1*, *2*], but for $V(\underline{r}) = \gamma \, |\underline{r}|^{-2}$ and for potentials decreasing more slowly one has in general $\bar{\sigma}(\lambda) = \infty$ [*3*]. For potentials of the latter type one may then ask how the scattering amplitude may be related to the scattering operator.

For reasonable potentials, one expects the non finiteness of $\bar{\sigma}(\lambda)$ to result from a strong singularity of the scattering amplitude in the forward direction. In Hilbert space language, this would mean that, although $R(\lambda)$ is not itself in the Hilbert-Schmidt class, $G_2 R(\lambda) G_1$ is Hilbert-Schmidt if G_1 and G_2 are suitable projection operators. We have in mind the case where G_i projects onto the subspace of states, at energy λ, having direction of momentum in a subset Θ_i of $S^{(2)}$, such that Θ_1 contains in its interior the direction of $\underline{p}_0$ and Θ_2 is disjoint from Θ_1. Each Θ_i determines a cone C_i with vertex at the origin. The derivation of the expression for the cross-section for scattering into C_2 with initial states f having momentum support in C_1 [*1*, Section 7-3] is valid without modification, provided (3) holds with $G_2R(\lambda)G_1$ instead of $R(\lambda)$. Moreover, the kernel $R(\lambda ; \omega_2, \omega_1)$ of the integral operator $G_2R(\lambda)G_1$ determines the scattering amplitude for $\omega_1 \in C_1$, $\omega_2 \in C_2$, as in eq. (1). If $G_2R(\lambda)G_1$ is Hilbert-Schmidt for all closed cones C_1, C_2 with $C_1 \cap C_2 = \{ 0 \}$, then the scattering amplitude $f(\lambda ; \underline{\omega}' \to \underline{\omega})$ is defined for almost all $\underline{\omega}$, $\underline{\omega}'$.

The purpose of this paper is to prove $G_2R(\lambda)G_1 \in \mathscr{B}_2(\mathscr{H}_0)$ if $C_1 \cap C_2 = \{ 0 \}$, as well as continuity in λ in $\| \cdot \|_2$ of $G_2R(\lambda)G_1$, for a class of slowly decreasing short-range potentials, and with non-relativistic kinematics. Our main result concerning cross-sections is given by Theorem 5. The proof uses an explicit expression for $R(\lambda)$ from time-independent scattering theory. The idea is to commute G_2 through $R(\lambda)$ in order to make use of the property $G_2G_1 = 0$. As a preliminary step we shall theorefore, in Section 2, study commutators of Hilbert-Schmidt type in an abstract setting. In particular, we obtain an abstract characterisation of the conditions under which estimates in $\mathscr{B}_2$ of commutators with a given bounded operator Φ may be expressed in terms of the corresponding estimates for another bounded operator T which is related to Φ.

The problem of estimating the behaviour of scattering cross-sections away from the forward direction for potentials of the generality considered here appears to have received little previous attention in the literature. We may, however, cite the work of Agmon [*4*], who, by different methods, is also able to treat the case of long range potentials. In the short range case, Agmon's methods place more stringent conditions on the class of potentials, while leading to more detailed conclusions on the behaviour of the scattering amplitude. We are grateful to A.-M. Berthier for drawing our attention to the results of Agmon.

2. COMMUTATORS OF HILBERT-SCHMIDT TYPE

Let $\mathscr{H}$ be a separable complex Hilbert space. We shall denote by $\mathscr{B}_2(\mathscr{H})$ the Hilbert space of all Hilbert-Schmidt operators from $\mathscr{H}$ to $\mathscr{H}$, with inner product

$$\langle A, B \rangle = \mathrm{tr}\,(A^*B). \tag{4}$$

We shall need the following simple results:

(H.S.1): Let $\{ E_i \}$ be a family of orthogonal projections satisfying $E_iE_j = \delta_{ij}E_i$, $\Sigma E_i = I$.

Then, for $A \in \mathscr{B}_2(\mathscr{H})$,

$$(\| A \|_2)^2 = \sum_{i,j} (\| E_iAE_j \|_2)^2 .$$

(H.S.2): Let $\{ B_n \}$ be a family of bounded self-adjoint operators from $\mathscr{H}$ to $\mathscr{H}$ such that $B_n \to B$ strongly as $n \to \infty$. Then, for $A \in \mathscr{B}_2(\mathscr{H})$,

(α) $AB_n \to AB$, $B_nA \to BA$ in $\| \cdot \|_2$, and hence, for example,

(β) $B_nAB_n \to BAB$ in $\| \cdot \|_2$.

(See [*1*], Lemma 8.23).

Given a bounded self-adjoint operator T from $\mathscr{H}$ to $\mathscr{H}$, define bounded operators $l(T)$, $r(T)$, $c(T)$ from $\mathscr{B}_2(\mathscr{H})$ to $\mathscr{B}_2(\mathscr{H})$ by

$$\left.\begin{aligned} l(T)A &= TA \\ r(T)A &= AT \\ c(T)A &= (l(T) - r(T))A = [T, A] \end{aligned}\right\} \tag{5}$$

It is straightforward to verify, using the inner product (4), that the operators defined by (5) are self-adjoint. Moreover, if T has purely continuous spectrum then so does $c(T)$. For suppose $c(T)$ had an eigenvalue λ_0, so that $c(T)A = \lambda_0 A$ for some $A \neq 0$. Then, $T^nA = A(T + \lambda_0)^n$, so that, for $f \in \mathscr{H}$,

$$\exp(i\beta T)Af = A \exp(i\beta(T + \lambda_0))f. \tag{6}$$

Since $(T + \lambda_0)$ has continuous spectrum, for each f one may find a sequence $\{ \beta_n \}$ such that $\beta_n \to \infty$ and $\exp(i\beta_n(T + \lambda_0))f$ converges weakly to zero. Since A is compact, setting $\beta = \beta_n$ the r. h. s. of (6) converges strongly to zero. But this is a contradiction, since $\| e^{i\beta_n T}Af \| = \| Af \| \neq 0$ for some f.

We shall henceforth assume that T has purely continuous spectrum. In that case, certainly zero cannot be an eigenvalue of $c(T)$, so that the self-adjoint operator $[c(T)]^{-1}$ may be defined. Since, as we shall show later, $[c(T)]^{-1}$ is necessarily unbounded, it will be useful to define subspaces of $\mathscr{B}_2$ which lie in the domain of $[c(T)]^{-1}$. With this in mind, let $\mathscr{M}(\Delta_1, \Delta_2)$ be the set of all Hilbert-Schmidt operators of the form $E(\Delta_1)AE(\Delta_2)$, where Δ_1, Δ_2 are non-intersecting open intervals, $E(\Delta)$ is the spectral projection of T for the interval Δ, and $A \in \mathscr{B}_2(\mathscr{H})$. Then $\mathscr{M}(\Delta_1, \Delta_2)$ is a closed subspace of $\mathscr{B}_2(\mathscr{H})$ and we have

LEMMA 1. — *i*) Let $\mathscr{M}$ be the closed linear manifold spanned by the $\mathscr{M}(\Delta_1, \Delta_2)$ as Δ_1, Δ_2 are varied (always with $\Delta_1 \cap \Delta_2 = \emptyset$).

Then $\mathscr{M} = \mathscr{B}_2(\mathscr{H})$.

ii) $\mathscr{M}(\Delta_1, \Delta_2) \subset \mathscr{D}([c(T)]^{-1})$ if dist. $(\Delta_1, \Delta_2) > 0$, and

$$\| [c(T)]^{-1} \restriction \mathscr{M}(\Delta_1, \Delta_2) \| \leqslant (\text{dist.}\ (\Delta_1, \Delta_2))^{-1}. \tag{7}$$

Proof. — *i*) Let Δ be a finite interval, and consider a sequence $\{ \Pi^{(n)} \}$ of partitions of Δ, each $\Pi^{(n)}$ being a set of disjoint open intervals $\Delta_i^{(n)}$ ($i = 1, 2, \ldots, k(n)$) such that $\cup_{i=1}^{k(n)} \overline{\Delta_i^{(n)}} = \Delta$. Choose the sequence such that each $\Delta_i^{(n+1)}$ is a subinterval of some $\Delta_j^{(n)}$, and such that the maximum length $| \Pi^{(n)} |$, as i varies, of the $\Delta_i^{(n)}$, converges to zero as n tends to infinity.

For $A \in \mathscr{B}_2(\mathscr{H})$, we have

$$E(\Delta)AE(\Delta) = \lim_{n\to\infty} \sum_{i\neq j} E(\Delta_i^{(n)})AE(\Delta_j^{(n)}) + \lim_{n\to\infty} \sum_{i} E(\Delta_i^{(n)})AE(\Delta_i^{(n)}),$$

so that, to prove *i*) of the Lemma, we have only to show that the second sum converges to zero in $\| \cdot \|_2$.

Since

$$\left\| \sum_i E(\Delta_i^{(n)})(A - B)E(\Delta_i^{(n)}) \right\|_2 \leqslant \| A - B \|_2,$$

and since operators of rank one span a dense linear manifold in $\mathscr{B}_2(\mathscr{H})$, it is only necessary to show this in the case $A = \langle g, \cdot \rangle f$. But, in that case,

$$\left\| \sum_i E(\Delta_i^{(n)})AE(\Delta_i^{(n)}) \right\|_2^2 = \sum_i \| E(\Delta_i^{(n)})f \|^2 \cdot \| E(\Delta_i^{(n)})g \|^2 \leqslant \max_i . \| E(\Delta_i^{(n)})f \|^2 \cdot \| E(\Delta)g \|^2 \to 0$$

as $n \to \infty$ since T has purely continuous spectrum.

ii) Note that $c(T)$ leaves $\mathscr{M}(\Delta_1, \Delta_2)$ invariant. Assume dist. $(\Delta_1, \Delta_2) > 0$. Taking partitions $\{ \Pi_1^{(n)} \}$, $\{ \Pi_2^{(n)} \}$ of Δ_1, Δ_2 respectively we have, for $A \in \mathscr{B}_2(\mathscr{H})$ and using (H.S.2),

$$c(T)E(\Delta_1)AE(\Delta_2) = \lim_{n\to\infty} \sum_{i,j} (\lambda_{i,1}^{(n)} - \lambda_{j,2}^{(n)})E(\Delta_{i,1}^{(n)})AE(\Delta_{j,2}^{(n)}) \tag{8}$$

in $\| \cdot \|_2$, where $\lambda_{i,1}^{(n)} \in \Delta_{i,1}^{(n)}$ and $\lambda_{j,2}^{(n)} \in \Delta_{j,2}^{(n)}$. Hence for $A \in \mathscr{M}(\Delta_1, \Delta_2)$ we have

$$(\| c(T)A \|_2)^2 = \lim_{n\to\infty} \sum_{i,j} (\lambda_{i,1}^{(n)} - \lambda_{j,2}^{(n)})^2 \| E(\Delta_{i,1}^{(n)})AE(\Delta_{j,2}^{(n)}) \|_2^2 \geqslant (\text{dist.}\ (\Delta_1, \Delta_2))^2 (\| A \|_2)^2 . \tag{9}$$

Now $(c(T) \restriction \mathscr{M}(\Delta_1, \Delta_2))^{-1}$ is self-adjoint. Since this operator is therefore closed, and from (9) is bounded, it follows that the operator is defined on the whole of $\mathscr{M}(\Delta_1, \Delta_2)$. Eq. (7) now follows.

Remark 1. — A similar argument shows that $\| c(T) \restriction \mathscr{M}(\Delta, \Delta) \| \leqslant | \Delta |$, (length of Δ), and one may verify that the spectrum of $c(T)$ is just

$$\sigma(c(T)) = \{ \lambda - \mu \,;\, \lambda, \mu \in \sigma(T) \}.$$

In particular, zero lies in $\sigma(c(T))$, so that $[c(T)]^{-1}$ is unbounded.

One of the main purposes of the present paper is to estimate, in $\| \cdot \|_2$, commutators with a second bounded self-adjoint operator Φ in terms of commutators with T, with the idea that the latter may be simpler to evaluate. In what circumstances can we use this method to estimate commutators with Φ? A natural mathematical expression of this relation between commutators is to suppose that $c(\Phi)[c(T)]^{-1}$ is bounded. This will be so provided

$$\| [\Phi, A] \|_2 \leqslant K \| [T, A] \|_2 \tag{10}$$

for all $A \in \mathscr{B}_2(\mathscr{H})$ and for some $K > 0$. We shall then say that $c(\Phi)$ is $c(T)$-bounded. That this is a very restrictive condition on Φ is shown by the following

THEOREM 1. — Let Φ, T be bounded self-adjoint operators in $\mathscr{H}$, let T have purely continuous spectrum, and suppose that $c(\Phi)$ is $c(T)$-bounded. Then $\Phi = \phi(T)$ (function of T) where the function ϕ satisfies, for $\lambda, \mu \in \sigma(T)$,

$$| \phi(\lambda) - \phi(\mu) | \leqslant \text{const} \, | \lambda - \mu | . \tag{11}$$

Furthermore, $c(\Phi)$ and $c(T)$ commute.

Proof. — *i*) Let $E(\Delta) \neq 0$ be the spectral projection of T for a finite interval Δ. Take f in the range of $E(\Delta)$, $\| f \| = 1$, and set $A = \langle f, \cdot \rangle f$ in (10).

With $\lambda \in \Delta$, and noting $\| (T - \lambda)E(\Delta) \| \leqslant | \Delta |$ (as operator norm) we have

$$\| [T, A] \|_2 = \| [(T - \lambda), A] \|_2 \leqslant 2 | \Delta | . \tag{12}$$

Hence (10) implies

$$\| [\Phi, A] \|_2 \leqslant 2K | \Delta | . \tag{12)'$$

$$\text{But } \| [\Phi, A]f \| \leqslant \| [\Phi, A] \|_2 ,$$

so that (12)' implies

$$\| \Phi f - \langle f, \Phi f \rangle f \| \leqslant 2K | \Delta | . \tag{13}$$

We write (13) in the form

$$\| \Phi f - \beta f \| \leqslant 2K | \Delta | . \tag{13)'$$

Note that β depends on f, and $| \beta | \leqslant \| \Phi \|$. We shall show that an inequality like (13)' holds (but with a slightly different bound on the r. h. s.) with a constant β which is independent of f in the range of $E(\Delta)$, but which depends on Δ.

ii) Let $f_1, f_2 \in$ range $(E(\Delta))$, with $\| f_1 \| = \| f_2 \| = 1$, and $\langle f_1, f_2 \rangle = 0$. (Note range $(E(\Delta))$ cannot have dimension 1.)

Then

$$\left.\begin{aligned} \| \Phi f_1 - \beta_1 f_1 \| &\leqslant 2K | \Delta | \\ \| \Phi f_2 - \beta_2 f_2 \| &\leqslant 2K | \Delta | \\ \| \Phi(f_1 + f_2) - \beta_{12}(f_1 + f_2) \| &\leqslant 2\sqrt{2}K | \Delta | \end{aligned}\right\} \tag{14}$$

Eqs. (14) imply

$$\| (\beta_1 - \beta_{12})f_1 + (\beta_2 - \beta_{12})f_2 \| \leqslant (4 + 2\sqrt{2})K \,|\Delta|\,;$$

since $\langle f_1, f_2 \rangle = 0$, this gives

$$|\beta_1 - \beta_{12}| \leqslant (4 + 2\sqrt{2})K\,|\Delta|$$

$$|\beta_2 - \beta_{12}| \leqslant (4 + 2\sqrt{2})K\,|\Delta|.$$

$$\therefore \quad \| \Phi f_2 - \beta_1 f_2 \| \leqslant (10 + 4\sqrt{2})K\,|\Delta|.$$

Taking fixed $f_1 \in$ range $(E(\Delta))$, any $f \in$ range $(E(\Delta))$ with $\| f \| = 1$ may be represented as $f = c_1 f_1 + c_2 f_2$ for some $f_2 \perp f_1$, $\| f_2 \| = 1$, with $c_1^2 + c_2^2 = 1$. In that case we find $\| \Phi f - \beta_1 f \| \leqslant (12 + 4\sqrt{2})K\,|\Delta|$, where the constant on the r. h. s. is by no means optimal!

Here β_1 is independent of f. For some $c > 0$ we have shown that, for any Δ such that $E(\Delta) \neq 0$, we can find $\beta(\Delta)$ (non-unique) such that

$$\| \Phi f - \beta(\Delta) f \| \leqslant cK\,|\Delta|\,\| f \| \tag{15}$$

for all $f \in$ range $(E(\Delta))$.

iii) Take now two such intervals Δ_a, Δ_b, which may overlap, with $\Delta_a \subset \Delta$, $\Delta_b \subset \Delta$.

For $f \in$ range $(E(\Delta_a)) \subset$ range $E(\Delta)$ we have both

$$\| \Phi f - \beta(\Delta) f \| \leqslant cK\,|\Delta|\,\| f \|$$

and

$$\| \Phi f - \beta(\Delta_a) f \| \leqslant cK\,|\Delta_a|\,\| f \| \leqslant cK\,|\Delta|\,\| f \|.$$

It follows that $|\beta(\Delta) - \beta(\Delta_a)| \leqslant 2cK\,|\Delta|$, with a similar estimate for Δ_b. Hence $\Delta_a, \Delta_b \subset \Delta \Rightarrow$

$$|\beta(\Delta_a) - \beta(\Delta_b)| \leqslant 4cK\,|\Delta|. \tag{16}$$

Now take $\lambda \in \sigma(T)$ and a decreasing sequence $\Delta_1 \supset \Delta_2 \supset \Delta_3 \ldots$ such that $\lambda \in \Delta_n$ and $\lim_{n\to\infty} |\Delta_n| = 0$. Since

$$|\beta(\Delta_m) - \beta(\Delta_n)| \leqslant 4cK\,|\Delta_m| \qquad (m < n) \tag{16}'$$

and hence converges to zero as $m, n \to \infty$, we can define a function ϕ by

$$\phi(\lambda) = \lim_{n\to\infty} \beta(\Delta_n). \tag{17}$$

Eq. (16) implies that $\phi(\lambda)$ is independent of the particular sequence of intervals converging to λ, and also that

$$|\phi(\lambda) - \phi(\mu)| \leqslant 4cK\,|\lambda - \mu|. \tag{18}$$

Eq. (16)′ implies that

$$|\beta(\Delta) - \phi(\lambda)| \leqslant 4cK\,|\Delta| \tag{18}'$$

for $\lambda \in \Delta \cap \sigma(T)$.

Taking a sequence $\{\Pi^{(n)}\}$ of partitions of a given interval Δ, with $\lim_{n\to\infty} |\Pi^{(n)}| = 0$, we have

$$\Phi E(\Delta)f = \sum_i \Phi E(\Delta_i^{(n)})f = s\text{-}\lim_{n\to\infty} \sum_i \beta(\Delta_i^{(n)})E(\Delta_i^{(n)})f$$

on using (15), since

$$\sum_i |\Delta_i^{(n)}| \, \| E(\Delta_i^{(n)})f \| \leqslant |\Delta| \max_i \| E(\Delta_i^{(n)})f \| \to 0$$

as $n \to \infty$. Hence

$$\begin{aligned} \| \Phi E(\Delta)f - \phi(T)E(\Delta)f \|^2 &= \lim_{n\to\infty} \left\| \sum_i (\beta(\Delta_i^{(n)}) - \phi(\lambda_i^{(n)}))E(\Delta_i^{(n)})f \right\|^2 (\lambda_i^{(n)} \in \Delta_i^{(n)}) \\ &= \lim_{n\to\infty} \sum_i |\beta(\Delta_i^{(n)}) - \phi(\lambda_i^{(n)})|^2 \, \| E(\Delta_i^{(n)})f \|^2 \\ &\leqslant \lim_{n\to\infty} (4cK |\Pi^{(n)}| \, \| f \|)^2 = 0 , \end{aligned}$$

on using (18)'. Thus $\Phi = \phi(T)$. The commutativity of $c(\Phi)$ and $c(T)$ is a simple algebraic consequence of the definition of these operators and the fact that $[\Phi, T] = 0$.

Remark 2. — The domain of definition of ϕ may be extended to the entire real line while preserving the inequality (11). $\mathbb{R} \setminus \sigma(T)$ is a countable union of disjoint open intervals (a_n, b_n). If $a_n, b_n \in \sigma(T)$ define ϕ to be linear on $[a_n, b_n]$, whereas if $b_n = +\infty$ take $\phi(\lambda) = \phi(a_n)$ for $\lambda \geqslant a_n$ (similarly if $a_n = -\infty$).

The converse to Theorem 1 also holds, namely

THEOREM 2. — Let $\Phi = \phi(T)$, where $|\phi(\lambda) - \phi(\mu)| \leqslant K |\lambda - \mu|$. Then, (10) holds for all $A \in \mathscr{B}_2(\mathscr{H})$.

Proof. — Take an interval Δ such that $E(\Delta) = I$, together with a sequence $\{\Pi^{(n)}\}$ of partitions of Δ, satisfying $\lim_{n\to\infty} |\Pi^{(n)}| = 0$.

Then using (H.S.1) we have

$$\| [T, A] \|_2 = \sum_{i,j} \| [T, E(\Delta_i^{(n)})AE(\Delta_j^{(n)})] \|_2^2 . \tag{19}$$

From (H.S.2), as in the proof of (8), we have

$$\lim_{n\to\infty} \sum_{i,j} (T - \lambda_i^{(n)})E(\Delta_i^{(n)})AE(\Delta_j^{(n)}) = 0 \quad \text{in} \quad \| \cdot \|_2$$

for $\lambda_i^{(n)} \in \Delta_i^{(n)}$, and similarly

$$\lim_{n\to\infty} \sum_{i,j} E(\Delta_i^{(n)})AE(\Delta_j^{(n)})(T - \lambda_j^{(n)}) = 0 \qquad \text{in} \quad \|\cdot\|_2 .$$

Hence

$$\| [T, A] \|_2^2 = \lim_{n\to\infty} \sum_{i,j} (\lambda_i^{(n)} - \lambda_j^{(n)})^2 \, \| E(\Delta_i^{(n)})AE(\Delta_j^{(n)}) \|_2^2 .$$

Using (11), an exactly similar result applies to $\| [\phi(T), A] \|_2$, with $\lambda_i^{(n)} - \lambda_j^{(n)}$ replaced by $\phi(\lambda_i^{(n)}) - \phi(\lambda_j^{(n)})$. A further application of (11) now gives (10).

Frequently the operators with which we commute T are bounded rather than Hilbert-Schmidt. We then need the following result:

THEOREM 3. — Suppose $c(\Phi)[c(T)]^{-1}$ is bounded and that $[T, W] \in \mathscr{B}_2(\mathscr{H})$ for some bounded operator W.

Then $[\Phi, W] \in \mathscr{B}_2(\mathscr{H})$.

Proof. — $c(\Phi)[c(T)]^{-1}$ may be extended by continuity onto the whole of $\mathscr{B}_2(\mathscr{H})$. Since $c(\Phi)$, $c(T)$ commute and $c(T)$ is self-adjoint, this extension is just

$$(c(\Phi)[c(T)]^{-1})^* = [c(T)]^{-1}c(\Phi) .$$

Now let

$$A = [c(T)]^{-1}c(\Phi)[T, W] \in \mathscr{B}_2(\mathscr{H}) . \tag{20}$$

Then $[T, A] = [\Phi, [T, W]] = [T, [\Phi, W]]$. We would like to conclude that $A = [\Phi, W]$.

Take two finite disjoint intervals Δ_1, Δ_2 and let

$$Z_{12} = E(\Delta_1)(A - [\Phi, W])E(\Delta_2) .$$

Then Z_{12} commutes with T, and hence with $E(\Delta_1)$. It follows that $Z_{12} = E(\Delta_1)Z_{12} = Z_{12}E(\Delta_1) = 0$.

I. e.
$$E(\Delta_1)AE(\Delta_2) = E(\Delta_1)[\Phi, W]E(\Delta_2). \tag{21}$$

Taking an interval Δ with $E(\Delta) = I$, the proof of Lemma 1 shows that

$$A = \lim_{n\to\infty} \sum_{i\neq j} E(\Delta_i^{(n)})AE(\Delta_j^{(n)}) \tag{22}$$

for a sequence $\{ \Pi^{(n)} \}$ of partitions of Δ, the limit being in $\|\cdot\|_2$. In particular, (22) holds as a strong limit in $\mathscr{H}$, and in view of (21), in order to prove $A = [\Phi, W]$, it suffices to show that

$$\lim_{n\to\infty} \left\| \sum_i E(\Delta_i^{(n)})[\Phi, W]E(\Delta_i^{(n)})f \right\|^2 = 0 \tag{23}$$

for each $f \in \mathscr{H}$. Taking $\lambda_i^{(n)} \in \Delta_i^{(n)}$, the l. h. s. of (23) is

$$\left\| \sum_i E(\Delta_i^{(n)})[\Phi - \phi(\lambda_i^{(n)}), W]E(\Delta_i^{(n)})f \right\|^2$$

$$= \sum_i \| E(\Delta_i^{(n)})[\Phi - \phi(\lambda_i^{(n)}), W]E(\Delta_i^{(n)})f \|^2$$

$$\leqslant 2 \sum_i \| (\Phi - \phi(\lambda_i^{(n)}))E(\Delta_i^{(n)}) \|^2 \| W \|^2 \| E(\Delta_i^{(n)})f \|^2$$

$$\leqslant 2K^2 | \Pi^{(n)} |^2 \| f \|^2 \| W \|^2$$

(using (11)), $\rightarrow 0$ as $n \rightarrow \infty$.

We may deduce that $A = [\Phi, W] \in \mathscr{B}_2(\mathscr{H})$, which concludes the proof of the theorem.

Remark 3. — Using $\| [\Phi, W] \|_2 \leqslant \| c(\Phi)[c(T)]^{-1} \| \cdot \| [T, W] \|_2$, it follows that, if $\{ W^{(\lambda)} \}$ is a one parameter family of bounded operators from $\mathscr{H}$ to $\mathscr{H}$, and if $[T, W^{(\lambda)}]$ is continuous in $\| \cdot \|_2$ as λ is varied, then $[\Phi, W^{(\lambda)}]$ will also be continuous.

In applying these results, one has an additional structure represented by a given self-adjoint operator, usually the free Hamiltonian H_0.

We assume that H_0 is absolutely continuous, has constant spectral multiplicity, and commutes with T (hence also with $\phi(T)$). We then have the representation

$$\mathscr{H} = L^2(\Lambda, \mathscr{H}_0), \tag{24}$$

Λ being some interval (more generally some Borel set).

Thus for $f \in \mathscr{H}$ we have the representation $f \leftrightarrow \{ f_\lambda \}$ for $\lambda \in \Lambda$, where $f_\lambda \in \mathscr{H}_0$ is defined for almost all λ and

$$\langle f, g \rangle = \int_\Lambda \langle f_\lambda, g_\lambda \rangle d\lambda$$

(the first inner product in $\mathscr{H}$, the second in $\mathscr{H}_0$). We then have, for $f \in \mathscr{D}(H_0)$, $(H_0 f)_\lambda = \lambda f_\lambda$. Since $[T, H_0] = 0$, there is a family $T(\lambda)$ of bounded self-adjoint operators from $\mathscr{H}_0$ to $\mathscr{H}_0$, such that

$$(Tf)_\lambda = T(\lambda) f_\lambda \qquad \text{(almost all } \lambda). \tag{25}$$

We shall assume that, in some open subinterval $\tilde{\Lambda}$ of Λ, the family $T(\lambda)$ may be defined for each value of λ in such a way that $T(\lambda)$ is strongly continuous in λ. We suppose also that each $T(\lambda)$ has purely continuous spectrum.

We may also represent operators A in $\mathscr{B}_2(\mathscr{H})$ by families of operators $\{ A_\lambda \}$ where each A_λ is in $\mathscr{B}_2(\mathscr{H}, \mathscr{H}_0)$, the space of Hilbert-Schmidt operators from $\mathscr{H}$ to $\mathscr{H}_0$. This representation is described in the following Lemma:

LEMMA 2. — Let $\mathscr{H}$, $\mathscr{H}_0$ be Hilbert spaces with $\mathscr{H} = L^2(\Lambda, \mathscr{H}_0, d\mu)$. Let $\mathscr{B}_0 = \mathscr{B}_2(\mathscr{H}, \mathscr{H}_0)$ and $\mathscr{B} = L^2(\Lambda ; \mathscr{B}_0, d\mu)$.

(Notice $\mathscr{B}$ is a Hilbert space; for each $A \in \mathscr{B}$, with $A \leftrightarrow \{ A_\lambda \}$, we may define an operator from $\mathscr{H}$ to $\mathscr{H}$ by $(Af)_\lambda = A_\lambda f$, $f \in \mathscr{H}$).

Then $\mathscr{B} = \mathscr{B}_2(\mathscr{H})$.

Proof. — Closely follows the standard proof that Hilbert-Schmidt operators in an L^2 space may be identified with integral operators. See [*5*], [*6*].

Remark 4. — If $\Phi = \phi(T)$, where the function ϕ satisfied (11), just as in (25) we may define a family $\{ \Phi(\lambda) \}$ of operators from $\mathscr{H}_0$ to $\mathscr{H}_0$, where $\Phi(\lambda)$ may be identified with $\phi(T(\lambda))$. To see this, let Δ be a finite interval such that $E(\Delta) = I$, and observe that, by virtue of Remark 2, ϕ may be uniformly approximated on Δ by a sequence $\{ P_n \}$ of polynomials. Then $P_n(T) \to \phi(T)$ and $P_n(T(\lambda)) \to \phi(T(\lambda))$, both in the uniform operator topology, and the identification of $\Phi(\lambda)$ with $\phi(T(\lambda))$ may be deduced from the corresponding result for polynomials. Similarly, since the convergence of $P_n(T(\lambda))$ to $\Phi(\lambda)$ is uniform in λ (in the uniform operator topology), and since $P_n(T(\lambda))$ is strongly continuous in λ for each n, it follows that $\Phi(\lambda)$ is strongly continuous in λ (for $\lambda \in \tilde{\Lambda}$).

Remark 5. — For clarity we summarise the notations for families of operators:

$$\left.\begin{array}{l} B(\lambda) \text{ denotes a bounded operator from } \mathscr{H}_0 \text{ to } \mathscr{H}_0 \\ B_\lambda \text{ denotes a Hilbert-Schmidt operator from } \mathscr{H} \text{ to } \mathscr{H}_0 \\ B^{(\lambda)} \text{ denotes a bounded operator from } \mathscr{H} \text{ to } \mathscr{H} \end{array}\right\}$$

Henceforth we shall use the notation $\mathscr{B}_0 = \mathscr{B}_2(\mathscr{H}, \mathscr{H}_0)$. Later, in Section 3, we shall introduce the further notion of constructing, for a suitable class of operators A from $\mathscr{H}$ to $\mathscr{H}$, a λ-dependent family of operators from $\mathscr{H}$ to $\mathscr{H}_0$, which are not necessarily Hilbert-Schmidt.

For each $\lambda \in \tilde{\Lambda}$, define a bounded self-adjoint operator from $\mathscr{B}_0$ to $\mathscr{B}_0$ by

$$c_0(T ; \lambda)A_0 = T(\lambda)A_0 - A_0T \tag{26}$$

for $A_0 \in \mathscr{B}_0$. Since $T(\lambda)$ is strongly continuous in λ, so is $c_0(T ; \lambda)$. (Meaning that $c_0(T ; \lambda)A_0$ is continuous in $\| \cdot \|_2$ for each A_0, by (H.S.*2*)). The $c_0(T ; \lambda)$ have purely continuous spectrum (c. f. the argument following eq. (6), which becomes

$$\exp(i\beta T(\lambda))A_0 = A_0 \exp(i\beta(T + \lambda_0)).)$$

Similarly $c_0(\Phi ; \lambda)$ is defined by replacing T by Φ in (26), and is also strongly continuous in λ. Moreover, $c_0(\Phi ; \lambda)$ commutes with $c_0(T ; \lambda)$.

Since $(Af)_\lambda = A_\lambda f$ $(f \in \mathscr{H})$, and using (25), we may verify that, for each $A \in \mathscr{B}_2(\mathscr{H})$ and for almost all λ,

$$(c(T)A)_\lambda = c_0(T\,;\lambda)A_\lambda\,. \tag{27}$$

We can now state

THEOREM 4. — In addition to the assumptions of Theorem 3, suppose that

i) $T(\lambda)$ is strongly continuous in λ $(\lambda \in \tilde{\Lambda})$, and each $T(\lambda)$ has purely continuous spectrum.

ii) The family $[T, W]_\lambda$, corresponding to the Hilbert-Schmidt operator $[T, W]$ by virtue of Lemma 2, is continuous in λ in $\| \cdot \|_2$ (Hilbert-Schmidt norm in $\mathscr{B}_0$); or, more precisely, suppose there exists a continuous family, denoted by $[T\,; W]_\lambda$ such that, for each $f \in \mathscr{H}$ and for almost all $\lambda \in \tilde{\Lambda}$, $([T, W]f)_\lambda = [T, W]_\lambda f$.

Then the family $[\Phi, W]_\lambda$ corresponding to $[\Phi, W]$ (again through Lemma 2) is also continuous in λ in $\| \cdot \|_2$, for $\lambda \in \tilde{\Lambda}$.

Proof. — *i*) Suppose $\| c(\Phi)[c(T)]^{-1} \| \leqslant K$. Then

$$\| c_0(\Phi\,;\lambda)[c_0(T\,;\lambda)]^{-1} \| \leqslant K \qquad (\lambda \in \tilde{\Lambda})\,.$$

For suppose $\| c_0(\Phi, \lambda_1)A_0 \|_2 > K \| c_0(T\,;\lambda_1)A_0 \|_2$ for some $A_0 \in \mathscr{B}_0$ and some $\lambda_1 \in \tilde{\Lambda}$. By continuity, this inequality must hold in some neighbourhood N of λ_1. Defining a family $\{ A_\lambda \}$ by

$$\begin{aligned} A_\lambda &= A_0\,; &\quad& \lambda \in N \\ &= 0\,; && \text{otherwise}, \end{aligned}$$

let A be the associated Hilbert-Schmidt operator in $\mathscr{B}_2$, by Lemma 2. Then, using (27),

$$\| c(\Phi)A \|_2^2 = \int_{\tilde{\Lambda}} \| c_0(\Phi, \lambda)A_0 \|_2^2 d\lambda > K^2 \| c(T)A \|_2^2\,,$$

which is in contradiction with the bound on $c(\Phi)[c(T)]^{-1}$.

ii) Next we prove that $[c_0(T\,;\lambda)]^{-1}c_0(\Phi\,;\lambda)$ is strongly continuous in λ. In fact, given $\lambda \in \tilde{\Lambda}$, $B_0 \in \mathscr{B}_0$, and $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\| E_\delta(\lambda)B_0 \|_2 < \varepsilon\,, \tag{28}$$

where $E_\delta(\lambda)$ is the spectral projection of $c_0(T\,;\lambda)$ for the interval $[-\delta, \delta]$. Since (by [7], p. 432) $E_\delta(\lambda)$ is strongly continuous in λ, we can choose δ such that (28) holds in a neighbourhood of λ. The continuity in λ of the spectral projections of $c_0(T\,;\lambda)$ also implies the continuity of a class of functions of $c_0(T\,;\lambda)$, provided e. g. these functions f may be derived from projections E_{Δ_i} by uniform limits of approximating Riemann sums $\Sigma f_i E_{\Delta_i}$. In particular, let

$$\left.\begin{aligned} f(z) &= \frac{1}{z}\,; && z \notin [-\delta, \delta] \\ &= 0\,; && z \in [-\delta, \delta]\,. \end{aligned}\right\} \tag{29}$$

We then obtain the continuity of $[c_0(T\,;\lambda)]^{-1}F_\delta(\lambda)$, where $F_\delta(\lambda) = I - E_\delta(\lambda)$. Having chosen δ, we can take $|\lambda - \lambda'|$ sufficiently small that

$$\begin{aligned} &\|\,[c_0(T\,;\lambda)]^{-1}c_0(\Phi\,;\lambda)B_0 - [c_0(T\,;\lambda')]^{-1}c_0(\Phi\,;\lambda')B_0\,\|_2 \\ &\quad\leqslant \|\,[c_0(T\,;\lambda)]^{-1}c_0(\Phi\,;\lambda)F_\delta(\lambda)B_0 - [c_0(T\,;\lambda')]^{-1}c_0(\Phi,\lambda')F_\delta(\lambda')B_0\,\|_2 \\ &\qquad + K\,\|\,E_\delta(\lambda)B_0\,\|_2 + K\,\|\,E_\delta(\lambda')B_0\,\|_2 < \varepsilon + 2K\varepsilon, \qquad \text{say.} \end{aligned}$$

iii) Now define a family $\{Z_\lambda\}$, $Z_\lambda \in \mathscr{B}_0$, by

$$Z_\lambda = [c_0(T\,;\lambda)]^{-1}c_0(\Phi\,;\lambda)[T, W]_\lambda \tag{30}$$

for λ in some closed subinterval N of $\tilde{\Lambda}$, with $Z_\lambda = 0$ for $\lambda \notin N$.

The Z_λ are continuous in $\|\cdot\|_2$ in the interior of N, and satisfy

$$c_0(T\,;\lambda)Z_\lambda = c_0(\Phi\,;\lambda)[T, \tilde{W}]_\lambda, \tag{31}$$

where $\tilde{W} = F(N)W$ and F(N) is the spectral projection of H_0 for the interval N.

Let $Z \in \mathscr{B}_2(\mathscr{H})$ be defined by the family $\{Z_\lambda\}$. From (27), and the corresponding equation with T replaced by Φ, (31) implies

$$c(T)Z = c(\Phi)[T, \tilde{W}]\,. \tag{31$'$}$$

Following closely the proof of Theorem 3, this means that we can identify $[\Phi, \tilde{W}]$ with Z. Hence for λ in the interior of N, $[\Phi, W]_\lambda = [\Phi, \tilde{W}]_\lambda$ and is continuous in λ in $\|\cdot\|_2$. Since N is an arbitrary closed subinterval of $\tilde{\Lambda}$, the theorem is proved.

3. CROSS-SECTIONS BETWEEN CONES

From now on, as described in the introduction, we restrict our attention to non-relativistic potential scattering. In that case, $\mathscr{H} = L^2(\mathbb{R}^3)$, $H_0 = -\Delta$, $\Lambda = [0, \infty]$ and $\mathscr{H}_0 = L^2(S^{(2)})$ (See [*1*], p. 225 or [*2*]).

We are interested in scattering from an initial momentum lying in some cone C_1 to final momentum in some cone C_2, where $C_1 \cap C_2 = \{0\}$. We take both cones to have apex at the origin, and for convenience choose C_1 to be a right circular cone with semi-angle less than $\pi/2$; take also the P_3 coordinate axis $\underline{n}_3$ to coincide with the axis of C_1.

Denote by Θ_1, Θ_2 respectively the intersection of C_1, C_2 with the unit sphere $S^{(2)}$. Then (assuming the cones do not touch) Θ_1 and Θ_2 will be separated by a positive distance. We consider a range of energies defined by some interval Σ, and define the cross-section for scattering from C_1 to C_2 by

$$\sigma(\lambda ; C_1 \to C_2) = \int_{\Theta_1} d\omega_1 \int_{\Theta_2} d\omega_2 \, | f(\lambda ; \underline{\omega}_1 \to \underline{\omega}_2) |^2 \tag{32}$$

Define the sets Σ_1, Σ_2 by

$$\Sigma_i = \{ \sqrt{\lambda} \underline{\omega} . \underline{n}_3 \,;\, \lambda \in \Sigma, \, \underline{\omega} \in \Theta_i \} \tag{33}$$

and choose Σ sufficiently small that Σ_1, Σ_2 have positive distance apart.

Taking a sufficiently large that $\mu(\mu^2 + a^2)^{-1}$ is monotonic for $\mu \in \Sigma_1 \cup \Sigma_2$, we can now find smooth functions ϕ_1, ϕ_2 from $\mathbb{R}$ to $\mathbb{R}$ having the following properties:

i) Each ϕ_i has compact support and satisfies (11).

ii)
$$\phi_1(\mu(\mu^2 + a^2)^{-1}) \equiv 1 \quad \text{for} \quad \mu \in \Sigma_1 ,$$
$$\phi_2(\mu(\mu^2 + a^2)^{-1}) \equiv 1 \quad \text{for} \quad \mu \in \Sigma_2 .$$

iii)
$$\text{supp. } \phi_1 \cap \text{supp. } \phi_2 = \emptyset .$$

Setting

$$T = P_3(P_3^2 + a^2)^{-1} , \tag{34}$$

we can define operators $\Phi_1 = \phi_1(T)$, $\Phi_2 = \phi_2(T)$ from $\mathcal{H}$ to $\mathcal{H}$, which from *i)* - *iii)* above, and from Theorem 2, satisfy

$$\| c(\Phi_i) [c(T)]^{-1} \| < \infty \tag{35}$$

$$\Phi_1 \Phi_2 = 0 \tag{36}$$

and

$$\Phi_i \chi_i = \chi_i , \tag{37}$$

where χ_i is the operator of multiplication in momentum space by the characteristic function of the set

$$\{ \underline{k} \,;\, \underline{k}^2 \in \Sigma \text{ and } \underline{k}/|\underline{k}| \in \Theta_i \} .$$

From (35), and the results of Section 2, we may use commutators with T to estimate commutators with Φ_i.

Since

$$T = \frac{1}{2}((P_3 + ia)^{-1} + (P_3 - ia)^{-1}), \tag{38}$$

it will be sufficient to consider commutators with $(P_3 \pm ia)^{-1}$.

At energy λ, the operator $R(\lambda)$ acts from $\mathcal{H}_0$ to $\mathcal{H}_0$. Let G_i denote the operator of multiplication by the characteristic function of Θ_i. Since $\sigma(\lambda ; C_1 \to C_2) = 4\pi^2\lambda^{-1} \| G_2 R(\lambda) G_1 \|_2^2$, we wish to show that $\{ G_2 R(\lambda) G_1 \}$ is a family of Hilbert-Schmidt operators continuous in $\| \cdot \|_2$. To do this, we shall use an explicit formula for $R(\lambda)$, which involves families of operators from $\mathcal{H}$ to $\mathcal{H}_0$. In constructing such a family from a given operator A from $\mathcal{H}$ to $\mathcal{H}$, we define the family $\{ A_\lambda \}$ of operators from $\mathcal{H}$ to $\mathcal{H}_0$

such that $(\mathrm{A}f)_\lambda = \mathrm{A}_\lambda f$, but no longer assume $\mathrm{A}_\lambda \in \mathscr{B}_0$. In this more general case we shall write $\mathrm{M}(\mathrm{A}\,;\lambda)$ instead of A_λ.

Thus $\mathrm{M}(\mathrm{A}\,;\lambda)$ is defined by

$$\mathrm{M}(\mathrm{A}\,;\lambda)f = (\mathrm{A}f)_\lambda . \tag{39}$$

This definition makes sense, for example, in the following cases:

If $\mathrm{A} \in \mathscr{B}_2(\mathscr{H})$, then $\mathrm{M}(\mathrm{A}\,;\lambda) = \mathrm{A}_\lambda$. Moreover, if $\Psi \equiv \psi(\underline{\mathrm{Q}})$ is the operator of multiplication in position space by $\psi(\underline{r})$, then

(M.1): if $\psi \in \mathrm{L}^2(\mathbb{R}^3)$, then $\mathrm{M}(\Psi\,;\lambda) \in \mathscr{B}_0$ for each $\lambda > 0$ and is continuous in λ in $\|\cdot\|_2$,

(M.2): if $|\psi(\underline{r})| \leqslant$ const. $(1 + |\underline{r}|)^{-\frac{1}{2}-\varepsilon}$ $(\varepsilon > 0)$, then $\mathrm{M}(\Psi\,;\lambda)$ is compact for each $\lambda > 0$ and continuous in λ for $\lambda > 0$ in operator norm (See [*1*], Chapter 10).

The trace operation also has the following properties:

$$\mathrm{M}(\mathrm{A} + \mathrm{B}\,;\lambda) = \mathrm{M}(\mathrm{A}\,;\lambda) + \mathrm{M}(\mathrm{B}\,;\lambda) \tag{40}$$

and

$$\mathrm{M}(\mathrm{AB}\,;\lambda) = \mathrm{M}(\mathrm{A}\,;\lambda)\mathrm{B} . \tag{41}$$

If $\rho(\underline{\mathrm{P}})$ is a function of the momentum operator $\underline{\mathrm{P}}$, then

$$\mathrm{M}(\rho(\underline{\mathrm{P}})\mathrm{A}\,;\lambda) = \rho(\underline{\mathrm{P}})(\lambda)\mathrm{M}(\mathrm{A}\,;\lambda), \tag{42}$$

where $\rho(\underline{\mathrm{P}})(\lambda) : \mathrm{L}^2(\mathrm{S}^{(2)}) \to \mathrm{L}^2(\mathrm{S}^{(2)})$ is given by

$$[\rho(\underline{\mathrm{P}})(\lambda)f](\underline{\omega}) = \rho(\sqrt{\lambda}\underline{\omega})f(\underline{\omega}) . \tag{43}$$

We can now write down the following formula for $\mathrm{R}(\lambda)$ ([*1*], Propositions 10.12, 10.23):

$$\mathrm{R}(\lambda) = -\,2\pi i \mathrm{M}(\mathrm{W}_2\,;\lambda)(\mathrm{I} + \mathscr{W}^{(\lambda)})^{-1}\mathrm{M}(\mathrm{W}_1\,;\lambda)^* . \tag{44}$$

Eq. (44) holds in most cases of interest for scattering by a short range potential V. Here W_1, W_2 are operators of multiplication by functions $\mathrm{W}_1(\underline{r})$, $\mathrm{W}_2(\underline{r})$ in position space and are relatively bounded with respect to H_0. The W_i satisfy $\mathrm{W}_1\mathrm{W}_2 = \mathrm{V}$, corresponding to a factorisation of the potential. This factorisation is not unique, though of course all factorisations lead to an identical $\mathrm{R}(\lambda)$.

The family $\{\mathscr{W}^{(\lambda)}\}$ is norm-continuous in λ, and each $\mathscr{W}^{(\lambda)}$ is defined as an operator from $\mathscr{H}$ to $\mathscr{H}$ by

$$\mathscr{W}^{(\lambda)} = u\text{-}\lim_{\varepsilon \to 0+} \mathrm{W}_1\mathrm{R}_0(\lambda + i\varepsilon)\mathrm{W}_2 , \tag{45}$$

where $\mathrm{R}_0(z) = (\mathrm{H}_0 - z)^{-1}$.

The inverse $(\mathrm{I} + \mathscr{W}^{(\lambda)})^{-1}$ exists as a bounded linear operator from $\mathscr{H}$ to $\mathscr{H}$ and is norm-continuous in λ, apart from values of λ in a closed set Γ_0 of measure zero. We assume that $\bar{\Sigma} \cap \Gamma_0 = \varnothing$.

The representation (44) of the scattering matrix, together with the above properties of the W_i and $\mathscr{W}^{(\lambda)}$, hold for example provided

$$| V(\underline{r}) | \leqslant \text{const.}\ (1 + | \underline{r} |)^{-1-\delta} \qquad (\delta > 0) \tag{46}$$

in which case a possible choice is ([*1*], p. 447)

$$\left. \begin{aligned} W_1 &= (1 + | \underline{r} |)^{-1/2-\delta/2} \\ W_2 &= V(1 + | \underline{r} |)^{1/2+\delta/2} \end{aligned} \right\} \tag{47}$$

One can also deal with potentials for which $V(1 + | \underline{r} |)^{1/2+\delta/2} \in L^2$, using the factorisation (47), or more generally with a sum of potentials in these two classes.

Since we are interested in $G_2R(\lambda)G_1$, and for a range of energies Σ, we may write down a representation of the scattering matrix between the cones C_1 and C_2, for $\lambda \in \Sigma$, by replacing W_i by $F(\Sigma)\chi_i W_i$ in (44), where $F(\Sigma)$ is the spectral projection of H_0 for the interval Σ. In view of (37) and (42) we may replace W_i in (44) by

$$\tilde{W}_i = F(\Sigma)\chi_i\Phi_i W_i \tag{48}$$

or rather by the closure of this operator which is bounded since $W_iF(\Sigma)$ is bounded.

Now

$$\tilde{W}_i = \chi_i[\Phi_i, \overline{F(\Sigma)W_i}] + \tilde{W}_i\Phi_i \tag{49}$$

and our first commutator estimate is given by

LEMMA 3. — Suppose $\dfrac{\partial W_i}{\partial x_3} \in L^2(\mathbb{R}^3)$ (defining the derivative first of all in the sense of distributions).

Then $[\Phi_i, \overline{F(\Sigma)W_i}] \in \mathscr{B}_2$ and $[\Phi_i, \overline{F(\Sigma)W_i}]_\lambda$ is continuous in λ in $\| \cdot \|_2$.

Proof. — First observe that the operator T defined by (34) corresponds to the family $\{ T(\lambda) \}$ of operators from $L^2(S^{(2)})$ to $L^2(S^{(2)})$, where each $T(\lambda)$ is just multiplication by $\lambda^{1/2}\underline{\omega}.\underline{n}_3(\lambda(\underline{\omega}.\underline{n}_3)^2 + a^2)^{-1}$ $(\underline{\omega} \in S^{(2)})$. It is simple to verify, e. g. by the Lebesgue dominated-convergence theorem, that the $T(\lambda)$ are strongly continuous in λ. Moreover, each $T(\lambda)$ has purely continuous spectrum.

Since the ϕ_i have been defined to satisfy (11), according to Theorems 2, 3 and 4 and eq. (38) we need consider only the commutator of $(P_3 \pm ia)^{-1}$ with $\overline{F(\Sigma)W_i}$.

Now, on a dense linear manifold of $L^2(\mathbb{R}^3)$ we have

$$[(P_3 \pm ia), W_i] = -i\frac{\partial W_i}{\partial x_3}.$$

Hence

$$[(P_3 \pm ia)^{-1}, \overline{F(\Sigma)W_i}] = iF(\Sigma)(P_3 \pm ia)^{-1}\frac{\partial W_i}{\partial x_3}(P_3 \pm ia)^{-1}. \tag{50}$$

This operator will be Hilbert-Schmidt, since

$$\left\| \frac{\partial W_i}{\partial x_3} F(\Sigma) \right\|_2 \leqslant c(\Sigma) \left\| \frac{\partial W_i}{\partial x_3} \right\|_{L^2(\mathbb{R}^3)} < \infty .$$

Since the family $\{ (P_3 \pm ia)^{-1}(\lambda) \}$ is strongly continuous in λ, it follows by (M.1) and (42) that

$$M\left(\left[(P_3 \pm ia)^{-1} \frac{\partial W_i}{\partial x_3} \right]; \lambda \right)$$

is continuous in $\| \cdot \|_2$ for λ in the interior of Σ, and this completes the proof of the Lemma.

We now have

$$G_2 R(\lambda) G_1 = G_2 \{ [\Phi_2, \overline{F(\Sigma)W_2}]_\lambda + M(W_2 ; \lambda)\Phi_2 \} \cdot (I + \mathscr{W}^{(\lambda)})^{-1} \{ [\Phi_1, \overline{F(\Sigma)W_1}]_\lambda^* + \Phi_1 M(W_1 ; \lambda)^* \} G_1 . \quad (51)$$

In view of Lemma 3 and (M1), (M2), the problem of proving (for suitable W_i) that $G_2R(\lambda)G_1$ is in $\mathscr{B}_2(\mathscr{H}_0)$ and is continuous in $\| \cdot \|_2$ reduces to that of proving that

$$\Phi_2(I + \mathscr{W}^{(\lambda)})^{-1}\Phi_1$$

is in $\mathscr{B}_2(\mathscr{H})$ and is continuous in $\| \cdot \|_2$. Using (36), we need consider only the commutator $[\Phi_1, (I + \mathscr{W}^{(\lambda)})^{-1}]$, and by Theorem 3 and Remark 3 we can replace this by the commutator $[T, (I + \mathscr{W}^{(\lambda)})^{-1}]$. Since $(I + \mathscr{W}^{(\lambda)})^{-1}$ is norm continuous and

$$[T, (I + \mathscr{W}^{(\lambda)})^{-1}] = (I + \mathscr{W}^{(\lambda)})^{-1} [\mathscr{W}^{(\lambda)}, T](I + \mathscr{W}^{(\lambda)})^{-1}$$

it is sufficient to derive these two properties for $[T, \mathscr{W}^{(\lambda)}]$, which in view of (38) we may replace by $[\mathscr{W}^{(\lambda)}, (P_3 \pm ia)^{-1}]$. However, from (45)

$$[\mathscr{W}^{(\lambda)}, (P_3 \pm ia)^{-1}] = - i(P_3 \pm ia)^{-1} \left\{ \underset{\varepsilon \to 0+}{u\text{-}\lim} \left(\frac{\partial W_1}{\partial x_3} R_0(\lambda + i\varepsilon) W_2 + W_1 R_0(\lambda + i\varepsilon) \frac{\partial W_2}{\partial x_3} \right) \right\} \cdot (P_3 \pm ia)^{-1} . \quad (52)$$

Now if Ψ_1, Ψ_2 are respectively multiplication operators by $\psi_1(\underline{r})$, $\psi_2(\underline{r})$ in $L^2(\mathbb{R}^3)$, one may show (cf. [*1*], p. 371-373) that $\underset{\varepsilon \to 0+}{u\text{-}\lim}\, \Psi_1 R_0(\lambda + i\varepsilon)\Psi_2$ is in $\mathscr{B}_2(\mathscr{H})$ and is continuous in λ in $\| \cdot \|_2$ provided *either*

i) $| \psi_1(r) | \leqslant \text{const.}\ (1 + |\underline{r}|)^{-1/2-\delta/2}$ for some $\delta > 0$ and

$$(1 + |\underline{r}|)^{-\delta/2}\psi_2 \in L^2(\mathbb{R}^3),$$

or

ii) $| \psi_1(\underline{r}) | \leqslant \text{const.}\ (1 + |\underline{r}|)^{-3/2-\delta/2}$ for some $\delta > 0$ and

$$(1 + |\underline{r}|)^{-1}\psi_2 \in L^2(\mathbb{R}^3).$$

This enables us to prove that the r. h. s. of (52) is in $\mathcal{B}_2(\mathcal{H})$ and is continuous in λ in $\|\cdot\|_2$, e. g. under the assumptions

$$\left.\begin{aligned} |W_1(\underline{r})| &\leqslant \text{const.}\,(1+|\underline{r}|)^{-1/2-\delta/2} \\ \left|\frac{\partial W_1(\underline{r})}{\partial x_3}\right| &\leqslant \text{const.}\,(1+|\underline{r}|)^{-3/2-\delta/2} \\ (1+|\underline{r}|)^{-\delta/2}\frac{\partial W_2(\underline{r})}{\partial x_3} &\in L^2(\mathbb{R}^3) \\ (1+|\underline{r}|)^{-1}W_2 &\in L^2(\mathbb{R}^3). \end{aligned}\right\} \tag{53}$$

Under the assumption (46), with W_1, W_2 defined by (47), conditions (53) amount simply to $(1+|\underline{r}|)^{1/2}\dfrac{\partial V}{\partial x_3}\in L^2(\mathbb{R}^3)$.

On the other hand, if we assume $W_2 \in L^2(\mathbb{R}^3)$ and $W_1 = (1+|\underline{r}|)^{-1/2-\delta/2}$, the total cross-section may be shown to be finite (even without restriction to disjoint cones), and it follows from (M1) and (M2) that the addition of such a contribution to W_2 does not affect the finiteness or continuity of $\| G_2R(\lambda)G_1 \|_2$ and $G_2R(\lambda)G_1$.

We have, then,

THEOREM 5. — Suppose $V(\underline{r}) = V_1(\underline{r}) + V_2(\underline{r})$ where, for some $\delta > 0$,

$$|V_1(\underline{r})| \leqslant \text{const.}\,(1+|\underline{r}|)^{-1-\delta/2}$$

$$(1+|\underline{r}|)^{1/2+\delta/2}\frac{\partial V_1(\underline{r})}{\partial x_3} \in L^2(\mathbb{R}^3),$$

and

$$(1+|\underline{r}|)^{1/2+\delta/2}V_2(\underline{r}) \in L^2(\mathbb{R}^3).$$

Then, if the axis $\underline{n}_3$ is in the direction of the axis of the right circular cone C_1, and $\bar{C}_1 \cap \bar{C}_2 = \{0\}$, it follows that

i) $G_2R(\lambda)G_1$ is Hilbert-Schmidt except possibly for $\lambda \in \Gamma_0$ (closed set of measure zero),

ii) $G_2R(\lambda)G_1$ is continuous in λ in $\|\cdot\|_2$.

In particular, the cross-section $\sigma(\lambda\,;\,C_1 \to C_2)$ for scattering from C_1 to C_2 is finite and continuous in λ.

Remark 6. — In the statement and proof of Theorem 5, the $\underline{n}_3$-axis is chosen in the direction of the incoming beam of particles. More generally, let $\underline{\omega}_1$ and $\underline{\omega}_2 \in S^{(2)}$ be two fixed but arbitrary directions. Then provided cones C_1, C_2 containing $\underline{\omega}_1$, $\underline{\omega}_2$ respectively are taken to be sufficiently small (in a sense dictated by the geometry), the conclusions of Theorem 5 remain valid if the $\underline{n}_3$-axis is chosen to be in any direction such that $\underline{n}_3.(\underline{\omega}_1 - \underline{\omega}_2) \neq 0$.

For example, with $V(\underline{r}) = (\cos x_1)/(1+|\underline{r}|)^{1+\varepsilon}$ $(\varepsilon > 0)$ the cross section for scattering from initial directions near $\underline{\omega}_1$ to final directions near $\underline{\omega}_2$

will be finite provided $(\underline{\omega}_1 - \underline{\omega}_2)$ is not parallel to the $\underline{n}_1$-axis. For this potential, the Born approximation to the kernel has a singularity like $|\lambda^{1/2}(\underline{\omega}_1 - \underline{\omega}_2) - \underline{n}_1|^{-2+\varepsilon}$, which for $\varepsilon \leqslant 1/2$ leads to a divergence in the Born approximation to the cross-section if the initial and final directions are related by $(\underline{\omega}_1 - \underline{\omega}_2) = \lambda^{-1/2}\underline{n}_1$. A similar phenomenon occurs for the potential $V(\underline{r}) = (\cos|\underline{r}|)/(1 + |\underline{r}|)^{1+\varepsilon}$ for $0 < \varepsilon < 1$, and by taking linear combinations of potentials of this type one can construct short range potentials for which, in Born approximation, the cross-section is infinite between *any* pair of cones. However, the Born contribution to the cross-section is probably of little significance in this case, and it is known that for rapidly oscillating potentials, even with slower decay than $1/|\underline{r}|^2$ at infinity, the total cross-section may be finite. The subject of cross-sections for rapidly oscillating potentials will be dealt with in a forthcoming note.

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