ANNALES DE L'I. H. P., SECTION A

MARTIN SCHECHTER

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Annales de l'I. H. P., section A, tome 30, nº 2 (1979), p. 109-127
<http://www.numdam.org/item?id=AIHPA_1979__30_2_109_0>

Ann. Inst. Henri Poincaré
Vol. XXX, n° 2, 1979, p. 109-127.

Section A :
Physique théorique.

Completeness of Wave Operators in Two Hilbert Spaces

by

Martin SCHECHTER

ABSTRACT. — We give a simple theory of scattering on two Hilbert spaces. The hypotheses are weaker and the proofs simpler than those found in the literature. Applications are considered.

1. INTRODUCTION

Let $\mathscr{H}_0$, $\mathscr{H}$ be Hilbert spaces and let J be a bounded linear operator from $\mathscr{H}_0$ to $\mathscr{H}$. Let $H_0(H)$ be a selfadjoint operator on $\mathscr{H}_0(\mathscr{H})$. We shall study the set of those $f \in \mathscr{H}_0$ such that the limits

$$(1.1) \qquad Wf = \lim_{t\to\infty} e^{itH} J e^{-itH_0} f$$

exist. Such f are said to be in the domain of the wave operator W. It is of interest in scattering theory that the domain and range of W be as large as possible. On the other hand, if f is an eigenelement corresponding to an eigenvalue of H_0, the limit (1.1) cannot exist unless H has the same eigenvalue and Jf is a corresponding eigenelement (cf. [*1*]). It is therefore unreasonable for one to expect the limit (1.1) to exist for such f. In addition, in all applications known to date, the orthogonal complement of the eigenelements of H is the subspace of absolute continuity $\mathscr{H}_{ac}(H_0)$ of H_0 consisting of those $f \in \mathscr{H}_0$ such that $(E_0(\lambda)f, f)$ is an absolutely continuous function of λ, where $\{ E_0(\lambda) \}$ denotes the spectral family of H_0. We shall say that the wave operator (1.1) *exists* if $\mathscr{H}_{ac}(H_0) \subset D(W)$. It is easily shown that W maps $\mathscr{H}_{ac}(H_0)$ into $\mathscr{H}_{ac}(H)$.

Research partially supported by USNSF grant.

Let $\mathscr{H}(J)$ be the smallest reducing subspace of H containing the range R(J) of J, and let H_1 be the restriction of H to $\mathscr{H}(J)$. Clearly we can replace H by H_1 in (1.1) without changing the limit. This shows that $R(W) \subset \mathscr{H}(J)$. It also shows that as far as (1.1) is concerned we might as well replace $\mathscr{H}$ by $\mathscr{H}(J)$ and H by H_1. We shall say that the operator W is *complete* if $\mathscr{H}_{ac}(H_0) \subset D(W)$ and R(W) is dense in $\mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$.

In this paper we present several theorems concerning existence and completeness of wave operators. Scattering theory in two Hilbert spaces has been developed by Belopolskii-Birman [*2*], Kato [*3*] [*4*], Birman [*5*], Deic [*6*] [*7*], Schechter [*8*], Suzuki [*9*], Schulenberger-Wilcox [*10*]. With the exception of Kato [*4*], all of these authors assumed that J is bijective. We do not make this assumption in the present paper. Kato [*4*] assumed

$$(1.2) \qquad a\int_0^\infty e^{-at}\, \| Je^{-itH_0}f \|^2 dt \to \| f \|_0^2 \qquad \text{as} \quad a \to 0$$

and

$$(1.3) \qquad \limsup_{t\to\infty} \| Je^{-itH_0}f \| \leqslant \| f \|_0$$

neither of which are needed here. These assumptions imply that W is a partial isometry. Kato also assumes that H_0 is spectrally absolutely continuous. We do not need this requirement.

In the first part of this paper we assume that H and H_0 are related by

$$(1.4) \qquad (Ju, Hv) - (JH_0u, v) = \langle Au, Bv \rangle$$

where A, B are appropriately defined operators. An application is given in Section 5. In the second part (Section 6) we work within the framework of Kato [*4*] assuming only

$$(1.5) \qquad JR_0(z) = R(z)G(z), \qquad \operatorname{Im} z > 0$$

on a subspace for some appropriate operator G(*z*). We obtain existence and completeness. To obtain the same conclusions, Kato [*4*] assumes (1.5) for $\operatorname{Im} z \neq 0$ and

$$(1.6) \qquad R(z) = JR_0(z)F(z), \qquad \operatorname{Im} z \neq 0$$

on some suitable subspace. For a comparison of our results (cf. Section 6, May 11, 1978).

2. THE ABSTRACT THEOREMS

As before, we consider Hilbert spaces $\mathscr{H}_0$, $\mathscr{H}$ and a bounded linear operator J mapping $\mathscr{H}_0$ into $\mathscr{H}$. If I, L are open subsets of the real line R, we shall write $L \subset\subset I$ when L is bounded and $\bar{L} \subset I$. Our theorems concern two selfadjoint operators, H_0 on $\mathscr{H}_0$ and H on $\mathscr{H}$. We shall denote their spectral measures by $E_0(I)$, E(I), and their subspaces of absolute continuity by $\mathscr{H}_{ac}(H_0)$, $\mathscr{H}_{ac}(H)$, respectively (cf. [*1*]). For $f \in \mathscr{H}_0$ we shall put $f_{0t} = e^{-itH_0}f$ and for $g \in \mathscr{H}$ we shall use the notation $g_t = e^{-itH}g$. The domain, range, adjoint and closure of an operator T will be denoted by D(T), R(T), T* and [T], respectively. For $I \subset R$, we put $CI = R - I$. The resolvents of H_0, H are given by $R_0(z) = (z - H_0)^{-1}$, $R(z) = (z - H)^{-1}$. When

$$e^{itH}Je^{-itH_0}f \to g \quad \text{as} \quad t \to \infty \tag{2.1}$$

we shall say that $f \in D(W(H, H_0, J))$ and $W(H, H_0, J)f = g$. Bounded operators from X to Y will be denoted by B(X, Y). If X is a Banach space, we let X' denote its dual space and for $x \in X$, $x' \in X'$ we write $x'(x) = \langle x, x' \rangle$ to indicate the duality.

Our first theorem makes the following assumptions.

I. There exist a Banach space $\mathscr{K}$ and linear operators A from $\mathscr{H}_0$ to $\mathscr{K}$ and B from $\mathscr{H}$ to $\mathscr{K}'$ such that $D(H_0) \subset D(A)$, $D(H) \subset D(B)$ and

$$(Ju, Hv) - (JH_0u, v) = \langle Au, Bv \rangle, \qquad u \in D(H_0), \quad v \in D(H) \tag{2.2}$$

II. BR(i) is a bounded operator, and there exist an open subset Λ of R and a dense subset S of $\mathscr{H}_{ac}(H_0)$ such that $C\Lambda$ has measure 0, and for each open $I \subset \subset \Lambda$, $AE_0(I)$ is a bounded operator and

$$\int_0^{\infty} (\| AE_0(I)f_{0t} \|_{\mathscr{K}}^2 + \| BE(I)g_t \|_{\mathscr{K}'}^2)dt < \infty \tag{2.3}$$

holds for all $f \in S$ and $g \in \mathscr{H}$.

THEOREM 2.1. — Under hypotheses I and II, $\mathscr{H}_{ac}(H_0) \subset D(W(H, H_0, J))$.

Note that Theorem 2.1 is almost symmetric in H_0 and H. In fact, if we replace II by

II'. $AR_0(i)$, $BR(i)$ are bounded operators, and there is an open set $\Lambda \subset R$ such that $C\Lambda$ has measure 0 and (2.3) holds for each $I \subset \subset \Lambda$, $f \in \mathscr{H}_0$ and $g \in \mathscr{H}$.

We have

COROLLARY 2.2. — Under hypotheses I and II', $\mathscr{H}_{ac}(H_0) \subset D(W(H, H_0, J))$ and $\mathscr{H}_{ac}(H) \subset D(W(H_0, H, J^*))$.

If we add the hypothesis

III. If $g \in \mathscr{H}_{ac}(H) \cap \mathscr{H}(J)$ and $J^*g_t \to 0$, then $g = 0$.

We have

THEOREM 2.3. — Under hypotheses I, II', III, the wave operator $W(H, H_0, J)$ is complete.

In applications it is sometimes more convenient to replace III by either one of the following criteria.

III'. The range of J is closed in $\mathscr{H}$ and has finite codimension in $\mathscr{H}(\mathrm{J}) \cap \mathscr{H}_{ac}(\mathrm{H})$.

III''. There is an operator $\mathrm{J}_1 \in \mathrm{B}(\mathscr{H}, \mathscr{H}_0)$ such that $(\mathrm{JJ}_1 - 1)g_t \to 0$ as $t \to \infty$ for each $g \in \mathscr{H}_{ac}(\mathrm{H}) \cap \mathscr{H}(\mathrm{J})$.

We have

LEMMA 2.4. — Hypothesis III' implies III'', and III'' implies III. A disadvantage of our results so far is that they require knowledge of both $\mathrm{E}_0(\mathrm{I})$ and $\mathrm{E}(\mathrm{I})$. Usually it is not easy in applications to obtain information on both of them. An ingenious method due to Kato-Kuroda [*11*] allows one to « transfer » the burden from H to H_0. We shall use the following hypotheses:

1. There exist a Banach space $\mathscr{K}$ and linear operators A from $\mathscr{H}_0$ to $\mathscr{K}$ and B from $\mathscr{H}_0$ to $\mathscr{K}'$ such that $\mathrm{D}(\mathrm{H}_0) \subset \mathrm{D}(\mathrm{A}) \cap \mathrm{D}(\mathrm{B})$, $\mathrm{D}(\mathrm{H}) \subset \mathrm{D}(\mathrm{BJ}^*)$ and

(2.4) $(\mathrm{J}u, \mathrm{H}v) - (\mathrm{JH}_0 u, v) = \langle \mathrm{A}u, \mathrm{BJ}^* v \rangle, \qquad u \in \mathrm{D}(\mathrm{H}_0), \quad v \in \mathrm{D}(\mathrm{H})$

2. There exists an open subset Λ of R such that $\mathrm{C}\Lambda$ has measure 0 and for each $\mathrm{I} \subset\subset \Lambda$ there is a constant C_I such that

(2.5) $a(\| \mathrm{AR}_0(s + ia) \|^2 + \| \mathrm{BR}_0(s + ia) \|^2) \leqslant \mathrm{C}_\mathrm{I}, \qquad a > 0, \quad s \in \mathrm{I}$

3. The operators $\mathrm{Q}_0(z) = [\mathrm{B}(\mathrm{AR}_0(\bar{z}))^*]$, $\mathrm{G}_0(z) = 1 - \mathrm{Q}_0(z)$ and $\mathrm{G}_0(z)^{-1}$ are bounded and everywhere defined on $\mathscr{K}'$ for $\mathrm{Im}\, z \neq 0$.

4. For each $\mathrm{I} \subset\subset \Lambda$, $\mathrm{Q}_0(z)$ is uniformly continuous in the region $\omega_\mathrm{I} = \{ z = s + ia \mid 0 < a < 1, s \in \mathrm{I} \}$.

5. There is a z_1 such that $[\mathrm{BR}_0(z)(\mathrm{AR}_0(z_1))^*]$ is a compact operator on $\mathscr{K}'$ when $\mathrm{Im}\, z \neq 0$.

6. $\mathrm{R}(\mathrm{B}^*)$ is dense in $\mathscr{H}_0$.

THEOREM 2.5. — Under hypothese 1-6, the wave operator $\mathrm{W}(\mathrm{H}, \mathrm{H}_0, \mathrm{J})$ exists and is complete.

Notice that the only hypothesis among 1-6 that refers to H is the first, and even this hypothesis does not refer to E(I), R(z) or $e^{-it\mathrm{H}}$. Theorem 2.1, 2.3 and Lemma 2.4 are proved in Section 3, and Theorem 2.5 is proved in Section 4.

Theorems 2.1, 2.3 and 2.5 have many applications, some of which will be published later. Here we shall give an illustration which we have simplified to avoid technical details and involved calculations. We consider the case $\mathscr{H}_0 = \mathscr{H} = \mathrm{L}^2(\mathrm{E}^n)$, $\mathrm{H}_0 = \mathrm{H} = -\Delta$ and $\mathrm{J} = b(x)$, a bounded function. We assume that there are functions a_1 and $a_2 = (a_{21}, a_{22}, \ldots, a_{2n})$ such that a_2 is bounded, a_1 is $\nabla-$ bounded and $a_1 a_2 = b^{-1}\nabla b$. Moreover, we assume that there are constants $\alpha \geqslant 0$, $p \geqslant 1$ such that $\alpha > 1 - (n/p)$

and $(1 + |x|)^{\alpha}(|a_1| + |a_2|)$ is in $L^p(E^n)$. Then the wave operators $W(H, H_0, J)$ exist and are complete. To see this we note that

$$(Ju, Hv) - (JH_0u, v) = (a_1u, ba_2 \cdot \nabla(b^{-1}J^*v)) - (a_2 \cdot \nabla u, a_1J^*v)$$

We can take

$$Au = \{ a_1u, a_2 \cdot \nabla u \}, \qquad Bv = \{ ba_2 \cdot \nabla(b^{-1}v), - a_1v \}$$

It is not difficult to verify that the hypotheses of Theorem 2.5 are satisfied (cf. [*8*]), and the result follows. One can also consider the case when $H = H_0 + V$ on some subset Ω of E^n, where V is a potential. Details will be given in a forth-coming publication.

Corollary 2.2 generalizes a result of Lavine-Kato (cf. [*13*], Theorem 2.3).

3. THE METHOD

In proving Theorems 2.1 and 2.3 we use a new approach which is a mixture of the time dependent and time independent methods.

Proof of Theorem 2.1. — First we note that for each $I \subset\subset \Lambda$ there is a constant C_I such that

$$(3.1) \qquad \int_0^\infty \| BE(I)g_t \|_{\mathcal{K}'}^2 dt \leqslant C_I \| g \|^2, \qquad g \in \mathcal{H}$$

This is a simple consequence of the closed graph theorem. In fact, (2.3) shows that the operator $Lg = BE(I)g_t$ mapping $\mathcal{H}$ into $L^2(0, \infty, \mathcal{K}')$ is defined on the whole of $\mathcal{H}$. It is easily verified that L is a closed operator. Hence it is bounded. Next we note that (2.2) implies

$$(3.2) \qquad (R(z)Jf - JR_0(z)f, g) = \langle AR_0(z)f, BR(\bar{z})g \rangle$$

Thus if $z = s + ia$, we have

$$(3.3) \quad a \int_I ([R(z)J - JR_0(z)]E_0(I)f, R(z)E(I)g)ds$$
$$= a \int_I \langle AR_0(z)E_0(I)f, BR(\bar{z})R(z)E(I)g \rangle ds$$
$$= -\pi i \int_0^\infty e^{izt} \int_I \langle AE_0(I)f_{0t}, BE(I)\delta_a(H - s)g \rangle dsdt$$
$$\to -\pi i \int_0^\infty \langle AE_0(I)f_{0t}, BE(I)g_t \rangle dt$$

where $\delta_a(\lambda) = a/\pi(\lambda^2 + a^2)$. Here we made use of the formulas

$$(3.4) \qquad R(z)h = -i \int_0^\infty e^{izt}h_t dt$$

and

$$(3.5) \qquad \int_I \varphi(z)(\delta_a(s - H)u, v)ds \to (\varphi(H)\tilde{E}(I)u, v) \quad \text{as} \quad a \to 0$$

where $\tilde{E}(I) = \frac{1}{2}[E(I) + E(\bar{I})]$ (cf. [12]). Another application of (3.5) shows that the limit

$$(3.6) \qquad (W_I f, g) = \lim_{0<a\to 0} \frac{a}{\pi}\int_I (JR_0(z)E_0(I)f, R(z)E(I)g)ds$$

exists for each $f \in S$, $g \in \mathscr{H}$, and

$$(3.7) \qquad ([W_I - JE_0(I)]f, E(I)g) = i\int_0^\infty \langle AE_0(I)f_{0t}, BE(I)g_t \rangle dt$$

By (3.6)

$$(3.8) \quad |(W_I f, g)|^2 \leqslant \limsup \frac{a}{\pi}\int_I \| JR_0(z)E_0(I)f \|^2 ds \times \frac{a}{\pi}\int_I \| R(z)E(I)g \|^2 ds \leqslant \| J \|^2 \| E_0(I)f \|^2 \| E(I)g \|^2$$

Moreover, the same argument shows that

$$a\int_{CI} (JR_0(z)E_0(I)f, R(z)E(I)g)ds \to 0 \quad \text{as} \quad a \to 0$$

Thus the integral in (3.6) may be taken over the whole real line. In view of (3.4), this implies by Parseval's identity

$$(3.9) \qquad (W_I f, g) = \lim_{a\to 0} 2a\int_0^\infty e^{-2at}(JE_0(I)f_{0t}, E(I)g_t)dt$$

From this we deduce readily that

$$(3.10) \qquad (W_I f_{0t}, g_t) = (W_I f, g)$$

Suppose L, I are intervals such that $L \subset\subset I \subset\subset \Lambda$. Then

$$\begin{aligned}(3.11) \quad & \| [W_I - J]E_0(L)f \|^2 \\ &= ([W_I - J]E_0(L)f, W_I E_0(L)f) - ([W_I - J]E_0(L)f, JE_0(L)f) \\ &= i\int_0^\infty \langle AE_0(L)f_{0t}, BE(I)[W_I E_0(L)f]_t \rangle dt + \| E(C\bar{I})JE_0(L)f \|^2 \\ &- i\int_0^\infty \langle AE_0(L)f_{0t}, BE(I)[JE_0(L)f]_t \rangle dt \leqslant \| E(C\bar{I})JE_0(L)f \|^2 \\ &+ C(\| W_I E_0(L)f \| + \| JE_0(L)f \|)\left(\int_0^\infty \| AE_0(L)f_{0t} \|_{\mathscr{K}}^2 dt\right)^{1/2} \\ &\leqslant \| E(C\bar{I})JE_0(L)f \|^2 + C_1 \| f \| \left(\int_0^\infty \| AE_0(L)f_{0t} \|_{\mathscr{K}}^2 dt\right)^{1/2}\end{aligned}$$

where the constant C_1 does not depend on f. Here we used (3.7), (3.8) and (3.1). We shall show that

$$(3.12) \qquad E(C\bar{I})JE_0(L)f_{0t} \to 0 \quad \text{as} \quad t \to \infty$$

Assuming this for the moment, we have by (3.11)

$$\| [W_I - J]E_0(L)f_{0t} \|^2 \leqslant \| E(C\bar{I})E_0(L)f_{0t} \|^2 + C_1 \| f \| \left(\int_t^\infty \| AE_0(L)f_{0\tau} \|_{\mathscr{K}}^2 d\tau \right)^{1/2} \to 0$$

Thus

$$[W_I - W(t)]E_0(L)f = e^{itH}[W_I - JE_0(L)]f_{0t} \to 0$$

This shows that $E_0(L)f \in D(W)$. Since $D(W)$ is a closed subspace of $\mathscr{H}_0$, we see that $E_0(I)f \in D(W)$ and consequently that $E_0(\Lambda)f \in D(W)$. Since $f \in \mathscr{H}_{ac}(H_0)$ and $C\Lambda$ has measure 0, $f = E_0(\Lambda)f$. Thus $f \in D(W)$. Since S is dense in $\mathscr{H}_{ac}(H_0)$, we have $\mathscr{H}_{ac}(H_0) \subset D(W)$, and the theorem follows. It remains only to prove (3.12). Let (c, d) be an interval such that $L \subset\subset (c, d) \subset\subset I$. Let $a > 0$ be a constant to be chosen later, and let C be the rectangle with vertices $(c \pm ia, d \pm ia)$. Then we have

$$\begin{aligned}(3.13) \quad (JE_0(L)f, E(C\bar{I})g) &= \frac{1}{2\pi i} \int_C (JR_0(z)E_0(L)f, E(C\bar{I})g)dz \\ &= \frac{1}{2\pi i} \int_C ([JR_0(z) - R(z)J]E_0(L)f, E(C\bar{I})g)dz \\ &= \frac{i}{2\pi} \int_C \langle AR_0(z)E_0(L)f, BR(\bar{z})E(C\bar{I})g \rangle dz\end{aligned}$$

where we used (3.2) and the fact that $E(CI)R(z)$ is analytic on the closure of C. Let $\varepsilon > 0$ be given, and take a so small that

$$(3.14) \qquad \int_{-a}^{a} \| E(C\bar{I})[JR_0(c + i\eta) - R(c + i\eta)J]E_0(L)f \| d\eta < \varepsilon$$

with a similar estimate holding for c replaced by d. Thus the integral over the two vertical sides of C in (3.13) is $< 2\epsilon \| g \|$. Since $BR(i)$ is bounded and $BR(z) = BR(i)\,(1 + (i - z)R(z))$, we see that $BR(z)$ is a continuous function for $\operatorname{Im} z \neq 0$. Thus there is a constant C_2 such that

$$\int_c^d \| BR(s \pm ia)g \|^2 ds \leqslant C_2 \| g \|^2$$

Hence the squares of the integrals in (3.4) over the horizontal sides of C are bounded by $C_3 \| g \|^2$ times

$$\int_{-\infty}^{\infty} (\| AE_0(L)R_0(z)f \|^2 + \| AE_0(L)R_0(\bar{z})f \|^2)ds$$

$$= \int_{-\infty}^{\infty} \| AE_0(L)[R_0(z) - R_0(\bar{z})]f \|^2 ds = 2\pi \int_{-\infty}^{\infty} e^{-2a|t|} \| AE_0(L)f_{0t} \|^2 dt$$

(cf. [*12*]). This implies

$$(3.15)\quad \| E(C\bar{I})JE_0(L)f_{0\sigma} \| \leqslant 2\varepsilon + C\left(\int_{-\infty}^{\infty} e^{-2a|t-\sigma|} \| AE_0(L)f_{0t} \|^2 dt\right)^{\frac{1}{2}}$$

in view of (3.14). Since $AE_0(L)$ is a bounded operator, we have

$$\int_{-\infty}^{0} e^{-2a|t-\sigma|} \| AE_0(L)f_{0t} \|^2 dt \leqslant \| AE_0(L) \|^2 \| f \|^2 \int_{-\infty}^{0} e^{-2a|t-\sigma|} dt \to 0 \quad \text{as} \quad t \to \infty$$

On the other hand, $\| AE_0(I)f_{0t} \| \in L^2(0, \infty)$ by (2.3) and $e^{-2a|t-\sigma|}$ is bounded and converges to 0 a. e. as $\sigma \to \infty$. Thus

$$\int_{0}^{\infty} e^{-2a|t-\sigma|} \| AE_0(I)f_{0t} \|^2 dt \to 0 \quad \text{as} \quad t \to \infty$$

This gives the desired result in view of (3.15). □

COROLLARY 2.2. — Is an immediate consequence of Theorem 2.1.

Proof of Theorem 2.3. — The first requirement of completeness follows from Theorem 2.1. To obtain the second, suppose $g \in \mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$ and $g \perp R(W(H, H_0, J))$. Then

$$(J^*g_t, f_{0t}) \to 0 \quad \text{as} \quad t \to \infty, \qquad f \in \mathscr{H}_{ac}(H_0)$$

By Corollary 2.2, $W(H_0, H, J^*)g = 0$. This is the same as saying $J^*g_t \to 0$. By hypothesis III, $g = 0$. Thus $R(W(H, H_0, J))$ is dense in $\mathscr{H}_{ac}(H) \cap \mathscr{H}(T)$. □

Proof of Lemma 2.4. — Note that J is a bijective operator from $N(J)^\perp$ to $R(J)$. Let J_1 be the inverse of J on $R(J)$ and let it vanish on $R(J)^\perp$. Now $1 - JJ_1$ is the orthogonal projection onto $R(J)^\perp$, and $R(J)^\perp \cap \mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$ is finite dimensional. Hence $JJ_1 - 1$ is a compact operator on $\mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$. On the other hand, if $g \in \mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$,

$$(g_t, h) = \int e^{-it\lambda} \frac{d}{d\lambda}(E(\lambda)g, h)d\lambda \to 0$$

Thus g_t converges weakly to 0 as $t \to \infty$. This implies that $(JJ_1 - 1)g_t \to 0$. This in turn implies $(g_t, (JJ_1 - 1)h_t) \to 0$ as $t \to \infty$, $h \in \mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$. Now if $J^*g_t \to 0$, then we also have $(g_t, JJ_1h_t) \to 0$, and consequently $(g, h) = (g_t, h_t) = 0$ for all $h \in \mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$. Thus $g = 0$ and III holds. □

4. THE KATO-KURODA METHOD

In proving Theorem 2.5 we shall make use of the following lemmas.

LEMMA 4.1. — If T is an operator such that $D(H) \subset D(T)$ and

$$(4.1)\qquad a \| TR(s + ia) \|^2 \leqslant C, \qquad a > 0, \quad s \in I$$

then there are constants C_1, C_2 such that

$$(4.2) \qquad a \| TE(I)R(s + ia) \|^2 \leqslant C_1, \qquad a > 0, \quad s \in R$$

and

$$(4.3) \qquad \int_{-\infty}^{\infty} \| TE(I)g_t \|^2 dt \leqslant C_2 \| g \|^2, \qquad g \in \mathscr{H}$$

Proof. — The proof of (4.2) was given in Lavine [*13*]. The identity

$$(4.4) \quad 2\pi \int_{-\infty}^{\infty} e^{-2a|t|} \| TE(I)g_t \|^2 dt = \int_{-\infty}^{\infty} \| TE(I)[R(z) - R(\bar{z})]g \|^2 ds$$
$$= 4a^2 \int_{-\infty}^{\infty} \| TE(I)R(z)R(\bar{z})g \|^2 ds$$

was proved in Kato [*12*], where $z = s + ia$. If we use (4.2), we see that the right hand side of (4.4) is bounded by

$$4C_1 a \int_{-\infty}^{\infty} \| R(\bar{z})g \|^2 ds \leqslant 4C_1 \| g \|^2$$

Letting $a \to 0$, we obtain (4.3). □

LEMMA 4.2. — Let $T(z)$ be a bounded operator on $\mathscr{H}$ for each $z \in \omega_I$ which depends analytically on z in ω_I and continuously on z in $\bar{\omega}_I$. Suppose that $K(z) = 1 - T(z)$ is compact and $T(z)$ has a bounded inverse for each $z \in \omega_I$. Then the set of those $s \in I$ for which $T(s)$ has no bounded inverse has measure 0.

The proof of Lemma 4.2 can be found in Kato-Kuroda [*11*]. Now we give the

Proof of Theorem 2.5. — We apply Theorem 2.1 with B replaced by BJ*. Hypothesis I follows from 1, and (2.5) implies that $AR_0(i)$ is bounded and

$$(4.5) \qquad \int_{-\infty}^{\infty} \| AE_0(I)f_{0t} \|^2 dt < \infty$$

for each $I \subset \subset \Lambda$ and $f \in \mathscr{H}_0$ in view of Lemma 4.1. Next we show that $BJ^*R(i)$ is bounded and there is an open subset Γ of Λ such that $C\Gamma$ has measure 0 and

$$(4.6) \qquad \int_{0}^{\infty} \| BJ^*E(I)g_t \|^2 dt < \infty$$

for each $I \subset \subset \Gamma$ and $g \in \mathscr{H}$. This will imply II′ and existence will follow from Theorem 2.1. To this end we note that (2.4) implies

$$(4.7) \qquad G_0(z)BJ^*R(z) = BR_0(z)J^*$$

(It is for this reason that we replaced B by BJ* in (2.2).) Let $G(z) = G_0(z)^{-1}$.

A simple argument shows that we may assume $\operatorname{Im} z_1 \neq 0$ in hypothesis 5. Then

$$G(z_1)G_0(z) = 1 + K(z) \tag{4.8}$$

where

$$K(z) = (z - z_1)G(z_1)BR_0(z)(AR_0(\bar{z}_1))^* \tag{4.9}$$

By hypothesis, $K(z)$ is compact for $\operatorname{Im} z \neq 0$. Moreover by (4.8), $T(z) = 1 + K(z)$ has a bounded inverse for $\operatorname{Im} z \neq 0$. Let $I \subset \subset \Lambda$ be given. Then $K(z)$ is uniformly continuous with respect to z in ω_I by (4.8) and hypothesis 4. Hence it can be extended bo be continuous in $\bar{\omega}_I$. Since $K(z)$ depends analytically on z in ω_I, we can apply Lemma 4.2 to conclude that the set of those $s \in I$ where $T(z)$ has no bounded inverse is closed and of measure 0. By (4.8), the same is true of $G_0(z)$. Hence there is an open subset Γ of Λ such that $C\Gamma$ has measure 0 and $G_0(s)$ has a bounded inverse for $s \in \Gamma$. It follows that $G(z)$ is uniformly bounded in each ω_I for $I \subset \subset \Gamma$. In view of (4.7) and (2.5), this implies

$$a \, \| BJ^*R(z) \|^2 \leqslant C_I, \qquad a > 0, \quad s \in I \subset \subset \Gamma \tag{4.10}$$

In particular $BJ^*R(i)$ is bounded, and (4.6) holds by Lemma 4.1.

To prove completeness, suppose $g \in \mathscr{H}(J) \cap \mathscr{H}_{ac}(H) \cap R(W)^\perp$. By Corollary 2.2, $g \in D(W(H_0, H, J^*))$ and $W(H_0, H, J^*)g = 0$. Moreover, by (4.7)

$$a \int_I (JR_0(z)B^*u, R(z)g)ds = a \int_I (R(z)JB^*G_0(z)^*u, R(z)g)ds$$
$$\to \pi \int_I m(s, JB^*G_0(s)^*u, g)ds, \qquad I \subset \subset \Lambda$$

where

$$m(s, f, g) = d(E(s)f, g)/ds$$

Thus by (3.6)

$$(B^*u, W_1\alpha(H)g) = \int \alpha(s)^* m(s, JB^*G_0(s)^*u, g)ds \tag{4.10}$$

where $W_1 = W(H_0, H, J^*)$ and $\alpha(s)$ is any function with support in Λ. Thus we have

$$m(s, JB^*G_0(s)^*u, g) = 0 \qquad \text{a. e.}$$

Since $G_0(s)$ has a bounded inverse for a. e. s., this implies

$$m(s, JB^*w, g) = 0 \qquad \text{a. e.,} \quad w \in D(B^*)$$

Thus

$$(JB^*w, \alpha(H)g) = 0, \qquad w \in D(B^*)$$

Since $R(B^*)$ is dense in $\mathscr{H}$ (hypothesis 6), we see that g is orthogonal to every element of the form $\alpha(H)Jh$. We note that $\mathscr{H}(J)$ is precisely the closed subspace generated by sums of elements of this form. Thus g is orthogonal to itself and must vanish. Hence R(W) is dense in $\mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$ and W is complete. □

In applying Theorem 2.5 it may not be a simple matter to verify that the operator $G_0(z)$ in hypothesis 3 has a bounded inverse. The following lemma gives a sufficient condition.

LEMMA 4.3. — Assume that J is a bijective bounded linear operator from $\mathscr{H}_0$ to $\mathscr{H}$ and there are closed linear operators A from $\mathscr{H}$ to $\mathscr{K}$ and B from $\mathscr{H}_0$ to $\mathscr{K}'$ such that $D(H_0) \subset D(AJ) \cap D(B)$, $D(H) \subset D(BJ^*) \cap D(A)$ and

$$(Ju, Hv) - (JH_0u, v) = \langle AJu, BJ^*v \rangle \tag{4.10}$$

holds for $u \in D(H_0)$, $v \in D(H)$ and for $Ju \in D(H)$, $J^*v \in D(H_0)$. If $Q_0(z) = [BR_0(z)J^*A^*]$, $Q(z) = [BJ^*R(z)A^*]$ are bounded operators and $G_0(z) = 1 - Q_0(z)$, $G(z) = 1 + Q(z)$, then $G(z) = G_0(z)^{-1}$.

Proof. — If we put $u = R_0(z)f$, $v = R(z)g$ in (4.10), we get

$$([R(z)J - JR_0(z)]f, g) = \langle AJR_0(z)f, BJ^*R(\bar{z})g \rangle$$

On the other hand, if we put $Ju = R(z)g$, $J^*v = R_0(\bar{z})f$, we get

$$([J^{-1}R(z) - R_0(z)J^{-1}]g, f)_0 = \langle AR(z)g, BR_0(\bar{z})f \rangle$$

Thus

$$R(z)J - JR_0(z) = [BJ^*R(z)]^*AJR_0(z) = J[BR_0(\bar{z})]^*AR(z)J \tag{4.11}$$

Now

$$\begin{aligned} Q(z) - Q_0(z) &= [B(J^*R(z) - R_0(z)J^*)A^*] \\ &= [BR_0(z)J^*A^*BJ^*R(z)A^*] = Q_0(z)Q(z) = Q(z)Q_0(z). \end{aligned}$$

This shows that $G_0(z)G(z) = G(z)G_0(z) = 1$. □

5. AN APPLICATION

Let H_0 be the selfadjoint operator associated with $L = -d^2/dx^2$ on $R = (-\infty, \infty)$ in $\mathscr{H}_0 = L^2(R)$ and let H be the selfadjoint operator associated with L on $R_+ = (0, \infty)$ in $\mathscr{H} = L^2(R_+)$ with the boundary condition $v(0) = 0$. Let $b(x)$ be a function on R which is bounded away from 0 and such that $b'(x) \in L^1(R_+)$. We take Ju to be the restriction of bu to R_+. A simple calculation shows that (2.2) holds with

$$\mathscr{K} = \mathbb{C} \oplus L^2(R_+) \oplus L^2(R_+)$$

and

$$Au = \{ b(0)u(0), V_1u, V_2u' \}, \qquad Bv = \{ v'(0), V_2^*v', -V_1v \}$$

where $V_1(x) = |b'(x)|^{\frac{1}{2}}$ and $V_2 = b'/V$. Now

$$R_0(z)f = \frac{1}{2\kappa i}\int_{-\infty}^{\infty} e^{i\kappa|x-y|}f(y)dy$$

where $\kappa^2 = z$, Im $\kappa > 0$. We take $\Lambda = R - \{0\}$. It is easily checked that $a\,\|AR_0(z)\|^2 \leqslant C_I$ for $I \subset\subset \Lambda$ provided $V_1, V_2 \in L^2(R_+)$. Moreover $R(z)g$ is the restriction to R_+ of $R_0(z)g_1$, where $g_1 = g$ in R_+ and $g_1(x) = -g(-x)$ in $R - R_+$. This shows that $a\,\|BR(z)\|^2 \leqslant C_I$ for $I \subset\subset \Lambda$ as well. These inequalities imply (2.3) (Lemma 4.1). Now $J^*v = b^*v$ in R_+ and vanishes in $R - R_+$. Thus $\|J^*g_t\|_0 \geqslant c_0\|g\|$, where $c_0 > 0$. Thus hypotheses I, II, III hold. Thus the wave operators exist and are complete.

6. A COMPARISON

In this section we want to show what results can be obtained by our methods within the framework considered by Kato [4]. Again we consider selfadjoint operators H_0 in $\mathscr{H}_0$, H in $\mathscr{H}$ and a bounded linear operator J from $\mathscr{H}_0$ to $\mathscr{H}$. We assume that there are linear manifolds $X_0 \subset \mathscr{H}_{ac}(H_0)$ and $X \subset \mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$ with their own norms and an open set $\Lambda \subset R$ with $C\Lambda$ having measure 0 such that

a) $m(\lambda, u) = d(E(\lambda)u, u)/d\lambda \leqslant C_I\|u\|_X^2, \quad u \in X, \quad \lambda \in I \subset\subset \Lambda$

b) $\mathscr{H}_{ac}(H_0)$ is the smallest subspace of $\mathscr{H}_0$ reducing H_0 and containing X_0, and $\mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$ is the smallest subspace of $\mathscr{H}$ reducing H and containing X.

c) For each $z = s + ia$, $s \in \Lambda$, $0 \leqslant a < 1$, there is a bounded linear map $G(z)$ of X_0 onto X which is strongly continuous in z and such that

(6.1) $$JR_0(z)u = R(z)G(z)u, \quad u \in X, \quad 0 < a < 1$$

We have

THEOREM 6.1. — Under hypotheses $a - c$, there is a bounded operator Ω from $\mathscr{H}_{ac}(H_0)$ to $\mathscr{H}_{ac}(H)$ such that

(6.2) $$a\int_0^\infty e^{-at}\,\|W(t)f - \Omega f\|^2 dt \to 0 \quad \text{as} \quad a \to 0, \quad f \in \mathscr{H}_{ac}(H_0)$$

and $R(\Omega)$ is dense in $\mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$. In particular, if $W(H, H_0, J)$ exists, it is complete.

THEOREM 6.2. — If, in addition, X is a Hilbert space, then

(6.3) $$(W(t)f, g) \to (\Omega f, g) \quad \text{as} \quad t \to \infty, \quad f \in \mathscr{H}_{ac}(H_0), \quad g \in \mathscr{H}$$

If

(6.4) $$\limsup_{t\to\infty}\|Jf_{0t}\|^2 = \limsup_{a\to 0}\, a\int_0^\infty e^{-at}\,\|Jf_{0t}\|^2 dt$$

as well, then the wave operator $W(H, H_0, J)$ exists and is complete.

In comparing our results with those of Kato [4], one should note that we do not have to assume (1.2), (1.3) or $R(z)v = JR_0(z)F(z)v$ for some suitable map $F(z)$ from X to X_0. We also do not require that

$$m_0(\lambda, u) = d(E_0(\lambda)u, u)/d\lambda$$

be accessible and continuous on $\Lambda \times X_0 \times X_0$. On the other hand, our hypothesis *a*, not assumed by Kato, is a consequence of his assumptions. In fact, it is a consequence of

$$(6.3) \qquad a \| R(z)u \|^2 \leqslant C_I \| u \|_X^2, \qquad u \in X, \quad \lambda \in I \subset \subset \Lambda$$

which is implied by them.

We prove Theorems 6.1 and 6.2 by means of a few simple lemmas.

LEMMA 6.3. — Let $\alpha(\lambda)$, $\beta(\lambda)$ be continuous functions with compact supports in Λ. Then

$$(6.4) \int a \quad (JR_0(z)\alpha(H_0)u, R(z)\beta(H)v)ds \to \pi \int \alpha(\lambda)\beta(\lambda)^* m(\lambda, G(\lambda)u, v)d\lambda \qquad \text{as} \quad a \to 0$$

for $u \in X_0$ and $v \in \mathscr{H}$.

Proof. — First we note that

$$(6.5) \quad a \int \alpha(s)(JR_0(z)u, R(z)\beta(H)v)ds = \pi \int \alpha(s) \int \delta_a(\lambda - s)\beta(\lambda)^* m(\lambda, G(z)u, v)d\lambda ds$$

converges to the right hand side of (6.4) since

$$\left(\int_I \int_I \delta_a(\lambda - s) \,|\, m(\lambda, G(z)u - G(\lambda)u, v) \,|\, d\lambda ds \right)^2$$
$$\leqslant \int_I \int_I \delta_a(\lambda - s) m(\lambda, G(z)u - G(\lambda)u) d\lambda ds \int_I \int_I \delta_a(\lambda - s) m(\lambda, v) d\lambda ds$$
$$\leqslant C_I \| v \|^2 \int_I \int_I \delta_a(\lambda - s) \| G(z)u - G(\lambda)u \|_X^2 d\lambda ds$$

where $I \subset \subset \Lambda$ contains the supports of α and β. This converges to 0 as $a \to 0$ by the continuity of $G(z)$. Finally, we note that the difference between the left hand sides of (6.4) and (6.5) converges to 0 as $a \to 0$ since

$$(6.6) \int a \quad \| JR_0(z)[\alpha(H_0) - \alpha(s)]u \|^2 ds \leqslant \pi \| J \|^2 \int\int \delta_a(s - \lambda) \,|\, \alpha(\lambda) - \alpha(s) \,|^2 m_0(\lambda, u) d\lambda ds$$

where

$$(6.7) \qquad m_0(\lambda, u) = d(\mathrm{E}_0(\lambda)u, u)/d\lambda$$

The right hand side of (6.6) converges to 0 as $a \to 0$. □

LEMMA 6.4. — Under the same hypotheses,

$$(6.8) \quad a \int (\mathrm{JR}_0(z)\alpha(\mathrm{H}_0)u, \mathrm{JR}_0(z)\beta(\mathrm{H}_0)v)ds \to \pi \int \alpha(\lambda)\beta(\lambda)^* m(\lambda, \mathrm{G}(\lambda)u, \mathrm{G}(\lambda)v)d\lambda$$

for $u, v \in \mathrm{X}_0$.

Proof. — Using the arguments of Preceding proof we show that

$$(6.9) \quad a \int \alpha(s)\beta(s)^*(\mathrm{JR}_0(z)u, \mathrm{JR}_0(z)v)ds = \pi \int \alpha(s)\beta(s)^* \int \delta_a(s-\lambda)m(\lambda, \mathrm{G}(z)u, \mathrm{G}(z)v)d\lambda ds$$

converges to the right hand side of (6.8). Then we use (6.6) to show that the difference between the left hand sides of (6.8) and (6.9) converges to 0 as $a \to 0$. □

LEMMA 6.5. — There is a unique bounded linear operator Ω from $\mathscr{H}_{ac}(\mathrm{H}_0)$ to $\mathscr{H}_{ac}(\mathrm{H})$ such that

$$(6.10) \qquad (\Omega\alpha(\mathrm{H}_0)v, \beta(\mathrm{H})v) = \int \alpha(\lambda)\beta(\lambda)^* m(\lambda, \mathrm{G}(\lambda)u, v)d\lambda$$

holds for $u \in \mathrm{X}_0$, $v \in \mathscr{H}_{ac}(\mathrm{H})$ and $\alpha(\lambda)$, $\beta(\lambda)$ bounded and having supports in Λ.

Proof. — For

$$(6.11) \qquad u = \Sigma\alpha_j(\mathrm{H}_0)u_j, \qquad v = \Sigma\beta_j(\mathrm{H})v_j$$

with $u_j \in \mathrm{X}_0$, $v_j \in \mathscr{H}_{ac}(\mathrm{H})$ and $\alpha(\lambda)$, $\beta(\lambda)$ continuous with compact supports in Λ, we define Ω by means of (6.10). For this definition to make sense, we must show that $u = 0$ implies $\Omega u = 0$. But

$$(6.12) \quad (\Omega u, v) = \sum \int \alpha_j(\lambda)m(\lambda, \mathrm{G}(\lambda)u_j, v)d\lambda = \lim \frac{a}{\pi} \int (\mathrm{JR}_0(z)u, \mathrm{R}(z)v)ds$$

by Lemma 6.3. Thus $\Omega u = 0$ when $u = 0$. This also show that

$$|(\Omega u, v)| \leqslant \| \mathrm{J} \| \; \| u \| \; \| v \|$$

showing that Ω is a bounded operator. Finally we note that elements u of the form (6.11) are dense in $\mathscr{H}_{ac}(\mathrm{H}_0)$ by hypotheses *b*. Thus Ω can be defined on the whole of $\mathscr{H}_{ac}(\mathrm{H}_0)$ by continuity. □

LEMMA 6.6. — For $u, v \in X_0$ and $\alpha(\lambda)$, $\beta(\lambda)$ continuous with compact supports in Λ

$$(6.13)\quad (\Omega\alpha(H_0)u, \Omega\beta(H_0)v) = \int \alpha(\lambda)\beta(\lambda)^* m(\lambda, G(\lambda)u, G(\lambda)v)d\lambda$$

Proof. — By (6.10) and (6.4) the left hand side of (6.13) equals

$$\int \alpha(\lambda)m(\lambda, G(\lambda)u, \Omega\beta(H_0)v)d\lambda$$

$$= \lim \frac{a}{\pi} \int (R(z)G(z)u, R(z)\alpha(H)^*\Omega\beta(H_0)v)ds$$

$$= \lim \iint \delta_a(\lambda - s)\alpha(\lambda)\beta(\lambda)^* m(\lambda, G(z)u, G(\lambda)v)d\lambda ds$$

This equals the right hand side of (6.13). □

COROLLARY 6.7. — Under the same hypotheses the right hand side of (6.8) converges to the left hand side of (6.13).

Proof. — Use Lemma 6.4. □

LEMMA 6.8. — For $u \in \mathscr{H}_{ac}(H_0)$, $v \in \mathscr{H}$

$$(6.14)\qquad a\int_0^\infty e^{-at}(W(t)u, v)dt \to (\Omega u, v) \qquad \text{as} \quad a \to 0$$

Proof. — If we replace a by $2a$, the left hand side of (6.14) equals

$$\frac{a}{\pi}\int (JR_0(z)u, R(z)v)ds$$

If u is of the form (6.11), this converges to $(\Omega u, v)$ by (6.12). Since such u are dense in $\mathscr{H}_{ac}(H_0)$, the result follows. □

LEMMA 6.9. — $R(\Omega)$ is dense in $\mathscr{H}(J) \cap \mathscr{H}_{ac}(H)$.

Proof. — Suppose $v \in \mathscr{H}_{ac}(H) \cap \mathscr{H}(J)$ is orthogonal to $R(\Omega)$. Then $(\Omega\alpha(H_0)u, \beta(H)v) = 0$ for all $u \in X_0$ and continuous $\alpha(\lambda)$, $\beta(\lambda)$ with compact supports in Λ. By Lemma 6.5

$$\int \alpha(\lambda)\beta(\lambda)^* m(\lambda, G(\lambda)u, v)d\lambda = 0$$

for all such u, α, β. Thus

$$(6.15)\qquad m(\lambda, G(\lambda)u, v) = 0 \qquad \text{a. e.,} \qquad u \in X_0$$

Since $R(G(\lambda)) = X$, we see that

$$(6.16)\qquad m(\lambda, w, v) = 0 \qquad \text{a. e.,} \qquad w \in X$$

Since

$$(\alpha(\mathrm{H})w, v) = \int \alpha(\lambda)m(\lambda, w, v)d\lambda$$

we see that v is orthogonal to all elements of the form $\alpha(\mathrm{H})w$, with $w \in \mathrm{X}$. Since linear combinations of such elements are dense in $\mathscr{H}(\mathrm{J}) \cap \mathscr{H}_{ac}(\mathrm{H})$ (hypotheses *b*), v is orthogonal to $\mathscr{H}(\mathrm{J}) \cap \mathscr{H}_{ac}(\mathrm{H})$. This gives the desired conclusion. □

LEMMA 6.10. — For $u \in \mathscr{H}_{ac}(\mathrm{H}_0)$

$$a \int e^{-at} \| \mathrm{J}u_{0t} \|^2 dt \rightarrow \| \Omega u \|^2 \quad \text{as} \quad a \rightarrow 0 \tag{6.17}$$

Proof. — If u is of the form (6.11), this is an immediate consequence of Corollary 6.7. Since such u are dense in $\mathscr{H}_{ac}(\mathrm{H}_0)$, the result follows. □

We have essentially given the

Proof of Theorem 6.1. — Everything has been proved except possibly (6.2). But this follows from Lemma 6.8 and 6.10. In fact the left hand side of (6.2) is

$$a \int_0^\infty e^{-at}(\| \mathrm{J}f_{0t} \|^2 - 2 \,\mathrm{Re}\, (\mathrm{W}(t)f, \Omega f) + \| \Omega f \|^2)dt$$

which converges to 0 by (6.14) and (6.17). □

Next we turn our attention to Theorem 6.2.

LEMMA 6.11. — For $\alpha(s) \in \mathrm{C}_0^\infty(\Lambda)$, $u, v \in \mathscr{H}_{ac}(\mathrm{H})$ we have

$$\int \alpha(s)(\mathrm{R}(z)u, v)ds \rightarrow -2\pi i(\alpha_+(\mathrm{H})u, v) \quad \text{as} \quad a \rightarrow 0 \tag{6.18}$$

where

$$\alpha_+(\lambda) = \frac{1}{2\pi} \int_0^\infty e^{-i\lambda\sigma} \int e^{is\sigma}\alpha(s)dsd\sigma \tag{6.19}$$

Proof. — The left hand side of (6.18) equals

$$\int \mathrm{L}(\lambda)m(\lambda, u, v)d\lambda$$

where

$$\mathrm{L}(\lambda) = \int \alpha(s)(z - \lambda)^{-1}ds = -i \int \alpha(s) \int_0^\infty e^{i(z-\lambda)t}dtds$$

$$= -i \int_0^\infty e^{-i\lambda t - at} \int e^{ist}\alpha(s)ds$$

This converges to the right hand side of (6.19) as $a \rightarrow 0$. □

LEMMA 6.12. — There is a locally bounded function $M(\lambda)$ on Λ with values in $B(X)$ such that

$$m(\lambda, u, v) = (M(\lambda)u, v)_X, \qquad u, v \in X \tag{6.20}$$

and

$$\int \alpha(s)(G(z)u, R(z)\beta(H)v)ds \to 2\pi i \int_0^\infty (F\alpha(\cdot)G(\cdot)u, F\beta(\cdot)M(\cdot)v)\, d\lambda \tag{6.21}$$

as $a \to 0$, where F is the Fourier transform

$$Fw(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda s} w(s)ds \tag{6.22}$$

Proof. — The existence of $M(\lambda)$ satisfying (6.20) follows from the fact that X is a Hilbert space, hypothesis a and the Riesz representation theorem. Now

$$\begin{aligned}
&\int \alpha(s)^*(R(z)\beta(H)v, G(z)u)ds \\
&\quad = \int \alpha(s)^* \int (z-\lambda)^{-1}\beta(\lambda)m(\lambda, v, G(z)u)d\lambda ds \\
&\quad = -i \int_0^\infty \alpha(s)^* \int e^{i(z-\lambda)t}\beta(\lambda)(M(\lambda)v, G(z)u)_X d\lambda ds dt \\
&\quad = -i \int_0^\infty e^{-at} \left(\int e^{-i\lambda t}\beta(\lambda)M(\lambda)v d\lambda, \int e^{-ist}\alpha(s)G(z)u ds \right)_X dt
\end{aligned}$$

This converges to the conjugate of the right hand side of (6.21) as $a \to 0$. □

LEMMA 6.13. —

$$(\Omega\alpha(H_0)u, \beta(H)v) = (J\alpha_+(H_0)u, \beta(H)v) + \frac{1}{2\pi i} T \tag{6.23}$$

where T is the right hand side of (6.21).

Proof. — We have

$$\begin{aligned}
a(JR_0(z)u, R(z)v) &= a(R(z)G(z)u, R(z)v) \\
&= \frac{1}{2} i([R(z) - R(\bar{z})]G(z)u, v) \\
&= \frac{1}{2} i(JR_0(z)u, v) - \frac{1}{2} i(G(z)u, R(z)v)
\end{aligned} \tag{6.24}$$

Thus

$$\begin{aligned}
a \int \alpha(s)(JR_0(z)u, R(z)\beta(H)v)ds &= \frac{1}{2} i \int \alpha(s)(JR_0(z)u, \beta(H)v)ds \\
&\quad - \frac{1}{2} i \int \alpha(s)(G(z)u, R(z)\beta(H)v)ds
\end{aligned} \tag{6.25}$$

Let $a \to 0$ and use Lemmas 6.3, 6.5, 6.11 and 6.12. □

LEMMA 6.14. — If $\alpha_-(\lambda) = \alpha(\lambda) - \alpha_+(\lambda)$ and $\alpha^t(\lambda) = e^{-it\lambda}\alpha(\lambda)$, then

$$(6.26) \qquad \alpha^t_-(\lambda) \to 0 \quad \text{as} \quad t \to \infty$$

and

$$(6.27) \qquad \alpha^t_-(H)u \to 0 \quad \text{as} \quad t \to \infty, \qquad u \in \mathscr{H}_{ac}(H)$$

Proof. — By (6.19)

$$\alpha^t_-(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{0} e^{-i\lambda\sigma}\int e^{is(\sigma-t)}\alpha(s)ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-t} e^{-i\lambda(\tau+t)}\bar{F}\alpha(\tau)d\tau$$

where $\bar{F}$ denotes the inverse Fourier transform. This proves (6.26). Since

$$\| \alpha^t_-(H)u \|^2 = \int | \alpha^t_-(\lambda) |^2 m(\lambda, u)d\lambda$$

and the $\alpha^t_-(\lambda)$ are bounded uniformly in t, we obtain (6.27). □

We are now ready for the

Proof of Theorem 6.2. — By Lemma 6.13

$$(6.28) \quad ([\Omega - J]\alpha^t(H_0)u, \beta^t(H)v) = -(\alpha^t_-(H_0)u, \beta^t(H)v) + \int_t^{\infty} (F\alpha(\cdot)G(\cdot)u, F\beta(\cdot)M(\cdot)v)_X d\lambda$$

By Lemma 6.14, the first term on the right of (6.28) tends to 0 as $t \to \infty$. The same is true of the second. The limit (6.3) now follows by (6.10). Now suppose (6.4) holds. Then

$$\| (\Omega - J)f_{0t} \|^2 = \| \Omega f \|^2 + \| Jf_{0t} \|^2 - 2 \operatorname{Re} (\Omega f_{0t}, Jf_{0t})$$

Thus by (6.3)

$$(6.29) \qquad \limsup \| (\Omega - J)f_{0t} \|^2 = \limsup \| Jf_{0t} \|^2 - \| \Omega f \|^2$$

This vanishes by (6.3) and Lemma 6.10. We now apply (6.10) again to obtain the existence of the wave operator $W(H, H_0, J)$. □

BIBLIOGRAPHY

[*1*] Tosio KATO, *Perturbation Theory for Linear Operators*, Springer, 1966.

[*2*] A. L. BELOPOLSKII and M. S. BIRMAN, The existence of wave operators in scattering theory for pairs of spaces, *Izv. Akad. Nauk SSSR*, t. **32**, 1968, p. 1162-1175.

[*3*] Tosio KATO, Scattering theory with two Hilbert spaces, *J. Func. Anal.*, t. **1**, 1967, p. 342-369.

[*4*] Tosio KATO, Two spaces scattering theory, *J. Fac. Sci. Univ. Tokyo*, t. **24**, 1977, p. 503-514.

[*5*] M. S. BIRMAN, Scattering problems for differential operators with perturbation of space, *Izv. Akad. Nauk SSSR*, t. **35**, 1971, p. 440-445.

[*6*] V. G. DEIC, The local stationary method in the theory of scattering with two spaces, *Dokl. Akad. Nauk SSSR*, t. **197**, 1971, p. 1247-1250.

[7] V. G. DEIC, Application of the method of nuclear perturbation in two space scattering theory, *Izv. Vyss. Ucebn. Zaved*, 1971, p. 33-42.
[8] Martin SCHECHTER, A unified approach to scattering, *J. Math. Pures Appl.*, t. **53**, 1974, p. 373-396.
[9] Tosio SUZUKI, Scattering theory for a certain nonsefadjoint operator, *Mem. Fac. L. b. Arts Educ.*, t. **23**, 1974, p. 14-18.
[10] J. R. SCHULENBERGER and C. H. WILCOX, Completeness of the wave operators for perturbations of uniformly propagative systems, *J. Funk. Anal.*, t. **7**, 1971, p. 447-474.
[11] Tosio KATO and S. T. KURODA, The abstract theory of scattering, *Rocky Mtn. J. Math.*, t. **1**, 1971, p. 127-171.
[12] Tosio KATO, Wave operators and similarity for some nonselfadjoint operators, *Math. Ann.*, t. **162**, 1966, p. 258-279.
[13] R. B. LAVINE, Commutators and scattering theory II, *Indiana Univ. Math. J.*, t. **21**, 1972, p. 643-656.

(Manuscrit reçu le 7 juillet 1978)