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Variational principle for quasi-local algebras over the lattice

by

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ABSTRACT. — It is shown that a variational principle holds for certain quasi-local algebras over the lattice.

1. INTRODUCTION AND DEFINITION

In general, as in Ruelle's book [*3*, 6.2.4], to describe the infinite systems of statistical mechanics over the lattice Z^ν, we associate finite-dimensional algebras $\mathfrak{A}_\Lambda$ with finite subsets Λ of Z^ν and we assume that:

a) If $\Lambda \subset \Lambda'$, an (identity-preserving) isomorphism $\alpha_{\Lambda'\Lambda}$ of $\mathfrak{A}_\Lambda$ into $\mathfrak{A}_{\Lambda'}$ is given such that if $\Lambda \subset \Lambda' \subset \Lambda''$, then $\alpha_{\Lambda''\Lambda} = \alpha_{\Lambda''\Lambda'} \circ \alpha_{\Lambda'\Lambda}$.

b) An isomorphism τ_n^Λ of $\mathfrak{A}_\Lambda$ onto $\mathfrak{A}_{\Lambda+n}$ is given for each translation $n \in Z^\nu$ and each Λ such that $\tau_{n+m}^\Lambda = \tau_n^{\Lambda+m} \circ \tau_m^\Lambda$; and if $\Lambda \subset \Lambda'$, then

$$\tau_n^{\Lambda'} \circ \alpha_{\Lambda'\Lambda} = \alpha_{\Lambda'+n,\Lambda+n} \circ \tau_n^\Lambda .$$

By using *a*) we define the C*-inductive limit $\mathfrak{A}$ of the family $\{ \mathfrak{A}_\Lambda, \alpha_{\Lambda'\Lambda} \}$, i. e. we have a unique C*-algebra $\mathfrak{A}$ and isomorphisms α_Λ of $\mathfrak{A}_\Lambda$ into $\mathfrak{A}$ such that if $\Lambda \subset \Lambda'$, $\alpha_{\Lambda'} \circ \alpha_{\Lambda'\Lambda} = \alpha_\Lambda$; and the union of $\alpha_\Lambda(\mathfrak{A}_\Lambda)$ is dense in $\mathfrak{A}$. By using *b*) we define a homomorphism τ of the group Z^ν into the automorphism group of $\mathfrak{A}$ such that $\tau_n \circ \alpha_\Lambda = \alpha_{\Lambda+n} \circ \tau_n^\Lambda$. The triple $(\mathfrak{A}, Z^\nu, \tau)$ is called the « quasi-local » algebra constructed from the « local » algebras $\mathfrak{A}_\Lambda$.

We further assume the following properties, the first part of which is commonly assumed:

c) If $\Lambda_1 \cap \Lambda_2 = \phi$, then $\alpha_{\Lambda\Lambda_1}(\mathfrak{A}_{\Lambda_1})$ is in the commutant of $\alpha_{\Lambda\Lambda_2}(\mathfrak{A}_{\Lambda_2})$, and $\alpha_{\Lambda\Lambda_1}(\mathfrak{A}_{\Lambda_1})$ and $\alpha_{\Lambda\Lambda_2}(\mathfrak{A}_{\Lambda_2})$ generate a subalgebra of $\mathfrak{A}_\Lambda$ whose relative commutant is its center, where $\Lambda = \Lambda_1 \cup \Lambda_2$.

The second part of *c*) also seems quite natural. It requires that observables in $\mathfrak{A}_\Lambda$ which commute with all strictly local ones, i. e. elements in $\alpha_{\Lambda\{n\}}(\mathfrak{A}_{\{n\}})$, $n \in \Lambda$, must be generated by strictly local observables.

Now, if $\Lambda_1 \cap \Lambda_2 = \phi$, we define a homomorphism $\Phi_{\Lambda_1\Lambda_2}$ of $\mathfrak{A}_{\Lambda_1} \otimes \mathfrak{A}_{\Lambda_2}$ into $\mathfrak{A}_\Lambda$ by:

$$\Phi_{\Lambda_1\Lambda_2}(\Sigma a_i \otimes b_i) = \Sigma \alpha_{\Lambda_1\cup\Lambda_2,\Lambda_1}(a_i)\alpha_{\Lambda_1\cup\Lambda_2,\Lambda_2}(b_i).$$

It is easily shown that $\Phi_{\Lambda_1\Lambda_2}$ is well-defined and satisfies:

*c'*1) the restriction of $\Phi_{\Lambda_1\Lambda_2}$ to $\mathfrak{A}_{\Lambda_1}$ (identified with $\mathfrak{A}_{\Lambda_1} \otimes 1$) is $\alpha_{\Lambda_1\cup\Lambda_2,\Lambda_1}$;

*c'*2) if Λ_1, Λ_2 and Λ_3 are mutually disjoint,

$$\Phi_{\Lambda_1\cup\Lambda_2,\Lambda_3} \circ \Phi_{\Lambda_1\Lambda_2} \otimes l = \Phi_{\Lambda_1,\Lambda_2\cup\Lambda_3} \circ l \otimes \Phi_{\Lambda_2\Lambda_3}$$

where l is the identity isomorphism;

*c'*3) the quotient $\mathfrak{A}_{\Lambda_1} \otimes \mathfrak{A}_{\Lambda_2}/\ker \Phi_{\Lambda_1\Lambda_2}$ is mapped into $\mathfrak{A}_{\Lambda_1\cup\Lambda_2}$ with multiplicity 1 under the induced isomorphism; and

*c'*4) $\tau_n^{\Lambda_1\cup\Lambda_2} \circ \Phi_{\Lambda_1\Lambda_2} = \Phi_{\Lambda_1+n,\Lambda_2+n} \circ \tau_n^{\Lambda_1} \otimes \tau_n^{\Lambda_2}$ for $n \in Z^\nu$.

In the rest of this note we show that the variational principle holds for the systems satisfying *a*), *b*) and *c*). In particular we show that if $\mathfrak{A}_\Lambda$ is generated by $\alpha_{\Lambda\{n\}}(\mathfrak{A}_{\{n\}})$, $n \in \Lambda$ for any Λ, then the systems are classical (or rather semi-quantum) and the ones considered by Ruelle, i. e. the systems, restricted to closed invariant subsets of the whole configuration space (see the announcement in [*4*]).

In section 2, we derive some results on the algebra $\mathfrak{A}$ from the condition *c*) and in section 3 we show the existence of thermodynamic quantities. The main result is shown in section 4 and examples are given in section 5.

2. STRUCTURE OF $\mathfrak{A}$

Suppose *a*), *b*) and *c*). If $\Lambda_1, \ldots, \Lambda_k$ are mutually disjoint, we inductively define a homomorphism $\Phi_{\Lambda_1,\ldots,\Lambda_k}$ of $\mathfrak{A}_{\Lambda_1} \otimes \ldots \otimes \mathfrak{A}_{\Lambda_k}$ into $\mathfrak{A}_\Lambda$ with $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_k$ by $\Phi_{\Lambda_1\cup\ldots\cup\Lambda_{k-1},\Lambda_k} \circ \Phi_{\Lambda_1,\ldots,\Lambda_{k-1}} \otimes l$, where if $k = 2$, $\Phi_{\Lambda_1} = l$ and $\Phi_{\Lambda_1\Lambda_2}$ is already defined. By using *c'*2) we have the identity:

$$\begin{aligned}\Phi_{\Lambda'\cup\Lambda_{k-1},\Lambda_k} \circ \Phi_{\Lambda_1,\ldots,\Lambda_{k-1}} \otimes l \\ &= \Phi_{\Lambda'\cup\Lambda_{k-1},\Lambda_k} \circ \Phi_{\Lambda',\Lambda_{k-1}} \otimes l \circ \Phi_{\Lambda_1,\ldots,\Lambda_{k-2}} \otimes l \otimes l \\ &= \Phi_{\Lambda',\Lambda_{k-1}\cup\Lambda_k} \circ \Phi_{\Lambda_1,\ldots,\Lambda_{k-2}} \otimes \Phi_{\Lambda_{k-1},\Lambda_k}\end{aligned}$$

where $\Lambda' = \Lambda_1 \cup \ldots \cup \Lambda_{k-2}$. Thus the homomorphism $\Phi_{\Lambda_1}, \ldots, \Lambda_k$ does not depend on the order of $\{ \Lambda_{k-1}, \Lambda_k \}$, and so, inductively, on the order of $\{ \Lambda_1, \ldots, \Lambda_k \}$. Note that the properties as in *c'*1), *c'*3) and *c'*4) still hold for $\Phi_{\Lambda_1,\ldots,\Lambda_k}$.

For a finite $\Lambda = \{ n_1, \ldots, n_k \}$ let $\tilde{\mathfrak{A}}_\Lambda$ be the tensor product of $\mathfrak{A}_{\{n_i\}}$, $i = 1, \ldots, k$, and let $\tilde{\Phi}_\Lambda = \Phi_{\{n_1\},\ldots,\{n_k\}}$. If $\Lambda \subset \Lambda'$, we have that $\alpha_{\Lambda'\Lambda} \circ \tilde{\Phi}_\Lambda = \tilde{\Phi}_{\Lambda'} \circ l$ where l is the natural embedding of $\tilde{\mathfrak{A}}_\Lambda$ into $\tilde{\mathfrak{A}}_{\Lambda'}$. In more general we have the following commutative diagram : if $\Lambda_1 \cap \Lambda_2 = \phi$,

$$\begin{array}{ccc} \tilde{\mathfrak{A}}_{\Lambda_1} \otimes \tilde{\mathfrak{A}}_{\Lambda_2} & = & \tilde{\mathfrak{A}}_{\Lambda_1 \cup \Lambda_2} \\ {\scriptstyle \tilde{\Phi}_{\Lambda_1} \otimes \tilde{\Phi}_{\Lambda_2}} \downarrow & & \downarrow {\scriptstyle \tilde{\Phi}_{\Lambda_1 \cup \Lambda_2}} \\ \mathfrak{A}_{\Lambda_1} \otimes \mathfrak{A}_{\Lambda_2} & \xrightarrow{\Phi_{\Lambda_1 \Lambda_2}} & \mathfrak{A}_{\Lambda_1 \cup \Lambda_2} \end{array}$$

This is shown by the induction on the cardinality of $\Lambda_1 \cup \Lambda_2$, and by the identities:

$$\begin{aligned} \Phi_{\Lambda_1\Lambda_2} \circ (\tilde{\Phi}_{\Lambda_1} \otimes \tilde{\Phi}_{\Lambda_2}) &= \Phi_{\Lambda_1\Lambda_2} \circ (\tilde{\Phi}_{\Lambda_1} \otimes (\Phi_{\Lambda_2'\{n\}} \circ \tilde{\Phi}_{\Lambda_2'} \otimes l)) \\ &= \Phi_{\Lambda_1\Lambda_2} \circ (l \otimes \Phi_{\Lambda_2'\{n\}}) \circ (\tilde{\Phi}_{\Lambda_1} \otimes \tilde{\Phi}_{\Lambda_2'} \otimes l) \\ &= \Phi_{\Lambda_1 \cup \Lambda_2' \{n\}} \circ ((\Phi_{\Lambda_1\Lambda_2'} \circ \tilde{\Phi}_{\Lambda_1} \otimes \tilde{\Phi}_{\Lambda_2'}) \otimes l) \end{aligned}$$

where $n \in \Lambda_2$ and $\Lambda_2' = \Lambda_2 \backslash \{ n \}$.

Let $\tilde{\mathfrak{A}}$ be the C*-inductive limit of $\tilde{\mathfrak{A}}_\Lambda$. We have a homomorphism $\tilde{\Phi}$ of $\tilde{\mathfrak{A}}$ into $\mathfrak{A}$ such that $\alpha_\Lambda \circ \tilde{\Phi}_\Lambda = \tilde{\Phi}$ on $\tilde{\mathfrak{A}}_\Lambda$. If $\tau^{\tilde{\mathfrak{A}}}$ denotes the action of Z^ν on $\tilde{\mathfrak{A}}$ extending $\bigotimes_{m \in \Lambda} \tau_n^{\{m\}}$ of $\tilde{\mathfrak{A}}_\Lambda$ into $\tilde{\mathfrak{A}}_{\Lambda+n}$, we have that $\tau_n \circ \tilde{\Phi} = \tilde{\Phi} \circ \tau_n^{\tilde{\mathfrak{A}}}$.

THEOREM 1. — Let $\mathfrak{A}_\Lambda$ satisfy *a*), *b*) and *c*). Further suppose that $\mathfrak{A}_\Lambda$ is generated by $\alpha_{\Lambda\{n\}}(\mathfrak{A}_{\{n\}})$, $n \in \Lambda$, for any Λ. Then the quasi-local algebra $(\mathfrak{A}, Z^\nu, \tau)$ is isomorphic to $(\tilde{\mathfrak{A}}/I, Z^\nu, \tau')$, where I is the kernel of $\tilde{\Phi}$ and τ' is the induced action on the quotient algebra $\tilde{\mathfrak{A}}/I$ from $\tau^{\tilde{\mathfrak{A}}}$.

If $\mathfrak{A}_{\{n\}}$ is commutative, say the algebra C(F) of (continuous) functions on a finite set F, then $\tilde{\mathfrak{A}} \simeq C(F^{Z^\nu})$ and $\tilde{\mathfrak{A}}/I \simeq C(\Omega)$ where

$$\Omega = \{ x \in F^{Z^\nu}; f(x) = 0, f \in I \}$$

is a translation invariant closed set. Hence, this system has a good thermodynamic property (cf. [*4*]). This is easily extended to the « semi-quantum » case, i. e. the case that $\mathfrak{A}_{\{n\}}$ is not commutative. In particular, we have a theorem similar to [*2*, 8.3].

In general there is a unique projection p of norm 1 of $\mathfrak{A}$ onto $\tilde{\Phi}(\tilde{\mathfrak{A}})$, such that $p(\mathfrak{A}_\Lambda) = \tilde{\Phi}(\tilde{\mathfrak{A}}_\Lambda)$. This is shown by using the fact

$$\mathfrak{A}_\Lambda \cap \tilde{\Phi}(\tilde{\mathfrak{A}}_\Lambda)' = \tilde{\Phi}(\tilde{\mathfrak{A}}_\Lambda) \cap \tilde{\Phi}(\tilde{\mathfrak{A}}_\Lambda)'.$$

Now we associate central projections e_Λ of $\tilde{\mathfrak{A}}_\Lambda$ with finite Λ such that $\ker \tilde{\Phi}_\Lambda = (1 - e_\Lambda)\tilde{\mathfrak{A}}_\Lambda$. If $\Lambda_1 \cap \Lambda_2 = \phi$, we have that $e_\Lambda \otimes e_{\Lambda_2} \geq e_{\Lambda_1 \cup \Lambda_2}$ since the kernel of $\tilde{\Phi}_{\Lambda_1 \cup \Lambda_2}$ is larger than that of $\tilde{\Phi}_{\Lambda_1} \otimes \Phi_{\Lambda_2}$. In particular, if $\Lambda \subset \Lambda'$, then $e_{\Lambda'} \leq e_\Lambda$ in $\tilde{\mathfrak{A}}$. And the kernel of $\tilde{\Phi}$ is generated by $1 - e_\Lambda$, with finite Λ.

Further we have that if $\Lambda_1 \cap \Lambda_2 = \phi$, $\tilde{\Phi}_{\Lambda_1} \otimes \tilde{\Phi}_{\Lambda_2}(e_{\Lambda_1 \cup \Lambda_2})$ is in the center of $\mathfrak{A}_{\Lambda_1} \otimes \mathfrak{A}_{\Lambda_2}$ and that

$$\ker \Phi_{\Lambda_1\Lambda_2} = \tilde{\Phi}_{\Lambda_1} \otimes \tilde{\Phi}_{\Lambda_2}(e_{\Lambda_1} \otimes e_{\Lambda_2} - e_{\Lambda_1 \cup \Lambda_2}) \cdot \mathfrak{A}_{\Lambda_1} \otimes \mathfrak{A}_{\Lambda_2}$$

Let $\mathscr{B}_{\{n\}}$ be a factor containing $\mathfrak{A}_{\{n\}}$ with multiplicity 1. Let $\mathscr{B}_\Lambda = \bigotimes_{n\in\Lambda} \mathscr{B}_{\{n\}}$ and let $\widehat{\mathscr{B}}_\Lambda = e_\Lambda \mathscr{B}_\Lambda e_\Lambda$. There is a unique subalgebra $\widehat{\mathfrak{A}}_\Lambda$ of $\widehat{\mathscr{B}}_\Lambda$ such that $\widehat{\mathfrak{A}}_\Lambda$ is isomorphic to $\mathfrak{A}_\Lambda$ by an isomorphism extending $\tilde{\Phi}_\Lambda$ of $\tilde{\mathfrak{A}}_\Lambda e_\Lambda (\subset \widehat{\mathscr{B}}_\Lambda)$ into $\mathfrak{A}_\Lambda$. If $\Lambda \subset \Lambda'$, there is a natural embedding of $\widehat{\mathfrak{A}}_\Lambda$ into $\widehat{\mathfrak{A}}_{\Lambda'}$ given by the multiplication of $e_{\Lambda'}$. At this point we do not know if there are isomorphisms β_Λ of $\mathfrak{A}_\Lambda$ onto $\widehat{\mathfrak{A}}_\Lambda$ satisfying the obvious consistency relations. But examples given in section 5 have the structure of $(\widehat{\mathfrak{A}}_\Lambda)$. So we give a remark: let $\mathscr{B}$ be the C*-inductive limit of $\mathscr{B}_\Lambda$ and let $\mathscr{D}$ be the C*-subalgebra generated by $\widehat{\mathfrak{A}}_\Lambda$ with all Λ, in $\mathscr{B}$. Let I be the ideal of $\mathscr{D}$ generated by $1 - e_\Lambda$ with all Λ. Then the C*-inductive limit $\widehat{\mathfrak{A}}$ of $\widehat{\mathfrak{A}}_\Lambda$ is isomorphic to the quotient $\mathscr{D}/\mathrm{I}$ (the action of Z^ν on $\widehat{\mathfrak{A}}$ should be the induced one from the natural translations on $\mathscr{B}$). Hence, $\widehat{\mathfrak{A}}$ has a good thermodynamic property, too, as maybe shown in the same way as in the classical case.

If $\Lambda_1, \ldots, \Lambda_k$ are mutually disjoint and if β_{Λ_i} are isomorphisms of $\mathfrak{A}_{\Lambda_i}$ onto $\widehat{\mathfrak{A}}_{\Lambda_i}$ with $\tilde{\Phi}_{\Lambda_i} \circ \beta_{\Lambda_i} = l$ on $\tilde{\Phi}(\tilde{\mathfrak{A}}_\Lambda)$, it is shown that there is an isomorphism β_Λ of $\mathfrak{A}_\Lambda$ onto $\widehat{\mathfrak{A}}_\Lambda$ with $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_k$ such that $\beta \circ \Phi_{\Lambda_1,\ldots,\Lambda_k} = e_\Lambda \cdot \beta_{\Lambda_1} \otimes \ldots \otimes \beta_{\Lambda_k}$. For instance if $k = 2$, and if $x \in \mathfrak{A}_{\Lambda_1} \otimes \mathfrak{A}_{\Lambda_2}$, $e_\Lambda \cdot \beta_{\Lambda_1} \otimes \beta_{\Lambda_2}(x) = \beta_{\Lambda_1} \otimes \beta_{\Lambda_2}(e_\Lambda x)$. So if $\Phi_{\Lambda_1\Lambda_2} \cdot e_\Lambda$ denotes the isomorphism of $e_\Lambda \mathfrak{A}_{\Lambda_1} \otimes \mathfrak{A}_{\Lambda_2}$ into $\mathfrak{A}_\Lambda$, β_Λ is an extension of $\beta_{\Lambda_1} \otimes \beta_{\Lambda_2} \circ (\Phi_{\Lambda_1\Lambda_2} \cdot e_\Lambda)^{-1}$.

Let t be the unique tracial state of $\mathscr{B}$. We define a trace t_Λ on $\mathfrak{A}_\Lambda$ by $t \circ \beta_\Lambda$, which does not depend on β_Λ, and takes the same value on each minimal projection of $\mathfrak{A}_\Lambda$. Note that $t_{\Lambda'}(1)^{-1} t_{\Lambda'} \circ \alpha_{\Lambda'\Lambda} = t_\Lambda(1)^{-1} t_\Lambda$ does not hold in general ($\Lambda \subset \Lambda'$).

3. THERMODYNAMIC QUANTITIES

Let ω be a translation invariant state of $\mathfrak{A}$. For each Λ let $\rho_\Lambda = \rho_\Lambda(\omega)$ be an element of $\mathfrak{A}_\Lambda$ satisfying that $t_\Lambda(\rho_\Lambda \mathrm{A}) = \omega(\mathrm{A})$ for all $\mathrm{A} \in \mathfrak{A}_\Lambda$ and set

$S(\Lambda) = - t_\Lambda(\rho_\Lambda \log \rho_\Lambda)$. If $\Lambda_1 \cap \Lambda_2 = \phi$, we have the subadditivity $S(\Lambda_1 \cup \Lambda_2) \leq S(\Lambda_1) + S(\Lambda_2)$, by the inequality [3, 2.5.3] :

$$t(\beta_\Lambda(\rho_\Lambda) \log \beta_\Lambda(\rho_\Lambda)) - t(\beta_\Lambda(\rho_\Lambda) \log \beta_{\Lambda_1}(\rho_{\Lambda_1}) \otimes \beta_{\Lambda_2}(\rho_{\Lambda_2})) \geq 0$$

where $\Lambda = \Lambda_1 \cup \Lambda_2$, and β_Λ, β_{Λ_1} and β_{Λ_2} satisfy that

$$\beta_\Lambda \circ \Phi_{\Lambda_1\Lambda_2} = e_\Lambda . \beta_{\Lambda_1} \otimes \beta_{\Lambda_2}.$$

Hence, we can define the mean entropy:

$$s(\omega) = \lim_{N} \mid \Lambda(N) \mid^{-1} S(\Lambda(N))$$

where $\Lambda(N) = \{ n \in Z^\nu ; 0 \leq n_i < N_i \}$ for $N \in Z^\nu$ with $N_i > 0$ and $\mid \Lambda(N) \mid$ is the cardinality of $\Lambda(N)$ (cf. [3, 7.2.11]).

Let Φ be a (translation-invariant) potential in $\mathfrak{A}$, i. e. Φ is a family of $\Phi(\Lambda) \in \mathfrak{A}_\Lambda$ with all non-empty finite subsets Λ of Z^ν satisfying that $\Phi(\Lambda)^* = \Phi(\Lambda)$, $\tau_n\Phi(\Lambda) = \Phi(\Lambda + n)$ and $\| \Phi \| \equiv \Sigma_{\Lambda \ni 0} \mid \Lambda \mid^{-1} \| \Phi(\Lambda) \| < \infty$. We set

$$U_\Lambda = U_\Lambda^\Phi = \sum_{\Gamma \subset \Lambda} \alpha_{\Lambda\Gamma}(\Phi(\Gamma)),$$

$$p_\Lambda = p_\Lambda(\Phi) = \mid \Lambda \mid^{-1} t_\Lambda(e^{-U_\Lambda})$$

If $\Lambda_1, \ldots, \Lambda_k$ are mutually disjoint, we have

$$t_\Lambda\left(\exp\left(-\sum_1^k \alpha_{\Lambda\Lambda_i}(U_{\Lambda_i})\right)\right) = t\left(e_\Lambda . \exp\left(-\sum \beta_{\Lambda_i}(U_{\Lambda_i})\right)\right)$$

$$\leq t\left(\exp\left(-\sum \beta_{\Lambda_i}(U_{\Lambda_i})\right)\right) = \prod_1^k t_{\Lambda_i}(e^{-U_{\Lambda_i}})$$

where $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_k$, and $\beta_\Lambda \circ \Phi_{\Lambda_1 \ldots \Lambda_k} = e_\Lambda . \beta_{\Lambda_1} \otimes \ldots \otimes \beta_{\Lambda_k}$.

If Φ is of finite range, we can show, as in the proof of [3, 2.3.1], that

$$p_{\Lambda(M)} \leq p_{\Lambda(N)} + \varepsilon_N + \delta_M(N)$$

where ε_N tends to zero (independently of M) as $N \to \infty$ and $\delta_M(N)$ tends to zero for each N as $M \to \infty$. Thus we have

$$\sup \lim p_{\Lambda(M)} \leq \inf \lim p_{\Lambda(N)}.$$

By the same reasoning as in [3, 2.3.3] we have the pressure $p(\Phi) = \lim p_{\Lambda(N)}(\Phi)$ for any Φ (with $\| \Phi \| < \infty$).

From the special case $\Phi = 0$, we have that $p(0) = \lim \mid \Lambda \mid^{-1} \log t_\Lambda(1)$. Hence, replacing t_Λ by the normalized $t_\Lambda(1)^{-1} t_\Lambda$ in the definition of entropy and pressure implies replacing $s(\omega)$ by $s(\omega) - p(0)$ and $p(\Phi)$ by $p(\Phi) - p(0)$.

For any invariant state ω of $\mathfrak{A}$ and any potential Φ, as easily shown, we have the mean energy

$$\omega(A_\Phi) = \lim \mid \Lambda(N) \mid^{-1} \omega\big(U^{\Phi}_{\Lambda(N)}\big)$$

where

$$A_\Phi = \sum_{\Lambda \ni 0} \mid \Lambda \mid^{-1} \Phi(\Lambda).$$

4. VARIATIONAL PRINCIPLE

Let ω be a translation invariant state of $\mathfrak{A}$ and let Φ be a potential. For each Λ we have

$$\log t_\Lambda\Big(e^{-U^{\Phi}_{\Lambda}}\Big) \geq - t_\Lambda(\rho_\Lambda(\omega) \log \rho_\Lambda(\omega)) - \omega\big(U^{\Phi}_{\Lambda}\big)$$

Thus, we obtain the variational inequality: $p(\Phi) \geq s(\omega) - \omega(A_\Phi)$.

Let $N \in Z^\nu$ with $N_i > 0$ and let $\mathfrak{A}(N) = \otimes_{n \in Z^\nu} \mathfrak{A}_{\Lambda(N)+nN}$ and so especially $\mathfrak{A}(1, \ldots, 1) = \tilde{\mathfrak{A}}$. In the same way as to construct $\tilde{\Phi}$ in section 2, we have a homomorphism Φ_N of $\mathfrak{A}(N)$ into $\mathfrak{A}$, extending

$$\Phi_{\Lambda(N)+n_1 N, \ldots, \Lambda(N)+n_k N} \; (\{ n_1, \ldots, n_k \} \subset Z^\nu).$$

Furthermore, we have the natural action of NZ^ν on $\mathfrak{A}(N)$ such that $\tau_{Nn} \circ \Phi_N = \Phi_N \circ \tau_{Nn}$.

Let M be also in Z^ν with $M_i > 0$. We have a homomorphism $\Phi_{N,M}$ of $\mathfrak{A}(N)$ into $\mathfrak{A}(NM)$ given by

$$\bigotimes_n \Phi_{\Lambda(N)+(nM+a)N, \ldots, \Lambda(N)+(nM+b)N}$$

with $\{ a, \ldots, b \} = \Lambda(M)$. We have that $\Phi_{NM} \circ \Phi_{N,M} = \Phi_N$; and

$$\tau_{NMn} \circ \Phi_{N,M} = \Phi_{N,M} \circ \tau_{NMn}.$$

Let Φ be a potential of finite range and let φ_N be a product state of $\mathfrak{A}(N) = \bigotimes_n \mathfrak{A}_{\Lambda(N)+Nn}$ such that

$$\rho_{\Lambda(N)+Nn}(\varphi_N) = e^{-U_{\Lambda(N)+Nn}} / t_{\Lambda(N)+Nn}\big(e^{-U_{\Lambda(N)+Nn}}\big).$$

Then φ_N is $\tau_{N\cdot}$-invariant. Let $\varphi_{N,M} = \varphi_{NM} \circ \Phi_{N,M}$, which is a $\tau_{NM\cdot}$-invariant state of $\mathfrak{A}(N)$, and let $\overline{\varphi}_{N,M}$ be the $\tau_{N\cdot}$-invariant state obtained by averaging $\varphi_{N,M}$ over the translations NZ^ν. Then, by using the product trace of $t_{\Lambda(N)+Nn}$ in the definition of entropy, we have,

$$\begin{aligned} s(\varphi_{NM}) &= \mid \Lambda \mid^{-1} S_\Lambda(\varphi_{NM}) \leq \mid \Lambda \mid^{-1} S_\Lambda(\varphi_{NM} \circ \Phi_{N,M}) \\ &= s(\varphi_{N,M}) = s(\overline{\varphi}_{N,M}) \end{aligned}$$

and

$$s(\varphi_{NM}) = |\Lambda|^{-1}t_\Lambda(e^{-U_\Lambda}) + |\Lambda|^{-1}t_\Lambda(U_\Lambda e^{-U_\Lambda})/t_\Lambda(e^{-U_\Lambda})$$

where $\Lambda = \Lambda(NM)$. A simple argument shows that there is a constant ε_N which tends to zero as $N \to \infty$ such that

$$s(\bar{\varphi}_{N,M}) \geq |\Lambda(NM)|^{-1}t_{\Lambda(NM)}(e^{-U_{\Lambda(NM)}}) + |\Lambda(N)|^{-1}\bar{\varphi}_{N,M}(U_{\Lambda(N)}) - \varepsilon_N.$$

Let ω_N be a weak limit point of $\bar{\varphi}_{N,M}$ as $M \to \infty$. By the upper semicontinuity of $s(.)$ we have

$$s(\omega_N) \geq p(\Phi) - |\Lambda(N)|^{-1}\omega_N(U_{\Lambda(N)}) - \varepsilon_N.$$

For any $m \in Z^\nu$ with $m_i > 0$, $\varphi_{N,M}(e_{\Lambda+nN}) = 1$ with $\Lambda = \Lambda(Nm)$ if $\Lambda + nN \subset \Lambda(NM)$. Thus, we have that $\omega_N(e_{\Lambda+n}) = 1$ for any $n \in NZ^\nu$. Since the kernel of Φ_N is generated by $1 - e_{\Lambda(Nm)+Nn}$, we have a unique ($\tau_{N\cdot}$-invariant) state $\widehat{\omega}_N$ of $\Phi_N(\mathfrak{A}(N))$ such that $\omega_N = \widehat{\omega}_N \circ \Phi_N$. We extend $\widehat{\omega}_N$ to a state of $\mathfrak{A}$, denoted by $\widehat{\omega}_N$ also, by using a unique projection of norm 1 of $\mathfrak{A}$ onto $\Phi_N(\mathfrak{A}(N))$ mapping $\mathfrak{A}_{\Lambda(NM)}$ onto $\Phi_N(\otimes_{m\in\Lambda(M)}\mathfrak{A}_{\Lambda(N)+mN})$. Let $\bar{\omega}_N$ be the τ-invariant state of $\mathfrak{A}$ obtained by averaging $\widehat{\omega}_N$ over Z^ν. Then, we have $s(\bar{\omega}_N) = s(\widehat{\omega}_N) = s(\omega_N)$. Thus,

$$s(\bar{\omega}_N) \geq p(\Phi) - |\Lambda(N)|^{-1}\widehat{\omega}_N(U_{\Lambda(N)}) - \varepsilon_N$$

where $U_{\Lambda(N)}$ is identified with $\alpha_{\Lambda(N)}(U_{\Lambda(N)})$. Again a simple argument shows that $\widehat{\omega}_N$ can be replaced by $\bar{\omega}_N$ in the above inequality with ε_N replaced by a different constant ε'_N tending to zero as $N \to \infty$. If ω is a weak limit point of $\bar{\omega}_N$ as $N \to \infty$, we have

$$s(\omega) \geq p(\Phi) - \omega(A_\Phi).$$

Hence, the equality holds and further this equality holds for any Φ (not only of finite range) (cf. [*3*, 7.4.1]).

THEOREM 2. — Let $(\mathfrak{A}_\Lambda)$ satisfy *a*) *b*) and *c*). Then, the thermodynamic qualities can be defined and the variational principle holds.

5. EXAMPLES

First we give a known example in classical case, i. e. lattice gas with hard core of radius 1. Let $F = \{0, 1\}$. For each finite Λ let Ω_Λ be a subset of F^Λ such that $\Omega_\Lambda = \{\xi \in F^\Lambda; \xi_n\xi_m = 0 \text{ if } |n - m| = 1\}$ where $|n| = \Sigma_1^\nu |n_i|$.

Let $\mathfrak{A}_\Lambda = C(\Omega_\Lambda)$. If $\Lambda \subset \Lambda'$, there is a natural injection of $\mathfrak{A}_\Lambda$ into $\mathfrak{A}_{\Lambda'}$ since the projection of $\Omega_{\Lambda'}$ into F_Λ is Ω_Λ. If $\Lambda_1 \cap \Lambda_2 = \phi$, it follows from $\Omega_{\Lambda_1} \times \Omega_{\Lambda_2} \supset \Omega_{\Lambda_1 \cup \Lambda_2}$ that there is a homomorphism of $\mathfrak{A}_{\Lambda_1} \otimes \mathfrak{A}_{\Lambda_2}$ ($\simeq C(\Omega_{\Lambda_1} \times \Omega_{\Lambda_2})$) onto $\mathfrak{A}_{\Lambda_1 \cup \Lambda_2}$, given by restriction. Further we have all properties given in *a*), *b*) and *c*).

The corresponding quantum model is constructed as follows: we associate a 2×2 matrix algebra $\mathscr{B}_{\{n\}}$ with each $n \in Z^\nu$ such that $\mathscr{B}_{\{n\}} \supset C(F_n)$ with $F_n = F$. Let $\partial\Lambda = \{ n \in \Lambda; \exists m \notin \Lambda \text{ s. t. } | n - m | = 1 \}$. With each Λ and $\xi \in \Omega_{\partial\Lambda}$ we associate a subfactor $\mathfrak{A}_\Lambda^\xi$ of $\mathscr{B}_\Lambda = \bigotimes_{n \in \Lambda} \mathscr{B}_{\{n\}}$, by $\mathfrak{A}_\Lambda^\xi = \chi_{\Omega_\Lambda^\xi} \mathscr{B}_\Lambda \chi_{\Omega_\Lambda^\xi}$ where $\chi_{\Omega_\Lambda^\xi}$ is the characteristic function of $\chi_{\Omega_\Lambda^\xi} = \{ \eta \in \Omega_\Lambda : \eta \mid \partial\Lambda = \xi \}$. Let $\mathfrak{A}_\Lambda$ be the algebra generated by $\mathfrak{A}_\Lambda^\xi$, $\xi \in \Omega_{\partial\Lambda}$; $\mathfrak{A}_\Lambda \simeq \oplus \mathfrak{A}_\Lambda^\xi$.

If $\Lambda \subset \Lambda'$ and $\xi \in \Omega_{\partial\Lambda}$ and $\eta \in \Omega_{\partial\Lambda'}$, the map $\alpha_{\Lambda'\Lambda}$ of $\mathfrak{A}_\Lambda^\xi$ into $\mathfrak{A}_{\Lambda'}^\eta$, is given by: $A \mapsto A\chi_{E_{\xi\eta}}$, where $E_{\xi\eta} = \{ \zeta \in \Omega_{\Lambda' \setminus \Lambda \cup \partial\Lambda}; \zeta \mid \partial_\Lambda = \xi, \zeta \mid \partial_{\Lambda'} = \eta \}$, which may be empty. Since $\bigcup_\eta E_{\xi\eta} \neq \phi$ for each $\xi \in \Omega_{\partial\Lambda}$, this map is injective.

Let $\Lambda_1 \cap \Lambda_2 = \phi$ and let $\xi_1 \in \Omega_{\partial\Lambda_1}$, and $\xi_2 \in \Omega_{\partial\Lambda_2}$. The map $\Phi_{\Lambda_1\Lambda_2}$ of $\mathfrak{A}_{\Lambda_1}^{\xi_1} \otimes \mathfrak{A}_{\Lambda_2}^{\xi_2}$ into $\mathfrak{A}_{\Lambda_1 \cup \Lambda_2}$ is given by: $A \mapsto A$ if $\xi_1 \times \xi_2 \in \Omega_{\partial\Lambda_1 \cup \partial\Lambda_2}$ and $A \mapsto 0$ otherwise. It is easily shown that all properties in *a*), *b*) and *c*) hold.

This is maximal in the sense that if there are a family $(\mathfrak{A}'_\Lambda)$ of local algebras satisfying *a*), *b*) and *c*) and a family (ϕ_Λ) of isomorphisms of $\mathfrak{A}_\Lambda$ into $\mathfrak{A}'_\Lambda$ with multiplicity 1 satisfying the obvious consistency relations, then all ϕ_Λ are surjective.

Hence, we can take as $\mathscr{D}$ in section 2 the C*-subalgebra of $\mathscr{B} = \otimes \mathscr{B}_{\{n\}}$ of elements which commute with all χ_{Ω_Λ}, and as I the ideal of $\mathscr{D}$ generated by all $1 - \chi_{\Omega_\Lambda}$. Then the family $(\mathfrak{A}_\Lambda)$ constructed above is isomorphic to $(q(\mathscr{D} \cap \mathscr{B}_\Lambda))$ where q is the quotient map of $\mathscr{D}$ onto $\mathscr{D}/I$.

We notice that if the distance between $\partial\Lambda$ and $\partial\Lambda'$ is larger than 1 in case $\Lambda \subset \Lambda'$, then $E_{\xi\eta} \neq \phi$ for any $\xi \in \Omega_{\partial\Lambda}$ and $\eta \in \Omega_{\partial\Lambda'}$. Thus each subfactor $\mathfrak{A}_\Lambda^\xi$ of $\mathfrak{A}_\Lambda$ is mapped into each subfactor $\mathfrak{A}_{\Lambda'}^\eta$ of $\mathfrak{A}_{\Lambda'}$. So the C*-inductive limit $\mathfrak{A}$ of $\mathfrak{A}_\Lambda$ is simple [*1*].

Both the classical and quantum models above satisfy: if $\Lambda_1 \cap \Lambda_2 = \phi$ and the distance between Λ_1 and Λ_2 is larger than 1, $\Phi_{\Lambda_1\Lambda_2}$ is an isomorphism of $\mathfrak{A}_{\Lambda_1} \otimes \mathfrak{A}_{\Lambda_2}$ onto $\mathfrak{A}_{\Lambda_1 \cup \Lambda_2}$.

Any finite-dimensional abelian algebra C can be a quasi-local algebra by setting $\mathfrak{A}_\Lambda = C$ for all Λ. This is maximal in the sense above but not simple.

There is an example of local algebras where $(\mathfrak{A}_\Lambda)$ satisfies *a*), *b*) and *c*) except the second part of *c*). Let $\mathscr{B}_\Lambda$ be a usual quantum lattice system and set $\mathfrak{A}_\Lambda = \mathscr{B}_{\Lambda^0}$ (or $\mathscr{B}_\Lambda \otimes \mathscr{B}_{\Lambda^0}$) with $\mathscr{B}_\phi = \mathbb{C}.1$, where Λ^0 is the interior of Λ, i. e. $\Lambda = \Lambda \setminus \partial\Lambda$.

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