ANNALES DE L'I. H. P., SECTION A

ENRICO MASSA
ANGELO MORRO

A dynamical approach to relativistic continuum thermodynamics

Annales de l'I. H. P., section A, tome 29, nº 4 (1978), p. 423-454
<http://www.numdam.org/item?id=AIHPA_1978__29_4_423_0>

Ann. Inst. Henri Poincaré,
Vol. XXIX, n° 4, 1978, p. 423-454.

Section A :
Physique théorique.

A dynamical approach to relativistic continuum thermodynamics

by

Enrico MASSA and Angelo MORRO
Istituto di Matematica, Università di Genova, Genova, Italy

ABSTRACT. — The role of the mass-energy equivalence principle in the construction of a relativistically invariant theory of continuum thermodynamics is examined. It is seen that, as a consequence of this principle, the study of the irreversible phenomena taking place within a given body $\mathscr{B}$ may be reduced to the study of the mutual interactions between two ideal subsystems of $\mathscr{B}$, called respectively the heat subsystem, and the material substratum. The analysis of this aspect, as well as of the limitations posed by the 2^{nd} law of thermodynamics, constitutes the essence of the « dynamical approach » proposed here. Among the consequences of the general theory, a particularly significant one is that the resulting description of the phenomena of heat conduction and of internal dissipation is automatically consistent with the existence of a finite wave speed both for the temperature and for the acceleration fields.

RÉSUMÉ. — Nous examinons le rôle du principe d'équivalence masse-énergie dans la création d'une théorie relativiste de la thermodynamique du continu. Comme conséquence de ce principe, l'étude des phénomènes irréversibles ayant lieu dans un corps $\mathscr{B}$ peut être réduit à l'étude des interactions mutuelles de deux sous-systèmes idéaux de $\mathscr{B}$, appelés respectivement sous-système thermique et support matériel. L' « approche dynamique » que nous proposons se base sur l'analyse de ce point, et des limitations imposées par le second principe de la thermodynamique. Une des conséquences spécialement remarquables de la théorie générale est que la description correspondante des phénomènes de la conduction de la chaleur

Work performed under the auspices of the National Research Council (C. N. R.-G. N. F. M.), Italy.

et de la dissipation interne est automatiquement consistante avec l'existence d'une vitesse finie pour les ondes des champs soit de la température, soit de l'accélération.

---

1. INTRODUCTION

The first significant contribution to a relativistically invariant formulation of continuum thermodynamics was given in 1940 by C. Eckart, who, in his pioneering paper [*1*], took over into the framework of relativity the thermodynamics of irreversible processes. The subsequent development of the theory reflects the simultaneous evolution of non-relativistic continuum thermodynamics. In this connection we recall, for instance, the paper by Grot and Eringen [*2*], which examines, in the relativistic context, the viewpoint of Coleman and Noll [*3*] about the interpretation of the 2nd law of thermodynamics. Analogously, the work by Curtis and Lianis [*4*] extends the classical thermodynamic analysis of Coleman [*5*] to a wide class of materials with memory. A different form of the entropy principle, which is the relativistic counterpart of the entropy principle proposed by Müller in [*6*], was considered in the papers by Müller [*7*], and Alts and Müller [*8*].

Among the basic properties of any acceptable relativistic theory, a fundamental one is the requirement that the perturbations of any physical field should propagate with a *finite* speed, not exceeding the speed of light. A drawback of Eckart's theory (and of many other theories as well) is that it *violates* the stated condition. The reason for this drawback is entirely classical, and is related to the well known result that Fourier's theory of heat conduction, and Navier-Stokes' theory of viscosity rule out the possibility of thermal and acceleration waves. In the case of thermal disturbances, a way out of the difficulty was shown in 1948 by Cattaneo [*9*], who proposed to replace Fourier's law by the more general constitutive equation

$$\underset{\sim}{q} + \tau \dot{\underset{\sim}{q}} = - \kappa \underset{\sim}{g}$$

where $\underset{\sim}{q}$ is the heat flux, $\underset{\sim}{g}$ is the temperature gradient, κ is the thermal conductivity, and τ is a characteristic time (¹). After the work by Cattaneo, many papers appeared on the theory of heat conduction with finite wave speed; among these, the works by Müller [*11*] and Lebon and Lambermont [*12*] can be viewed within the framework of an extended thermo-

(¹) Sometimes, the equation proposed by Cattaneo is referred to as the Maxwell-Cattaneo equation, since it was obtained previously by Maxwell [*10*] in 1867. However, Maxwell was not interested in wave propagation, and thus neglected the term $\tau\dot{\underset{\sim}{q}}$, as « the rate of conduction will rapidly establish itself ».

dynamics of irreversible processes; the papers by Green and Laws [*13*] and Müller [*14*] account for the existence of temperature waves, starting from a quite general expression for the second law of thermodynamics. It seems, however, that most of the literature on the subject follows the viewpoint of Coleman and Noll [*3*, *5*]. For instance, concerning materials with instantaneous response, we mention the papers by Bogy and Naghdi [*15*] (temperature-rate dependence), and by Kosinski [*16*] (internal variables). Rigid heat conductors with fading memory are considered by Gurtin and Pipkin [*17*], McCarthy [*18*] and Morro [*19*], while a general linear theory of heat conduction is given by Meixner in [*20*], starting with a basic inequality that does not make use of non-equilibrium entropy.

Within the relativistic context, beyond the already cited papers [*7*, *8*], it is worth recalling the works by Bressan [*21*], Kranys [*22*], Boillat [*23*], Maugin [*24*] and Lianis [*25*], all dealing—by different procedures—with theories of heat conduction consistent with temperature propagation with finite speed.

On the contrary, both in the classical and in the relativistic context, much less attention has been paid to the analogous problem of seeking a constitutive equation for the viscous stress tensor, eliminating the paradox of the impossibility of wave propagation for the acceleration field. In this connection, we mention first the relation written by Maxwell [*10*]

$$\underset{\sim}{\mathrm{F}} + \mathrm{T}\dot{\underset{\sim}{\mathrm{F}}} = \mathrm{TE}\dot{\underset{\sim}{\mathrm{S}}}$$

where, in Maxwell's notation, $\underset{\sim}{\mathrm{F}}$ is the stress tensor, $\dot{\underset{\sim}{\mathrm{S}}}$ is the time derivative of strain, T and E are suitable constants. As it is well known, this equation is compatible with the existence of acceleration waves. In the case of materials with memory, a far reaching analysis of wave propagation within the class of Maxwellian materials is given by Coleman, Greenberg and Gurtin [*26*]. Finally, within the relativistic context, we recall the work by Kranys [*27*], based on an *ad hoc* phenomenological postulate, according to which the constitutive equations for non-stationary processes are obtained from the corresponding equations valid in the quasi-stationary case through the action of a suitable differential operator, called the relaxation operator.

Although largely incomplete, the previous list of references points out the wide variety of themes involved in the construction of a relativistically invariant theory of continuum thermodynamics. Strangely enough, however, until now no systematic attempt has been made in order to embody in the mathematical foundations of the theory another basic law of relativistic physics, namely the mass-energy equivalence principle. This law, which is at least as fundamental as the existence of an upper bound to the speed of propagation of physical disturbances, is reflected in the fact that « in relativistic mechanics an energy flux necessarily involves a mass flux » ([*28*], § 127), and is therefore very likely to have a direct relevance in the study of the phenomenon of heat conduction. In a non-relativistic context, an idea in this sense is already present in the work by Kaliski [*29*], where the suggestion is made that, in the case of a heat flow of marked non-stationary character, account must be taken of the inertia of time variability of the heat flow, or « thermal inertia ».

In this paper, we propose to push forward this viewpoint, by discussing a *dynamical* approach to relativistic continuum thermodynamics, in which the mass-energy equivalence principle is set at the basis of all subsequent developments. As a preliminary step in this analysis, in § 2 we present a brief review of the main concepts and definitions involved in the construction of a *mathematical model* for a material continuum $\mathscr{B}$. The argument is completed by a relativistic formulation of the 2nd law of thermodynamics (interpreted in the sense of Coleman and Noll, i. e. as an *a priori* restriction on the constitutive characterization of the model), and by a subsequent description of the so called *purely mechanical scheme*, in which all irreversible phenomena taking place within the body $\mathscr{B}$ are explicitly neglected.

The role of the mass-energy equivalence principle in the construction of a relativistically invariant theory of irreversible thermodynamics is discussed extensively in § 3. As we shall see, the main role of this principle is that it leads quite naturally to associate with the heat flux $\underset{\sim}{q}$ a set of mechanical attributes (kinetic energy density, momentum density, momentum-flux density), summarized into a symmetric tensor field $\underset{\sim}{Q}$. The simplest conjecture is then to regard $\underset{\sim}{Q}$ as the energy-momentum tensor of an ideal physical system $\mathscr{Q}$, called the *heat subsystem* of the continuum $\mathscr{B}$. This viewpoint is completed by representing $\mathscr{B}$ as a *binary system*, formed by two interacting subsystems: the heat subsystem $\mathscr{Q}$, and a further subsystem $\mathscr{S}$ called the *material substratum* of $\mathscr{B}$. In this way, the central problem of irreversible thermodynamics may be viewed as the study of the *mutual interactions* between the subsystems $\mathscr{S}$ and $\mathscr{Q}$, and may therefore be handled with the standard techniques of relativistic continuum mechanics. In particular, as a result of the interactions, the subsystem $\mathscr{Q}$ will generally be set *in motion* with respect to $\mathscr{S}$, thus giving rise to the phenomenon of heat conduction. In addition to this, we may have various effects of *heat production*, either by direct energy exchanges between $\mathscr{S}$ and $\mathscr{Q}$, or by other causes (external effects, internal dissipation). After these preliminaries, the rest of § 3 is devoted to a detailed analysis of the nature of the various interactions, as well as of the *a priori* constraints posed by the 2nd law of thermodynamics. The results so obtained are applied in § 4 to the study of two typical irreversible effects, namely the phenomenon of heat conduction in non-dissipative media, and the phenomenon of internal dissipation in non-conducting materials. Both arguments are merely sketched for illustrative purposes, leaving aside all unnecessary details, and looking only to the essential aspects of the theory. The important conclusion to be drawn from this analysis is that the dynamical scheme developed in § 3 is automatically consistent with the requirement of wave propagation both for the temperature and for the acceleration field, as shown by the fact that it yields back Maxwell-Cattaneo's equation for heat conduction, and Maxwell's equation for viscosity as non relativistic limits of the general theory.

2. MATHEMATICAL FOUNDATIONS

2.1. Preliminaries

Let $\mathscr{V}_4$ be a space-time manifold, with fundamental form (²)

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta.$$

In $\mathscr{V}_4$, let Ξ denote the congruence of stream lines of a material continuum $\mathscr{B}$; also, let $\underset{\sim}{V}$ denote the corresponding four-velocity field, with normalization $V^\alpha V_\alpha \doteq -1$. As discussed extensively in Refs. [*30-32*], under suitable smoothness conditions, the congruence Ξ determines a corresponding *physical frame of reference* in $\mathscr{V}_4$ (in this connection, see also [*33-36*]). The latter will be indicated by $[\Xi]$, and will be called the co-moving frame of reference associated with $\mathscr{B}$, or, more synthetically, the *material rest-frame*. Strictly associated with $[\Xi]$ is a pair of projection operators $\mathscr{P}$ (temporal projection) and $\mathscr{N}$ (spatial projection), defined on vector fields by

$$\mathscr{P}(\underset{\sim}{A}) = -cV_\alpha A^\alpha\,; \qquad \mathscr{N}(\underset{\sim}{A}) = (\delta^\alpha{}_\beta + V^\alpha V_\beta)A^\beta \doteq h^\alpha{}_\beta A^\beta \qquad (2.1)$$

and extended in the obvious way to tensor fields of arbitrary rank. Through a systematic use of the operators (2.1) it is possible to replace the generally covariant formulation of physical laws by an equivalent formulation, having a strictly *relative* character, i. e. involving only *space vectors* and *space tensors* relative to $[\Xi]$. All this is well known (see, e. g., the already cited references [*30-32*]), and will not be repeated here.

With these preliminaries, let us now analyse the construction of a *dynamical model* for the given continuum $\mathscr{B}$. In this connection, a very general line of approach may be traced as follows:

1) Introduction of the space of thermokinetic processes of the system $\mathscr{B}$. The latter is defined as the abstract space $\mathfrak{T}$ formed by the totality of pairs $(\Xi, \theta(\cdot))$, where Ξ is a time-like congruence over $\mathscr{V}_4$, identified with the

(²) Greek indices run from 0 to 3. Einstein's summation convention is used throughout. The signature of the metric is $- + + +$. Partial derivatives are indicated by a comma; covariant derivatives by a semicolon. The symbol $\doteq$, in place of $=$, indicates that the corresponding relation is meant as a *definition*.

congruence of *stream lines* of $\mathscr{B}$, and $\theta(\cdot)$ is a strictly positive scalar field, identified with the *temperature field* inside the body $\mathscr{B}$.

2) Introduction of an abstract space $\mathfrak{A}^*$ (henceforth called the *representative space* of $\mathscr{B}$), defined by the condition that *all* physical attributes of $\mathscr{B}$ may be expressed as *maps* of $\mathfrak{A}^*$ into the algebra of tensor fields over $\mathscr{V}_4$. In general, each element $\mathfrak{E} \in \mathfrak{A}^*$ will consist of a collection $(\Xi, \theta(\cdot), \lambda(\cdot), \ldots, \nu(\cdot))$, where $\Xi, \theta(\cdot)$ are the thermokinetic variables already introduced in 1), while $\lambda(\cdot), \ldots, \nu(\cdot)$ are $n(\geqslant 0)$ auxiliary fields, all having a direct relevance to the problem in study.

3) Representation of the *internal response* of the continuum $\mathscr{B}$, by means of a set of *constraints*, expressed by functional equations of the form

$$\mathscr{F}_{(k)}(\mathfrak{E}) = \mathscr{F}_{(k)}(\Xi, \theta(\cdot), \lambda(\cdot), \ldots, \nu(\cdot)) = 0 \quad (k = 1, \ldots, m) \qquad (2.2)$$

where the $\mathscr{F}_{(k)}$'s are suitable maps of the space $\mathfrak{A}^*$ into the algebra of tensor fields over $\mathscr{V}_4$. In connection with equations (2.2), the basic requirement is that they express the *evolution* of the fields $\lambda(\cdot), \ldots, \nu(\cdot)$ in terms of the thermokinetic variables $\Xi, \theta(\cdot)$. Quite generally, this has the effect of restricting the choice of the elements $\mathfrak{E} \in \mathfrak{A}^*$ to a distinguished subspace $\mathfrak{A} \subset \mathfrak{A}^*$, called the space of *admissible processes* of $\mathscr{B}$. Depending on the nature of the constraints (2.2), the models may be further classified into *finite* (or holonomic) ones, in which eqs. (2.2) determine the fields $\lambda(\cdot), \ldots, \nu(\cdot)$ *uniquely* in terms of $\Xi, \theta(\cdot)$, thus setting up a $1 - 1$ correspondence between the space $\mathfrak{T}$ of thermokinetic processes and the subspace $\mathfrak{A} \subset \mathfrak{A}^*$ of admissible processes, and *differential* (or non-holonomic) ones, in which, for each choice of $\Xi, \theta(\cdot)$, the determination of the fields $\lambda(\cdot), \ldots, \nu(\cdot)$ relies on the solution of a Cauchy problem, and thus involves the introduction of a suitable set of initial data. Exactly as it happens in classical Lagrangian dynamics, in the holonomic case one can get rid of all unnecessary redundancies, by starting at the outset with the identification $\mathfrak{A}^* = \mathfrak{T}$. In general, such a simplification does not occur in the non-holonomic case, unless one regards the initial data as fixed once for all (e. g. in the asymptotic limit $t \to -\infty$), in which case all differences between the finite models and the differential ones disappear.

4) Characterization of the dynamical response of the continuum $\mathscr{B}$ under the effect of the interactions with the external world. In a relativistic context, this is achieved by assigning to $\mathscr{B}$ a corresponding *energy-momentum tensor*, defined as a map $\mathbb{T}$ of the space $\mathfrak{A}^*$ into the module of symmetric tensor fields over $\mathscr{V}_4$. The effect of $\mathbb{T}$ on an arbitrary element $\mathfrak{E} \in \mathfrak{A}^*$ will be indicated by $\underset{\sim}{\mathrm{T}}$ i. e.

$$\underset{\sim}{\mathrm{T}} \doteq \mathbb{T}(\mathfrak{E}).$$

With this in mind, the whole content of relativistic continuum mechanics relies on the following

AXIOM 2.1. — In every *physical* process, the field $T = \mathbb{T}(\underset{\sim}{\mathfrak{C}})$ is related to the total four-force density $\underset{\sim}{b}$ acting on $\mathscr{B}$ as a consequence of the interactions with the external world by the *balance equations*

$$T^{\alpha\beta}{}_{;\beta} = b^{\alpha} \tag{2.3}$$

Conversely, for each choice of the external four-force $\underset{\sim}{b}$, eqs. (2.3), together with the constraints (2.2), are sufficient to determine the *evolution* of $\mathscr{B}$ (i. e. a unique admissible process $\mathfrak{C} \in \mathfrak{A}$) from given initial data (³).

When the system $\mathscr{B}$ is *isolated*, the balance equations (2.3) take the simpler form

$$T^{\alpha\beta}{}_{;\beta} = 0. \tag{2.4}$$

The study of the solutions of eqs. (2.4)—or, more generally, of eqs. (2.3)—within the class of admissible processes, constitutes the essence of the so called *problem of motion* for the continuum $\mathscr{B}$. In this connection, see, e. g., [*34*, *37*].

For later use, it is also worth noticing that, in view of eq. (2.3), the temporal projection $\mathscr{P}(T^{\alpha\beta}{}_{;\beta})$ coincides with the *external power density* acting on $\mathscr{B}$ in its own rest-frame. In this sense, the quantity $\mathscr{P}(T^{\alpha\beta}{}_{;\beta})$ has always a *non-mechanical* nature, and accounts for all effects of *heat transfer* to the system $\mathscr{B}$ from the external sources, or of *heat production* within $\mathscr{B}$, due to external causes. The corresponding equation

$$\mathscr{P}(T^{\alpha\beta}{}_{;\beta}) = \mathscr{P}(\underset{\sim}{b}) = -\, cV_{\alpha}b^{\alpha} \tag{2.5 a}$$

(energy balance equation) is often referred to as *the 1st law of thermodynamics*. Similarly, the spatial projection of eqs. (2.3) in the material rest-frame [Ξ] gives rise to the momentum balance equation

$$\mathscr{N}(T^{\alpha\beta}{}_{;\beta}) = \mathscr{N}(\underset{\sim}{b}) = h^{\alpha}{}_{\beta}b^{\beta}. \tag{2.5 b}$$

2.2. Entropy principle and related topics

i) As it is clear from the previous discussion, once the nature of the space $\mathfrak{A}^*$ has been fixed, the construction of a mathematical model for the continuum $\mathscr{B}$ relies on the introduction of a suitable set of *constitutive equations*, defined as *maps* of $\mathfrak{A}^*$ into the algebra of tensor fields over $\mathscr{V}_4$. These are explicitly involved in two basic operations:

a) representation of the *internal response* of $\mathscr{B}$, through the introduction of

(³) Strictly speaking, this assertion, as well as the whole characterization of the system $\mathscr{B}$ discussed above, is valid only when the geometry of $\mathscr{V}_4$ is regarded as *given*, i. e. whenever the contribution of $\mathscr{B}$ to the total gravitational fields is regarded as *negligible* (as it happens e. g. in Special Relativity). In the opposite case, one would have to resort to more general techniques that are outside the scope of this work. For a geometrical approach to the problem of motion in General Relativity, see e. g. [*38*, *39*].

the constraints (2.2), and the subsequent definition of the class $\mathfrak{A} \subset \mathfrak{A}^*$ of admissible processes of $\mathscr{B}$;

b) representation of the attributes of $\mathscr{B}$ which are relevant to the problem in study (We recall that, in a dynamical context, these must necessarily include the energy-momentum tensor $\underset{\sim}{\mathrm{T}} = \mathbb{T}(\mathfrak{E})$).

Both operations *a*) and *b*) together will be said to provide a *constitutive characterization* of the continuum $\mathscr{B}$ over the representative space $\mathfrak{A}^*$. In this connection, an *a priori* condition, valid for *all* types of materials, comes from a fundamental law of Physics, known as the *2nd law of Thermodynamics*. Following the modern trend, initiated by Coleman and Noll [*3*, *5*], we shall regard the latter as a restriction on the choice of the constitutive equations, expressed in the form of an inequality that must hold *identically*, over the whole subspace $\mathfrak{A} \subset \mathfrak{A}^*$. In this sense, the 2nd law plays the role of a *selection rule*, that allows to discriminate between models that *may be* physically significant, and models that are surely *unrealistic*. In a relativistic context, a quite general formulation is obtained by including among the relevant attributes of $\mathscr{B}$ two further quantities, namely:

1) the *entropy-flux four vector*, defined as a map $\underset{\sim}{s} = \mathfrak{s}(\mathfrak{E})$ of the space $\mathfrak{A}^*$ into the module of vector fields over $\mathscr{V}_4$;

2) the *dynamical temperature* ϑ, defined as a map of $\mathfrak{A}^*$ into the class of strictly positive functions over $\mathscr{V}_4$.

The basic constraint is then expressed by the following

AXIOM 2.2 (dissipation principle). — For each $\mathfrak{E} \in \mathfrak{A}^*$, define the *entropy production* associated with $\mathfrak{E}$ as the quantity

$$\sigma \doteqdot s^{\alpha}{}_{;\alpha} - \frac{1}{\vartheta}\mathscr{P}(\mathrm{T}^{\alpha\beta}{}_{;\beta}) = s^{\alpha}{}_{;\alpha} + \frac{c}{\vartheta}\mathrm{V}_{\alpha}\mathrm{T}^{\alpha\beta}{}_{;\beta} \tag{2.6 a}$$

with $\underset{\sim}{s} = \mathfrak{s}(\mathfrak{E})$, $\vartheta = \theta(\mathfrak{E})$, $\underset{\sim}{\mathrm{T}} = \mathbb{T}(\mathfrak{E})$. Then, a necessary condition for a mathematical model of $\mathscr{B}$ to be physically admissible is that the inequality

$$\sigma \geqslant 0 \tag{2.6 b}$$

be satisfied identically over the whole subspace $\mathfrak{A} \subset \mathfrak{A}^*$. In particular, every process $\mathfrak{E} \in \mathfrak{A}$ for which the associated entropy production (2.6 *a*) is identically zero is called a *reversible process* of $\mathscr{B}$.

Axiom 2.2 provides a formulation of the 2nd law especially suited to the type of interpretation discussed above (in this connection see also [*2*]). In fact, according to our previous definitions, every process $\mathfrak{E} \in \mathfrak{A}$ is determined —up to initial data—by the knowledge of the thermokinetic variables Ξ, $\theta(\cdot)$. The content of Axiom 2.2 is therefore that, in any *realistic* model of $\mathscr{B}$, the inequality (2.6 *b*) should hold identically, for *all* choices of Ξ, $\theta(\cdot)$ and of the initial data. In this sense, the requirement (2.6 *b*) does indeed imply an effective restriction on the nature of the constitutive equations, in complete agreement with the interpretation of the 2nd law in the sense of Coleman and Noll. Notice also that, when the system $\mathcal{B}$ is *isolated*, the inequality (2.6 *b*) reduces to the simpler condition $s^{\alpha}{}_{;\alpha} \geqslant 0$, valid for all processes $\mathfrak{C} \in \mathfrak{A}$ *consistent* with the balance equations (2.4). In connection with this aspect of the 2nd law, see e. g. [7].

ii) Besides the 2nd law of thermodynamics, another fundamental constraint on the choice of a physically admissible model for the continuum $\mathcal{B}$ comes from the law of *baryon conservation* [40]. When the class of phenomena in study does not include quantum effects (particle-antiparticle creation or annihilation, β decay, etc.), the latter may be replaced by a stronger assumption, concerning the existence of a *material content* of $\mathcal{B}$, conserved in time· Again, this property is expressed relativistically by assigning to $\mathcal{B}$ a *material density* $m = \mathfrak{m}(\mathfrak{C})$, such that the associated four current $m\underset{\sim}{\mathrm{V}}$ satisfies the balance equation

$$(m\mathrm{V}^{\alpha})_{;\alpha} = 0. \tag{2.7}$$

In view of eq. (2.7), the density m is essentially a *geometrical* quantity, related, up to a time-independent factor, to the reciprocal of the volume element in the material rest-frame [Ξ]. In this sense, the constitutive equation for m is not really significant, and, as such, it will be no longer mentioned in the following. As a further comment on eq. (2.7) we remark once again that the definition of m is strictly connected with a *macroscopic* picture of physical reality, in which the concept of « material content » of $\mathcal{B}$ has a primitive character. In the presence of quantum effects, one would have to resort to a different type of density n, called the *baryon number density* (number of baryons per unit proper volume, with anti-baryon, if any, counted negatively). With this definition, the balance equation $(n\mathrm{V}^{\alpha})_{;\alpha} = 0$ has then general validity [40].

iii) In most cases of actual physical interest, the construction of a mathematical model for the continuum $\mathcal{B}$ is subject to a further constraint, of structural nature, expressed in the form of an algebraic relation among the fields $\underset{\sim}{\mathrm{T}}$, $\underset{\sim}{s}$ and ϑ. To see this point, for each $\mathfrak{C} \in \mathfrak{A}^*$, let us express the spatial resolution of the entropy-flux four vector $\underset{\sim}{s} = \underset{\sim}{\mathfrak{s}}(\mathfrak{C})$ in the material rest-frame [Ξ] in terms of a scalar field $\eta = \mathbb{N}(\mathfrak{C})$ and a spatial vector field $\underset{\sim}{q} = \mathfrak{q}(\mathfrak{C})$ according to the equation

$$\underset{\sim}{s} \doteq mc\eta\underset{\sim}{\mathrm{V}} + \frac{\underset{\sim}{q}}{\vartheta} \tag{2.8}$$

Following the standard terminology, we call the map $\eta = \mathbb{N}(\mathfrak{C})$ the *specific entropy* of $\mathcal{B}$. In a similar way, we define the heat flux within the body $\mathcal{B}$ as the map sending each $\mathfrak{C} \in \mathfrak{A}^*$ into the spatial field $\underset{\sim}{q} = \underset{\sim}{\mathfrak{q}}(\mathfrak{C})$. In this way, except for the presence of the dynamical temperature ϑ in place of θ, the representation (2.6 *a*) of the entropy production σ agrees with the one commonly adopted in non-relativistic continuum thermodynamics (4). Moreover, in view of our previous definitions, the heat flux $\underset{\sim}{q}$ is dimensionally omogeneous with the *energy flux* in the material rest-frame, expressed in terms of the energy-momentum tensor $\underset{\sim}{\mathrm{T}} = \mathbb{T}(\mathfrak{C})$ as $\mathscr{P} \otimes \mathscr{N}(\underset{\sim}{\mathrm{T}})$.

With this in mind, we now state.

DEFINITION 2.1. — A mathematical model of $\mathscr{B}$ is said to be *constitutively simple* if and only if the equality

$$\underset{\sim}{q} = \mathscr{P} \otimes \mathscr{N}(\underset{\sim}{\mathrm{T}}) \tag{2.9}$$

holds identically, for all $\mathfrak{C} \in \mathfrak{A}^*$.

As it is clear from eq. (2.8), under the simplifying assumption (2.9) the spatial projection $\mathscr{N}(\underset{\sim}{s})$ of the entropy-flux four vector is determined uniquely in terms of ϑ and $\underset{\sim}{\mathrm{T}}$, thus reducing the number of independent quantities involved in the constitutive characterization of the model.

In the following, we shall systematically embody the requirement of constitutive simplicity in the set of *a priori assumptions* concerning the nature of the models in study. From a physical viewpoint, this means essentially that from here on, we shall restrict our attention to an ideal class of continua, in which the only contribution to the energy flux in the material rest-frame [Ξ] comes from the phenomenon of heat conduction. In this connection, see also [*1*, *2*].

2.3. **The purely mechanical scheme**

As the simplest application of the concepts outlined in § 2.2, we shall now discuss the so called *purely mechanical scheme* of $\mathscr{B}$. Besides being physically interesting on its own, the latter will also provide a useful reference for all subsequent developments. By definition, we shall regard the purely mechanical model as being completely characterized by the following structural properties:

1) Identification of the representative space $\mathfrak{A}^*$ with the class $\mathfrak{T}$ of thermokinetic processes (corresponding to the assumption of holonomy of the model, see § 2.1).

2) Identification of the dynamical temperature ϑ with the absolute temperature, i. e. choice of the constitutive equation $\vartheta\,(\mathfrak{C}) = \theta$.

3) Vanishing of the heat flux $\underset{\sim}{q}$ for *all* processes $\mathfrak{C} \in \mathfrak{A}^*$. Recalling eqs. (2.8), (2.9), this gives rise to the representations

$$s^\alpha = mc\eta \mathrm{V}^\alpha \tag{2.10}$$

$$\widehat{\mathrm{T}}_{\alpha\beta} = m(c^2 + \varepsilon)\mathrm{V}_\alpha \mathrm{V}_\beta - \mathrm{S}_{\alpha\beta} \tag{2.11}$$

(4) On the occurrence of the same temperature ϑ in both expressions (2.6 *a*), (2.8), see e. g. [*41*].

with $S_{\alpha\beta}V^{\beta} = 0$. The notation in eq. (2.11) is conventional; the scalar field ε and the spatial tensor field $\underset{\sim}{S}$ are called respectively the *specific internal energy* and the *stress tensor* of $\mathscr{B}$.

4) Vanishing of the entropy production σ for all processes $\mathfrak{C} \in \mathfrak{A}^*$. Recalling eq. (2.6 *a*) and properties 2), 3) stated above, this condition is expressed by the constraint

$$c^{-1}\sigma = m\eta_{,\alpha}V^{\alpha} + \frac{1}{\theta}V_{\alpha}T^{\alpha\beta}{}_{;\beta} \equiv 0 \qquad (2.12)$$

for all $\mathfrak{C} \in \mathfrak{A}^*$. An equivalent formulation may be given by introducing the *specific free energy*

$$\psi = \varepsilon - \theta\eta. \qquad (2.13)$$

By eqs. (2.10) (2.13) we have then

$$m\psi_{,\alpha}V^{\alpha} = S^{\alpha\beta}V_{\alpha;\beta} - m\eta\theta_{,\alpha}V^{\alpha} \qquad (2.14)$$

which is easily recognized as the condition for the stress $\underset{\sim}{S}$ to be entirely *recoverable* [*42*].

The previous properties put rather severe restrictions on the class of phenomena that can be fitted into the purely mechanical model; for example, it is self evident that eqs. (2.10), (2.11), (2.14) preclude at the outset any possibility of accounting for such effects as heat conduction, viscosity, etc. The deep reason for such a restrictive interpretation of the model will become apparent after the analysis of § 3.

Collecting all previous results, we conclude that, within the purely mechanical scheme, the most general model of the continuum $\mathscr{B}$ relies on the choice of three constitutive equations, namely

$$\varepsilon = \mathbb{E}(\Xi, \theta(\cdot)), \qquad \eta = \mathbb{N}(\Xi, \theta(\cdot)), \qquad \underset{\sim}{S} = \mathbb{S}(\Xi, \theta(\cdot)) \qquad (2.15)$$

in which $\mathbb{E}$, $\mathbb{N}$ and $\mathbb{S}$ denote suitable maps defined over the space $\mathfrak{T}$ of thermokinetic processes, i. e. suitable *functionals* of the arguments Ξ, $\theta(\cdot)$, subject to the necessary requirements of causality and differentiability, as well as to the *a priori* constraint expressed by eq. (2.12) ([5]).

Remark 2.1. — To get an idea of the wide variety of models embodied by the representation (2.15) of the constitutive equations notice that, in principle, the functional dependence on the congruence Ξ may imply the dependence on any kinematical or geometrical object associated with Ξ (deformation gradient, rate-of-strain tensor, etc.), with the only limitations posed by the constraint (2.12), and by the general principles of continuum

([5]) A more precise representation of the constitutive equations, accounting explicitly for the *causality* requirement, would be $\varepsilon(x) = \mathbb{E}(\Xi^x, \theta^x(\cdot))$, etc., where, for each event $x \in \mathscr{V}_4$, the symbols Ξ^x, $\theta^x(\cdot)$ denote the *restrictions* of the congruence Ξ and of the temperature field $\theta(.)$ to the closed region bounded by the past null cone with vertex at x.

mechanics (causality, frame indifference, etc.). Similarly, the dependence on the field $\theta(\cdot)$ may involve the dependence on the value $\theta(x)$ of the temperature at any event $x \in \mathscr{V}_4$, or on the temperature gradient $\theta_{,\alpha}(x)$, etc., or even on the whole past history $\theta^x(\cdot)$ of the temperature field.

3. THE DYNAMICAL APPROACH

3.1. Formulation of the problem

i) The model described in § 2.3 represents a higly idealized situation, in which all irreversible effects are purposely neglected. In order to set up a more complete picture of the continuum $\mathscr{B}$, we shall now examine how the phenomena of *heat conduction*, and *heat production* by internal dissipation can be framed within the mathematical scheme developed in § 2.1.

To start with, let us consider first the case in which the only possible cause of irreversibility comes from *heat conduction*. In this case, the spatial resolution of the energy-momentum tensor of $\mathscr{B}$ in the material rest frame $[\Xi]$ reads

$$T_{\alpha\beta} = m(c^2 + \varepsilon^*)V_\alpha V_\beta - S^*_{\alpha\beta} + \frac{1}{c}(q_\alpha V_\beta + q_\beta V_\alpha) \tag{3.1}$$

with $q_\alpha V^\alpha = 0$, $S^*_{\alpha\beta}V^\beta = 0$. In strictly logical terms, eq. (3.1) is nothing but a *definition* of the quantities ε^*, $\underset{\sim}{q}$, and S^*. In particular, the spatial vector field $\underset{\sim}{q} = \mathscr{P} \otimes \mathscr{N}(\mathrm{T})$ describes the *energy flux* associated with $\mathscr{B}$ in the material rest frame. Following Eckart [*1*], we shall identify the latter with the *heat flux* within the body $\mathscr{B}$, in complete agreement with the notation already employed in § 2.2. As far as the other terms involved in the representation (3.1) are concerned, it is usually admitted that, in the absence of internal dissipation, they retain the same physical meaning as in the purely mechanical model. A closer investigation, however, reveals that such a viewpoint is not completely satisfactory, and can be regarded at most as an *approximation* of a more realistic scheme. In fact, on the basis of the mass-energy equivalence principle, the presence of a heat flux $\underset{\sim}{q}$ in the material rest-frame implies the simultaneous existence of an associated momentum density $c^{-2}\underset{\sim}{q}$ (as implicit in eq. (3.1)), and thus also of a *kinetic energy density*, and of a *momentum-flux density*. In principle, all these terms should be present in the representation (3.1) of the energy-momentum tensor, thus affecting the whole set of components $T_{\alpha\beta}$, and not merely the mixed projection $\mathscr{P} \otimes \mathscr{N}(\mathrm{T})$.

This viewpoint, that in a sense generalizes Kaliski's idea of *thermal inertia* [*29*], can be given a quantitative meaning by observing that, on the basis of the previous discussion, the phenomenon of heat conduction is unavoidably endowed with a set of *mechanical* attributes. To account for these —and, in particular, to assign definite expressions both to the kinetic energy density and to the momentum-flux density associated with $\underset{\sim}{q}$—it is convenient to regard heat conduction as an effective *transport phenomenon*, expressing the fact that, under suitable circumstances, *a certain amount* $m\delta_0$ *of energy density of the continuum* $\mathscr{B}$ *can be set in motion with velocity* $\underset{\sim}{v}$ *relative to the material rest frame* Ξ (6). Setting as usual $\gamma \doteq (1 - v^2)^{-\frac{1}{2}}$, $\delta \doteq \gamma\delta_0$, the previous argument is synthesized by the equation

$$\underset{\sim}{q} = \gamma m\delta_0 c\underset{\sim}{v} = m\delta c\underset{\sim}{v} \tag{3.2}$$

with $v_\alpha V^\alpha = 0$. The kinetic energy density is then given by $(\gamma - 1)m\delta_0$, and the momentum-flux density by $m\delta v_\alpha v_\beta$. Therefore, denoting by $\eta = \mathbb{N}(\Xi, \theta(\cdot))$ the specific entropy of $\mathscr{B}$, we may tentatively replace the representation (3.1) of the energy-momentum tensor by the equivalent expression

$$T_{\alpha\beta} = \widehat{T}_{\alpha\beta} + (\gamma - 1)m\delta_0 V_\alpha V_\beta + m\delta v_\alpha v_\beta + \frac{1}{c}(q_\alpha V_\beta + q_\beta V_\alpha) + \widehat{W}_{\alpha\beta} \tag{3.3}$$

where $\underset{\sim}{\widehat{T}} = \widehat{\mathbb{T}}(\Xi, \theta(\cdot))$ has the same formal structure as the energy-momentum tensor involved in the purely mechanical picture of $\mathscr{B}$, namely

$$\widehat{T}_{\alpha\beta} = m(c^2 + \varepsilon)V_\alpha V_\beta - S_{\alpha\beta} \tag{3.4a}$$

with the *a priori* constraint

$$m\eta_{,\alpha}V^\alpha + \frac{1}{\theta}V_\alpha \widehat{T}^{\alpha\beta}{}_{;\beta} = 0 \tag{3.4b}$$

while $\underset{\sim}{\widehat{W}}$ collects all other contributions to $\underset{\sim}{T}$ not explicitly included in the other terms at the right-hand side of eq. (3.3). In particular, the requirement of consistency of eq. (3.3) with eq. (3.1) implies

$$\mathscr{P} \otimes \mathscr{N}(\underset{\sim}{\widehat{W}}) = 0. \tag{3.5a}$$

To this we add the further constraint

$$\underset{\sim}{\widehat{W}} = 0 \qquad \text{for} \qquad \underset{\sim}{v} \equiv 0 \tag{3.5b}$$

expressing the condition that, in the *absence* of heat conduction, the representation (3.3) of the total energy-momentum tensor should agree with the one employed in the purely mechanical model of $\mathscr{B}$ (i. e. $\underset{\sim}{T} = \underset{\sim}{\widehat{T}}$ for $\underset{\sim}{v} \equiv 0$).

The previous scheme is easily generalized to the case when the class of

(6) For notational convenience, we shall measure all velocities in units of c (c = speed of light in vacuo), so that $\underset{\sim}{v}$ is a dimensionless spatial vector field subject to the constraint $v^2 < 1$.

irreversible phenomena that may take place within the body $\mathscr{B}$ includes possible effects of *heat production* by internal dissipation. In this case, the simplest conjecture is to admit that the presence of these effects contributes *additively* to the total energy-momentum tensor $\underset{\sim}{\mathrm{T}}$, through a *dissipative term* ${}^{\mathrm{D}}\underset{\sim}{\mathrm{W}}$, to be added to the other terms at the right-hand side of eq. (3.3). Setting for simplicity

$$\underset{\sim}{\mathrm{W}} \doteq \underset{\sim}{\widehat{\mathrm{W}}} + {}^{\mathrm{D}}\underset{\sim}{\mathrm{W}} \tag{3.6}$$

we have then the complete expression

$$\mathrm{T}_{\alpha\beta} = \widehat{\mathrm{T}}_{\alpha\beta} + (\gamma - 1)m\delta_0 \mathrm{V}_\alpha \mathrm{V}_\beta + m\delta v_\alpha v_\beta + \frac{1}{c}(q_\alpha \mathrm{V}_\beta + q_\beta \mathrm{V}_\alpha) + \mathrm{W}_{\alpha\beta} \tag{3.7}$$

Comparison with eq. (2.9) shows that, likewise $\underset{\sim}{\widehat{\mathrm{W}}}$, also the tensor ${}^{\mathrm{D}}\underset{\sim}{\mathrm{W}}$ is subject to the *a priori* constraint

$$\mathscr{P} \otimes \mathscr{N}({}^{\mathrm{D}}\underset{\sim}{\mathrm{W}}) = 0 \tag{3.8}$$

expressing the fact that, even in the more general case, the only contribution to the energy flux in the material rest-frame $[\Xi]$ should come from *heat conduction.*

ii) From a *constitutive* viewpoint, the previous arguments can be put on rational grounds by identifying the *representative space* $\mathfrak{A}^*$ of $\mathscr{B}$ with the abstract space formed by the totality of elements of the form $(\Xi, \theta(\cdot), \delta_0(\cdot), \underset{\sim}{v}(\cdot))$ (see § 2.1, 2). The *internal response* of $\mathscr{B}$ is then completely described in terms of *four* constraints

$$\mathscr{F}_{(\alpha)}(\Xi, \theta(\cdot), \delta_0(\cdot), \underset{\sim}{v}(\cdot)) = 0 \qquad (\alpha = 0, \ldots, 3) \tag{3.9}$$

subject to the conditions already oulined in § 2.1, 3. In particular, a model will be said to be *separable* if and only if the associated class $\mathfrak{A} \subset \mathfrak{A}^*$ of admissible processes admits at least one representation of the form (3.9) in which one of the functionals (say $\mathscr{F}_{(0)}$) depends only on the arguments $\Xi, \theta(\cdot), \delta_0(\cdot)$, but *not* on $\underset{\sim}{v}(\cdot)$ (while the remaining functionals will generally involve the whole set $(\Xi, \theta(\cdot), \delta_0(\cdot), \underset{\sim}{v}(\cdot))$. The situation is formalized by introducing a further abstract space $\mathfrak{A}_0^*$, generated by the totality of elements of the form $(\Xi, \theta(\cdot), \delta_0(\cdot))$. Then, a model is separable if and only if eqs. (3.9) may be splitted into the system

$$\begin{cases} \mathscr{F}_{(0)}(\Xi, \theta(\cdot), \delta_0(\cdot))) = 0 \\ \mathscr{F}_{(i)}(\Xi, \theta(\cdot), \delta_0(\cdot), \underset{\sim}{v}(\cdot)) = 0 \qquad (i = 1, 2, 3) \end{cases}$$

in which $\mathscr{F}_{(0)}$ is a map of the space $\mathfrak{A}_0^*$ into the class of scalar fields over $\mathscr{V}_4$.

With this in mind, let us now analyse the construction of a mathematical model for the continuum $\mathscr{B}$, along the lines already indicated in § 2. In this connection, a possible line of approach is suggested by the representation (3.2) of the heat flux. In four-dimensional language, the latter has its natural counterpart in the introduction of an ideal physical system $\mathscr{Q}$—henceforth called the *heat subsystem* of $\mathscr{B}$—characterized by the energy-momentum tensor

$$Q_{\alpha\beta} = m\delta^0 U_\alpha U_\beta \tag{3.10 a}$$

with

$$\delta^0 \doteq \delta_0/\gamma \quad , \quad \underset{\sim}{U} \doteq \gamma(\underset{\sim}{V} + \underset{\sim}{v}) \tag{3.10 b}$$

(whence $U^\alpha U_\alpha = -1$, $U^\alpha V_\alpha = -\gamma$). In view of eqs. (3.2), (3.10 *b*), the tensor (3.10 *a*) admits the equivalent expression

$$Q_{\alpha\beta} = m\delta V_\alpha V_\beta + m\delta v_\alpha v_\beta + \frac{1}{c}(q_\alpha V_\beta + q_\beta V_\alpha)$$

in which all terms—except $m\delta V_\alpha V_\beta$—represent meaningful quantities that, on the basis of eq. (3.7), should contribute to the *total* energy-momentum tensor $\underset{\sim}{T}$. In this respect, the term $m\delta V_\alpha V_\beta$ is partly exceptional since, besides the kinetic energy density $(\gamma - 1)m\delta_0$, it includes also an additional contribution to the total energy density in the frame of reference [Ξ], due to the rest energy $m\delta_0$ associated with the heat subsystem.

To account for this fact, the next logical step is to regard the entire physical system $\mathscr{B}$ as a *binary system*, formed by two interacting subsystems: the heat subsystem $\mathscr{Q}$, and a further subsystem $\mathscr{S}$, called the *material substratum* of $\mathscr{B}$. The natural suggestion coming from the previous discussion is then to assign to $\mathscr{S}$ the energy-momentum tensor

$$M_{\alpha\beta} = \widehat{T}_{\alpha\beta} - m\delta_0 V_\alpha V_\beta \doteq m(c^2 + \chi)V_\alpha V_\beta - S_{\alpha\beta} \tag{3.11}$$

In this way, taking eqs. (3.4 *a, b*) into account, it is easily seen that the description of the substratum $\mathscr{S}$ relies on the choice of three constitutive equations

$$\chi = \mathscr{X}(\Xi, \theta(\cdot), \delta_0(\cdot))\, ; \qquad \underset{\sim}{S} = \mathbb{S}(\Xi, \theta(\cdot))\, ; \qquad \eta = \mathbb{N}(\Xi, \theta(\cdot)) \tag{3.12}$$

subject to the *a priori* constraints

$$\mathscr{X}(\Xi, \theta(\cdot), \delta_0(.)) = \mathscr{X}(\Xi, \theta(\cdot), 0) - \delta_0 \doteq \mathbb{E}(\Xi, \theta(\cdot)) - \delta_0 \tag{3.13 a}$$

$$m\theta\eta_{,\alpha}V^\alpha + V_\alpha M^{\alpha\beta}{}_{;\beta} = (m\delta_0 V^\alpha)_{;\alpha} \tag{3.13 b}$$

In view of eqs. (3.11)-(3.13), all attributes of $\mathscr{S}$ are expressed solely in terms of the variables Ξ, $\theta(\cdot)$, $\delta_0(\cdot)$. From this, recalling our previous definitions, we conclude that the concept of material substratum has a primitive character (i. e. it admits a constitutive characterization of its own, independently of the *kinematical* behaviour of the heat subsystem) if and only if the resulting model of the continuum $\mathscr{B}$ is *separable*. Under the stated assumption, the representative space of $\mathscr{S}$ may be identified with the abstract space $\mathfrak{A}_0^*$ introduced above, while the corresponding constitutive characterization is given explicitly by eqs. (3.11)-(3.13), completed by the constraint

$$\mathscr{F}_{(0)}(\Xi, \theta(\cdot), \delta_0(\cdot)) = 0 \tag{3.14}$$

relating the evolution of the field $\delta_0(\cdot)$ to the thermokinetic variables Ξ, $\theta(\cdot)$.

Collecting all previous results, we now observe that, on the basis of eqs. (3.10 *a*, *b*), (3.11), the *total* energy-momentum tensor (3.7) may be written synthetically as

$$\underset{\sim}{T} = \underset{\sim}{M} + \underset{\sim}{Q} + \underset{\sim}{W} \tag{3.15}$$

where, likewise $\underset{\sim}{M}$ and $\underset{\sim}{Q}$, also the tensor $\underset{\sim}{W}$ is to be expressed in terms of the variables Ξ, $\theta(\cdot)$, $\delta_0(\cdot)$, $\underset{\sim}{v}(\cdot)$. Eq. (3.15) is perfectly consistent with the binary representation of the system $\mathscr{B}$ discussed above, and suggests a natural interpretation of $\underset{\sim}{W}$ as an *interaction tensor*, accounting for the possible contributions to $\underset{\sim}{T}$ coming from the *mutual interactions* between the subsystems $\mathscr{S}$ and $\mathscr{Q}$.

3.2. Dynamics of the heat subsystem

i) As a subsequent step in our analysis, we shall now examine the nature of the constraints (3.9) involved in the representation of the internal response of the continuum $\mathscr{B}$. As pointed out in § 2.1, the argument relies on the introduction of a suitable characterization of the *evolution* of both fields $\delta_0(\cdot)$, $\underset{\sim}{v}(\cdot)$ in terms of the thermokinetic variables Ξ, $\theta(\cdot)$. In this respect, it is then clear that the choice of the constraints (3.9) has its physical counterpart in the study of the *dynamics* of the heat subsystem $\mathscr{Q}$, or, more generally, of the *mutual interactions* between $\mathscr{Q}$ and the substratum $\mathscr{S}$. Both aspects of the problem, the constitutive one and the dynamical one, can be dealt with on a unified basis by introducing the quantities

$$f^\alpha \doteq Q^{\alpha\beta}{}_{;\beta} = (m\delta^0 U^\alpha U^\beta)_{;\beta} \tag{3.16 a}$$

$$\zeta \doteq -\mathscr{P}(M^{\alpha\beta}{}_{;\beta}) = cV_\alpha M^{\alpha\beta}{}_{;\beta} \tag{3.16 b}$$

expressing respectively the four-force density acting on $\mathscr{Q}$, and the power density *lost* by the substratum $\mathscr{S}$ in the material rest-frame [Ξ]. The analysis may then be splitted into two parts, according to the following scheme:

1) Define the *transverse part* of the four-force $\underset{\sim}{f}$ as

$$f^\alpha_\perp \doteq (\delta^\alpha{}_\lambda + U^\alpha U_\lambda)f^\lambda = (\delta^\alpha{}_\lambda + U^\alpha U_\lambda)Q^{\alpha\beta}{}_{;\beta} = m\delta^0 U^\alpha{}_{;\beta}U^\beta \tag{3.17}$$

The spatial resolution of the four-vector (3.17) in the material rest-frame [Ξ] may then be written synthetically as

$$\underset{\sim}{f_\perp} = \underset{\sim}{\tilde{f}} + \frac{\pi}{c}\underset{\sim}{V} \tag{3.18 a}$$

where, on account of the identity $U_\alpha f^\alpha_\perp = 0$, the temporal component π/c satisfies

$$\frac{\pi}{c} = \tilde{f}^\alpha v_\alpha \tag{3.18 b}$$

as it is easily checked from eqs. (3.10 *b*), (3.18 *a*) by direct computation. From a physical viewpoint, the fields $\tilde{\underset{\sim}{f}}$ and π define respectively the mechanical force density and the mechanical power density acting on $\mathscr{Q}$ in the frame of reference [Ξ]. Moreover, in view of eqs. (3.17), (3.18 *a*, *b*), it follows at once that the study of the evolution of the four-velocity field $\underset{\sim}{\mathrm{U}}(\cdot)$ (or, what is the same, of the spatial velocity $\underset{\sim}{v}(\cdot)$) is mathematically equivalent to the introduction of a suitable conjecture concerning the nature of the spatial force $\tilde{\underset{\sim}{f}}$, i. e. to the representation of $\tilde{\underset{\sim}{f}}$ as a functional

$$\tilde{\underset{\sim}{f}} = \tilde{\mathbb{F}}(\Xi, \theta(\cdot), \delta_0(\cdot), v(\cdot)) \tag{3.19}$$

over the space $\mathfrak{A}^*$.

2) In a similar way, one should expect the evolution of δ_0 to be determined by the knowledge of the *longitudinal component* $-\mathrm{U}_\alpha f^\alpha = (m\delta^0\mathrm{U}^\alpha)_{;\alpha}$ of the four-force (3.16 *a*). For constitutive purposes, however, a more direct line of approach is provided by the energy-balance equation (3.16 *b*). In view of eqs. (3.12), (3.13 *b*), the latter is already of the required form (3.9), provided only that we express the power density ζ as a functional over the space $\mathfrak{A}^*$, namely

$$\zeta = \mathbb{Z}(\Xi, \theta(\cdot), \delta_0(\cdot), \underset{\sim}{v}(\cdot)). \tag{3.20}$$

In particular, if we regard the quantity ζ as determined solely by the substratum $\mathscr{S}$, independently of the kinematical behaviour of the heat subsystem, eq. (3.20) simplifies to

$$\zeta = \mathbb{Z}(\Xi, \theta(\cdot), \delta_0(\cdot)) \tag{3.20'}$$

corresponding to a representation of ζ as a functional over the space $\mathfrak{A}_0^*$. With this ansatz, eq. (3.16 *b*) is automatically of the form (3.14), thus giving rise to a *separable* model of $\mathscr{B}$, in the sense clarified in § 3.1.

In any case, collecting all previous results, we conclude that the choice of the constitutive equations (3.19), (3.20) (or (3.19), (3.20')) is sufficient to provide a complete characterization of the internal response of the continuum $\mathscr{B}$, in the sense required by eqs. (3.9).

ii) From a physical viewpoint, the meaning of the quantity ζ is further clarified by an analysis of the *energy exchanges* between the subsystems $\mathscr{S}$ and $\mathscr{Q}$. Among other advantages, the argument will also provide a deeper insight into the nature of the interaction tensor $\underset{\sim}{\mathrm{W}}$ involved in the representation (3.15) of the total energy-momentum tensor T.

To start with, we observe that, in view of eqs. $(\tilde{2}.1)$, (3.15), (3.16 *a*), (3.17), (3.18 *a*), the temporal projection of the four-force $\underset{\sim}{f}$ in the material rest-frame [Ξ] may be expressed in either forms

$$\mathscr{P}(\underset{\sim}{f}) = -c\mathrm{V}_\alpha \mathrm{Q}^{\alpha\beta}{}_{;\beta} = -c\mathrm{V}_\alpha(\mathrm{T}^{\alpha\beta}{}_{;\beta} - \mathrm{M}^{\alpha\beta}{}_{;\beta} - \mathrm{W}^{\alpha\beta}{}_{;\beta})$$

$$\mathscr{P}(\underset{\sim}{f}) = -c\mathrm{V}_\alpha f_\perp^\alpha - \gamma c \mathrm{U}_\alpha f^\alpha = \pi + \gamma c(m\delta^0\mathrm{U}^\alpha)_{;\alpha}.$$

From these, taking eq. (3.16 *b*) and the definition (3.6) of $\underset{\sim}{W}$ into account, we get the further relation

$$\gamma c(m\delta^0 U^\alpha)_{;\alpha} = \mathscr{P}(\underset{\sim}{f}) - \pi = - cV_\alpha T^{\alpha\beta}{}_{;\beta} + {}^D\pi + \gamma\zeta^* \qquad (3.21)$$

where, for simplicity, we have introduced the notation

$${}^D\pi \doteq cV_\alpha {}^DW^{\alpha\beta}{}_{;\beta} \qquad (3.22\,a)$$

$$\gamma\zeta^* \doteq cV_\alpha(M^{\alpha\beta}{}_{;\beta} + \widehat{W}^{\alpha\beta}{}_{;\beta}) - \pi = \zeta + cV_\alpha\widehat{W}^{\alpha\beta}{}_{;\beta} - \pi. \qquad (3.22\,b)$$

Due to the definition of π given by eq. (3.18 *b*), it is then clear that the difference $\mathscr{P}(\underset{\sim}{f}) - \pi$ represents the *non-mechanical part* of the power density transferred to the heat subsystem $\mathscr{Q}$ in the material rest-frame [Ξ]. In this respect, all terms at the right-hand side of eq. (3.21) may therefore be interpreted as effective contributions to the process of *heat production* taking place within the body $\mathscr{B}$, as confirmed by the fact that their sum coincides, up to the factor γc, with the four-divergence of the heat four-current $m\delta^0\underset{\sim}{U}$. The role of the terms $- cV_\alpha T^{\alpha\beta}{}_{;\beta}$ and ${}^D\pi$ is then quite obvious. Indeed, on the basis of the energy balance equation (2.5 *a*), the quantity $- cV_\alpha T^{\alpha\beta}{}_{;\beta}$ coincides with the *external power density* $\mathscr{P}(\underset{\sim}{b})$ acting on $\mathscr{B}$ in its own rest frame. In this sense, the latter has always a strictly non-mechanical nature, and may be taken as a description of the *external contributions* to the phenomenon of heat production. In a similar way, the presence of the term ${}^D\pi$ at the right-hand side of eq. (3.21) is fully justified by eq. (3.22 *a*), and by the identification of ${}^D\underset{\sim}{W}$ with the *dissipative part* of the interaction tensor $\underset{\sim}{W}$.

The nature of the remaining term $\gamma\zeta^*$, however, is quite different, and involves a precise analysis of the *energy exchanges* between the subsystems $\mathscr{B}$ and $\mathscr{Q}$. This viewpoint is supported by the fact that, in the *absence* of heat conduction ($\underset{\sim}{v} \equiv 0$), eqs. (3.5 *b*), (3.18 *b*), (3.22 *b*) would imply $\zeta^* = \zeta$, thus showing that, in this special case, the whole amount ζ of power density *lost* by the substratum $\mathscr{S}$ in the material rest-frame [Ξ] would be converted *entirely* into invariant mass density of the heat subsystem. In the general case ($\underset{\sim}{v} \neq 0$), the situation is more delicate, due to the necessity of accounting also for the *mechanical* interactions between $\mathscr{S}$ and $\mathscr{Q}$. This means that, in principle, part of the power density lost by the material substratum may be transferred to $\mathscr{Q}$ in the form of mechanical work, and has therefore to be *subtracted* from the heat production mechanism. In full generality, the argument may be formalized by admitting the existence of a linear relation of the form

$$\zeta = \zeta^* + \alpha\pi \qquad (3.23\,a)$$

in which the coefficient $\alpha(0 \leqslant \alpha \leqslant 1)$ determines the exact contribution of the substratum $\mathscr{S}$ to the mechanical power density π. Comparison with eq. (3.22 *b*) then yields the identification

$$cV_\alpha\widehat{W}^{\alpha\beta}{}_{;\beta} = (\gamma - 1)\zeta^* + (1 - \alpha)\pi \qquad (3.23\,b)$$

expressing the dynamical contribution of the tensor $\widehat{\underset{\sim}{W}}$ to the energy exchanges between $\mathcal{S}$ and $\mathcal{Q}$. On the basis of eqs. (3.23 *a*, *b*), the type of interaction mechanism giving rise to the effect described by eq. (3.22 *b*) may then be understood as follows: the substratum $\mathcal{S}$ and the tensor $\widehat{\underset{\sim}{W}}$, together, supply the whole amount of mechanical power density π acting on $\mathcal{Q}$ in the material rest-frame [Ξ]. The respective contributions are determined by the coefficient α and are respectively $\alpha\pi$ and $(1 - \alpha)\pi$. The remaining amount $\zeta^* = \zeta - \alpha\pi$ of power density lost by $\mathcal{S}$ and not converted into mechanical work is entirely « transformed into heat », thus determining a corresponding increase in the invariant mass density of the heat subsystem (compare eq. (3.21) with eq. (3.22 *b*), both divided by the common factor γ). The heat so produced, however, is delivered by $\mathcal{S}$ *at rest* in the frame of reference [Ξ] (as implicit in the definition of ζ^* given by eq. (3.23 *a*)), and has therefore to be *accelerated* to the velocity $\underset{\sim}{v}$, in order to be embodied by the heat subsystem. The necessary amount of mechanical power—corresponding to a supply of kinetic energy per unit time equal to $(\gamma - 1)\zeta^*$—is provided entirely by the interaction tensor $\widehat{\underset{\sim}{W}}$, through the first term at the right-hand side of (3.23 *b*). As a result, the total amount of non-mechanical power density transferred to the heat subsystem $\mathcal{Q}$ in the frame of reference [Ξ] through the above process is $\zeta + V_\alpha \widehat{W}^{\alpha\beta}{}_{;\beta} - \pi = \zeta^* + (\gamma - 1)\zeta^* = \gamma\zeta^*$, in complete agreement with eq. (3.22 *b*).

The previous discussion has its natural counterpart in the representation of the *total* four-force density $\underset{\sim}{f}$ acting on the heat subsystem $\mathcal{Q}$. Taking eqs. (3.16 *a*), (3.17), (3.21) into account, the latter may be expressed synthetically as

$$\underset{\sim}{f} = \underset{\sim}{f}_\perp + (m\delta^0 U^\alpha)_{;\alpha}\underset{\sim}{U} = \underset{\sim}{k} + \gamma^{-1}\left(- V_\alpha T^{\alpha\beta}{}_{;\beta} + \frac{{}^{D}\pi}{c}\right)\underset{\sim}{U} \qquad (3.24)$$

where, in view of the stated assumptions, the contribution

$$\underset{\sim}{k} \doteq \underset{\sim}{f}_\perp + \zeta^* \underset{\sim}{U} \qquad (3.25)$$

has to be regarded as coming entirely from the substratum $\mathcal{S}$, and from the interaction tensor $\widehat{\underset{\sim}{W}}$. According to this viewpoint, the four-vector $\underset{\sim}{k}$ may be further splitted into

$$k^\alpha = \widehat{k}^\alpha - \widehat{W}^{\alpha\beta}{}_{;\beta} \qquad (3.26)$$

in which the contribution $\widehat{\underset{\sim}{k}}$ is now due entirely to the material substratum. The nature of $\widehat{\underset{\sim}{k}}$ is then completely determined by the pair of identifications

$$\mathcal{P}(\widehat{\underset{\sim}{k}}) = cV_\alpha M^{\alpha\beta}{}_{;\beta} = \zeta \qquad (3.27\,a)$$

$$\widehat{k}^\lambda_\perp \doteq (\delta^\lambda{}_\beta + U^\lambda U_\beta)\widehat{k}^\beta = \alpha f^\lambda_\perp \qquad (3.27\,b)$$

expressing respectively the conditions *a*) that the total amount of power density ζ lost by the substratum $\mathcal{S}$ is transferred *entirely* to the heat subsystem $\mathcal{Q}$, and *b*) that the substratum $\mathcal{S}$ supplies only an amount $\alpha\pi$ of the mechanical power, and thus also an amount $\alpha\tilde{\underset{\sim}{f}}$ of the mechanical force acting on $\mathcal{Q}$ in the material rest-frame $[\Xi]$. Taking eqs. (3.18 *a, b*) into account, the content of eqs. (3.27 *a, b*) may be summarized into the single relation

$$\hat{k}^\beta = \alpha f_\perp^\beta + \gamma^{-1}\left(\frac{\zeta}{c} - \alpha\frac{\pi}{c}\right)U^\beta = \alpha(\delta^\beta{}_\lambda - v^\beta v_\lambda)\tilde{f}^\lambda + \gamma^{-1}\frac{\zeta}{c}U^\beta. \quad (3.28)$$

In view of eqs. (3.23 *a*), (3.25), the latter may be written in the equivalent form

$$\hat{\underset{\sim}{k}} = \alpha\underset{\sim}{f}_\perp + \gamma^{-1}\frac{\zeta^*}{c}\underset{\sim}{U} = \gamma^{-1}\underset{\sim}{k} + (\alpha - \gamma^{-1})\underset{\sim}{k}_\perp$$

with inverse

$$\underset{\sim}{k} = \gamma\hat{\underset{\sim}{k}} + (\alpha^{-1} - \gamma)\hat{\underset{\sim}{k}}_\perp. \quad (3.29)$$

Moreover, on the basis of eq. (3.28) it is easily seen that, once the structure of the coefficient α has been fixed, the choice of the constitutive eq. (3.19), (3.20) is mathematically equivalent to the introduction of a corresponding representation

$$\hat{\underset{\sim}{k}} = \hat{\mathbb{K}}(\Xi, \theta(\cdot), \delta_0(\cdot), \underset{\sim}{v}(\cdot))$$

expressing the four-force $\hat{\underset{\sim}{k}}$ as a functional over the space $\mathfrak{A}^*$.

iii) Concerning the physical interpretation of the coefficient α, a possible suggestion comes from eq. (3.29). The latter shows that the definition of α is directly related to the introduction of a suitable conjecture concerning the relation between the four-vectors $\hat{\underset{\sim}{k}}$ and $\underset{\sim}{k}$.

To analyse this aspect, let us denote temporarily by $[\mathcal{Q}]$ the « heat rest-frame », i. e. the co-moving frame of reference associated with the heat subsystem $\mathcal{Q}$.

Then, given an arbitrary infinitesimal portion Δ of $\mathcal{Q}$, at any instant t' relative to $[\mathcal{Q}]$ we can determine a corresponding portion Δ_0 of material substratum, such that the spatial regions occupied by Δ and Δ_0 at the instant t' in the frame of reference $[\mathcal{Q}]$ coincide (see fig. 1). In this way, the proper volume $\delta\Sigma$ of Δ is by definition identical to the *relativistic* volume of Δ_0 as measured in $[\mathcal{Q}]$. Therefore, denoting by $\delta\Sigma_0$ the *proper* volume of Δ_0, the Lorentz contraction formula gives

$$\delta\Sigma_0 = \gamma\delta\Sigma \quad (3.30)$$

We now recall that, according to our previous definitions, the Minkowsky four-force *supplied* by the portion Δ_0 of material substratum at the instant t' is precisely $\hat{\underset{\sim}{k}}\delta\Sigma_0$, while the corresponding amount of four-force *absorbed* by the portion Δ of heat subsystem is $\underset{\sim}{k}\delta\Sigma$. The simplest conjecture is then

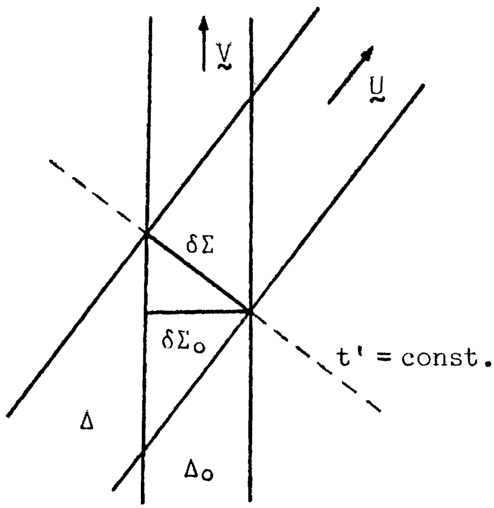

FIG. 1. — Two dimensional space-time diagram of the relation between Δ and Δ_0.

to assume the *equality* between these four-forces. Comparison with eqs. (3.29), (3.30) shows that this is equivalent to the ansatz

$$\underset{\sim}{k} = \gamma \underset{\sim}{\widehat{k}} \tag{3.31 a}$$

corresponding to the identification

$$\alpha = \gamma^{-1}. \tag{3.31 b}$$

According to this viewpoint, the contribution of the tensor $\underset{\sim}{\widehat{W}}$ to the mutual interactions between the subsystems $\mathcal{S}$ and $\mathcal{Q}$ is then completely explained in terms of the transformation law for spatial volumes, i. e. it accounts for a *purely relativistic* effect, with no classical analogue. In particular, in view of eqs. (3.26), (3.31 *a*), we have the identification

$$-\widehat{W}^{\alpha\beta}{}_{;\beta} = k^\alpha - \widehat{k}^\alpha = (\gamma - 1)\widehat{k}^\alpha \tag{3.32}$$

The latter, together with eq. (3.27 *a*), points out another important feature of the ansatz (3.31 *a*). In fact, taking eqs. (2.1), (3.27 *a*), (3.32) and the representation (3.11)-(3.13) of the material substratum into account, we have the identity

$$V_\alpha \widehat{W}^{\alpha\beta}{}_{;\beta} = (\gamma - 1) V_\alpha M^{\alpha\beta}{}_{;\beta} = \gamma \{ (m\delta_0 V^\alpha)_{;\alpha} - m\theta\eta_{,\alpha}V^\alpha \} + m\chi_{,\alpha}V^\alpha - S^{\alpha\beta}V_{\alpha;\beta}.$$

Setting for simplicity

$$\widehat{\psi} \doteq \chi + \gamma(\delta_0 - \theta\eta) \tag{3.33}$$

the latter may be written in the equivalent form

$$m\left(\widehat{\psi}_{,\alpha} - \frac{\partial\widehat{\psi}}{\partial\gamma}\gamma_{,\alpha}\right)V^\alpha = V_\alpha \widehat{W}^{\alpha\beta}{}_{;\beta} + S^{\alpha\beta}V_{\alpha;\beta} - \gamma m\eta\theta_{,\alpha}V^\alpha$$

expressing a sort of « generalized recoverability condition » for the tensor $\widehat{\underset{\sim}{W}}$, in the sense that, likewise $S^{\alpha\beta}V_{\alpha;\beta}$, also the power density $V_\alpha\widehat{W}^{\alpha\beta}{}_{;\beta}$ (or, more precisely, the sum $V_\alpha\widehat{W}^{\alpha\beta}{}_{;\beta} + S^{\alpha\beta}V_{\alpha;\beta}$) is derivable from the generalized potential $\widehat{\psi}$ defined by eq. (3.33).

As a concluding remark we observe that, with the identification (3.31 *b*), the representation (3.28) of the four-force $\widehat{\underset{\sim}{k}}$ simplifies to

$$\widehat{k}^\alpha = \gamma^{-1}\left\{(\delta^\alpha{}_\beta - v^\alpha v_\beta)\tilde{f}^\beta + \frac{\zeta}{c}U^\alpha\right\} \tag{3.34}$$

while the content of eqs. (3.24), (3.31 *a*), (3.32) is summarized into the pair of relations

$$Q^{\alpha\beta}{}_{;\beta} = \gamma\widehat{k}^\alpha + \gamma^{-1}\left\{\left(-V_\lambda T^{\lambda\beta}{}_{;\beta} + \frac{{}^D\pi}{c}\right)\right\}U^\alpha \tag{3.35 a}$$

$$\begin{aligned} M^{\alpha\beta}{}_{;\beta} &= T^{\alpha\beta}{}_{;\beta} - Q^{\alpha\beta}{}_{;\beta} - {}^DW^{\alpha\beta}{}_{;\beta} - \widehat{W}^{\alpha\beta}{}_{;\beta} = \\ &= \mathcal{N}(T^{\alpha\beta}{}_{;\beta} - {}^DW^{\alpha\beta}{}_{;\beta}) + \left(V_\lambda T^{\lambda\beta}{}_{;\beta} - \frac{{}^D\pi}{c}\right)v^\alpha - \widehat{k}^\alpha. \end{aligned} \tag{3.35 b}$$

These, together with the balance equations (2.3), provide the formal basis for the study of the *problem of motion* for both subsystems $\mathcal{S}$ and $\mathcal{Q}$, under the effect of the *mutual* interactions, and of the interactions with the external world.

3.3. **Entropy inequality**

To complete our mathematical scheme, we still need an explicit formulation of the 2nd law of thermodynamics for the class of models in study. In view of eqs. (2.6 *a*, *b*), the argument relies essentially on the introduction of a suitable assumption concerning the nature of the dynamical temperature ϑ. The analysis is greatly simplified by introducing the dimensionless quantity

$$\nu \doteq \frac{\gamma\theta}{\vartheta} \tag{3.36}$$

In this way, taking eqs. (2.8), (3.2) into account, we get the explicit representation

$$\underset{\sim}{s} = mc\eta\underset{\sim}{V} + \frac{\nu\underset{\sim}{q}}{\gamma\theta} = mc\eta\underset{\sim}{V} + \frac{\nu cm\delta_0\underset{\sim}{v}}{\theta} \tag{3.37}$$

whence, recalling eq. (2.7)

$$s^\alpha{}_{;\alpha} = mc\eta_{,\alpha}V^\alpha + \frac{\nu c}{\theta}\left\{(m\delta_0 v^\alpha)_{;\alpha} + m\delta_0 v^\alpha\left(\frac{\nu_{,\alpha}}{\nu} - \frac{\theta_{,\alpha}}{\theta}\right)\right\}.$$

Also, on the basis of eqs. (3.10 *b*), (3.13 *b*), (3.16 *b*), (3.21), (3.23 *a*), (3.31 *b*), (3.36) we have the identity

$$\frac{c}{\vartheta} V_\alpha T^{\alpha\beta}{}_{;\beta} = \frac{v}{\gamma\theta}\{ - \gamma c(m\delta^0 U^\alpha)_{;\alpha} + {}^D\pi + \gamma\zeta^* \}$$

$$= - \frac{v}{\theta}\left\{ c(m\delta_0 V^\alpha)_{;\alpha} + c(m\delta_0 v^\alpha)_{;\alpha} - \frac{{}^D\pi}{\gamma} + \frac{\pi}{\gamma} - \zeta \right\}$$

$$= - vcm\eta_{,\alpha}V^\alpha - \frac{v}{\theta}\left\{ c(m\delta_0 v^\alpha)_{;\alpha} - \frac{{}^D\pi}{\gamma} + \frac{\pi}{\gamma} \right\}$$

so that the entropy production (2.6 *a*) may be written explicitly as

$$\sigma = (1 - v)mc\eta_{,\alpha}V^\alpha + \frac{c}{\theta} m\delta_0 v^\alpha v_{,\alpha} + \frac{v}{\gamma\theta}\left(- \gamma mc\delta_0 v^\alpha \frac{\theta_{,\alpha}}{\theta} + {}^D\pi - \pi \right) \quad (3.38)$$

From this it follows at once that the simplest conjecture concerning the nature of the coefficient v is provided by the ansatz

$$v \equiv 1$$

corresponding to the identification

$$\vartheta = \gamma\theta. \quad (3.39)$$

Then recalling eq. (3.2) and the representation (3.18 *b*) of the mechanical power π, eq. (3.38) reduces to

$$\sigma = \frac{1}{\vartheta}\left\{ {}^D\pi - q^\alpha\left(\frac{\theta_{,\alpha}}{\theta} + \frac{\tilde{f}_\alpha}{m\delta}\right) \right\} \quad (3.38')$$

so that the entropy inequality (3.6 *b*) takes the simple form

$$ {}^D\pi - q^\alpha\left(\frac{\theta_{,\alpha}}{\theta} + \frac{\tilde{f}_\alpha}{m\delta}\right) \geqslant 0 \quad (3.40)$$

which is almost identical to the corresponding classical expression, except for the presence of the *mechanical force* $\underset{\sim}{\tilde{f}}$ at the left-hand side of eq. (3.40).

According to the viewpoint of Coleman and Noll, the inequality (3.40) must hold identically, for all admissible processes of the system $\mathscr{B}$. From this, taking the representation (3.22 *a*) of the dissipative power ${}^D\pi$ into account, we conclude that the 2nd law restricts the class of possible models of $\mathscr{B}$, by posing an *a priori* constraint on the choice of the functionals

$$\underset{\sim}{\tilde{f}} = \tilde{\mathbb{F}}(\Xi, \theta(\cdot), \delta_0(\cdot), \underset{\sim}{v}(\cdot)) ; \qquad {}^D\underset{\sim}{W} = {}^D\mathbb{W}(\Xi, \theta(\cdot), \delta_0(\cdot), \underset{\sim}{v}(\cdot)) \quad (3.41)$$

On the contrary, the inequality (3.40) does not provide any information on the nature of the quantity ζ; in particular, it does not indicate any preferred choice between the alternatives expressed respectively by eqs. (3.20) and (3.20'). From an axiomatic viewpoint, this means essentially that the 2nd law leaves us the freedom of *completing* the mathematical scheme developed so far, by adding a further *hypothesis* on the structure of the models in study. The natural suggestion coming from the analysis of § 3.1, 3.2 is then to include among the properties of the model the requirement of *separability*, as expressed by the ansatz

$$\zeta = \mathbb{Z}(\Xi, \theta(\cdot), \delta_0(\cdot)). \tag{3.42}$$

As pointed out in § 3.2, this corresponds to assigning to the substratum $\mathscr{S}$ the role of a *primitive* concept, completely characterized by the *four* functionals

$$\begin{array}{ll} \chi = \mathscr{X}(\Xi, \theta(\cdot), \delta_0(\cdot) \quad , & \zeta = \mathbb{Z}(\Xi, \theta(\cdot), \delta_0(\cdot)) \\ \underset{\sim}{S} = \mathbb{S}(\Xi, \theta(\cdot)) & \eta = \mathbb{N}(\Xi, \theta(\cdot)) \end{array} \tag{3.43}$$

all defined over the representative space $\mathfrak{A}_0^*$, and subject to the *a priori* constraints expressed by eqs. (3.13 *a*, *b*), (3.16 *b*). Even more important than that, the freedom in the choice of the functional (3.42) is reflected in the fact that, in the construction of a mathematical model for the continuum $\mathscr{B}$, the constitutive characterization of the material substratum must be given *a priori*, independently of any restriction coming from the 2nd law.

4. APPLICATIONS

For illustrative purposes, we shall now indicate how the results established in § 3 can be applied to the study of two typical situations of actual physical interest, namely heat conduction in non-dissipative media, and internal dissipation in non-conducting materials. In both cases we shall simply sketch the main ideas involved in the development of the theory, leaving it to a subsequent paper to fill the necessary details.

A) In the case of a non-dissipative material, the mathematical scheme developed in § 3 is completed by the further constraint

$$^{D}\underset{\sim}{W} \equiv 0 \tag{4.1}$$

As a result, the entropy inequality (3.40) simplifies to

$$-\, q^{\alpha}\left(\frac{\theta_{,\alpha}}{\theta} + \frac{\tilde{f}_{\alpha}}{m\delta}\right) \geqslant 0 \tag{4.2}$$

For *isotropic* media, the simplest ansatz consistent with the requirement (4.2) is

$$q^{\alpha} = -\,\kappa g^{\alpha\beta}\left\{ \mathscr{N}(\theta_{,\beta}) + \theta\,\frac{\tilde{f}_{\beta}}{m\delta} \right\} \tag{4.3}$$

κ being a non-negative constant, depending on the properties of the material, and called the *thermal conductivity* (7). Using the simplified notation $\tilde{\theta}_\alpha \doteq \mathcal{N}(\theta_{,\alpha})$, eq. (4.3) implies

$$\tilde{f}_\alpha = -\frac{m\delta}{\theta}\left(\frac{q_\alpha}{\kappa} + \tilde{\theta}_\alpha\right) = -\frac{\gamma m\delta_0}{\theta}\left(\frac{q_\alpha}{\kappa} + \tilde{\theta}_\alpha\right) \tag{4.4}$$

From this, recalling our previous results, we conclude that, once the constitutive characterization (3.43) of the material substratum has been fixed, eqs. (4.1), (4.4), together with eqs. (3.34), (3.35 *a*, *b*), provide a complete picture of the *dynamics* of heat conduction in isotropic, non dissipative media.

As already stated above, a detailed analysis of the general problem will be dealt with in a subsequent paper. For the present purposes it is sufficient to remark that, in the classical limit, the spatial force $\tilde{f}$ may be identified with the (substantial) time derivative of the momentum density $c^{-2}\,\underset{\sim}{q}$ associated with the heat flux. In this case, setting for simplicity

$$\tau_c \doteq \frac{\kappa\theta}{m\delta_0 c^2} \tag{4.5}$$

and using the standard notation of Classical Mechanics, eq. (4.4) reduces to

$$q_\alpha + \tau_c \dot{q}_\alpha = -\kappa\tilde{\theta}_\alpha$$

which coincides with the equation originally proposed by Cattaneo [*9*] (see also Vernotte [*43*], Chester [*44*]) to account for the finite speed of propagation of thermal waves in material media. Besides illustrating a relevant feature of the dynamical scheme developed so far, the previous result is also important in order to get an idea of the order of magnitude of the energy density $m\delta_0$ associated with the heat subsystem. In fact, if we accept as a phenomenological result the viewpoint that τ_c is connected with the communication time between phonons (phonon-phonon collisions), a reasonable estimate seems to be $\tau_c \simeq 10^{-10}$ sec for most common metals [*44*]. The identification (4.5) then yields $m\delta_0 \simeq 3.10^{-9}$ cal.cm^{-3}, thus showing that the contribution of thermal inertia to the total energy-momentum tensor of $\mathcal{B}$ is indeed very small.

As a further comment on eq. (4.3) we observe that, for $\underset{\sim}{v} \equiv 0$, eqs. (3.10 *b*), (3.17), (3.18 *a*, *b*) yield immediately

$$\tilde{f}^\alpha = f_\perp^\alpha = m\delta \mathrm{V}^\alpha{}_{;\beta}\mathrm{V}^\beta \tag{4.6 a}$$

whence also

$$\tilde{\theta}_\alpha + \frac{\theta\tilde{f}_\alpha}{m\delta} = \tilde{\theta}_\alpha + \theta\mathrm{V}^\alpha{}_{;\beta}\mathrm{V}^\beta. \tag{4.6 b}$$

(7) A more general choice, accounting also for possible anisotropies, would be $q^\alpha = -\mathrm{K}^{\alpha\beta}(\theta_{,\beta} + \theta\tilde{f}_\beta/m\delta)$, $\mathrm{K}^{\alpha\beta} = \mathrm{K}^{\alpha\beta}(\Xi, \theta(\cdot))$ being any *positive definite* spatial tensor field over $\mathcal{V}_4$.

The term at the right-hand side of eq. (4.6 *b*) is usually referred to as the *pocket temperature*, and is considered as the natural relativistic extension of the temperature gradient [*2*, *21*, *24*]. This viewpoint is confirmed by eq. (4.3). In fact, for $\kappa \neq 0$, the latter shows that a necessary condition for thermal equilibrium within the body $\mathscr{B}$ is the vanishing of the pocket temperature (4.6 *b*) throughout the whole evolution of $\mathscr{B}$.

B) In the case of non-conducting materials, the mathematical scheme developed in § 3 is completed by the *a priori* constraint

$$\underset{\sim}{v} \equiv 0. \tag{4.7}$$

Under the stated assumption, the representative space of $\mathscr{B}$ may be identified at the outset with the abstract space $\mathfrak{A}_0^*$ introduced in § 3.1. The constitutive characterization of the spatial force $\underset{\sim}{\tilde{f}}$ is then trivial (see e. g. (4.6 *a*)), so that the whole problem is reduced to the study of the internal dissipation tensor ${}^{\mathrm{D}}\underset{\sim}{\mathrm{W}}$. In view of eq. (3.8), the latter may be represented symbolically as

$${}^{\mathrm{D}}\mathrm{W}_{\alpha\beta} = m\xi_0 \mathrm{V}_\alpha \mathrm{V}_\beta - \acute{\mathrm{S}}_{\alpha\beta} \tag{4.8}$$

with $\acute{\mathrm{S}}_{\alpha\beta} = \acute{\mathrm{S}}_{\beta\alpha}$, $\acute{\mathrm{S}}_{\alpha\beta}\mathrm{V}^\beta = 0$. This, together with eqs. (3.2), (3.22 *a*), (4.7), allows to express the entropy inequality (3.40) for the class of material in study in the form

$${}^{\mathrm{D}}\pi = -\, c(m\xi_0 \mathrm{V}^\alpha)_{;\alpha} + \acute{\mathrm{S}}^{\alpha\beta}\mathrm{D}_{\alpha\beta} \geqslant 0 \tag{4.9}$$

where, for simplicity, we are now indicating by

$$\mathrm{D}_{\alpha\beta} \doteq c\mathcal{N} \otimes \mathcal{N}\mathrm{V}_{(\alpha;\beta)} = c\left\{ \mathrm{V}_{(\alpha;\beta)} + \mathrm{V}_{(\alpha}\mathrm{V}_{\beta);\lambda}\mathrm{V}^\lambda \right\} \tag{4.10}$$

the *rate-of-strain tensor* of $\mathscr{B}$ ([8]).

In the case $\xi_0 = 0$, it would be natural to identify $\underset{\sim}{\acute{\mathrm{S}}}$ with the *dissipative stress tensor* of $\mathscr{B}$, and $-\, c\acute{\mathrm{S}}^{\alpha\beta}{}_{;\beta}\mathrm{V}_\alpha = \acute{\mathrm{S}}^{\alpha\beta}\mathrm{D}_{\alpha\beta}$ with the associated *dissipative power density*. What makes the present situation different, however, is the presence of the extra term $m\xi_0$ in both equations (4.8), (4.9). In view of eq. (4.8), this term contributes *additively* to the total energy density of $\mathscr{B}$ in the material rest-frame [Ξ]. Moreover, on the basis of eqs. (3.21), (4.9), it is easily seen that the contribution of the internal dissipation to the process of *heat production* is not expressed solely by the power density $\acute{\mathrm{S}}^{\alpha\beta}\mathrm{D}_{\alpha\beta}$ associated with the stress tensor $\underset{\sim}{\acute{\mathrm{S}}}$, but includes also the additional term $-\, c(m\xi_0\mathrm{V}^\alpha)_{;\alpha}$. Both aspects can be given a rational interpretation by introducing the concept of *thermal relaxation*. Quite generally, this consists in

---

([8]) Here and in the following, round brackets will indicate symmetrization of the indices. The presence of the factor *c* in eq. (4.10), as well as in the definition of the « time derivatives » introduced below, is due to the normalization of the four-velocity field $\underset{\sim}{\mathrm{V}}(\cdot)$ adopted in the text.

admitting that the mechanical work done by the stress tensor $\acute{\tilde{S}}$ is not converted *directly* into heat, but is temporarily *stored* by the system $\mathscr{B}$ in the form of an extra-energy density $m\xi_0$, *at rest* in the frame of reference [Ξ]. This accounts for the presence of the term $m\xi_0 V_\alpha V_\beta$ at the right-hand side of eq. (4.8). More precisely, the previous argument suggests that we regard ${}^D\tilde{W}$ as the energy-momentum tensor of a *third* ideal subsystem $\mathscr{D}$ of $\mathscr{B}$, completely characterized by its own invariant energy density $m\xi_0$, and its own mechanical stress $\acute{\tilde{S}}$. In view of eq. (3.22 *a*), the quantity ${}^D\pi$ may then be identified with the power density *lost* by $\mathscr{D}$ in the material rest frame [Ξ]. From eq. (3.21) we already know that this quantity is transferred entirely to the heat subsystem $\mathscr{Q}$. What is left to do in order to obtain a precise characterization of the energy exchanges between the subsystems $\mathscr{D}$ and $\mathscr{Q}$ is then to introduce a suitable assumption concerning the nature of ${}^D\pi$. This is precisely the role of the thermal relaxation hypothesis: in its simplest formulation, the latter consists in the conjecture that the subsystem $\mathscr{D}$ is intrinsically unstable, i. e. it has a natural tendency to *decay*, by transferring its energy to the heat subsystem at a rate $m\xi_0/\tau$, τ being a characteristic parameter of the material in study, called the *relaxation time.*

In view of the stated results, the previous hypothesis is summarized into the energy balance equation

$$cV_\alpha {}^D W^{\alpha\beta}{}_{;\beta} = -\, c(m\xi_0 V^\alpha)_{;\alpha} + \acute{S}^{\alpha\beta} D_{\alpha\beta} = \frac{m\xi_0}{\tau} \tag{4.11}$$

corresponding to the ansatz

$$^D\pi = \frac{m\xi_0}{\tau}\,. \tag{4.12}$$

Eq. (4.11) may be expressed more conveniently by introducing the notation

$$^D S_{\alpha\beta} \doteq \acute{S}_{\alpha\beta} - m\xi_0(g_{\alpha\beta} + V_\alpha V_\beta) = -\,{}^D W_{\alpha\beta} - m\xi_0 g_{\alpha\beta} \tag{4.13}$$

In this way, setting for simplicity

$$^D\dot{\pi} \doteq c\,{}^D\pi_{,\alpha} V^\alpha$$

and recalling eqs. (3.22 *a*), (4.12), we can replace eq. (4.11) by the equivalent relation

$$^D\pi + \tau\,{}^D\dot{\pi} = {}^D S^{\alpha\beta} D_{\alpha\beta}. \tag{4.14}$$

In the following, we shall call eq. (4.14) the *thermal relaxation equation.* The latter, together with eq. (4.12) and with the dissipation inequality

$$^D\pi \geqslant 0 \tag{4.15}$$

summarizes all aspects discussed above. In particular, if we exclude as *unphysical* any situation in which the tensor ${}^D\tilde{W}$ *diverges* in the asymptotic past $t \to -\infty$, it is easily seen that eq. (4.14) determines the field ${}^D\pi(\cdot)$ *uniquely* in terms of ${}^D\tilde{S}$ and of the kinematical behaviour of the material substratum. From this it follows at once that the inequality (4.15) provides an effective constraint on the nature of the tensor ${}^D\underset{\sim}{S}$. Moreover, in the limit $\tau \to 0$, eqs. (4.12), (4.14) imply ${}^D\pi = {}^DS^{\alpha\beta}D_{\alpha\beta}$, $\xi_0 = 0$, ${}^D\underset{\sim}{W} = -\acute{\underset{\sim}{S}} = -{}^D\underset{\sim}{S}$, thus showing that, in the absence of thermal relaxation, the scheme developed above yields back the familiar results of Continuum Mechanics.

The previous arguments strongly suggest that we identify ${}^D\underset{\sim}{S}$ with the *dissipative* stress tensor of the continuum $\mathscr{B}$. Among the various choices of ${}^D\underset{\sim}{S}$ consistent with the inequality (4.15), a particularly significant one is provided by the Maxwell-type constitutive relation

$$ {}^DS_{\alpha\beta} + 2\tau\tilde{\nabla}_0^*({}^DS_{\alpha\beta}) = \lambda D^\sigma{}_\sigma h_{\alpha\beta} + 2\mu D_{\alpha\beta} \tag{4.16} $$

where λ and μ are two constants, characteristic of the material in study, called respectively the *bulk viscosity* and the *shear viscosity*, $h_{\alpha\beta} \doteq g_{\alpha\beta} + V_\alpha V_\beta$ is the *spatial metric tensor* in the frame of reference $[\Xi]$, already introduced in eq. (2.1), and $\tilde{\nabla}_0^*$ is the so called standard time derivative relative to $[\Xi]$, whose action on spatial covariant tensor fields is expressed in terms of the Lie derivative $\mathscr{L}_{\underset{\sim}{V}}$ and of the tensor $\underset{\sim}{D}$ according to the equation (9)

$$ \tilde{\nabla}_0^*(Z_{\alpha\beta\dots\gamma}) = c\mathscr{L}_{\underset{\sim}{V}}(Z_{\alpha\beta\dots\gamma}) - D_\alpha{}^\lambda Z_{\lambda\beta\dots\gamma} - D_\beta{}^\lambda Z_{\alpha\lambda\dots\gamma} - \dots - D_\gamma{}^\lambda Z_{\alpha\beta\dots\lambda} \tag{4.17} $$

The ansatz (4.16) may be stated more synthetically by introducing the *relaxed stretching tensor* $\overset{*}{\underset{\sim}{D}}$, defined in terms of $\underset{\sim}{D}$ and of the relaxation time τ by the equation

$$ \overset{*}{\underset{\sim}{D}} + 2\tau\tilde{\nabla}_0^*(\overset{*}{\underset{\sim}{D}}) = \underset{\sim}{D} \tag{4.18} $$

completed by the requirement of regularity in the asymptotic past $t \to -\infty$. With this definition, eq. (4.16) may be written in the equivalent form

$$ {}^DS_{\alpha\beta} = \lambda\overset{*}{D}{}^\sigma{}_\sigma h_{\alpha\beta} + 2\mu\overset{*}{D}_{\alpha\beta} \tag{4.19 a} $$

while the thermal relaxation equation (4.14) is transformed into

$$ {}^D\pi + \tau\,{}^D\dot{\pi} = \lambda\overset{*}{D}{}^\sigma{}_\sigma D^\nu{}_\nu + 2\mu\overset{*}{D}{}^{\alpha\beta}D_{\alpha\beta} $$

which admits

$$ {}^D\pi = \lambda(\overset{*}{D}{}^\sigma{}_\sigma)^2 + 2\mu\overset{*}{D}{}^{\alpha\beta}\overset{*}{D}_{\alpha\beta} \tag{4.19 b} $$

as the unique solution that does not diverge in the limit $t \to -\infty$. A formal proof of eqs. (4.19 *a*, *b*) will be given in the Appendix.

From a physical viewpoint it is worth noticing that both eqs. (4.19 *a*, *b*)

---

(9) The geometrical meaning of the operator (4.17), as well as its role in the construction of a spatial tensor analysis in a given frame of reference $[\Xi]$ are discussed extensively in refs. [*30*, *31*].

are formally identical to the corresponding classical equations involved in the Navier-Stokes theory of viscosity, except for the presence of the relaxed stretching tensor $\overset{*}{\underset{\sim}{D}}$ in place of $\underset{\sim}{D}$. In particular, the requirement of consistency of eq. (4.19 *b*) with the inequality (4.15) is summarized into the pair of conditions

$$\mu \geqslant 0, \qquad 3\lambda + 2\mu \geqslant 0 \tag{4.20}$$

which agree with the classical inequalities of Duhem and Stokes for the shear viscosity μ and the bulk viscosity λ (see, e. g. [*45*], chap. 2).

As a concluding remark we recall that, for $\tau = 0$ (corresponding to the identification $\overset{*}{\underset{\sim}{D}} = \underset{\sim}{D}$) the inequalities (4.20) are at the basis of Duhem's theorem, according to which acceleration waves are impossible in a linearly viscous (Newtonian) fluid. On the contrary, for $\tau \neq 0$, the type of constitutive relation expressed by eq. (4.16) does no longer preclude the existence of acceleration waves. A detailed analysis of this point is beyond the scope of the present work. In this connection, the reader is referred to the extensive analysis of Coleman, Greenberg and Gurtin [*26*], concerning wave propagation in the general context of Maxwellian materials.

APPENDIX

i) Definition and formal properties of the standard time derivative.

Quite generally, a time derivative in the frame of reference [Ξ] is defined as a derivation $\tilde{\nabla}_0$ of the tensor algebra over $\mathscr{V}_4$, subject to the following requirements [30] :

a) $\tilde{\nabla}_0(f) = cf_{,\alpha}\, V^\alpha \doteq \dot{f}$ for all differentiable functions,

b) $\tilde{\nabla}_0(A^\alpha B_\alpha) = \tilde{\nabla}_0(A^\alpha)B_\alpha + A^\alpha \tilde{\nabla}_0(B_\alpha)$ (i. e. $\tilde{\nabla}_0$ commutes with contractions),

c) $\tilde{\nabla}_0(g_{\alpha\beta}) = 0$, $\tilde{\nabla}_0(V_\alpha) = 0$, whence also $\tilde{\nabla}_0(h_{\alpha\beta}) = 0$ (i. e. $\tilde{\nabla}_0$ commutes with the process of raising and lowering the tensor indices, as well as with the spatial resolution process in the frame of reference [Ξ], based on the projection operators (2.1)).

Denoting by $\mathscr{L}_{\underset{\sim}{V}}$ the Lie derivative in the direction $\underset{\sim}{V}$, properties *a)* and *b)* together imply that the action of the operator $\tilde{\nabla}_0$ on an arbitrary tensor field $\underset{\sim}{Z}$ has the general structure

$$\tilde{\nabla}_0\left(Z^\alpha{}_{\beta\ldots\gamma}\right) = c\mathscr{L}_{\underset{\sim}{V}}\left(Z^\alpha{}_{\beta\ldots\gamma}\right) + \Sigma^\alpha{}_\lambda Z^\lambda{}_{\beta\ldots\gamma} - \Sigma^\lambda{}_\beta Z^\alpha{}_{\lambda\ldots\gamma} - \ldots - \Sigma^\lambda{}_\gamma Z^\alpha{}_{\beta\ldots\lambda} \quad \text{(A.1)}$$

in which the quantities $\Sigma^\alpha{}_\beta$ form the components of a mixed tensor field of rank 2. Moreover, taking the identities

$$\mathscr{L}_{\underset{\sim}{V}}(g_{\alpha\beta}) = V_{\alpha;\beta} + V_{\beta;\alpha} \quad , \quad \mathscr{L}_{\underset{\sim}{V}}(V_\alpha) = V_{\alpha;\beta}V^\beta$$

into account, the content of property *c)* is summarized into the pair of relations

$$c(V_{\alpha;\beta} + V_{\beta;\alpha}) = \Sigma_{\alpha\beta} + \Sigma_{\beta\alpha}$$
$$cV_{\alpha;\beta}V^\beta = \Sigma^\lambda{}_\alpha V_\lambda.$$

From this, recalling the representation (4.10) of the rate-of-strain tensor $\underset{\sim}{D}$, it follows at once that the most general choice of $\Sigma_{\alpha\beta}$ consistent with the stated requirements is given explicitly by

$$\Sigma_{\alpha\beta} = D_{\alpha\beta} + A_{\alpha\beta} - cV_\alpha V_{\beta;\lambda}V^\lambda \quad \text{(A.2)}$$

where $\underset{\sim}{A}$ denotes an *arbitrary* antisymmetric spatial tensor field, dimensionally homogeneous with $\underset{\sim}{D}$. The *standard time derivative* $\tilde{\nabla}^*_0$ relative to [Ξ] is then defined as the distinguished time derivative determined by the ansatz $\underset{\sim}{A} = 0$. With this choice, the action of $\tilde{\nabla}^*_0$ on an arbitrary covariant spatial tensor field $\underset{\sim}{Z}$ is represented in the form

$$\tilde{\nabla}^*_0(Z_{\alpha\beta\ldots\gamma}) = c\mathscr{L}_{\underset{\sim}{V}}(Z_{\alpha\beta\ldots\gamma}) - D^\lambda{}_\alpha Z_{\lambda\beta\ldots\gamma} - D^\lambda{}_\beta Z_{\alpha\lambda\ldots\gamma} - \ldots - D^\lambda{}_\gamma Z_{\alpha\beta\ldots\lambda}$$

which agrees with eq. (4.17).

ii) Proof of eqs. (4.19 *a, b*).

In view of the definition (4.18) of the relaxed stretching tensor $\overset{*}{\underset{\sim}{D}}$, taking the properties of the operator $\tilde{\nabla}^*_0$ into account, the constitutive equation (4.16) may be written in the equivalent form

$${}^{D}S_{\alpha\beta} + 2\tau\tilde{\nabla}^*_0\left({}^{D}S_{\alpha\beta}\right) = \lambda\left\{\overset{*}{D}{}^\sigma{}_\sigma + 2\tau\tilde{\nabla}^*_0(\overset{*}{D}{}^\sigma{}_\sigma)\right\}h_{\alpha\beta} + 2\mu\left\{\overset{*}{D}_{\alpha\beta} + 2\tau\tilde{\nabla}^*_0(\overset{*}{D}_{\alpha\beta})\right\}$$

which admits

$$ {}^{\mathrm{D}}\mathrm{S}_{\alpha\beta} = \lambda \overset{*}{\mathrm{D}}{}^{\sigma}{}_{\sigma} h_{\alpha\beta} + 2\mu \overset{*}{\mathrm{D}}_{\alpha\beta} $$

as the unique solution that does not diverge in the limit $t \to -\infty$. This establishes eq. (4.19 *a*). In a similar way, on the basis of eqs. (4.18), (4.19 *a*), the thermal relaxation equation (4.14) reads

$$ \begin{aligned} {}^{\mathrm{D}}\pi + \tau \tilde{\nabla}^{*}_{0}({}^{\mathrm{D}}\pi) &= \left(\lambda \overset{*}{\mathrm{D}}{}^{\sigma}{}_{\sigma} h^{\alpha\beta} + 2\mu \overset{*}{\mathrm{D}}{}^{\alpha\beta}\right) \left\{ \overset{*}{\mathrm{D}}_{\alpha\beta} + 2\tau \tilde{\nabla}^{*}_{0}(\overset{*}{\mathrm{D}}_{\alpha\beta}) \right\} \\ &= \lambda \overset{*}{\mathrm{D}}{}^{\sigma}{}_{\sigma} \left\{ \overset{*}{\mathrm{D}}{}^{\nu}{}_{\nu} + 2\tau \tilde{\nabla}^{*}_{0}\left(\overset{*}{\mathrm{D}}{}^{\nu}{}_{\nu}\right) \right\} + 2\mu \overset{*}{\mathrm{D}}{}^{\alpha\beta} \left\{ \overset{*}{\mathrm{D}}_{\alpha\beta} + 2\tau \tilde{\nabla}^{*}_{0}(\overset{*}{\mathrm{D}}_{\alpha\beta}) \right\} \\ &= \left\{ \lambda\left(\overset{*}{\mathrm{D}}{}^{\sigma}{}_{\sigma}\right)^2 + 2\mu \overset{*}{\mathrm{D}}{}^{\alpha\beta} \overset{*}{\mathrm{D}}_{\alpha\beta} \right\} + \tau \tilde{\nabla}^{*}_{0} \left\{ \lambda\left(\overset{*}{\mathrm{D}}{}^{\sigma}{}_{\sigma}\right)^2 + 2\mu \overset{*}{\mathrm{D}}{}^{\alpha\beta} \overset{*}{\mathrm{D}}_{\alpha\beta} \right\} \end{aligned} $$

which admits

$$ {}^{\mathrm{D}}\pi = \lambda\left(\overset{*}{\mathrm{D}}{}^{\sigma}{}_{\sigma}\right)^2 + 2\mu \overset{*}{\mathrm{D}}{}^{\alpha\beta} \overset{*}{\mathrm{D}}_{\alpha\beta} $$

as the unique solution which is regular at $t \to -\infty$. □

REFERENCES

[1] C. Eckart, *Phys. Rev.*, t. **58**, 1940, p. 919-924.
[2] R. A. Grot and A. C. Eringen, *Int. J. Engng Sci.*, t. **4**, 1966, p. 611-638.
[3] B. D. Coleman and W. Noll, *Arch. Rational Mech. Anal.*, t. **13**, 1963, p. 167-178.
[4] H. D. Curtis and G. Lianis, *Int. J. Engng Sci.*, t. **9**, 1971, p. 451-468.
[5] B. D. Coleman, *Arch. Rational Mech. Anal.*, t. **17**, 1964, p. 1-46.
[6] I. Müller, *Arch. Rational Mech. Anal.*, t. **40**, 1971, p. 1-36.
[7] I. Müller, *Arch. Rational Mech. Anal.*, t. **34**, 1969, p. 259-282.
[8] T. Alts and I. Müller, *Arch. Rational Mech. Anal.*, t. **48**, 1972, p. 245-273.
[9] C. Cattaneo, *Atti Sem. Mat. Fis. Univ. Modena*, t. **3**, 1948, p. 83-101.
[10] J. C. Maxwell, *Phil. Trans. Roy. Soc. London*, t. **157**, 1867, p. 49-88.
[11] I. Müller, *Zeit. Phys.*, t. **198**, 1967, p. 329-344.
[12] G. Lebon and J. Lambermont, *J. Mécan.*, t. **15**, 1976, p. 579-594.
[13] A. E. Green and N. Laws, *Arch. Rational Mech. Anal.*, t. **45**, 1972, p. 47-53.
[14] I. Müller, *Arch. Rational Mech. Anal.*, t. **41**, 1971, p. 319-332.
[15] D. B. Bogy and P. M. Naghdi, *J. Math. Phys.*, t. **11**, 1970, p. 917-923.
[16] W. Kosinski, *Arch. Mech. Stos.*, t. **27**, 1975, p. 733-748.
[17] M. E. Gurtin and A. C. Pipkin, *Arch. Rational Mech. Anal.*, t. **31**, 1968, p. 113-126.
[18] M. F. McCarthy, *Int. J. Engng Sci.*, t. **8**, 1970, p. 467-474.
[19] A. Morro, *Meccanica*, t. **12**, 1977, p. 73-77.
[20] J. Meixner, *Arch. Rational Mech. Anal.*, t. **39**, 1970, p. 108-130.
[21] A. Bressan, *Rend. Sem. Mat. Univ. Padova*, t. **34**, 1964, p. 1-73.
[22] M. Kranys, *Il Nuovo Cimento*, t. **B 42**, 1966, p. 51-70.
[23] G. Boillat, *Lettere al Nuovo Cimento*, t. **10**, 1974, p. 295-300.
[24] G. A. Maugin, *J. Phys. A : Math. Nucl. Gén.*, t. **7**, 1974, p. 465-484.
[25] G. Lianis, *Arch. Rational Mech. Anal.*, t. **55**, 1974, p. 300-331.
[26] B. D. Coleman, J. M. Greenberg and M. E. Gurtin, *Arch. Rational Mech. Anal.*, t. **22**, 1966, p. 333-354.
[27] M. Kranys, *Il Nuovo Cimento*, t. **B 50**, 1967, p. 48-63.
[28] L. Landau and E. Lifchitz, *Mécanique des Fluides*, Mir, Moscou, 1971.
[29] S. Kaliski, *Bull. Acad. Polon. Sci.*, Sér. sci. techn., t. **13**, 1965, p. 211-219.
[30] E. Massa, *General Relativity and Gravitation*, t. **5**, 1974, p. 555-572.

[31] E. MASSA, *General Relativity and Gravitation*, t. **5**, 1974, p. 573-591.
[32] E. MASSA, *General Relativity and Gravitation*, t. **5**, 1974, p. 715-736.
[33] F. EASTBROOK and H. WAHLQUIST, *J. Math. Phys.*, t. **5**, 1964, p. 1629-1644.
[34] C. MØLLER, *The Theory of Relativity*, Clarendon Press, Oxford, 1952.
[35] C. CATTANEO, *Il Nuovo Cimento*, t. **10**, 1958, p. 318-337.
[36] C. CATTANEO, *Ann. Mat. Pura Appl.*, t. **48**, 1959, p. 361-386.
[37] J. L. SYNGE, *Relativity: the Special Theory, North Holland*, Amsterdam, 1956.
[38] E. MASSA, *Commun. Math. Phys.*, t. **20**, 1971, p. 279-290.
[39] E. MASSA, *Commun. Math. Phys.*, t. **22**, 1971, p. 321-337.
[40] C. W. MISNER, K. S. THORNE and J. A. WHEELER, *Gravitation*, Freeman and Company, San Francisco, 1973.
[41] M. E. GURTIN, *Z. Angew. Math. Phys.*, t. **27**, 1976, p. 775-779.
[42] G. A. MAUGIN, *J. Phys. A : Math. Nucl. Gén.*, t. **6**, 1973, p. 306-321.
[43] P. VERNOTTE, *Compt. Rend. Acad. Sci.*, t. **246**, 1958, p. 3154-3155.
[44] M. CHESTER, *Phys. Rev.*, t. **131**, 1963, p. 2013-2015.
[45] C. TRUESDELL, *Rational Thermodynamics*, McGraw-Hill, New York, 1969.

(Manuscrit reçu le 29 septembre 1978)