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Filter theory and covering law

by

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ABSTRACT. — The main aim of this paper is to compare the operational approach to quantum axiomatics with the well-known lattice theoretic one, known under the name « quantum logic approach », and to show some advantages of the former, which consist in resolving the old troubles connected with quantum logics: the question of the complete lattice structure of the logic, the atomicity, and the validity of the covering law in the logic of propositions. By the way some equivalent forms of the covering property in a general orthomodular orthoposet are established and its geometrical sense is clarified in some details. Also the dimension theory for AC-orthoposets is developed.

1. INTRODUCTION

Among main attempts to axiomatize quantum theory (here we have in mind the C*-algebraic and the quantum logic theoretical frameworks) the so-called « operational approach » (1), in which the basic significance is attached to the set of states and the set of operations transforming the former one into itself, seems to be very promising and, at the same time, physically natural.

It is not a purpose of this paper to describe the operational approach in its all aspects. Our aim is much more modest; we want to compare it with the well-known quantum logic axiomatic scheme, and to show some advan-

(1) This axiomatic method has been advocated and developed (mainly) by Gunson [*12*], Pool [*26*], [*27*], Mielnik [*21*], [*22*], [*23*], Davies and Lewis [*4*], [*5*], [*6*], Edwards [*7*], [*8*], Srinivas [*28*], and others.

tages of the operational approach. Especially, we are aimed here to show that in the framework of the operational axiomatics we are in a position to resolve the well-known troubles of the quantum logic approach. Namely, we are able to answer the questions concerning the complete lattice structure, atomicity, and the validity of the covering law in the logic of propositions. It must be emphasized here that although the complete lattice property and atomicity can also be justified in the framework of the quantum logic approach by a suitable extension of the propositional logic (see, e. g. [*3*], [*16*]), the covering law, as yet, do not admit a unquestionable physical justification, although many attempts have been made to clarify its significance (see, e. g. [*17*], [*24*], [*2*]). Here (see Section 4) it appears as a consequence of the assumed axioms, any of which being of a direct physical significance. By the way, some equivalent forms of the covering law in a general orthomodular orthoposet are established, and its geometrical sense is clarified in some detail (see Section 3). Also, the dimension theory for AC-orthoposets is developed (Section 3).

2. AXIOMS, DEFINITIONS, NOTATION

With each physical system we shall associate a triple (B, F, d) consisting of two non-empty sets: an abstract set B, whose members will be interpreted as *beams*, and the set F of *filters*, whose members are mappings from B into itself (i. e., any filter transforms beams into beams), and the function d from B into non-negative reals ($d : \mathrm{B} \to \mathrm{R}_+$), called the *intensity* (or *strength*) *functional.* The properties that we shall require for the triple (B, F, d) will be formulated below as axioms.

A. Basic axioms (the first group)

AXIOM 1 $$\forall_{a\in\mathrm{F}} a^2 = a.$$

AXIOM 2 $$\forall_{a\in\mathrm{F}}\forall_{m\in\mathrm{B}} d(am) \leqslant d(m).$$

The two postulates above express physically obvious facts that every filter must act on beams as an idempotent, and that there is no any new beam produced by the filter a itself, hence the intensity of the beam am must be necessarily not greater than that of m.

AXIOM 3 $$[\forall_{a\in\mathrm{F}} d(am_1) = d(am_2)] \Rightarrow m_1 = m_2.$$

It is convenient to adjoin to the set of beams some fictitious beam, called the *zero beam*, the intensity of which is zero. This is the content of the next axiom:

AXIOM 4 $\exists_{m_0 \in B} d(m_0) = 0.$

By axioms 2 and 3 the beam m_0 satisfying $d(m_0) = 0$ is necessarily unique; we denote it by 0. Of course, $a0 = 0$ for all $a \in F$.

Also, it will be useful to adjoin to the set of filters the identity and zero transformations from B to B:

AXIOM 5. — The transformations $I, 0 : B \to B$ defined by $Im = m$ and $0m = 0$ for all $m \in B$, respectively, belong to F. We call them the *identity* and the *zero* filter, respectively.

AXIOM 6 $\forall_{m_1, m_2 \in B} \forall_{t_1, t_2 \geqslant 0} \exists_{m \in B} \forall_{a \in F} d(am) = t_1 d(am_1) + t_2 d(am_2).$

The beam m in the axiom above is, owing to axiom 3, unique; we denote it by $t_1 m_1 + t_2 m_2$ and call the *mixture of* m_1 *and* m_2 *in the proportion* $t_1 : t_2$. Axiom 6 imposes on the set B the structure of a convex cone. From this axiom one finds that

$$\forall_{a \in F} \forall_{m_1, m_2 \in B} \forall_{t_1, t_2 \geqslant 0} d(a(t_1 m_1 + t_2 m_2)) = t_1 d(am_1) + t_2 d(am_2) ;$$

in particular, if we put $a = I$, we get

$$d(t_1 m_1 + t_2 m_2) = t_1 d(m_1) + t_2 d(m_2)$$

for all $m_1, m_2 \in B$ and all $t_1, t_2 \geqslant 0$.

Hence

$$d(a(t_1 m_1 + t_2 m_2)) = d(t_1(am_1) + t_2(am_2))$$

for all $a \in F$, $m_1, m_2 \in B$ and $t_1, t_2 \geqslant 0$.

B. Partial ordering and orthogonality

By an *operation* on the set B of beams we shall mean any mapping from B to B. Of particular importance for us will be the set $0_i(B)$ of idempotent operations on B, as it contains the set F of filters as a subset.

For any two idempotent operations $P, Q \in 0_i(B)$ we define

$$P \leqslant Q : \Leftrightarrow QP = P = PQ.$$

The above-defined relation $\leqslant$ is, of course, a partial ordering in $0_i(B)$. Note also that the definition of the partial ordering above requires every comparable operations to be compatible (in the sense that they commute), which, for filters, is physically obvious.

Further, for $P, Q \in 0_i(B)$ we define

$$P \perp Q : \Leftrightarrow QP = 0 = PQ.$$

Let us note the following properties of the relation $\perp$:

i) $P \perp P$ implies $P = 0$,

ii) $P \perp Q \Leftrightarrow P + Q \in 0_i(B)$ and $P \perp Q$ implies $P + Q = P \vee Q$, where $\vee$ is used to denote the least upper bound,

iii) $P \perp Q$ and $Q \geqslant R$ imply $P \perp R$,

iv) $PQ = QP$ implies $PQ = P \wedge Q$, where $\wedge$ denotes the greater lower bound in $0_i(B)$.

C. **Further axioms (the second group)**

AXIOM 7. — For any two orthogonal filters $a, b \in F$, $a + b$ is also a filter.

As a consequence of the axiom above one gets immediately that

$$a \perp b(a, b \in F) \Rightarrow \forall_{m \in B} d(am) + d(bm) \leqslant d(m).$$

Note also that

$$a \leqslant b \Rightarrow \forall_{m \in B} d(am) \leqslant d(bm).$$

AXIOM 8 $$\forall_{a \in F} \exists_{b \in F, b \perp a} \forall_{m \in B} d(am) + d(bm) = d(m).$$

AXIOM 9 $$[\forall_{m \in B} d(am) = d(bm)] \Rightarrow a = b.$$

From the axioms above one easily finds that:

(1) The filter b in the axiom 8 is unique; we denote it by a'.

(2) For any $a \in F$ we have $a + a' = I$ and $a'' = a$.

(3) $0' = I$ and $I' = 0$.

(4) $a \perp b$ iff $a \leqslant b'$.

(5) $a \leqslant b \Rightarrow b' \leqslant a'$.

(6) F is orthomodular, that is $a \leqslant b(a, b \in F)$ implies $b = a \vee c$ for some $c \in F$, $c \perp a$ (²).

(7) Two filters a and b are compatible (³), $a \leftrightarrow b$, if and only if $ab \in F$ and $ab = ba$. Furthermore, if $a \leftrightarrow b$, then $ab = a \wedge b$ and there exists also $a \vee b$.

Therefore, we find $(F, \leqslant, \perp, ', 0, I)$ to be an orthomodular orthoposet (the abbreviation « orthoposet » one should read: orthocomplemented partially ordered set).

D. **Last axioms (the third group)**

AXIOM 10. — *i*) For every non-zero filter $a \in F$ there exists a homogeneous beam (⁴) p such that $d(ap) = d(p)$; moreover:

(²) As we know (see, e. g. [*29*]), c is uniquely determined by a and b, as $c = b \wedge a'$; we denote it by $b - a$.

(³) The compatibility relation $\leftrightarrow$ is defined, after Mackey [*18*], as follows: $a \leftrightarrow b$ iff $a = a_1 \vee c$ and $b = b_1 \vee c$ for some mutually orthogonal a_1, b , $c \in F$.

(⁴) A non-zero beam m is said to be *homogeneous*, if it cannot be written in the form $m = t_1 m_1 + t_2 m_2$, where t_1, t_2 are positive real numbers, and m_1, m_2 are two other non-zero beams, being not proportional to one another.

ii) If, at the same time, $b \not\leqslant a$, then the beam p can be chosen in such a way that $d(bp) > 0$.

Formally, the axiom 10 can be written as follows:

$$\forall_{a\in F, a\neq 0} \forall_{b\in F, b \not\perp a} \exists_{p\in B_h} d(ap) = d(p) \ \&\ d(bp) > 0,$$

where B_h stands for the set of all homogeneous beams.

The first part of the axiom above assumes that B_h, the set of homogeneous beams, is not only non-empty, but also sufficiently large. The second part of this axiom we easily find to be equivalent to the following statement, which was taken as a postulate by Gudder [*10*]:

(*) If for every pure state (5) p with $d(ap) = 1$ we have also $d(bp) = 1$ (where $a, b \in F$), then $a \leqslant b$.

In fact, assume the first part of the statement (*) and suppose that $a \not\leqslant b$. Then $a \not\perp b'$, and by axiom 10 there exists a pure state p with $d(ap) = 1$ and $d(b'p) > 0$, hence $d(bp) < 1$, which contradicts our assumption. Conversely, assume the validity of the first part of the axiom 10; then (*) implies *ii*). Indeed, let $b \not\perp a$, $a \neq 0$; then $a \not\leqslant b'$, and by (*) there exists a pure state p such that $d(ap) = 1$ and $d(b'p) < 1$, the latter being equivalent to $d(bp) > 0$ (The existence of at least one pure state p with $d(ap) = 1$ is guaranteed by *i*)).

Therefore, our axiom 10 may be formulated in the following equivalent form (see [*16*]):

AXIOM 10'. — *i)* For every non-zero filter $a \in F$ there exists a pure state p such that $d(ap) = 1$; moreover:

ii) If for each pure state p, for which $d(ap) = 1$, one has also $d(bp) = 1$ for some $b \in F$, then $a \leqslant b$.

We complete our list of axioms by postulating the following (compare [*19*]):

AXIOM 11. — For every homogeneous beam $p \in B_h$ there exists a filter $a \in F$ such that $d(ap) = d(p)$ and $d(aq) < d(q)$ for all homogeneous beams q, which are not proportional to p.

The above axiom asserts that pure beams can be realized in the laboratory: there exists a filter (measuring device) $a \in F$ answering the experimental question « Is a physical system in the pure state $p/d(p)$? ».

(5) By a *state* we mean any normalized beam $m \in B$ (that is, satisfying $d(m) = 1$). Any homogeneous state will also be called a *pure state*. For states we will also write $m(a)$ instead of $d(am)$.

3. COVERING LAW AND THE MINIMAL SUPERPOSITION PRINCIPLE

Suppose F to be an atomic orthoposet. We shall say that the *covering law* holds in F, or that F possesses the *covering property*, if

i) for any $a \in F$ and any atom $e \in F$ there exists $a \vee e$ in F,

ii) $a \vee e \geqslant b \geqslant a$ implies either $b = a$ or $b = a \vee e$ (that is, $a \vee e$ covers a, provided $e \nleqslant a$).

THEOREM. — Assume the condition *i)* to hold in F, F being an atomic orthomodular orthoposet. Then, the following statements about F are mutually equivalent [6]:

(1) *Minimal superposition principle* (abbreviated to MSP; compare Gudder [*11*]):

Let G be a finite set of atoms of F, say, $G = \{ e_1, \dots, e_n \}$, and let $e \in F$ be an atom such that $e \leqslant \bigvee G$ and $e \nleqslant \bigvee (G \setminus \{ e_j \})$ for each $j = 1, \dots, n$. Then, for any partition $I \cup J = \{ 1, 2, \dots, n \}$, $I \cap J = \emptyset$, of the index set $\{ 1, 2, \dots, n \}$ there exists an atom $f \in F$ such that

$$f \leqslant \Big(e \vee \bigvee_{i \in I} e_i\Big) \quad \text{and} \quad f \leqslant \bigvee_{j \in J} e_j.$$

(2) *MSP restricted* (briefly: MSPR; see [*15*]):

If e, e_1, e_2, e_3 are any four atoms of F (not necessarily all distinct !) such that $e \leqslant e_1 \vee e_2 \vee e_3$, $e \neq e_3$, and $e \neq e_1 \vee e_2$, then there exists an atom $f \in F$ such that $f \leqslant e \vee e_3$ and $f \leqslant e_1 \vee e_2$.

(3) *Weak MSP* + *Weak AEP* [7] (abbreviated to WMSP and WAEP, respectively):

i) Weak MSP: If e, e_1, e_2, e_3 are any four atoms of F (not necessarily all distinct) such that $e \leqslant e_1 \vee e_2 \vee e_3$, $e \neq e_3$, and $e \perp e_1 \vee e_2$, then there exists an atom $f \in F$ such that $f \leqslant e \vee e_3$ and $f \leqslant e_1 \vee e_2$.

ii) Weak AEP: If $e \leqslant g \vee h$, $e \perp g$, $e \neq h$ (e, g, h being atoms), then $h \leqslant g + e$ [8].

(4) *Jauch-Piron's condition* (see Jauch and Piron [*17*]):

For any $a \in F$ and any atom $e \in F$ not contained in a, $a \vee e - a$ is also an atom.

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(6) For the case, where F is a complete lattice, the equivalence of the conditions (1), (4), (5), (6) and (7) was shown by Bugajska and Bugajski [2].

(7) The abbreviation « AEP » one should read: « Atomic Exchange Property ».

(8) The symbol + is used to denote the least upper bound for orthogonal elements.

(5) *Varadarajan's Property* (see Varadarajan [*30*]):

For any $a \in F$, $0 < a < I$, and any atom $e \in F$ there exist two atoms $e_1, e_2 \in F$ such that $e_1 \leqslant a$, $e_2 \perp a$ and $e \leqslant e_1 \vee e_2$.

(6) *Covering Law*:

For any $a \in F$ and any atom $e \in F$, $a \vee e \geqslant b \geqslant a$ implies either $b = a$ or $b = a \vee e$.

(7) *Zierler's Condition* (see Zierler [*31*]):

For any three finite elements (9) $a, b, c \in F$ such that $b \wedge c = 0$ and $a \leqslant c$ one has $a = (a \vee b) \wedge c$, provided $(a \vee b) \wedge c$ there exists in F (10).

Remark 1. — Note that the Weak MSP implies the following property of F: (**) If $e \leqslant g \vee h$, $e \perp g$, $e \neq h$ (e, g, h being atoms), then $g \leqslant e \vee h$. Also, it will be useful to note the following consequence of the Covering Law (6), which is known under the name Atomic Exchange Property (AEP, in short):

$$f \leqslant e \vee a, f \nleqslant a (a \in F;\ e, f \text{ being atoms}) \Rightarrow e \leqslant f \vee a.$$

Remark 2. — Note that (2) admits a simple geometrical interpretation. To see this, two cases should be considered (compare [*15*]):

CASE I: $e_1 = e_2$. Then $f = e_1$ and the property (2) reduces to the following:

$$e \leqslant e_1 \vee e_3, \qquad e \neq e_1, e_3 \qquad \text{imply} \qquad e_1 \leqslant e \vee e_3,$$

which means that the line (11) $e_1 \vee e_3$ is also determined by another pair of its points (= atoms), e and e_3.

CASE II: $e_1 \neq e_2$. As, according to (2), $e \leqslant e_1 \vee e_2 \vee e_3$ and $e \nleqslant e_1 \vee e_2$, we have $e_3 \nleqslant e_1 \vee e_2$ (in particular, e_1, e_2, e_3 are all distinct). By (2) then $f \leqslant e_1 \vee e_2$ and $f \leqslant e \vee e_3$ for some atom f. (This means that any two distinct lines lying in the same plane (here, the lines $e_1 \vee e_2$ and $e \vee e_3$, which lie on the plane $e_1 \vee e_2 \vee e_3$) have always a common point (When the lines $e_1 \vee e_2$ and $e \vee e_3$ are parallel, f becomes the point at infinity). In other words, case II tells us that two lines in the same plane always cut.

Proof of the theorem. — The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are straightforward. To prove the implication (3) $\Rightarrow$ (4), let us consider an element $e \vee a - a \in F$, where $a \in F$, $e \in A(F)$, $e \nleqslant a$ (A(F) stands for the set of all atoms of F). Then, of course, $e \vee a - a \neq 0$, and, therefore, there exists at least one atom contained in $e \vee a - a$. Let us suppose, in contrary to the statement (4), that there are two distinct atoms e_1, e_2 such that $e_1, e_2 \leqslant e \vee a - a$, and consider the element $e_1 \vee e_2 \vee e$. One can assume, without any loss of generality, that $e \nleqslant a'$, as the inequality $e \leqslant a'$ implies, by the

(9) A filter $a \in F$ is said to be *finite*, if it is a join of a finite number of atoms.

(10) Note that $a \vee b$ always exists for finite $a, b \in F$.

(11) By *lines* we mean such elements of F which cover atoms.

orthomodularity of F, that $e \vee a - a = e \in A(F)$. Note that $e \nleqslant e_1 \vee e_2$, as the inequality $e \leqslant e_1 \vee e_2$ would imply $e \leqslant a'$, which contradicts our assumption. Note further that, owing to the orthomodularity of F, $e_1 \vee e_2 \vee e - e_1 \vee e_2 \neq 0$, hence, by the atomicity of F, there is an atom $e_3 \leqslant e_1 \vee e_2 \vee e - e_1 \vee e_2$, hence $e_3 \leqslant e_1 \vee e_2 \vee e$ and $e_3 \perp e_1 \vee e_2$. Let us now apply (4) to the set $\{ e_3, e_1, e_2, e \}$. Note that $e \neq e_3$, as required in (4), since $e = e_3 \perp e_1 \vee e_2$ would imply $e \vee a - a$ to contain an atom orthogonal to e (e_1, for instance), which is impossible. In fact, $e_1 \leqslant e \vee a - a \perp a$ and $e_1 \perp e$ imply $e_1 \perp e \vee a$ (as $e_1 \perp a$), a contradiction with $e_1 \leqslant e \vee a$. By (4) we see that there is an atom e_4 such that $e_4 \leqslant e_3 \vee e$ and $e_4 \leqslant e_1 \vee e_2$. Note that $e_4 \perp e_3$, as $e_1 \vee e_2 \perp e_3$, and that $e_4 \neq e$, as $e \nleqslant e_1 \vee e_2$. Obviously, $e_4 < e_1 \vee e_2$, since the equality $e_4 = e_1 \vee e_2$ leads to $e_1, e_2 \leqslant e_3 \vee e$ (where $e_1 \perp e_3, e_1 \neq e$), hence, by WAEP, $e \leqslant e_3 \vee e_1$, hence $e_2 \leqslant e_3 \vee e \leqslant e_3 \vee e_1$ (where $e_2 \perp e_3$, $e_2 \neq e_1$), which leads, by (**), to $e_3 \leqslant e_1 \vee e_2$, a contradiction with $e_3 \perp e_1 \vee e_2$. Note further that, by orthomodularity, $e_1 \vee e_2 - e_4 \neq 0$, thus there exists (by atomicity) an atom $e_5 \leqslant e_1 \vee e_2 - e_4$, hence $e_5 \perp e_4$. Also $e_5 \perp e_3$, as $e_5 \leqslant e_1 \vee e_2$ and $e_1 \vee e_2 \perp e_3$. But, by WAEP, $e_4 \leqslant e_3 \vee e$ (where $e_4 \perp e_3$, $e_4 \neq e$) implies $e \leqslant e_3 \vee e_4$, where $e_3 \vee e_4 \perp e_5$ (as e_3 and e_4 are both orthogonal to e_5), hence $e \perp e_5$. Also $e_5 \leqslant e_1 \vee e_2 \leqslant e \vee a - a$, and therefore we have shown that $e \vee a - a$ contains an atom e_5 orthogonal to e, which, as we already proved, is impossible. This completes the proof of the implication (3) $\Rightarrow$ (4).

We shall now show that (4) implies (5). Let $e \in A(F)$, $a \in F$, $a \neq 0$, I; then, by (4), $e \vee a - a \in A(F)$ or $= 0$ (the latter is when $e \leqslant a$). When $e \vee a - a = 0$, one gets $e \leqslant a$ (by orthomodularity), and we thus have $e \leqslant e + f$, where f is an arbitrary atom $\leqslant a'$. When $e \vee a - a \neq 0$, we meet two possibilities:

a) $e \vee a - a = e \Leftrightarrow e \leqslant a'$,

and then $e \leqslant e + f$, where f is an arbitrary atom $\leqslant a$; and

b) $e \vee a - a = f \in A(F), f \neq e \Leftrightarrow e \nleqslant a'$.

In the case *b*) we have, by (4), $e \vee a' - a' \in A(F)$. In fact, $e \vee a' - a' = 0$ would imply $e \leqslant a'$, hence $e \vee a - a = e$, which contradicts our assumption *b*). Applying now (twice) the dual version ([12]) of the following lemma due to Varadarajan [*29*]:

LEMMA 1. — Let L be an orthomodular orthoposet, and let $a, a_1, a_2, \ldots \in L$. If $a \leftrightarrow a_i$ for each $i = 1, 2, \ldots$, and if $\bigvee_{i=1}^{\infty} a_i$ and $\bigvee_{i=1}^{\infty} (a \wedge a_i)$ both exist, then

$$a \leftrightarrow \bigvee_{i=1}^{\infty} a_i$$

([12]) The dual form of the lemma 1 one obtains by replacing the symbols $\vee$ and $\wedge$ by $\wedge$ and $\vee$, respectively.

and

$$a \wedge \left(\bigvee_{i=1}^{\infty} a_i\right) = \bigvee_{i=1}^{\infty} (a \wedge a_i)$$

one easily finds that

$$\begin{aligned}(e \vee a' - a') + f &= [(e \vee a' - a') \vee (e \vee a)] \wedge [(e \vee a' - a') \vee a'] \\ &= (e \vee a) \wedge ((e \vee a') \wedge (a \vee a')) = (e \vee a) \wedge (e \vee a') \geqslant e,\end{aligned}$$

which proves the implication (4) $\Rightarrow$ (5).

Now, we shall show that (6) is a consequence of (5). This will be done in two steps: first we shall prove that (5) implies (4), and next that (4) implies (6).

Suppose (5) to hold in F, and let $e \in A(F)$, $a \in F$, $e \nleqslant a$. One can assume, without any loss of generality, that $a \neq 0$. Then, by (5), there exist two atoms $e_1 \leqslant a$ and $e_2 \leqslant a'$ such that $e \leqslant e_1 \vee e_2$; hence $e \vee a \leqslant (e_1 \vee e_2) \vee a = e_2 + a$, and therefore

$$e \vee a - a \leqslant (e_2 + a) - a = e_2$$

by the lemma 1, hence $e \vee a - a = e_2$, as $e \vee a - a \neq 0$. This proves the implication (5) $\Rightarrow$ (4).

Assume now the validity of (4) and then prove (6). One needs to show that $e \vee a$ covers a, provided $e \nleqslant a$. Let $e \vee a \geqslant b > a$; then

$$e \vee a - a \geqslant b - a > 0,$$

hence, as we find $e \vee a - a$ to be an atom by (4),

$$e \vee a - a = b - a,$$

hence, by orthomodularity,

$$e \vee a = (e \vee a - a) + a = (b - a) + a = b.$$

This proves that $e \vee a$ covers a indeed, and therefore the implication (4)$\Rightarrow$(6) is established.

Before proving the next implication, the implication (6) $\Rightarrow$ (7), one needs to develop the dimension theory for an atomic orthomodular orthoposet F with the covering law holding in it. Such an orthoposet will be called below the *AC – orthoposet.*

Let $A \subseteq A(F)$. We shall say that the subset A is *independent*, if for every $e \in A$ one has $e \nleqslant \bigvee(A \backslash \{ e \})$.

LEMMA 2. — Let $e_1, \ldots, e_n$ be a finite set of distinct atoms of F, where $n > 1$. Then $\{ e_1, \ldots, e_n \}$ is independent if ans only if

$$e_i \nleqslant \bigvee_{j=1}^{i-1} e_j \qquad (3.1)$$

for each $i = 2, 3, \ldots, n$.

Proof. — It needs to be shown only the « if » part of the lemma, as the part « only if » is an immediate consequence of the definition of independence. To prove it we use arguments of Varadarajan (see [*30*], Lemma 2.6). Assume the validity of (3.1) for the set $\{ e_1, \ldots, e_n \}$ and suppose, in contrary, that this set is not independent, i. e. that $e_j \leqslant \bigvee_{i \neq j} e_i$ for some j, $1 \leqslant j \leqslant n$. By (3.1) one finds that $j \neq n$.

We set, for any $m \leqslant n$,

$$f_m := \bigvee_{\substack{i \leqslant m \\ i \neq j}} e_i ;$$

note that $e_j \leqslant f_n$. Let k be the smallest positive integer $\leqslant n$ such that $e_j \leqslant f_k$ (In view of (3.1) one finds $k > j$). Then $e_j \nleqslant f_{k-1}$. But, as $k - 1 \geqslant j$, one has $e_j \vee f_{k-1} = \bigvee_{i=1}^{k-1} e_i$ and hence, by (3.1), $e_k \nleqslant e_j \vee f_{k-1}$, hence, by AEP,

$$e_j \nleqslant e_k \vee f_{k-1} = e_k \vee \bigvee_{\substack{i \leqslant k-1 \\ i \neq j}} e_i = f_k,$$

a contradiction. The lemma is thus proved.

As a consequence of Lemma 2 one gets:

LEMMA 3. — Let $a \in \mathrm{F}$ be finite, say $a = \bigvee_{i=1}^{n} e_i$, where e_i are atoms, and let $\{ f_1, \ldots, f_m \}$ be an independent set of atoms $\leqslant a$. Then $m \leqslant n$.

Proof. — Apply the arguments used in the proof of Lemma 2.7 in [*30*].

As an immediate consequence of the lemma 3 we obtain :

COROLLARY 1. — If $\{ e_1, \ldots, e_n \}$ and $\{ f_1, \ldots, f_m \}$ are two independent sets of atoms such that $\bigvee_{i=1}^{n} e_i = \bigvee_{j=1}^{m} f_j$, then $n = m$.

COROLLARY 2. — If $a \in \mathrm{F}$ is finite and $b < a$, $b \neq 0$, then also b is finite.

Proof. — Apply the orthomodularity and the atomicity of F, and next use the lemma 3.

COROLLARY 3. — Any finite element of F is a join of independent atoms. Moreover, these atoms can be chosen as pairwise orthogonal.

Proof. — Apply the orthomodularity and the atomicity of F, and next use the lemma 3.

For any finite element $a \in \mathrm{F}$ we define its *dimension* to be, as usually, the number of elements of any independent set of atoms, whose lattice sum is a. By Corollary 1, this number depends only on a, and not on the independent set in question. We write $d(a)$ for this number, and call d the *dimension function* on F_f, F_f denoting the set of all finite elements of F. For $a = 0$ we set, by definition, $d(0) = 0$. Note that when $a \neq 0$, there is, by the atomicity, at least one atom contained in a, so that then $d(a) \geqslant 1$.

By Lemma 3 we see that for any finite $a \in F$ one has

$$d(a) = \min \left\{ n : a = \bigvee_{i=1}^{n} e_i, \ e_i \in A(F) \right\}.$$

We shall now prove that d is a true dimension function on F_f, i. e., that except the obvious property

i) $d(0) = 0$, $d(a) \geqslant 0$ for all $a \in F_f$ ([13]),

the following holds for d:

ii) d is strictly increasing, i. e., $a < b (a, b \in F_f)$ implies $d(a) < d(b)$,

iii) $d(a \vee b) + d(a \wedge b) = d(a) + d(b)$ for all $a, b \in F_f$, for which there exists $a \wedge b (a \wedge b$ is then finite, by Corollary 2, provided $a \wedge b \neq 0)$.

Note that *ii*) is a consequence of the fact that $d(a) > 0$ for $a \neq 0$, and of *iii*). In fact, $a < b$ implies, by orthomodularity, $b = a + c$ for some non-zero c, $c \perp a$, hence by *iii*) and *i*)

$$d(b) = d(a) + d(c) > d(a),$$

as $d(c) > 0$.

It now remains to prove *iii*). This will be done in three steps. First, let $a, b \in F_f$ be orthogonal: $a \perp b$; we shall show that

$$d(a \vee b) = d(a) + d(b). \tag{3.2}$$

Let

$$a = \bigvee_{i=1}^{n} e_i, \quad b = \bigvee_{j=1}^{m} f_j, \quad \text{where} \quad \{ e_1, \ldots, e_n \} \quad \text{and} \quad \{ f_1, \ldots, f_m \}$$

are independent sets of atoms. To prove (3.2) it is now sufficient to note that the set $\{ e_1, \ldots, e_n, f_1, \ldots, f_m \}$, having $a \vee b$ as its lattice sum, is independent. Indeed, otherwise some member of this set has to be contained in the lattice sum of the preceding members (by Lemma 3), but since such a member cannot belong to $\{ e_1, \ldots, e_n \}$, as the latter set is independent, one then has $f_j \leqslant a \vee \bigvee_{i=1}^{j-1} f_i$ for some j, $1 \leqslant j \leqslant m$, hence $\bigvee_{i=1}^{j} f_i \leqslant a \vee \bigvee_{i=1}^{j-1} f_i$, which implies

$$\bigvee_{i=1}^{j} f_i = \bigvee_{i=1}^{j-1} f_i, \tag{3.3}$$

([13]) Also 0 will be regarded, by the definition, as finite, i. e. $0 \in F_f$.

as $\bigvee_{i=1}^{j} f_i \perp a$ (in fact, to prove (3.3) one needs to use the following simple implication: $x \leqslant y$, $y \perp z \Rightarrow x = (x \vee z) \wedge y$, in which one puts $x = \bigvee_{i=1}^{j-1} f_i$, $y = \bigvee_{i=1}^{j} f_i$, $z = a$), which contradicts the independence of the set $\{f_1, \ldots, f_m\}$. Thus the set $\{e_1, \ldots, e_n, f_1, \ldots, f_m\}$ is independent indeed, and therefore $d(a \vee b) = n + m = d(a) + d(b)$.

COROLLARY 4. — For any finite $a, b \in F_f$, $a > b$, also $a - b$ is finite, and

$$d(a - b) = d(a) - d(b).$$

Proof. — Finiteness of $a - b$ follows from Corollary 2. Further, by orthomodularity of F one has $a = (a - b) + b$, which allows to be applied the orthoadditivity (3.2) of the dimension function d, by which $d(a) = d(a - b) + d(b)$, which proves the corollary.

Our second step consists in the proof of the inequality:

$$d(a \vee b) \leqslant d(a) + d(b)$$

for finite a, b. Before proving it, one needs to show the following statement:

LEMMA 4. — For two filters $a > b$, if $a = b \vee \bigvee_{i=1}^{n} e_i$, where e_i are atoms, there exist at most n atoms $f_i \leqslant a$ orthogonal with each other and orthogonal to b such that $a = b + \bigvee_i f_i$.

Proof (by induction). — Let $n = 1$, and let $a = b \vee e_1$, $e_1 \in A(F)$. Let f_i be atoms such that $f_i \leqslant a$, $f_i \perp b$ and suppose that f_i are pairwise orthogonal. Then, as $b < b + f_i \leqslant b \vee e_1$, one deduces from the covering law that $b \vee e_1 = b + f_i$, hence, by the orthomodularity, $f_i = b \vee e_1 - b$ for all i, and thus we see that i cannot be greater than 1.

Suppose now the lemma to be true for n, and show that it is true for $n + 1$. Since

$$b \vee \bigvee_{i=1}^{n+1} e_i = \left(b \vee \bigvee_{i=1}^{n} e_i\right) \vee e_{n+1},$$

there exists, as we proved above, at most one atom $f_0 \leqslant b \vee \bigvee_{i=1}^{n+1} e_i$ orthogonal to $b \vee \bigvee_{i=1}^{n} e_i$. Then

$$b \vee \bigvee_{i=1}^{n} e_i < \left(b \vee \bigvee_{i=1}^{n} e_i\right) + f_0 \leqslant b \vee \bigvee_{i=1}^{n+1} e_i,$$

which implies, owing to the covering law,

$$\left(b \vee \bigvee_{i=1}^{n} e_i\right) + f_0 = b \vee \bigvee_{i=1}^{n+1} e_i.$$

Now, by the inductive assumption one can find at most n mutually orthogonal atoms f_i, being, at the same time, orthogonal to b, such that

$$b \vee \bigvee_{i=1}^{n} e_i = b + \bigvee_{i} f_i,$$

and therefore

$$b \vee \bigvee_{i=1}^{n+1} e_i = \left(b \vee \bigvee_{i=1}^{n} e_i\right) + f_0 = \left(b + \bigvee_{i} f_i\right) + f_0 = b + \bigvee_{j \geqslant 0} f_j,$$

where, of course, the atoms from the set $\{f_0, f_1, f_2, \ldots\}$ are mutually orthogonal and all orthogonal to b. Thus, we have proved the lemma for $n + 1$, as desired.

COROLLARY 5. — For finite $a, b \in \mathrm{F}$ one has

$$d(a \vee b - b) \leqslant d(a). \tag{3.4}$$

Proof. — Let $a, b \in \mathrm{F}_f$, and let $c = a \vee b - b$. Suppose $c \neq 0$; then, by Corollary 2, c is finite (as $c \leqslant a \vee b \in \mathrm{F}_f$) and from Corollary 3 we know that c can be written as a join of pairwise orthogonal atoms, whose number, being the dimension of c, is by Lemma 4 (as $a \vee b = b + c$) not greater than $d(a)$. When $c = 0$, the statement (3.4) is trivial. The corollary is thus proved.

From $a \vee b = (a \vee b - b) + b$, where $a, b \in \mathrm{F}_f$, one finds by (3.2)

$$d(a \vee b) = d(a \vee b - b) + d(b),$$

hence, by Corollary 5,

$$d(a \vee b) \leqslant d(a) + d(b), \tag{3.5}$$

as claimed.

In the last, third step, we prove that for finite $a, b \in \mathrm{F}$

$$d(a \vee b) + d(a \wedge b) = d(a) + d(b),$$

provided $a \wedge b$ there exists.

Let $c = a \wedge b$; we have on applying twice the lemma 1

$$a \vee b - c = (a \vee b) \wedge c' = (a - c) \vee (b - c),$$

as $a, b \leftrightarrow c'$, and at the same time

$$a \vee b - c = (a \vee b) \wedge (a' \vee b') = (a \vee b - a) \vee (a \vee b - b),$$

as $a \vee b \leftrightarrow a', b'$.

Hence, by using (3.5) and Corollary 4 we find

$$d(a \vee b) - d(c) = d(a \vee b - c) \leqslant d(a - c) + d(b - c) \\ = d(a) - d(c) + d(b) - d(c),$$

which leads to

$$d(a \vee b) + d(c) \leqslant d(a) + d(b), \tag{3.6}$$

and, at the same time,

$$d(a \vee b) - d(c) = d(a \vee b - c) \leqslant d(a \vee b - a) + d(a \vee b - b) \\ = d(a \vee b) - d(a) + d(a \vee b) - d(b),$$

hence

$$d(a \vee b) + d(c) \geqslant d(a) + d(b),$$

which is the inequality opposite to (3.6). Summarizing, we have shown the property *iii*).

COROLLARY 6 (Zierler's condition (7)). — For any finite a, b, c satisfying $a \leqslant c$ and $b \wedge c = 0$ we have $a = (a \vee b) \wedge c$, provided $(a \vee b) \wedge c$ there exists in F.

Proof. — Applying *iii*) and taking into account that $b \wedge a = 0$ and $b \wedge c = 0$, we find

$$d(a) + d(b) + d(c) = d(a \vee b) + d(c) = d(a \vee b \vee c) + d((a \vee b) \wedge c) \\ = d(b \vee c) + d((a \vee b) \wedge c) = d(b) + d(c) + d((a \vee b) \wedge c),$$

hence

$$d(a) = d((a \vee b) \wedge c).$$

Hence, by the property *ii*) of the dimension function d we get $a = (a \vee b) \wedge c$.

To close the proof of the theorem it remains to be shown the implication (7) $\Rightarrow$ (1).

Suppose (7) to hold, and assume that $e \leqslant \bigvee G$ and $e \nleqslant \bigvee (G \backslash \{ e_j \})$ for all $j = 1, 2, \ldots, n$, where $G = \{ e_1, \ldots, e_n \}$, $e, e_1, \ldots, e_n \in A(F)$. Suppose next, in contrary to (1), that there is no an atom f such that

$$f \leqslant e \vee \bigvee_{i \in I} e_i \quad \text{and} \quad f \leqslant \bigvee_{j \in J} e_j,$$

$I \cup J$ being some partition of the index set $\{ 1, 2, \ldots, n \}$, $I \cap J = \varnothing$, i. e. that

$$\left(e \vee \bigvee_{i \in I} e_i\right) \wedge \left(\bigvee_{j \in J} e_j\right) = 0.$$

Then, if we apply (7) to

$$a = \bigvee_{i \in I} e_i, \qquad b = \bigvee_{j\ J} e_j \qquad \text{and} \qquad c = e \vee \bigvee_{i \in I} e_i,$$

we shall obtain

$$\bigvee_{i\in I} e_i = \left(\left(\bigvee_{i\in I} e_i\right) \vee \left(\bigvee_{j\in J} e_j\right)\right) \wedge \left(e \vee \bigvee_{i\in I} e_i\right) = e \vee \bigvee_{i\in I} e_i,$$

hence

$$e \leqslant \bigvee_{i\in I} e_i,$$

which contradicts the assumptions of (1). This proves the implication (7) $\Rightarrow$ (1) and, at the same time, completes the proof of the theorem.

4. FILTERS AS PURE OPERATIONS

It is physically reasonable to assume for filters the following property:

AXIOM 12. — Any filter $a \in F$ transforms homogeneous beams into homogeneous ones.

As a consequence of the axiom above one obtains:

LEMMA 5. — Let $p \in B_h$, B_h being the set of homogeneous beams ([14]), and $a \in F$. Then:

i) ap is proportional to a pure state q such that $q(a) = 1$, with $d(ap)$ as the coefficient of proportionality, i. e.

$$ap = d(ap)q,$$

where $q \in B_h$ satisfies $d(aq) = d(q) = 1$.

ii) $ep_e = p_e$, where p_e denotes the unique pure state, whose carrier is e ([15]), that is, $p_e = \text{carr}^{-1}\, e$.

iii) Every atomic filter $e \in F$ is a positively-homogeneous mapping of the set B_h into itself, i. e.

$$e(sp) = s.ep$$

for all $p \in B_h$ and $s \in R_+$.

Proof. — *Ad. i).* — By axiom 12 one can write $ap = sq$, where q is a pure state and $s \in R_+$, which implies $s = d(ap)$, as $d(q) = 1$. If $ap \neq 0$, then $d(ap) \neq 0$ and we have $q = ap/d(ap)$, hence $q(a) = d[a(ap/d(ap))] = 1$; if $ap = 0$, then q may be chosen as an arbitrary pure state such that $q(a) = 1$.

Ad. ii). — By *i)* we get $ep_e = p_e(e)q = q$, as $p_e(e) = 1$. Hence $q(e) = d(e^2p_e) = d(ep_e) = p_e(e) = 1$, hence carr $q = e =$ carr p_e, which implies $q = p_e$,

([14]) We put, by definition, $0 \in B_h$ for the zero beam 0.

([15]) See Section 5.

as the mapping carr is one-to-one (see Section 5). Thus we have indeed $ep_e = p_e$.

Ad. iii). — Let $e \in A(F)$, $p \in B_h$ and $s \in R_+$. By *i)* one finds

$$e(sp) = d(e(sp))q = sd(ep)q,$$

where q is *a* pure state satisfying $q(e) = 1$, hence (see Section 5) $q = p_e$, and therefore

$$e(sp) = sd(ep)p_e = s.ep,$$

as claimed.

THEOREM. — Let $a \in F$, $a \neq I$, and let e be an arbitrary atom not contained in a. Then, there exists $a \vee e$, $a \vee e - a$ is an atom, and

$$(a \vee e - a)p_e = a'p_e,$$

where p_e is the unique pure state with carrier e.

Proof. — To prove that $\underline{a \vee e}$ there exists, let us assume for some $c \in F$ to satisfy $c \geqslant a, e$. Then, as $\underline{ep_e} = p_e$ by Lemma 5 *ii)*, we have

$$cp_e = cep_e = ep_e = p_e,$$

hence

$$a'p_e = a'cp_e = ca'p_e,$$

since $a' \leftrightarrow c$(16) (see the property (7) on page 360).

Hence, as by Lemma 5 *i)*

$$a'p_e = p_e(a')p$$

for some pure state p with $p(a') = 1$ (hence carr $p \leqslant a'$), one finds

$$p_e(a')p = c(p_e(a')p), \tag{4.1}$$

hence, after applying the functional d to (4.1), one gets

$$p_e(a') = p_e(a')p(c), \tag{4.2}$$

hence

$$p(c) = 1, \tag{4.3}$$

since $p_e(a') \neq 0$ (Indeed, $p_e(a') = 0$ would lead to $e = \text{carr}\, p_e \leqslant a$, which contradicts our assumption that $e \nleqslant a$). The equality (4.3) leads immediately to carr $p \leqslant c$.

Note now that there exists $a \vee \text{carr}\, p$, as $a \perp \text{carr}\, p$ (see above), and that $\underline{a \vee} \text{carr}\, p \leqslant c$, as $a \leqslant c$ and carr $p \leqslant c$. Since $(\text{carr}\, p)' \leftrightarrow a'$ (16), one finds by the property (7) on page 360 that

$$((\text{carr}\, p)' \wedge a')p_e = (\text{carr}\, p)'a'p_e = (\text{carr}\, p)'(p_e(a')p), \tag{4.4}$$

(16) This follows from the well-known facts (see, e. g. [*29*]) that in any orthomodular orthoposet $x \leqslant y$ implies $x \leftrightarrow y$, and $x \leftrightarrow y$ implies $x \leftrightarrow y'$.

hence, by applying the functional d to (4.4) we get

$$p_e((\mathrm{carr}\, p)' \wedge a') = p_e(a')p((\mathrm{carr}\, p)') = 0,$$

which implies

$$e = \mathrm{carr}\, p_e \leqslant ((\mathrm{carr}\, p)' \wedge a')' = \mathrm{carr}\, p + a.$$

Collecting the inequalities

$$e \leqslant \mathrm{carr}\, p + a,$$
$$a \leqslant \mathrm{carr}\, p + a,$$

$\mathrm{carr}\, p + a \leqslant c =$ any upper bound for a and e, we find that $\mathrm{carr}\, p + a = a \vee e$, hence $a \vee e - a = \mathrm{carr}\, p \in \mathrm{A(F)}$.

Finally, using the inequality $\mathrm{carr}\, p \leqslant a'$ and the Lemma 5 *iii)* and 5 *ii)* one finds that

$$(\mathrm{carr}\, p)p_e = (\mathrm{carr}\, p)a'p_e = (\mathrm{carr}\, p)(p_e(a')p) = p_e(a')(\mathrm{carr}\, p)p$$
$$= p_e(a')p = a'p_e,$$

or that

$$(a \vee e - a)p_e = a'p_e,$$

as claimed.

Corollary 7. — F is an AC-orthoposet.

Corollary 8 (17). — If $a \in \mathrm{F}$, $a \neq 0$, and if e is any atom $\nleqslant a'$, then there exists $e \vee a'$, $e \vee a' - a'$ is an atom, and

$$ap_e = (e \vee a' - a')p_e = p_e(a)p_{e \vee a' - a'}.$$

Proof. — Only the last equality has to be proved. By Lemma 5 *i)* we get

$$(e \vee a' - a')p_e = p_e(e \vee a' - a')p$$

for some pure state p satisfying $p(e \vee a' - a') = 1$, hence $p = p_{e \vee a' - a'}$, as $e \vee a' - a' \in \mathrm{A(F)}$.

On the other hand, $p_e(e \vee a' - a') = p_e(e \vee a') - p_e(a') = 1 - p_e(a') = p_e(a)$, which completes the proof.

We thus see that the action of a filter $a \in \mathrm{F}$ on pure states may be equivalently described as the action on atomic filters $e \in \mathrm{A(F)}$ defined by the formula:

$$a : e \mapsto e \vee a' - a' = (e \vee a') \wedge a.$$

This is the so-called Sasaki projection on the filter logic F.

5. TWO EMBEDDINGS OF THE FILTER LOGIC

We shall say that two states m_1 and m_2 are *mutually exclusive* or *orthogonal* [*10*], and write $m_1 \perp m_2$, if for some filter $a \in \mathrm{F}$ one has $m_1(a) = 1$ and $m_2(a) = 0$. This orthogonality relation is, obviously, symmetric.

(17) Under another axiom system a similar theorem was proved by Gunson [*12*].

The set P of all pure states endowed with the above-defined orthogonality plays a very essential role in quantum axiomatics (see, e. g. [*1*], [*14*], [*15*]). We shall call the pair $(P, \perp)$, $\perp$ being the orthogonality defined above restricted to P, the *phase space* of the physical system.

Let $S \subseteq P$; define $S^\perp$ to be the set of all pure states such that $p \perp S$ (read: $p \perp q$ for all $q \in S$), and write S^- instead of $S^{\perp\perp}$. Obviously, $S \subseteq S^-$. When $S = S^-$, we call the set S *closed*, or, to be more precise, $\perp$ – *closed*. The family $C(P, \perp)$ of all closed subsets of P we shall call the *phase geometry* associated with the physical system under study (see [*14*]). It is not difficult to check [*14*] that, under set inclusion, $C(P, \perp)$ becomes a complete lattice, whose joins and meets are given by

$$\bigvee_j S_j = \left(\bigcup_j S_j\right)^- \quad \text{and} \quad \bigwedge_j S_j = \bigcap_j S_j \tag{5.1}$$

($\{ S_j \}$ being an arbitrary family of closed subsets of P), and that the correspondence $S \mapsto S^\perp (S \in C(P, \perp))$ defines an orthocomplementation in $C(P, \perp)$ ([18]). Moreover, the axioms that we have assumed in Section 2 imply the following [*16*]:

i) The filter logic F is atomic (actually, it is also atomistic, see [*16*]), and there is a one-to-one mapping $p \mapsto \text{carr}\, p$, $p \in P$, of the set P of pure states onto the set A(F) of atoms of F such that $\text{carr}\, p \leqslant a$ iff $p(a) = 1$;

ii) The phase geometry $C(P, \perp)$ is atomistic;

iii) For every $a \in F$ the set $a^1 := \{ p \in P : p(a) = 1 \}$ belongs to $C(P, \perp)$, and the mapping $a \mapsto a^1$ is an orthoinjection of the filter logic F into the phase geometry $C(P, \perp)$.

Another embedding of the logic F into an orthocomplemented complete lattice can be realized by forming the so-called completion by cuts of F (see [*3*] for details). Alternatively, this embedding one can construct as follows. Define for any $M \subseteq F$

$$M^\perp := \{ a \in F : a \perp b \quad \text{for all} \quad b \in M \}$$

and let $\tilde{F} = \{ M \subseteq F : M = M^{\perp\perp} \}$; obviously, $M \subseteq M^{\perp\perp}$ for any $M \subseteq F$.

With respect to the set inclusion $\tilde{F}$ becomes a complete lattice with joins and meets given by formulae identical with (5.1) and with the orthocomplementation given by $M \mapsto M^\perp$. $\tilde{F}$ coincides with the usual completion by cuts of F (see, e. g. [*19*], Theorem 2.4), as for every $M \subseteq F$ one has $M^{\perp\perp} = M^{\nabla\triangle}$, where

$$M^\nabla := \{ a \in F : a \geqslant b \quad \text{for all} \quad b \in M \}$$
$$M^\triangle := \{ a \in F : a \leqslant b \quad \text{for all} \quad b \in M \}$$

(18) For the empty set $\emptyset$ we put, by definition, $\emptyset^\perp = P$, which leads immediately to $\emptyset, P \in C(P, \perp)$.

(we use here the notation of Bugajska and Bugajski [3]). Note also that [3]

$$M^{\perp} = M'^{\triangle},$$

where

$$M' = \{ a' : a \in M \}.$$

The embedding of F into $\tilde{F}$ is given by the mapping $a \mapsto \{ a \}^{\triangle} = \{ a \}^{\perp\perp}$, which, as it may easily be seen [3], has the desired properties of an orthoinjection.

We shall now state some facts about $\tilde{F}$, which follows from the axioms and from the properties of F.

PROPOSITION 1. — $\tilde{F}$ is atomic and satisfies MSPR.

Proof. — Atomicity of $\tilde{F}$ is obvious, as it follows directly from the atomicity of F. Indeed, let $M \in \tilde{F}$, $M \neq \{ 0 \}$; then there is a non-zero $a \in M$, and therefore, by atomicity of F, $a \geqslant e$ for some $e \in A(F)$, hence $\{ 0, e \} \subseteq M$ (we use here the fact that, for $M = M^{\nabla\triangle}$, $a \in M$ and $b \leqslant a$ imply $b \in M$), which proves the atomicity of $\tilde{F}$, since the subsets of the form $\{ 0, e \}$ (and only these subsets), where $e \in A(F)$, are the atoms of F.

Similarly, MSPR holding for F implies the validity of MSPR for F. In fact, let $\{ 0, e \} \subseteq (\{ 0, e_1 \} \cup \{ 0, e_2 \} \cup \{ 0, e_3 \})^{\nabla\triangle}$, where $\{ 0, e \} \neq \{ 0, e_3 \}$ and let $\{ 0, e \} \nsubseteq (\{ 0, e_1 \} \cup \{ 0, e_2 \})^{\nabla\triangle}$. This, as it may easily be seen, is equivalent to the following assumption: $e \leqslant e_1 \vee e_2 \vee e_3$, $e \neq e_3$, $e \nleqslant e_1 \vee e_2$, which implies (see the MSPR for F) the existence of an atom $f \in A(F)$ such that $f \leqslant e \vee e_3$ and $f \leqslant e_1 \vee e_2$. This may equivalently be written as

$$f \in \{ e, e_3 \}^{\nabla\triangle} \quad \text{and} \quad f \in \{ e_1, e_2 \}^{\nabla\triangle},$$

or as

$$\{ 0, f \} \subseteq (\{ 0, e \} \cup \{ 0, e_3 \})^{\nabla\triangle} \quad \text{and} \quad \{ 0, f \} \subseteq (\{ 0, e_1 \} \cup \{ 0, e_2 \})^{\nabla\triangle},$$

which shows that MSPR holds in $\tilde{F}$. This completes the proof of the proposition.

Now we will want to show that the orthomodularity of F implies the orthomodularity of $\tilde{F}$. However, in order to prove such a statement, we have to assume an additional postulate (19):

AXIOM 13. — For each sequence $\{ e_i \}_{i=1}^{\infty}$ of pairwise orthogonal atomic filters there exists a filter $a \in F$ such that $\sum_{i=1}^{\infty} d(e_i p) = d(ap)$ for all homogeneous beams $p \in B_h$.

(19) This is a weak form of the so-called « orthogonality postulate », the latter being commonly accepted in the quantum logic axiomatics.

Note that the filter a in the axiom above is unique, as $a = \bigvee_{i=1}^{\infty} e_i$. Indeed, as $e_i^1 \subseteq a^1$ for all i, we find by using the property (*) from Section 2 that $e_i \leqslant a$ for all i. If now $b \geqslant e_i (i = 1, 2, \dots)$, we get $b \geqslant \bigvee_{i=1}^{n} e_i$ for all $n = 1, 2, \dots$, hence

$$d(bp) \geqslant d\left(\left(\bigvee_{i=1}^{n} e_i\right)p\right) = \sum_{i=1}^{n} d(e_i p), \qquad \text{all } p \in \mathrm{B}_h,$$

which leads to

$$d(bp) \geqslant \sum_{i=1}^{\infty} d(e_i p) = d(ap), \qquad \text{all } p \in \mathrm{B}_h.$$

Hence $a^1 \subseteq b^1$, which implies $a \leqslant b$ by (*). This shows that $a = \bigvee_{i=1}^{\infty} e_i$, as claimed.

Now, having proved that for any countable family of pairwise orthogonal atoms there exists its lattice sum in F, we are in a position to show the orthomodularity of $\tilde{\mathrm{F}}$. This can be done in an exactly the same way as the proof of Corollary 3 in [*3*]. Thus we can write:

PROPOSITION 2. — $\tilde{\mathrm{F}}$ is orthomodular.

Summarizing, $\tilde{\mathrm{F}}$ becomes a complete atomic orthomodular lattice satisfying MSPR, hence we find $\tilde{\mathrm{F}}$ to be also atomistic (by atomicity and orthomodularity) and possessing the covering property. Therefore, our axioms 1-13, implying the above-mentioned properties of $\tilde{\mathrm{F}}$, are sufficient to get the well-known Piron-MacLaren's representation theorem for $\tilde{\mathrm{F}}$ (and therefore for the filter logic F also)—see, e. g., [*25*], [*19*], [*30*], [*20*]—if, of course, we assume $\tilde{\mathrm{F}}$ to be irreducible and of the projective dimension not smaller than 4.

Note that the irreducibility of $\tilde{\mathrm{F}}$ is not a restrictive assumption, as if it does not hold, then any irreducible part of $\tilde{\mathrm{F}}$ may be taken into consideration in place of $\tilde{\mathrm{F}}$. Moreover, the irreducibility of $\tilde{\mathrm{F}}$ can also easily be understood from the physical point of view, as it may be formulated in the form of the so-called « superposition principle » for F (compare [*9*]). This is the content of the following statement:

PROPOSITION 3. — $\tilde{\mathrm{F}}$ is irreducible if and only if every line in F has at least three distinct points (= atoms) lying on it.

Proof. — $\tilde{F}$ is irreducible if and only if every line in $\tilde{F}$ has at least three distinct points lying on it (see, e. g. [*25*]), but, as the atoms of $\tilde{F}$ are precisely of the form $\{0, e\}$, where $e \in A(F)$, this will be fulfilled when and only when any line in F contains at least three distinct points.

Note that the physical significance may be assigned to the following corollary to the proposition above (see [*13*], [*14*]):

PROPOSITION 4. — $\tilde{F}$ is irreducible if and only if the set P of pure states possesses the following property ([20]): For any pair p, q of distinct pure states there is a pure state $r \neq p, q$ such that $r \in \{p, q\}^{-}$.

Proof. — Replace pure states by their carriers, and then apply the proposition 3.

Remark. — Note that the two embeddings of the filter logic F described in this section are, in fact, the same, as one can easily find $C(P, \perp)$ to be orthoisomorphic with $\tilde{F}$. To prove the latter it suffices to replace pure states by their carriers and then apply theorems 2.4 and 2.5 from [*19*].

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