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Topology of vortices

by

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ABSTRACT. — A conceptual new approach to field theory is given in terms of unifield intrinsic field quantities which consistently describe interacting elementary particle systems. These non-local field quantities provide a classification of vortices in terms of homotopy theory and the Rham's cohomology framework. By way of such a classification scheme, interacting vortex lines, i. e. interacting particles associated with these vortex lines, are described in terms of a topological linkage property. A topological scattering set-up of this type displays similar features as Born-scattering.

I. INTRODUCTION

Local relativistic field theories with interaction are known to be divergent Therefore, an approach to elementary particle physics in terms of non-local field quantities is given. Two-component fields, consisting of a physical component $\overset{p}{\omega}$ and a topological component c_p are introduced as follows [*1*]:

$$(1) \qquad (\overset{p}{\omega}, c_p) \quad \overset{p}{\omega} \in \mathrm{F}^p(\mathrm{M}),\ c_p \in \mathrm{C}_p(\mathrm{M}) \qquad (p = 0, 1, \ldots, n = \dim \mathrm{M})$$

$\mathrm{F}^p(\mathrm{M})$ denotes the vector space of differentiable p-forms and $\mathrm{C}_p(\mathrm{M})$ the space of differentiable p-chains on the configuration space M [*2*], [*3*]. With the fields (1) can be associated physical observables (integral laws) as can be illustrated in the case of Gauss's law of electrostatics:

$$(2) \qquad (\overset{2}{\omega}, c_2) \mapsto 4\pi e = \int_{c_2} \overset{2}{\omega} \qquad (e \text{ : electrical charge})$$

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whenever $\overset{2}{\omega} = E_{ij}dx^i \wedge dx^j \in F^2(M)$, stands for the electrostatic field $\vec{E} = (E_1, E_2, E_3)$; $c_2 \in C_2(M)$, $M = \mathbb{R}^3$ $(i, j = 1, 2, 3)$. The role of the physical field component of (1) is displayed in table A where conventional (quantum) field theory (QFT) is compared to a relativistic hydrodynamic string theory with continuous degrees of freedom.

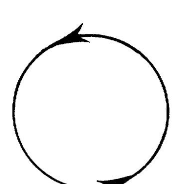
TABLE A.

Conventional QFT	Relativistic hydrodynamical string theory
Particles are singularities of fields such as $\varphi(x)$, $A_\mu(x)$, $F_{\mu\nu}(x)$ ... etc	*Particles* are singular strings (vortex lines or streamlines) af a relativistic fluid, corresponding to fields (1), such as $\overset{0}{\omega} = \varphi(x) \in F^0(M^4)$, $\overset{1}{\omega} = \sum_\mu A_\mu(x)dx^\mu \in F^1(M^4)$, $\overset{2}{\omega} = \sum F_{\mu\nu}(x)dx^\mu \wedge dx^\nu \in F^2(M^4)$ wich are differential forms of degree $p = 0, 1, 2, \ldots$
Interactions are mediated by vector currents $j_\mu(x)$ is terms of the principle of minimal interaction $L = ej_\mu A^\mu$	*Interactions* are determined by the curvature of a gauge geometry, i. e. Yang-Mills fields. Vector current-1-forms $j = \Sigma j_\mu(x).dx^\mu$ are defined in terms of the Yang-Mills eq. $d{*}\overset{2}{\omega} = 4\pi{*}j$; $*$: Hodge star operator [2], [3]; d : Exterior derivative [2], [3]; $d : F^p(M) \to F^{p+1}(M)$
Dynamics of a system is specified by a characteristic function, the Lagrangian $L(\varphi, A_\mu)$, by means of a variational principle for the action I : $\delta\int_{c_4} L d^4x = \delta I = 0$, $c_4 \in C_4(M^4)$	*Dynamics.* There is a variational principle for the streamlines of a relativistic fluid [3], [4] which accounts for the dynamics of strings.

Unified intrinsic fields of the type (1), when related to string theory, are those with $p = 1$. The corresponding topological « particle units » are 1-chains or 1-cycles (closed 1-chains or loops) of the following type:

$c_1 \in C_1(M)$

FIG. 1 *a*.



$c_1 \in \overset{\circ}{C}_1(M)$

FIG. 1 *b*.

II. STRINGS AS VORTEX LINES

String solutions of relativistic field equations are vortex type solutions whenever the strings are closed or infinite (fig. 1). Such relativistic vortex lines occur in the case of the Higgs model with Lagrangian density

$$(3)\quad L := -1/4F_{\mu\nu}^2 - \left| \left(\frac{\partial}{\partial x^\mu} - i\frac{e}{c}A_\mu \right)\varphi \right| - \lambda_1 |\varphi|^2 - \lambda_2 |\varphi|^4$$

λ_1, λ_2 are constants
$\mu = 0, 1, 2, 3$

Alternately, such a model can be given in terms of fields (1) on account of table A. The physical field of (1) is then a Higgs vector-current 1-form $\overset{1}{\omega}_j = j$ which derives from (3); the topological component c_1 stands for a closed or an infinite string (fig. 1), so

$$(4)\quad \overset{1}{\omega}_j = j = \sum_{\mu=0}^{3} j_\mu(x)dx^\mu = \frac{2e^2}{\hbar c^2}\bar{\varphi}\varphi A_\mu^{\cdot} dx^\mu + \frac{i.e}{\hbar c}(\bar{\varphi}d\varphi - \varphi d\bar{\varphi}) \in F^1(M^4)$$

where

$$(5)\quad \overset{1}{\omega}_A = \Sigma A_\mu dx^\mu \in F^1(M^4) \qquad \text{and} \qquad (6)\quad \varphi = Re^{iS/\hbar} \in F^0(M^4)$$

stand for a Yang-Mills potential $A_\mu(x)$ and a Higgs scalar field φ, respectively. Vortex lines occur when quantized flux is defined in terms of the differential action 1-form field dS which is associated with (4) by:

$$d\varphi = dR \wedge e^{iS/\hbar} + R\left(\frac{i}{\hbar}\right) \wedge dSe^{i\frac{S}{\hbar}} \quad \Rightarrow \bar{\varphi}d\varphi - \varphi d\bar{\varphi} = \frac{2iR^2}{\hbar}dS$$

$$d\bar{\varphi} = dR \wedge e^{-iS/\hbar} - R\left(\frac{i}{\hbar}\right) \wedge dSe^{-iS/\hbar}$$

$$\Rightarrow j_\mu^{\cdot} dx^\mu = \frac{2e^2}{\hbar c^2}\bar{\varphi}\varphi A_\mu dx^\mu + \frac{2ie^2R^2}{\hbar^2 c}dS$$

hence

$$(7)\qquad dS = \left(e/cA_\mu - \frac{\hbar^2}{2e\varphi\bar{\varphi}}\cdot j_\mu \right).dx^\mu \in F^1(M^4)$$

Since the field (6) must be uniform, the phase must change by a multiple of 2π along any loop c_1. Hence quantized flux is associated with a 1-field of type (1), i. e. is an integral law or observable of type (2):

$$(1')\qquad (\overset{1}{\omega}, c_1) \mapsto \phi = \int_{c_1} dS = 2\pi\hbar n,\ n \in \mathbb{Z}, \qquad \overset{1}{\omega} = dS \in F^1(M^4)$$

Vortex lines occur for solutions where $n \neq 0$.

Given the 1-field (7), i. e. consider the quantized flux (1′) associated with dS, then arises the following

Problem. — Determine the dynamics of relativistic strings (cf. table A) which gives rise to the quantum condition (1′).

The dynamics of strings is defined in terms of some dynamical system $\Sigma := \Sigma(f, B)$, where $f : P \to \mathbb{R}$ is any observable on phase space P and $B = B_{\mu\nu} dx^\mu \wedge dx^\nu \in F^2(M^4)$ a Yang-Mills field. Then the Euler-Lagrange 1-form on M^4, $\overset{1}{\omega}_f$, accounts for the dynamics, i. e.

$$(8) \quad \overset{1}{\omega}_f = E_\mu(f) dx^\mu = K_\mu dx^\mu \in F^1(M^4), \qquad E_\mu(f) := \frac{d}{d\tau}\frac{\partial f}{\partial \dot{x}^\mu} - \frac{\partial f}{\partial x^\mu}$$

and

$$(9) \quad \overset{1}{\omega}_L = e/c\widehat{u} \,\lrcorner\, B = e/c B_{\mu\nu} \dot{x}^\nu dx^\mu = K_\mu dx^\mu \in F^1(M^4)$$

$$(\lrcorner \text{ contraction symbol}) \qquad \widehat{u} = u^\mu = \frac{dx^\mu}{d\tau}$$

is the Lorentz force field.

Define the circulation Γ of $\overset{1}{\omega}$ to be the contour integral

$$(10) \qquad \Gamma := \int_{c_1} \overset{1}{\omega} \quad ; \quad \overset{1}{\omega} := \frac{\partial f}{\partial \dot{x}^\mu} dx^\mu$$

then the dynamical system $\Sigma(f, B)$, corresponding to (8) may be characterized in terms of the following

PROPOSITION 1. — Let

$$(11) \qquad \overset{2}{\Omega} := d\left(\frac{\partial f}{\partial \dot{x}^\mu} dx^\mu\right) + \frac{e}{c} B_{\mu\nu} dx^\mu \wedge dx^\nu \in F^2(M^4) \text{ [5]}$$

and let c_1 *and* c_1' *be two homotopic 1-cycles embrassing a 2-dimensional tube* $T = M^2$ *which is generated by the trajectories of the differential system*

$$\frac{dx^0}{u^0} = \frac{dx^1}{u^1} = \frac{dx^2}{u^2} = \frac{dx^3}{u^3} \qquad u^\mu = \dot{x}^\mu = \frac{dx^\mu}{d\tau} \qquad \mu = 0, 1, 2, 3.$$

Then

$$(12) \quad \frac{e}{c}\int_{c_2} B = \int_{c_1'} \overset{1}{\omega} - \int_{c_1} \overset{1}{\omega} \qquad c_1 - c_1' = \partial c_2 \quad ; \quad \partial : C_p(M^2) \to C_{p-1}(M^2)$$

$$p = 0, 1, 2$$

Proof. — Let

$$(13) \qquad S = \int_{\tau_0}^{\tau_1} f \, d\tau \quad ; \quad (\tau \text{ a parameter, i. e. } \tau \in \mathbb{R})$$

be the action integral along the arc $\widehat{x_0x_1}$ of the trajectory $\gamma:[\tau_0, \tau_1] \to M^4$ $\tau \mapsto \gamma(\tau) = x^\mu(\tau)$. Then the variation δS is evaluated to be

$$\delta S = \frac{\partial f}{\partial \dot{x}^\mu}\delta x^\mu \Big|_{x_0}^{x_1} - \int_{\tau_0}^{\tau_1} E_\mu(f)\delta x^\mu d\tau = \frac{\partial f}{\partial \dot{x}^\mu}\delta x^\mu \Big|_{x_0}^{x_1} - \int_{\tau_0}^{\tau_1} K_\mu \delta x^\mu d\tau \quad ; \quad \delta\dot{x}^\mu = \frac{d}{d\tau}\delta x^\mu$$

$$\Rightarrow \delta S = \frac{\partial f}{\delta \dot{x}^\mu}\delta x^\mu \Big|_{x_0}^{x_1} - \frac{e}{c}\int_{\tau_0}^{\tau_1} B_{\mu\nu}\delta x^\mu dx^\nu$$

since

$$K_\mu d\tau = \frac{e}{c}B_{\mu\nu}\dot{x}^\nu d\tau = \frac{e}{c}B_{\mu\nu}dx^\nu$$

If the two-chain $c_2 \in C_2(M^2)$ denotes the portion of $T = M^2$ which is limited by c_1 and c_1', integration along trajectories of Σ then yields

$$\int_{c_1} \frac{\partial f}{\partial \dot{x}^\mu} dx^\mu - \int_{c_1'} \frac{\partial f}{\partial \dot{x}^\mu} dx^\mu = \frac{e}{c}\int_{c_2} B_{\mu\nu} dx^\mu \wedge dx^\nu \qquad \blacksquare$$

Discussion. — If $\overset{2}{\Omega}$ is closed, i. e. $d\overset{2}{\Omega} = 0$, then $\overset{2}{\Omega}$ assumes the form

$$\overset{2}{\Omega} = d\left(\frac{\partial f}{\partial \dot{x}^\mu}dx^\mu + \frac{e}{c}A_\mu dx^\mu\right) \in \overset{0}{F}{}^2(M^4) \tag{14}$$

In fact: $\overset{2}{\Omega} \in \overset{0}{F}{}^2(M^4)$ yields

$$d\Omega = 1/3\,!\; C_{\mu\nu\lambda}dx^\mu \wedge dx^\nu \wedge dx^\lambda + e/c\frac{\partial B_{\mu\nu}}{\partial \dot{x}^\lambda}dx^\mu \wedge dx^\nu \wedge dx^\lambda = 0$$

Then

$$C_{\mu\nu\lambda} = \frac{\partial B_{\nu\lambda}}{\partial x^\mu} + \frac{\partial B_{\lambda\mu}}{\partial x^\nu} + \frac{\partial B_{\mu\nu}}{\partial x^\lambda} = 0 \qquad \text{and} \qquad \frac{\partial B_{\mu\nu}}{\partial \dot{x}^\lambda} = 0$$

are the necessary and sufficient conditions for $d\overset{2}{\Omega} = 0$ to hold. By de Rham's theorem there exists locally a potential A such that

$$B = B_{\mu\nu}dx^\mu \wedge dx^\nu = dA, \quad B_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \quad \text{and} \quad \overset{1}{\omega}_A = A = \Sigma A_\mu dx^\mu \text{ (éq. 5)}$$

Therefore (14') $\overset{2}{\Omega} = d\bar{\omega}$ which means that the trajectories of the given dynamical system admit the integral invariant

$$\bar{\omega} = \left(\frac{\partial f}{\partial \dot{x}^\mu} + \frac{e}{c}A_\mu\right)dx^\mu \quad ; \quad \left(S = \int_{\tau_0}^{\tau_1} f d\tau = \int_{\tau_0}^{\tau_1}\left(\bar{f} + \frac{e}{c}A_\mu\frac{dx^\mu}{d\tau}\right)d\tau\right) \tag{15}$$

The physical interpretation of (15) and (12) is the following. The 1-form (15) equals (7) iff $\frac{\partial f}{\partial \dot{x}^\mu} = p_\mu = \frac{-\hbar^2}{2e\bar{\varphi}\varphi} \cdot j_\mu$. Hence Quantisation of relativistic strings amounts to quantizing the « space-time-flux » (12):

$$(12\,bis)\quad \bar{\phi}_{\text{string}} = \bar{\phi} = \frac{e}{c}\int_{c_2} \mathrm{B} = \int_{c_1} \overset{1}{\omega} - \int_{c_1'} \overset{1}{\omega} = 2\pi\bar{n}\hbar \qquad \bar{n} := n_1 - n_2 \in \mathbb{Z}$$

whenever a generalized Bohr-Sommerfeld quantum condition

$$(12\,ter)\qquad \int_{c_1} \overset{1}{\omega} = \int_{c_1} p_\mu dx^\mu = 2\pi n_1 \hbar \qquad n_1 \in \mathbb{Z}$$

is postulated. Therefore clearly $n = \bar{n}$

$$\Rightarrow \bar{\phi}_{\text{string}} = \phi = \int_{c_1} d\mathrm{S}$$

is quantized flux (required quantum condition) associated with string dynamics.

Case of static vortices. — In this section we classify pure vortices without monopoles (a classification with monopoles is given in section III) i. e. we exhibit a vortex solution in the static cylindrically symmetric case. Let (r, ϑ, z) be cylindrical coordinates and take $\mathrm{A}_0 = \mathrm{A}_r = \mathrm{A}_z = 0$, $\mathrm{A}_\vartheta \neq 0$. For an infinitely long static vortex line lying along the z-axis (fig. 2), having ϑ as the azimuthal angle, the axial symmetry reduces the *problem to a two-dimensional* one. The field (6) is written as a differentiable map (loop)

$$(6')\qquad \varphi : [0, 1] \subset \mathbb{R} \dashrightarrow \mathrm{S}^1 \subset \mathbb{C} \quad ; \quad \varphi = |\varphi| e^{2\pi\, \mathrm{i}nt} \quad , \quad |\varphi| = 1$$

where $\mathrm{S}^1 := \{ z \in \mathbb{C} : |z| = 1 \}$. On account of $2\pi t = \alpha$, $n\alpha = \vartheta$ the field (6′) is interpreted as being "parallel transferred" along the circle S^1, and one may write $\mathrm{U}(\alpha) = e^{\mathrm{i}n\alpha}$ (6″). We give 2 types of vortex classifications. A homology classification and a homotopy classification.

II.1. Homology classification of static vortices

Define a 1-field of the type (1), $(\overset{1}{\omega}, c_1)$ in terms of

$$(16)\quad \overset{1}{\omega} = d\vartheta = \frac{x.dy - y.dx}{x^2 + y^2} \in \mathrm{F}^1(\mathbb{R}^2 - \{0\}), \quad c_1 \in \overset{0}{\mathrm{C}}_1(\mathbb{R}^2 - \{0\})$$

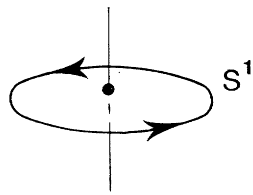


FIG. 2.

Let $H^1(S^1) = F^1(S^1)/dF^0(S^1) := \{ a.d\vartheta \mid a \in \mathbb{R} \}$ (closed 1-forms modulo exact 1-forms on S^1) denote the first de Rham group and $H_1(S^1) =: H_1$ be the first homology group of S^1 [3]. Then, by de Rham's first theorem there exists a nondegenerate bilinear form ß, given by [3]:

$$(17) \qquad ß : H^1 x H_1 \dashrightarrow \mathbb{R} \; ; \; (\overset{1}{\omega}, c_1) \mapsto \text{const.} \int_{c_1} \frac{xdy - ydx}{x^2 + y^2}$$

If $c_1 := S^1$ and $n \in \mathbb{Z} \subset \mathbb{R}$ we have on account of a smooth map

$$\psi : S^1 \to \mathbb{R}^2 - \{ 0 \}$$

$$(18) \qquad w_0(\psi) = n = \frac{1}{2\pi} \int_{S^1} \overset{1}{\omega} \in \mathbb{Z}$$

which is referred to as the winding number of ψ about 0. It states how many times $c_1 = S^1$ winds around the origine and which stands for the *vortex number* corresponding to c_1. In defining the quantity

$$(19) \qquad dS = \hbar . d\vartheta \qquad (\hbar : \text{Planck's constant})$$

the number $n \in \mathbb{Z}$ may be regarded as quantum number associated with the field $(\overset{1}{\omega}, c_1) \in H^1 \times H_1$. That is, n is associated with the Higgs-field $\varphi = | \varphi | e^{in\alpha}$ which is parallel transferred around $c_1 \in C_1(S^1)$, and hence

$$(1'') \qquad (\overset{1}{\omega}, c_1) \mapsto \varnothing = \int_{c_1} dS = 2\pi\hbar n \qquad \text{(cf. éq. (1'))}$$

II.2. Homotopy classification of static vortices

Vortices are physical objects exhibiting homotopic conservation laws. Let $M = S^1$. Then the vortex number $n \in \mathbb{Z}$ as given by (18) which is a topologically conserved quantity, can be associated with the Poincaré group (first homotopy group) of S^1, $\Pi_1(S^1)$, by

$$(20) \qquad \Pi_1(S^1) \simeq \mathbb{Z}$$

which is isomorphic with the additive group of integers. That $\Pi_1(S^1)$ is homomorphic to $\mathbb{Z}$ is displayed by the degree $\deg(\varphi) = d(\varphi)$ of (6'):

$$\deg(\varphi) = \frac{1}{2\pi i}\int_0^1 \frac{d\varphi}{\varphi}, \text{ where } \varphi \text{ is a differentiable loop.} \tag{21}$$

In fact, if φ_1, φ_2 are two differentiable loops as given by (6'), then

$$d : \Pi_1(S^1, e_1) \to \mathbb{Z} \quad , \quad e_1 = (1, 0) \in S^1$$

and

$$d(\varphi_1\varphi_2) = \int_0^{\frac{1}{2}} \frac{d\varphi_1(2\vartheta)}{\varphi_1(2\vartheta)} + \int_{\frac{1}{2}}^1 \frac{d\varphi_2(2\vartheta - 1)}{\varphi_2(2\vartheta - 1)} = \int_0^1 \frac{d\varphi_1(t)}{\varphi_1(t)} + \int_0^1 \frac{d\varphi_2(t)}{\varphi_2(t)}$$
$$= d(\varphi_1) + d(\varphi_2)$$

The homomorphism d is surjective: If $n \in \mathbb{Z}$, then the loop

$$\varphi : t \mapsto e^{2\pi\,\text{int}} = z(t)$$

satisfies $\deg(\varphi) = \dfrac{1}{2\pi i}\displaystyle\int_0^1 \frac{2\pi i n e^{2\pi\,\text{int}}}{e^{2\pi\,\text{int}}}\,dt$. Hence

$$\deg(\varphi) = n \qquad \text{in agreement with (18)} \tag{18'}$$

This is consistent with the fact that the integrand of (21) is related to the 1-form (16) in the following way:

$$\frac{1}{2\pi i}\frac{dz}{z} = \frac{1}{2\pi i}\frac{dx + idy}{x + iy} = \frac{1}{2\pi i}\frac{xdx + ydy}{x^2 + y^2} + \frac{1}{2\pi}\frac{xdy - ydx}{x^2 + y^2} \tag{22}$$

Hence the 1-field (16) is given to be

$$\overset{1}{\omega} = d\vartheta = \operatorname{Im}\left(\frac{dz}{z}\right)$$

TO SUMMARIZE. — *The scalar Higgs-field $\varphi = |\varphi| e^{2\pi\,\text{int}}$ (eqs. (6) and (6')) determines the degree $\deg(\varphi) = n$, which is an injective and surjective homomorphism $\Pi_1(S^1) \to \mathbb{Z}$. Hence homotopic conservation laws are obtained in terms of this isomorphism. The inverse of this isomorphism is the map $n \to [\varphi_n]$(1). Vortices are classified by the fundamental group $\Pi_1(S^1) \simeq \mathbb{Z}$. Each vortex is labelled by a homotopic invariant, the integer $n \in \mathbb{Z}$.*

(1) The class of the differentiable map $t \to e^{2\pi\,\text{int}}$ is a generator of $\pi_1(S^1, e_1)$.

III. CLASSIFICATION OF STRINGS WITH MONOPOLES

Within the frame of Lagrangian formalism a physical action field $S \in F^0(M)$ (cf. table A) may be related to a string model [5], [6] by

$$S = \int L dt = \int \hat{L} dl dt \tag{23}$$

where l is the measure of length along a string, L is a line density and $L dt \in F^1(M)$ the corresponding Lagrangian 1-form field. The Euler-Lagrange eq. of motion are [5], [6]:

$$\frac{\partial \hat{L}}{\partial x^k} - \frac{\partial}{\partial l}\left(\frac{\partial \hat{L}}{\partial\left(\frac{\partial x^k}{\partial l}\right)}\right) - \frac{\partial}{\partial t}\frac{\partial \hat{L}}{\partial\left(\frac{\partial x^k}{\partial t}\right)} = 0 \tag{24}$$

where the vector-valued map x: $(l, \tau) \to x^i(l, \tau)$, τ = ict, is regarded as a string in space, which, as time changes, sweeps out a 2-dimensional Riemannian manifold $M^2 := (M^2, ds^2)$. The string-model given by (23)-(24) may be cast in terms of topological fields $(\overset{p}{\omega}, c_p)$ (eq. (1)), whenever $p = 2$ as follows: Let

$$x : U \subset \mathbb{R}^2 \to \mathbb{R}^3 ; \quad (u^1, u^2) \mapsto x^i(u^1, u^2) \quad (U \text{ open in } \mathbb{R}^2) \tag{25}$$

be a vector-valued map of class C^k ($k \geqslant 1$) and define a topological action field $S \in F^0(M^2)$ associated with the topological 2-field

$$(\overset{2}{\omega}, c_2);\ \overset{2}{\omega} := k\sqrt{g}\, du^1 \wedge du^2 \quad , \quad k \in \mathbb{R}, g = \det(g_{ij}) \quad , \quad ds^2 = g_{ij} du^i du^j \quad c_2 \in C_2(M^2) \tag{26}$$

in terms of the assignment

$$(\overset{2}{\omega}, c_2) \mapsto S := \int_{c_2} \overset{2}{\omega} = k \int \sqrt{g}\, du^1 du^2 \tag{27}$$

The field (27) generalizes (23), since with the particular parametrisation $u^1 = l$, $u^2 = \tau$ = ict one recovers the physical action field (23). The problem then arises if there are any topological charges (conserved quantities), playing a similar role as (1') or (18) and being associated with fields $(\overset{2}{\omega}, c_2)$ (27).

III.1. Homology classification of Higgs-fields

Define a 2-field of the type (1) $(\overset{2}{\omega}, c_2)$, where

$$\overset{2}{\omega} = \sin \vartheta d\vartheta \wedge d\varphi \tag{26'}$$

and

$$c_2 := S^2 \quad , \quad c_2 \in \overset{0}{C}_2(\mathbb{R}^3 - \{0\})$$

Consider a gauge theory [1], [3] with Yang-Mills group G = SO (3), Yang-Mills field A^k_μ ($\mu = 0, \ldots, 3$, $k = 1, \ldots, 3$) and a Higgs-field ϕ given by

(28) $$\begin{pmatrix}\phi^1\\ \phi^2\\ \phi^3\end{pmatrix} = \begin{pmatrix}\phi(r)\sin\vartheta\cos(n\varphi)\\ \phi(r)\sin\vartheta\sin(n\varphi)\\ \phi(r)\cos\vartheta\end{pmatrix} = \phi \qquad \phi((r) \text{ expresses the radial dependence of } \phi)$$

The construction of a topological quantum number associated with (28) can be given in terms of cohomology as follows. Let

(29) $\widehat{\phi} : S^2(r) \to S^2_\phi \; ; \; (\widehat{\phi}^1, \widehat{\phi}^2) = (\widehat{\phi}^1(\vartheta, \varphi), \widehat{\phi}^2(\vartheta, \varphi)) = (\vartheta, n\varphi)$

be a smooth map, where

(30) $$\widehat{\phi}(\vec{x}) = \frac{\phi}{\| \phi \|}$$

stands for the normalized Higgs-field (28). With the 2-field $\overset{2}{\omega} = \sin\vartheta d\vartheta \wedge d\varphi$ (eq. (26')) can be associated the Brouwer degree in terms of the following commutative diagram:

(31) $$\begin{array}{ccc} H^n(S^2(r)) & \overset{\widehat{\phi}^*}{\longleftarrow} & H^n(S^2_\phi) \\ {\scriptstyle I_{S^2(r)} = \int_{S^2(r)}}\downarrow & & \downarrow{\scriptstyle I_{S^2_\phi} = \int_{S^2_\phi}} \\ \mathbb{R} & \overset{f_{\widehat{\phi}}}{\longleftarrow} & \mathbb{R} \end{array} \qquad n \in \{0, 1, 2\}$$

where

$$\int_{S^2(r)} : H^n(S^2(r)) \xrightarrow{\cong} \mathbb{R}$$

and

$$\int_{S^2_\phi} : H^n(S^2_\phi) \xrightarrow{\cong} \mathbb{R}$$

are linear isomorphisms which determine a unique linear map $f_{\widehat{\phi}} \in \mathscr{L}(\mathbb{R}, \mathbb{R})$

$$f_{\widehat{\phi}} : \mathbb{R} \to \mathbb{R} \quad , \quad t \mapsto kt \qquad k \in \mathbb{R}.$$

Then

(32) $$\deg(\widehat{\Phi}) = f_{\widehat{\phi}}(1)$$

is by definition the Brouwer degree. Since $I_{S^2(r)} \circ \widehat{\phi}^* = f_{\widehat{\phi}} \circ I_{S^2_\phi}$ and since $\overset{2}{\omega} \in [\overset{2}{\omega}] \in H^2(S^2(r))$ one obtains from (31)

(33) $$\int_{S^2(r)} \widehat{\phi}^* \overset{2}{\omega} = k \int_{S^2_\phi} \overset{2}{\omega}$$

This is just the topological action field as given by eq. (27), whenever $c_2 = S^2$. Hence conserved quantum numbers may be associated with the action field S. — — Let $x = (\vartheta, \varphi) \in S^2(r)$ be a regular point of $\widehat{\phi}$ so that $d\widehat{\phi}(x)$: $T_x(S^2(r)) \dashrightarrow T_{\widehat{\phi}(x)}(S^2_\phi)$ is a linear isomorphism between oriented vector spaces. Then $\widehat{\phi}$ determines an integer valued function

$$\text{sgn}\,(J_x(\widehat{\phi})) = \begin{cases} +1 & \text{if} \quad d\widehat{\phi}_x \text{ preserves the orientation} \\ -1 & \text{if} \quad d\widehat{\phi}_x \text{ reverses the orientation} \end{cases}$$

where

$$J_x(\widehat{\phi}) = \det\,(d\widehat{\phi}(x)) = \det\left(\frac{\partial\widehat{\phi}^j}{\partial x^i}(x)\right)$$

is the Jacobian of $\widehat{\phi}$. So

$$\int_{S^2(r)} \widehat{\phi}^* \overset{2}{\omega} = \left(\sum_{x \in \widehat{\phi}^{-1}(y)} \text{sgn}\,(\mathscr{J}_x(\widehat{\phi}))\right)\int_{S^2_\phi} \overset{2}{\omega} = k\int_{S^2_\phi} \overset{2}{\omega} \quad : \quad \widehat{\phi}^{-1}(y) = \{x^1, \dots x^n\}$$

Hence

$$\deg(\widehat{\phi}) = k = \sum_{x \in \widehat{\phi}^{-1}(y)} \text{sgn}\,(\mathscr{J}_x(\widehat{\phi})) \in \mathbb{Z} \tag{34}$$

is an integer [9]. On account of the Higgs field (28) the integer k equals n. In fact

$$\left(\frac{\partial\widehat{\phi}^j}{\partial x^i}(x)\right) = \begin{pmatrix} \dfrac{\partial\widehat{\phi}^1}{\partial x^1}(x) & \dfrac{\partial\widehat{\phi}^1}{\partial x^2}(x) \\ \dfrac{\partial\widehat{\phi}^2}{\partial x^1}(x) & \dfrac{\partial\widehat{\phi}^2}{\partial x^2}(x) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial\widehat{\phi}^1}{\partial \vartheta}(x) & \dfrac{\partial\widehat{\phi}^1}{\partial \varphi}(x) \\ \dfrac{\partial\widehat{\phi}^2}{\partial \vartheta}(x) & \dfrac{\partial\widehat{\phi}^2}{\partial \varphi}(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$$

Hence, by formula (34) one obtains

$$\deg\,(\widehat{\phi}) = \sum_i \text{sgn}\,\det\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} = \sum_i (+1) = n \tag{35}$$

TO SUMMARIZE. — *By virtue of (33)-(35) one may associate with a topological 2-field* $(\overset{2}{\omega}', c_2)$ *a conserved integer quantum number n, in setting:*

$$\overset{2}{\omega}' = \phi^* \overset{2}{\omega} \quad , \quad S^2(r) = c_2 \quad , \quad \int_{S^2_\phi} \overset{2}{\omega} = 4\pi \tag{33 bis}$$

Thus

$$(\overset{2}{\omega}', c_2) \mapsto n = \deg \widehat{\phi} = \frac{1}{4\pi}\int_{S^2(r)} \widehat{\phi}^* \overset{2}{\omega} \tag{36}$$

It turns out [8], that from the topological structure of the Higgs-field $(\emptyset_1, \emptyset_2, \emptyset_3)$ originates magnetic charge. In fact, let

$$(37)\quad \begin{cases} \overset{3}{\omega} = j_0 dx^1 \wedge dx^2 \wedge dx^3 - j_1 dx^0 \wedge dx^2 \wedge dx^3 - j_2 dx^0 \wedge dx^1 \wedge dx^3 \\ \qquad - j_3 dx^0 \wedge dx^1 \wedge dx^2 \in F^3(M^4) \\ d\overset{3}{\omega} = 0 \in F^4(M^4) \end{cases}$$

where

$$(38)\quad j_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} \quad , \quad \partial_\mu j^\mu = 0$$

then

$$(39)\quad M = \frac{1}{4\pi}\int j_0(t, x^1, x^2, x^3)dx^1 dx^2 dx^3$$

is magnetic charge, whenever (cf. [10]):

$$F_{\mu\nu} = \hat{\phi}^k G^k_{\mu\nu} - \frac{1}{e}\varepsilon_{klm}\hat{\phi}^k \nabla_\mu \hat{\phi}^l \nabla_\nu \hat{\phi}^m \qquad \hat{\phi}^k = \frac{\phi^k}{\| \phi^k \|}$$

$$G^k_{\mu\nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu + e\varepsilon^{klm} A^l_\mu A^m_\nu \quad ; \quad \nabla_\mu \hat{\phi}^k = \partial_\mu \phi^k + e\varepsilon^{klm} A^l_\mu \hat{\phi}^m$$

As shown in [8], the magnetic charge M is given to be

$$(40)\quad M = \frac{1}{e}\,.\deg(\hat{\phi})$$

III.2. Homotopy classification of strings

A homotopy classification of strings makes again use of the degree $\deg(\varphi)$ of a map (cf. sect. II.2). In fact, if $\varphi, \psi: S^2 \to S^2$ are smooth maps, such that $\deg(\varphi) = \deg(\psi)$, then φ and ψ are homotopic. Otherwise stated. The degree $d(\varphi) = \deg(\varphi)$ depends only on the homotopy class $[\varphi]$ of φ, and d induces an isomorphism $d: \Pi_2(S^2) \to \mathbb{Z}$ (cf. (20), section II.2). Hence, by (40), Higgs fields belonging to the same homotopy class have the same magnetic charge M.

IV. CHARACTERIZATION OF INTERACTING VORTEX LINES

Consider interacting vortex-line like particle structures $c_1, c_1' \in \overset{0}{C}_1(M)$ (cf. fig. 1 *b*) which are assumed to be interlinked as shown in figure 3. It is the aim of this section to show that such an interaction set up (i. e. topological scattering) results from a combination of Higgs-scalar fields of type (6) with Higgs-vector fields (28)-(29)-(30), i. e. in terms of homology classifications as given in section II.1 and III.1. Referring to the non-relativistic

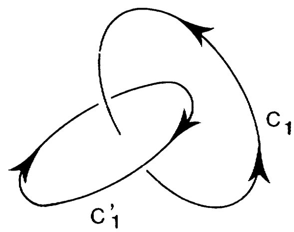


FIG. 3.

case of a Higgs-field of the type (6'), i. e. $\psi = Re^{i\frac{S}{\hbar}}$; $R = R(\vec{x}, t)$, $S = S(\vec{x}, t)$ where $\rho = e\bar{\psi}\psi = eR^2$, we obtain for the velocity of the corresponding non-relativistic fluid flow:

$$(41) \qquad \sum_k v_k dx^k = -\frac{i\hbar}{2m}\left(\frac{d\psi}{\psi} - \frac{d\bar{\psi}}{\bar{\psi}}\right) = \frac{\hbar}{m} d\vartheta = \frac{1}{m} . dS \in F^1(\mathbb{R}^3)$$

and hence, for the circulation around a vortex-line (non-rel. particle):

$$(42) \qquad \Gamma = \frac{\hbar}{m}\int_{c_1} d\vartheta = \frac{2\pi n\hbar}{m} \; ; \; n = \pm 1, \pm 2...$$

where

$$(43) \qquad \Gamma_q = \frac{h}{m}$$

denotes, what we refer to as *quantum of circulation*. So (42) stands for *Thomson's law of vorticity*. This law is susceptible to an interpretation in terms of the topological linking number $l(c_1, c_1')$ which describes a linkage property of the type as displayed in figure 3, whenever c_1 is a closed vortex line of an incompressible fluid and c_1' a loop which winds around the vortex tube displayed in the figure 4.

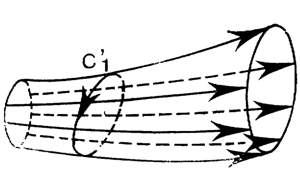


FIG. 4. — Vortex tube.

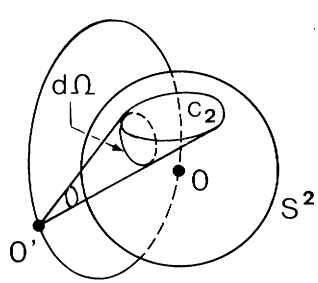


FIG. 5. — Topological scattering device.

Then one can show that

$$(44) \qquad l(c_1, c_1') = \int_{c_1'} v_k dx^k = \Gamma\ ;$$

where $\vec{v}$ must satisfy Biot-Savart's law (45) and relation (41) (cf. subsequent remark 1).

$$(45) \qquad \vec{v} = \frac{\Gamma}{4\pi} \int_{c_1} \frac{d\vec{x}' \wedge (\vec{x} - \vec{x}')}{\| \vec{x} - \vec{x}' \|^3}$$

First quantisation for the interlinked field quantities c_1 and c_1' is then given in combining (42) and (44), that is

$$(46) \qquad l(c_1, c_1') = n \frac{h}{m} \qquad n = \pm 1, \pm 2, \ldots$$

PROPOSITION 2. — *Let $f\colon S^1 \dashrightarrow \mathbb{R}^3, f(S^1) = c_1 = \partial c_2$ (fig. 6) be a loop which is the orbit of a material particle of an incompressible fluid. Then the circulation around the loop c_1' is given in terms of*

$$(44') \qquad l(c_1, c_1') = \int_{c_1} \overset{1}{\omega} \qquad \overset{1}{\omega} = \Sigma v_k dx^k \in F^1 \quad ; \quad d^* \overset{1}{\omega} = 0$$

where $\vec{v}$, $div\ \vec{v} = 0$, is the corresponding velocity field.

Proof. — Let $f\colon S^1 \dashrightarrow \mathbb{R}^3, f(S^1) = \partial c_2$ (c_2 a compact oriented 2-manifold with boundary, figure 6) and $g\colon S^1 \dashrightarrow \mathbb{R}^3$, $g(S^1) = c_1'$, such that $f(S^1) \cap g(S^1) = \emptyset$ consider the map

$$(47) \qquad \chi : S^1 \times S^1 \dashrightarrow S^2 \subset \mathbb{R}^3 \backslash \{0\} \quad ; \quad (s, t) \mapsto \frac{g(s) - f(t)}{\| g(s) - f(t) \|}$$

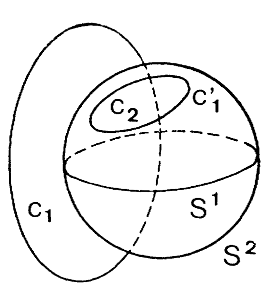


FIG. 6.

In substituting $S^1 \times S^1$ to $S^2(r)$ in the commutative diagram (31), section III.1 one obtains the formula

$$\deg(\chi) = l(c_1, c_1') = \frac{1}{4\pi} \int_{S^1 \times S^1} \chi^* \overset{2}{\omega}.$$

On account of the map (47) and formulae (44′)-(45) it suffices then to prove the following relationship:

$$l(c_1, c_1') = -\frac{1}{4\pi}\int \frac{A(s, t)dsdt}{\| g(s) - f(t) \|^3}$$

where

$$A(s, t) = \begin{vmatrix} g_1(s) - f_1(t) & g_2(s) - f_2(t) & g_3(s) - f_3(t) \\ g_1'(s) & g_2'(s) & g_3'(s) \\ f_1'(t) & f_2'(t) & f_3'(t) \end{vmatrix}$$

$$A(s, t) = -\Sigma(-1)^{i+1}(g_i(s) - f_i(t))[g_j'(s)f_k'(t) - g_k'(s)f_j'(t)] \quad i \neq j \neq k \quad j < k$$

It remains to compute the 2-form $\chi^* \overset{2}{\omega}$. Let $\xi(s, t) := g(s) - f(t)$, then

$$\omega = \omega'/\| \xi \|^3 \in F^2(\mathbb{R}^3\backslash\{0\} \quad ; \quad \omega' = \Sigma a_{jk} d\xi^j d\xi^k \equiv \Sigma(-1)^{i+1}\xi_i d\xi^j d\xi^k \in F^2(S^2)$$

and hence

$$\chi^*\omega = \frac{1}{\| g(s) - f(t) \|^3}\Sigma(-1)^{i+1}(g_i(s) - f_i(t))\left[\frac{\partial \xi_j}{\partial t}\frac{\partial \xi_k}{\partial s} - \frac{\partial \xi_k}{\partial t}\frac{\partial \xi_j}{\partial s}\right] ds \wedge dt \quad \blacksquare$$

REMARK 1. — Relations (41) and (45) for $\vec{v}$ yield

$$v_1 = \frac{1}{m}\frac{\partial S}{\partial x} \# \int_{S^1} f^*\left(\frac{(y - y')dz - (z - z')dy}{r^3}\right)$$
$$v_2 = \frac{1}{m}\frac{\partial S}{\partial y} \# \int_{S^1} f^*\left(\frac{(z - z')dx - (x - x')dz}{r^3}\right)$$
$$v_3 = \frac{1}{m}\frac{\partial S}{\partial z} \# \int_{S^1} f^*\left(\frac{(x - x')dy - (y - y')dx}{r^3}\right)$$

whenever

$$\vec{x} - \vec{x}' = (x - x', y - y', z - z') \quad , \quad \| \vec{x} - \vec{x}' \|^3 = r^3$$
$$d\vec{x}' = (dx', dy', dz')$$

Discussion. — The topological scattering device of figure 5 is similar to conventional scattering as exhibited in figure 7:

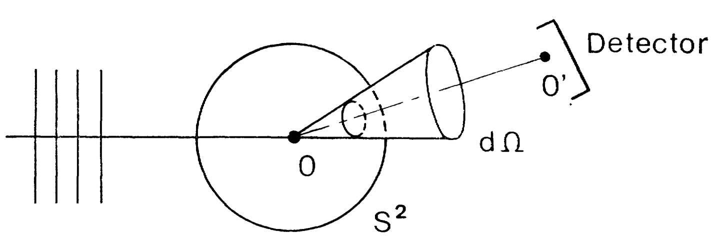


FIG. 7. — Conventional scattering device.

To see this similarity between topological and physical scattering consider a scattering field of type (1), $p = 2$ i. e. $(\overset{2}{\omega}, c_2)$ where $c_2 = S^2(r)$ is the « geometry of collision ». By virtue of eq. (41), section IV, we may write:

$$(41') \qquad j = \Sigma j_k dx^k = -\frac{ie\hbar}{2m}(\bar{\psi} d\psi - \psi d\bar{\psi}) \in F^1(\mathbb{R}^3) \qquad \text{(cf. table A)}$$

$$d\psi,\ d\bar{\psi} \in F^1(\mathbb{R}^3)$$

With respect to the basis $\{ \alpha^1 = dr, \alpha^2 = rd\vartheta, \alpha^3 = r.\ \sin\vartheta . d\varphi \}$ of $(\mathbb{R}^3)^*$ write:

$$j = -\frac{ie\hbar}{2m}\left[\bar{\psi}\left(\frac{\partial\psi}{\partial r}dr + \frac{\partial\psi}{\partial\vartheta}d\vartheta + \frac{\partial\psi}{\partial\varphi}d\varphi\right) - \psi\left(\frac{\partial\bar{\psi}}{\partial r}dr + \frac{\partial\bar{\psi}}{\partial\vartheta}d\vartheta + \frac{\partial\bar{\psi}}{\partial\varphi}d\varphi\right)\right]$$

$$= -\frac{ie\hbar}{2m}\left[\left(\bar{\psi}\frac{\partial\psi}{\partial r} - \psi\frac{\partial\bar{\psi}}{\partial r}\right)dr + \left(\bar{\psi}\frac{\partial\psi}{\partial\vartheta} - \psi\frac{\partial\bar{\psi}}{\partial\vartheta}\right)d\vartheta + \left(\bar{\psi}\frac{\partial\psi}{\partial\varphi} - \psi\frac{\partial\bar{\psi}}{\partial\varphi}\right)d\varphi\right]$$

$$(48) \qquad \Rightarrow j_r = -\frac{ie\hbar}{2m}\left(\bar{\psi}\frac{\partial\psi}{\partial r} - \psi\frac{\partial\bar{\psi}}{\partial r}\right)\alpha^1 \qquad (\vartheta, \varphi : \text{const.})$$

The 1-form (48) relates to the scattering field $(\overset{2}{\omega}, S^2(r))$, since $\overset{2}{\omega} = r^2.$ $\sin\vartheta d\vartheta \wedge d\varphi = r^2 d\Omega$ (cf. eq. (26')) is the solid angle 2-form. In fact, applying the Hodge star operator [3], $*: F^p \dashrightarrow F^{n-p}$, we obtain:

$$(48') \qquad {}^*j_r = -\frac{ie\hbar}{2m}\left[\bar{\psi}\frac{\partial\psi}{\partial r} - \psi\frac{\partial\bar{\psi}}{\partial r}\right]\alpha^2 \wedge \alpha^3 \quad ; \quad \alpha^2 \wedge \alpha^3 = d\Omega$$

i. e.

$$(48'') \qquad {}^*j_r = \frac{k}{r^2}d\Omega \qquad k = |f(\vartheta, \varphi)|^2$$

which is, up to a factor, the number of particles dn scattered into the solid angle $d\Omega$, that is

$$(48''') \qquad dn = kd\Omega \# {}^*j_r \qquad \text{(cf. remark 2 below)}$$

Within an approach to scattering in terms of fields $(\overset{p}{\omega}, c_p)$ (eq. (1)) there is a *Born-field* $(\overset{3}{\text{ß}}, c_3)$, such that

$$(49) \quad \text{ß} : H^3(M) \times H_3(M) \rightarrow \mathbb{R} \ ; \ (\overset{3}{\text{ß}}, c_3) \mapsto -\frac{2m}{\hbar^2}\int_{c_3} V(x, y, z)\psi\alpha^1 \wedge \alpha^2 \wedge \alpha^3$$

is a non-degenerate bilinear map [3]; (H^3, H_3 stand for the 3rd cohomology and homology group of $M: = \mathbb{R}^3 - \{ 0 \}$, respectively). Schrödinger's eq. for Born scattering becomes then a relation between differential 3-forms:

$$(50) \qquad d\sigma + \overset{3}{\text{ß}} = \kappa(*\psi) \quad ; \quad \kappa : = \frac{2mE}{\hbar^2}$$

whenever

$$\overset{3}{\beta} := \frac{2m}{\hbar^2} V\psi\alpha^1 \wedge \alpha^2 \wedge \alpha^3 \in \overset{0}{F}{}^3(M) \quad ; \quad \sigma := * d\psi \in F^2(M)$$

In fact, applying Hodge's star operator to both sides of Schrödinger's eq.

(50′) $-\frac{\hbar^2}{2m}\Delta\psi + V\psi = E\psi$ yields with respect to the basis $\{\alpha^1, \alpha^2, \alpha^3\}$

$$* \left[-\frac{\hbar^2}{2m}\Delta\psi + V\psi - E\psi = 0 \right]$$

$$= -\frac{\hbar^2}{2m}\Delta\psi\alpha^1 \wedge \alpha^2 \wedge \alpha^3 + V\psi\alpha^1 \wedge \alpha^2 \wedge \alpha^3 - E\psi\alpha^1 \wedge \alpha^2 \wedge \alpha^3 = * 0$$

where

$$\Delta\psi\alpha^1 \wedge \alpha^2 \wedge \alpha^3 = d(* d\psi).$$

REMARK 2. — The formula $dn = k . d\Omega$ (48‴) of particles dn scattered into the solid angle $d\Omega$ has its geometrical counterpart. Assume that $c_2 \subset S^2$ (fig. 5) is pierced by n closed vortex lines (cf. fig. 1 *b*) $c_1^i \in \overset{0}{C}_1(M)$, $M = c_2$ $c_1^i = g_i$, g_i: $S^1 \dashrightarrow \mathbb{R}^3$. Then the number $n(g_i)$ is given by

(51) $$n = n^+ - n^- = -\frac{1}{4\pi}\int_{g(S^1)} d\Omega = l(c_1, c_1') \quad \text{(prop. 11 in [2])}\cdot$$

n^+ is the number of intersections $g(S^1) \cap c_2$ (fig. 5) where $\frac{dg}{dt}$ points in the same direction as $w_x \in T_x(\mathbb{R}^3) - T_x(c_2)$, $x = g(t) \in c_2$. n^- is the number of other intersections. Note that the quantized version of (51) is given by $l(c_1, c_1') = n\frac{h}{m}$ (cf. eq. (46)).

V. MAGNETIC MONOPOLE CHARGE AS COUPLING CONSTANT

As proved in [2], proposition 14, the linkage property $l(c_1, c_1')$ between interacting 1-cycles $c_1, c_1' \in \overset{0}{C}_1(M)$ is related to the topological winding number

(52) $$w_0(\zeta) = \frac{1}{4\pi}\int_{\zeta(S^2)} \overset{2}{\omega} \quad ; \quad \overset{2}{\omega} = \sin\vartheta d\vartheta \wedge d\varphi \quad , \quad \zeta \in C^k(S^2, S^2) \; ; [2], [3]$$

which measures how many times S^2 is winding around the origine $0 \in \mathbb{R}^3$. This relationship (cf. proposition 14, p. 424, ref. [2]) is given by

(53) $$l(c_1, c_1') = kw_0(\zeta) \qquad k \in \mathbb{Z}$$

Since eq. (52) is just relationship (33), whenever $\zeta = \widehat{\phi}$, that is $w(\widehat{\phi}) \# \deg(\widehat{\phi})$ (by eq. (36)), so $w(\widehat{\phi}) = \mathrm{M}.e$ by eq. (40). This is consistent with the fact, that eq. (52) is a geometrization of Gauss's law of electrostatics (cf. eq. (2), sect. I). Hence the following interpretation for eq. (53) holds:

The interaction between vortex-line-like particle structures c_1 and c_1' which are associated with potentials $(\overset{1}{\omega}, c_1)$ and $(\overset{1}{\omega}', c_1')$ of the type (1), section I, equals k times the coupling constant M.*e*.

Hence it turns out that (53) is the topological counter-part to the Lagrangian $\mathrm{L} = ej_\mu \mathrm{A}^\mu$ corresponding to the principle of minimal interaction (cf. table A, sect. I).

TO SUMMARIZE. — A set-up of topological fields (1) in conjunction with eq. (53) describes interaction with a coupling constant, provided there are finite vortex-lines (fig. 1 *a*) *terminating at Dirac monopoles* (fig. 8 in [*2*]).

VI. CONCLUSION

A description of an elementary particle approach has been given which consists of the following elements:

1) Non-linear interacting fields or particles are characterized in terms of a « non-linear » configuration space M.

2) Field quantities are non-local.

3) The interaction set-up underlying this particle description is *not* based on the conventional Lagrangian approach to Field theory.

4) Topological « scattering » is similar to Born scattering.

5) Quantal effects are characterized in terms of first quantization only. Effects due to second quantization are « simulated » by the geometrical structure of the configuration space.

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