ANNALES DE L'I. H. P., SECTION A

A. M. ANILE
A. GRECO

Asymptotic waves and critical time in general relativistic magnetohydrodynamics

Annales de l'I. H. P., section A, tome 29, nº 3 (1978), p. 257-272
<http://www.numdam.org/item?id=AIHPA_1978__29_3_257_0>

Ann. Inst. Henri Poincaré,
Vol. XXIX, n° 3, 1978, p. 257-.272

Section A :
Physique théorique.

Asymptotic waves and critical time in General Relativistic Magnetohydrodynamics

by

A. M. ANILE
Seminario Matematico, Università di Catania
and Osservatorio Astrofisico di Catania

and

A. GRECO
Istituto Matematico della Facoltà di Ingegneria
Università di Palermo

SUMMARY. — A general method devised to construct oscillatory approximate solutions is applied to the system of General Relativistic Magnetohydrodynamics (G. R. M. H.), assuming the phase to be a solution of the magnetosonic wave equation.

The growth equation is established and the case of propagation into a constant state is explicitely worked out.

Finally the critical time is evaluated for:

i) flat spacetime in general;
ii) conformally flat spacetimes in the case of a radiative fluid.

RÉSUMÉ. — On applique au système de la Magnétohydrodynamique en Relativité Générale (M. H. R. G.) une méthode générale pour construire des solutions approchées oscillatoires, en prenant la phase solution de l'équation des ondes magnétosoniques.

On établit l'équation qui règle le premier terme de la perturbation et on considère en particulier le cas de propagation dans un état constant.

Enfin le temps critique est évalué dans les deux cas suivants : pour une métrique minkowskienne et pour une métrique conformément plate dans le cas du fluide radiatif.

1. INTRODUCTION

In a recent paper Y. Choquet-Bruhat [*1*] gives a general method for constructing asymptotic solutions to quasi-linear partial differential systems. These solutions are formally given as series expansions around a known solution in terms of a real parameter $\omega \gg 1$, the frequency, and a scalar real function, the phase. For consistency the latter must be a solution of the characteristic equation of the system corresponding to the chosen unperturbed state.

Essentially, let $\underset{(0)}{u}$ be the solution corresponding to the unperturbed state (assumed to be known). One seeks a solution for the perturbed state in the form

$$u = \underset{(0)}{u} + \frac{1}{\omega} \underset{(1)}{u}(x, \omega\varphi) + \frac{1}{\omega^2} \underset{(2)}{u}(x, \omega\varphi) + \dots$$

(where the x's are the independent variables, $\omega\varphi = \xi$ is a numerical parameter and $\underset{(1)}{u}, \underset{(2)}{u}, \dots$ are the perturbation terms to be determined).

Here this method will be applied to the system of perfect (infinite conductivity) general relativistic magnetohydrodynamics (G. R. M. H.) in order to find first-order approximate solutions:

$$u = \underset{(0)}{u} + \frac{1}{\omega} \underset{(1)}{u}(x, \omega\varphi) + O(\omega^{-2})$$

where u is the ten components field vector and $\underset{(0)}{u}$ is an unperturbed state independent of $\xi = \omega\varphi$. φ is chosen to obey the magnetosonic wave equation. In general these waves correspond to simple roots of the characteristic equation and are not exceptional [*2*] for a generic state equation.

Then the first-order perturbation term will be governed by only one function $\Pi = \Pi(x, \xi)$ obeying a non-linear partial differential equation (the growth equation). The non-linearity gives rise to the conspicuous phenomenon of the distortion of signals and to the occurrence of non-linear shocks at the so-called critical time.

The main results obtained in this paper are:

i) an explicit form is given for the growth equation, embodying an earlier result of Lichnerowicz [*3*] for the propagation of weak discontinuities;

ii) in flat spacetime the critical time is evaluated in the case of propagation into a constant state;

iii) a general method for generating solutions to the conservation equations in conformally flat spacetimes is given in the case of traceless energy-tensors. By applying this method to the G. R. M. H. system one can evaluate the critical time for a radiative fluid in a conformally flat spacetime. The important Robertson-Walker models are included as particular cases. It is noteworthy that lately radiative fluids have attracted the interest of many people working in the fields of Cosmology and relativistic astrophysics [4], [5], [6].

The scheme of the paper is the following.

In Section 2 first the basic system for perfect G. R. M. H. is recalled. Then, after choosing the phase to be a solution of the magnetosonic wave equation, the method is carried out up to the determination of the zeroth order approximate waves.

In Section 3 the growth equation for the first perturbation term is explicitely derived.

In Section 4 the case of propagation into a constant state is investigated in some detail.

In Section 5 the critical time for plane waves is evaluated.

In Section 6 the system for a radiative magnetofluid in a conformally flat spacetime is investigated.

It is found that, by a suitable transformation, it can be reduced to an equivalent system in flat spacetime. Finally the results of the previous sections are applied leading to an evaluation of the critical time for a radiative fluid in the Robertson-Walker models.

Notation. — Spacetime is assumed to be a four-dimensional manifold $\mathcal{M}^4$ whose normal hyperbolic metric, ds^2, with signature $+ - - -$, can be expressed in local coordinates in the form $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$; the metric tensor is assumed to be of class C^1, piecewise C^2; the four-velocity is defined as $u^\alpha = \frac{dx^\alpha}{ds}$ which implies $u^\alpha u_\alpha = 1$. ∇_α is the operator of covariant differentiation with respect to the given metric.

∂_α or a comma denotes the operator of ordinary differentiation.

Everywhere the units are such that the velocity of light is unity, $c = 1$.

2. ZEROTH ORDER APPROXIMATION

Let one consider a perfect charged relativistic thermodynamical fluid with constant magnetic permeability μ and infinite conductivity. Following Lichnerowicz [3] the field equations can be written:

$$\begin{aligned} \nabla_\alpha T^{\alpha\beta} &= 0 \\ \nabla_\alpha(u^\alpha b^\beta - u^\beta b^\alpha) &= 0 \\ \nabla_\alpha(ru^\alpha) &= 0 \end{aligned} \tag{1}$$

where $T^{\alpha\beta}$ is the energy tensor:

$$T^{\alpha\beta} = (rf + |\,b\,|^2)u^\alpha u^\beta - \left(p + \frac{1}{2}|\,b\,|^2\right)g^{\alpha\beta} - b^\alpha b^\beta \tag{2}$$

r being the proper material density (particle number), $f = 1 + i$ the index of the fluid, i the specific enthalpy, p the pressure; u^α the unitary 4-velocity ($u^\alpha u_\alpha = 1$) and $| b |^2 = - b^\alpha b_\alpha$ with $b^\alpha = \sqrt{\mu}{}^*F^{\alpha\beta}u_\beta$, ${}^*F^{\alpha\beta}$ being the dual of the electromagnetic tensor $F_{\mu\nu}$; obviously b^α is a spacelike vector ($b^\alpha u_\alpha = 0$). Furthermore the relation

$$df = \frac{1}{r} dp + T dS \tag{3}$$

which comes from thermodynamic principles, is assumed to hold. T and S are the proper temperature and specific entropy of the fluid respectively.

In the following it will be assumed that p and S are the independent thermodynamic variables.

By straightforward calculations the systems (1), taking into account eq. (3) can be transformed into the following equivalent one

$$\lambda u^\alpha \nabla_\alpha u^\beta - b^\alpha \nabla_\alpha b^\beta + (g^{\alpha\beta} - 2u^\alpha u^\beta) b_\rho \nabla_\alpha b^\rho - \frac{1}{\eta}[\eta\gamma^{\alpha\beta} + \gamma | b |^2 u^\alpha u^\beta - b^\alpha b^\beta]\partial_\alpha p = 0 \tag{4}$$

$$b^\alpha \nabla_\alpha u^\beta - u^\alpha \nabla_\alpha b^\beta + \frac{1}{\eta}(\gamma u^\alpha u^\beta - u^\beta b^\alpha)\partial_\alpha p = 0 \tag{5}$$

$$r \nabla_\alpha u^\alpha + r'_p u^\alpha \partial_\alpha p = 0 \tag{6}$$

$$u^\alpha \partial_\alpha S = 0 \tag{7}$$

where $\eta \equiv rf$, $\lambda \equiv \eta + | b |^2$, $\gamma^{\alpha\beta} \equiv g^{\alpha\beta} - u^\alpha u^\beta$ is the projection tensor and $\gamma \equiv f r'_p$ with the prime denoting partial differentiation with respect to the subscripted variable.

Now one looks for the solution $u = (u^\alpha, b^\alpha, p, S)$ as a series expansion of the form

$$u = \sum_{q=0}^{\infty} \omega^{-q} \underset{(q)}{u}(x, \omega\varphi) \tag{8}$$

$\omega \gg 1$ being a real parameter, φ a real scalar function, $\xi = \omega\varphi$ a numerical parameter, and where $\underset{(0)}{u} = (a_0^\alpha, b_0^\alpha, p_0, S_0)$ is solution of (4)-(7) independent of ξ which is assumed to be known.

One says that the series (8) is an asymptotic wave [1] for the system (4)-(7) if, when formally substituted into these equations (in which the coefficients have been developed in Taylor series in a neighbourhood. I of the given solution $\underset{(0)}{u}$, the resulting series $\sum_{q=-1}^{\infty} \omega^{-q} \underset{(q)}{F}(x, \omega\varphi(x))$ has all the coefficients $\underset{(q)}{F}$ identically vanishing with respect to x and $\xi = \omega\varphi$.

One says that the truncated expansion $u = \sum_{q=0}^{m} \omega^{-q} \underset{(q)}{u}(x, \omega\varphi)$ is an approximate wave of order $m - 1$ if the following condition holds:

$$\| \mathrm{M}(u) \| \leqslant \mathrm{C}\omega^{-(m+1)}$$

where $u \to \mathrm{M}(u)$ is the operator defined by eqs. (4-7) (i. e. such that these equations can be written in the form $\mathrm{M}(u) = 0$) and $\| \quad \|$ is a suitable norm [1].

For this to hold it is sufficient that the functions $\underset{(q)}{u}$, $q = 0, 1, \ldots, m$ belong to I, be bounded together with their derivatives with respect to x, ξ, and verify $\underset{(q)}{\mathrm{F}} = 0$ for $q < m$.

In the case under investigation $\underset{(-1)}{\mathrm{F}} = 0$ follows from $\underset{(0)}{\dot{u}} = 0$ (where a dot denotes the derivative with respect to ξ) because $\underset{(0)}{u}$ is a solution of eqs. (4-7) assumed to be independent of ξ.

The condition $\underset{(0)}{\mathrm{F}} = 0$, in its turn, gives

$$(9) \quad \lambda_0 \mathrm{U}_0 \dot{u}_1^\beta - \mathrm{B}\dot{b}_1^\beta + (\varphi^\beta - 2\mathrm{U}_0 u_0^\beta) b_{\rho_0} \dot{b}_1^\rho - \frac{1}{\eta_0}[\eta_0 \varphi^\beta + (\gamma_0 \mid b_0 \mid^2 - \eta_0)\mathrm{U}_0 u_0^\beta - \mathrm{B}_0 b_0^\beta]\dot{p}_1 = 0$$

$$(10) \quad \mathrm{B}_0 \dot{u}_1^\beta - \mathrm{U}_0 \dot{b}_1^\beta + \frac{1}{\eta_0}(\gamma_0 \mathrm{U}_0 b_0^\beta - \mathrm{B}_0 u_0^\beta)\dot{p}_1 = 0$$

$$(11) \quad r_0 \varphi_\alpha \dot{u}_1^\alpha + r'_{p_0} \mathrm{U}_0 \dot{p}_1 = 0$$

$$(12) \quad \mathrm{U}_0 \dot{\mathrm{S}}_1 = 0$$

where $\mathrm{U} \equiv u^\alpha \varphi_\alpha$, $\mathrm{B} \equiv b^\alpha \varphi_\alpha$, $\varphi_\alpha \equiv \partial_\alpha \varphi$.
The suffix 0 denotes that the quantity is evaluated in the unperturbed state. In the sequel we shall omit it whenever it does not lead to any confusion.

In order to have non trivial solutions for $\underset{(1)}{u}$ the phase φ must be a solution of the characteristic equation

$$(13) \quad \Omega \equiv \det \mathscr{A} = 0$$

where

$$(14) \quad \mathscr{A} \equiv \begin{pmatrix} \lambda \mathrm{U}\delta_\alpha^\beta & , & -\mathrm{B}\delta_\alpha^\beta + (\varphi^\beta - 2\mathrm{U}u^\beta)b_\sigma & , & d^\beta & , & 0^\beta \\ \mathrm{B}\delta_\alpha^\beta & , & -\mathrm{U}\delta_\alpha^\beta & , & c^\beta & , & 0^\beta \\ r\varphi_\alpha & , & 0_\alpha & , & \mathrm{U}r'_p & , & 0 \\ 0_\alpha & , & 0_\alpha & , & 0 & , & \mathrm{U} \end{pmatrix}$$

with

$$c^\beta = \frac{1}{\eta}(\gamma U b^\beta - B u^\beta) \quad , \quad d^\beta = -\frac{1}{\eta}[\eta\varphi^\beta - (\eta - \gamma \mid b \mid^2) U u^\beta - B b^\beta]$$

The contravariant index β is at the same time the row index while the covariant index α is also the column index. By expanding det$\mathscr{A}$ one finds

$$\Omega = \frac{1}{\eta}\lambda U^2 A^2 N_4 \tag{15}$$

where $A^2 \equiv \lambda U^2 - B^2$ and

$$\begin{aligned} N_4 &\equiv \eta(\gamma - 1)U^4 + (\eta + \gamma \mid b \mid^2)U^2 G - B^2 G \\ G &\equiv g^{\alpha\beta}\varphi_\alpha\varphi_\beta \end{aligned} \tag{16}$$

Henceforth only the case when φ is a solution of $N_4 = 0$ will be considered. It corresponds to the magnetosonic waves which, in general, are simple roots of $\Omega = 0$.

With this choice it is a well known result [1] that $\underset{(1)}{u}$ can be expressed as follows

$$\underset{(1)}{u} = \Pi(x, \xi) h(x) \tag{17}$$

where h is the right eigenvector of the matrix $\mathscr{A}$ corresponding to the chosen solution φ and Π is an arbitrary differentiable function.

One finds for $h \equiv \{ h^\alpha, h^{3+\alpha}, h^8, h^q \}$

$$\begin{aligned} h^\alpha &= \lambda U^2[Bc^\alpha - Ud^\alpha] + \frac{\lambda}{\eta}U(B^2 - \gamma \mid b \mid^2 U^2)(2Uu^\alpha - \varphi^\alpha) \\ U h^{3+\alpha} &= B h^\alpha + c^\alpha(\lambda U^2 A^2) \\ h^8 &= \lambda U^2 A^2 \\ h^9 &= 0 \end{aligned} \tag{18}$$

One has thus obtained the zeroth order approximate wave $u = \underset{(0)}{u} + \frac{1}{\omega}\underset{(1)}{u}$ with $\underset{(1)}{u}$ given by (17) where the values (18) have been substituted for h.

3. THE GROWTH EQUATION

In order to obtain the 1st order approximate waves $\underset{(1)}{F}$ is required to vanish. This yields

$$\lambda U \dot{u}_2^\beta - B\dot{b}_2^\beta + (\varphi^\beta - 2Uu^\beta) b_\rho \dot{b}_2^\rho - \frac{1}{\eta}[\eta\varphi^\beta + (\gamma \mid b \mid^2 - \eta)Uu^\beta + - Bb^\beta]\dot{p}_2 + \underset{(1)}{g^\beta} = 0 \tag{19}$$

$$\text{(20)} \qquad B\dot{u}_2^\beta - U\dot{b}_2^\beta + \frac{1}{\eta}(\gamma U b^\beta - B u^\beta)\dot{p}_2 + \underset{(1)}{g}^{3+\beta} = 0$$

$$\text{(21)} \qquad r\varphi_\alpha \dot{u}_2^\alpha + r'_p U \dot{p}_2 + \underset{(1)}{g}^8 = 0$$

$$\text{(22)} \qquad U\dot{S}_2 + \underset{(1)}{g}^9 = 0$$

The explicit expressions for $\underset{(1)}{g} \equiv \{ \underset{(1)}{g}^\beta, \underset{(1)}{g}^{3+\beta}, \underset{(1)}{g}^8, \underset{(1)}{g}^9 \}$ are rather complicated and are given elsewhere [7].

The algebraic compatibility conditions for the system (19-22) in $\underset{(2)}{\dot{u}}$ require the orthogonality condition

$$\text{(23)} \qquad \bar{h}.\underset{(1)}{g} = 0$$

where $\bar{h} \equiv \{ \bar{h}_\alpha, \bar{h}_{3+\alpha}, \bar{h}_8, \bar{h}_9 \}$ is the left eigenvector of $\mathscr{A}$ corresponding to the chosen φ. When evaluating $\underset{(1)}{g}$, $\underset{(1)}{u}$ must be expressed as in eq. (17).

One finds for $\bar{h}$

$$\bar{h}_\alpha = \frac{G - 2U^2}{\lambda U} B b_\alpha - U\varphi_\alpha$$

$$\bar{h}_{3+\alpha} = B\varphi_\alpha - (G - 2U^2) b_\alpha$$

$$\bar{h}_8 = \frac{A^2}{r}$$

$$\bar{h}_9 = 0$$

Then, after a long and tedious calculation, one obtain from (23)

$$\text{(24)} \qquad k\Omega^\alpha \partial_\alpha \Pi(x, \xi) + \alpha(x)\Pi(x, \xi)\dot{\Pi}(x, \xi) + \beta(x)\Pi(x, \xi) = 0$$

where

$$\text{(25)} \qquad k\Omega^\alpha = 4\lambda \frac{UA^2}{\eta} N^\alpha$$

$N^\alpha \equiv \frac{1}{4}\frac{\partial N_4}{\partial \varphi_\alpha}$ being the magnetosonic ray direction

$$\text{(26)} \qquad \alpha(x) = \lambda U^4 A^2 \bar{\alpha}$$

$$\bar{\alpha} \equiv [\eta\gamma'_p + (\gamma - 1)(3 - 5\gamma)]U^2 + \left[|b|^2\left(\gamma'_p - 2\gamma\frac{\gamma - 1}{\eta}\right) - 3(\gamma - 1)\right] G$$

In the general case the expression for β is extremely complicated and is given in the Appendix.

Thus one obtains the first order approximate wave $u = \underset{(0)}{u} + \frac{1}{\omega}\underset{(1)}{u} + \frac{1}{\omega^2}\underset{(2)}{u}$

if $\underset{(1)}{u}$ is given by eq. (17) with Π a solution of eq. (24) which is bounded together with its derivatives with respect to x and ξ.

Eq. (24) is the growth equation which governs the intensity Π of the first perturbation term.

Its apparent non-linearity gives rise to the well known phenomena of distorsion of the signals and breaking of the waves. Of course this does not occur when $\alpha = 0$ which corresponds to the exceptional case.

The expression for α given here corresponds exactly to that previously obtained by one of the authors [2] in a study directly concerned with the exceptionality conditions.

4. PROPAGATION INTO A CONSTANT STATE

The growth equation derived in the preceeding section simplifies considerably when the unperturbed state is a constant solution of the system (1).

Before considering the G. R. M. H. system one wants to comment briefly on the general case.

Let one consider a general 1st order quasi-linear hyperbolic system, which we write in the form

$$a_i^{j\lambda}(x, u)\partial_\lambda u^i + b^j(x, u) = 0 \tag{27}$$

For an explanation of the symbols the reader is referred to Choquet-Bruhat's paper [1].

The aim of what follows is to find a rather suggestive expression for the coefficient β of the linear term in the growth equation appropriate for the system (27). It turns out that this is analogous to that of Boillat [8] in the noncovariant formalism.

The general expression for β is [1]

$$\beta(x) = \bar{h}_j \left\{ a_{i0}^{j\lambda}\partial_\lambda h^i + \left(a_{il}^{j\lambda}\partial_\lambda \underset{(0)}{u^i} + b^j{}_l\right)h^l \right\}$$

which, in the case of a constant unperturbed solution, $\underset{(0)}{u^i}$ = constant, reduces to

$$\beta(x) = \widehat{\beta}(x) + \check{\beta}(x) \tag{27 a}$$

where

$$\widehat{\beta}(x) \equiv \bar{h}_j a_{i0}^{j\lambda}\partial_\lambda h^i \quad ; \quad \check{\beta}(x) \equiv \bar{h}_j b^j{}_l h^l \tag{28}$$

Now one has

$$\partial_\lambda h^i = \frac{\partial h^i}{\partial \varphi_\mu}\varphi_{\mu\lambda} \quad ; \quad \varphi_{\mu\lambda} \equiv \partial_\lambda \varphi_\mu$$

because $\underset{(0)}{u^i}$ = const.

It follows that $\widehat{\beta}(x)$ can be written in the form

$$\widehat{\beta}(x) = Q^{\mu\nu}\varphi_{\mu\nu} \tag{29}$$

with

$$Q^{\mu\nu} = \frac{1}{2}\bar{h}_j\left[a^{j\mu}_{i0}\frac{\partial h^i}{\partial\varphi_\nu} + a^{j\nu}_{i0}\frac{\partial h^i}{\partial\varphi_\mu}\right] \tag{30}$$

Moreover it is [1]

$$\bar{h}_j h^j = k A^i{}_j \tag{31}$$

where $A^i{}_j$ is the element belonging to the j-th row and i-th column of the adjoint matrix of $a^{j\lambda}_{i0}\varphi_\lambda$ and k is a normalization factor.

By contracting eq. (31) with h^j one obtains

$$h^i = \frac{k}{N}A^i{}_j h^j \tag{32}$$

where $N = \bar{h}_j h^j \neq 0$ because the system (27) is hyperbolic.

From (32) it follows $a^{j\lambda}_{i0}h^i = \frac{k}{N}a^{j\lambda}_{i0}A^i{}_q h^q$, but $a^{j\lambda}_{i0}A^i{}_r = \frac{\partial\Omega}{\partial\varphi_\lambda}\delta^j{}_r$, where $\Omega = \det\left(a^{j\lambda}_{i0}\varphi_\lambda\right)$, hence

$$a^{j\lambda}_{i0}h^i = \frac{k}{N}\frac{\partial\Omega}{\partial\varphi_\lambda}h^j \tag{33}$$

By differentiating (33) one gets

$$2Q^{\lambda\mu} = N\left\{\frac{\partial}{\partial\varphi_\mu}\left(\frac{k}{N}\frac{\partial\Omega}{\partial\varphi_\lambda}\right) + \frac{\partial}{\partial\varphi_\lambda}\left(\frac{k}{N}\frac{\partial\Omega}{\partial\varphi_\mu}\right) + \frac{k}{N}\bar{h}_j\left(\frac{\partial\Omega}{\partial\varphi_\lambda}\frac{\partial h^j}{\partial\varphi_\mu} + \frac{\partial\Omega}{\partial\varphi_\mu}\frac{\partial h^j}{\partial\varphi_\lambda}\right)\right\} \tag{34}$$

The rays system associated with the characteristic equation is

$$\frac{dx^\alpha}{d\sigma} = \frac{\partial\Omega}{\partial\varphi_\alpha} \quad ; \quad \frac{d\varphi_\alpha}{d\sigma} = -\frac{\partial\Omega}{\partial x^\alpha} \tag{35}$$

For a constant unperturbed state one has $\frac{\partial\Omega}{\partial x^\alpha} = 0$ whence it follows

$$\varphi_{\alpha\lambda}\frac{\partial\Omega}{\partial\varphi_\alpha} = \varphi_{\alpha\lambda}\frac{dx^\alpha}{d\sigma} = \frac{d\varphi_\lambda}{d\sigma} = 0 \tag{36}$$

Finally, substituting (34) into (29), with the help of (36) yields

$$\widehat{\beta} = 2k\,\tilde{\Box}\varphi \tag{37}$$

where

$$\tilde{\Box}\varphi = \frac{1}{2}\frac{\partial^2\Omega}{\partial\varphi_\mu\partial\varphi_\nu}\varphi_{\mu\nu} \tag{38}$$

Now let one return to the system (4-7) in the case of propagation into a constant state.

One finds

$$\beta = \widehat{\beta} = \frac{2\lambda \mathrm{U}\mathrm{A}^2}{\eta} \underset{(4)}{\Box}\varphi$$

with

$$\underset{(4)}{\Box}\varphi \equiv \frac{1}{2} \frac{\partial^2 \mathrm{N}_4}{\partial\varphi_\mu \partial\varphi_\nu} \varphi_{\mu\nu}$$

Therefore the growth equation, with the positions of Section 3, writes

$$\mathrm{N}^\mu \partial_\mu \Pi + (\lambda \mathrm{U}^4 \mathrm{A}^2)\overline{\alpha} \Pi \dot{\Pi} + \frac{1}{2} \underset{(4)}{\Box}\varphi = 0 \tag{39}$$

5. PLANE WAVES

It is convenient to introduce a minkowskian comoving frame,

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}\,(1, -1, -1, -1)\,; \qquad u^\alpha = \delta^\alpha_0, \qquad \mathrm{U} = \varphi_0.$$

Without loss of generality the spatial coordinate x^1 can be chosen along the unperturbed magnetic field so that $b^\alpha = (0, b, 0, 0)$ and $\mathrm{B} = b\varphi_1$.

One looks for solutions of $\mathrm{N}_4 = 0$ in the form of plane waves $\varphi = l_\alpha x^\alpha$, $l_\alpha = \varphi_\alpha =$ constant. Then one obtains

$$\gamma\lambda(l_0)^4 - [|\,b\,|^2\,|\,l\,|^2 \cos^2\theta + (\eta + \gamma\,|\,b\,|^2)\,|\,l\,|^2]\,(l_0)^2 + |\,b\,|^2\,|\,l\,|^4 \cos^2\theta = 0 \tag{40}$$

where $|\,l\,|^2 = \sum_{i=1}^{3} (l_i)^2$ and θ is the angle between the magnetic field b^α and the spatial direction defined by l^α, $\mathrm{B} = |\,b\,|\,|\,l\,| \cos\theta$.

It is convenient to introduce the following quantities

$$\begin{aligned} \Delta &= \eta + (\gamma + \cos^2\theta)\,|\,b\,|^2 \\ \mathrm{F} &= \{\,\Delta^2 - 4\gamma\lambda\,|\,b\,|^2 \cos^2\theta\,\}^{1/2} \\ g_\pm &= \frac{1}{2\gamma\lambda}\{\,\Delta \pm \mathrm{F}\,\} \end{aligned}$$

Then the solutions to eq. (40) write

$$(l_0)^2 = |\,l\,|^2 g_\pm \tag{41}$$

where the double sign refers to the fast and slow magnetosonic waves respectively.

A straightforward calculation gives for the magnetosonic ray direction

$$\mathrm{N}^0 = \pm\frac{1}{2} l_0\,|\,l\,|^2 \mathrm{F} \quad ; \quad \mathrm{N}^i = \frac{1}{2}(l_0)^2 \Delta l_i \tag{42}$$

and for $\bar{\alpha}$

(43) $$\bar{\alpha} = |l|^2\Psi_\pm$$

with

(44) $$\Psi_\pm = g_\pm\left[\lambda\gamma'_p - \gamma(\gamma-1)\left(5+2\frac{|b|^2}{\eta}\right)\right] - \left[|b|^2\left(\gamma'_p - 2\gamma\frac{\gamma-1}{3}\right) - 3(\gamma-1)\right]$$

Also, $\beta = 0$, because for plane waves $\varphi_{\mu\nu} = 0$.

Eq. (41) gives two solutions for the ratio for each mode, corresponding to forward and backward propagation.

In the case of forward propagation one has $l_0 = |l|g_\pm$ and the growth equation writes

(45) $$\pm F\partial_0\Pi + (\sqrt{g_\pm})\Delta n_i\partial_i\Pi + 2\lambda l_0 |l|^4(g_\pm)^2A^2\Psi_\pm\Pi\dot{\Pi} = 0$$

where n_i is the spatial unit vector in the direction of l^α

i. e. $$l_i = |l|n_i \quad , \quad \sum_{i=1}^{3}(n_i)^2 = 1.$$

The bicharacteristics of eq. (45) are defined by

(46) $$\frac{dt}{d\sigma} = \pm F \quad ; \quad \frac{dx^i}{d\sigma} = \sqrt{g_\pm}\Delta n_i$$
$$\frac{d\xi}{d\sigma} = 2\lambda l_0 |l|^4(g_\pm)^2A^2\Psi_\pm\Pi \quad ; \quad \frac{d\Pi}{d\sigma} = 0$$

By integration of (46.3) one gets

(47) $$\xi = \mu + \{ 2 W\lambda l_0 |l|^4(g_\pm)^2A^2\Psi_\pm \}\sigma$$

where μ and W are the initial values of ξ and Π respectively.

Let one assume that the initial perturbation has a sinusoidal profile

$$W = W_0 \sin\mu, \qquad W_0 = \text{constant}.$$

It is physically suggestive to express W_0 in terms of the initial value of the relative pressure perturbation δ,

$$\delta \equiv \left(\frac{p_1}{p_0}\right)_{t=0}$$

Then, by using (46.1), eq. (47) writes

(48) $$\xi = \mu \mp \left(2p_0\frac{\delta|l|}{F}\sqrt{g_\pm}\Psi_\pm\right)t \sin\mu$$

From eq. (48) one obtains both the signal distortion and the critical time.

In order to obtain the signal distortion one proceeds as follows.

Eq. (46.4) yields

$$\Pi = W = W_0 \sin \mu, \qquad W_0 = \text{const.} \tag{49}$$

where of course μ must be expressed in function of ξ.

If ones writes

$$e = \left(2p_0 \frac{\delta \mid l \mid}{F} \sqrt{g_\pm} \Psi_\pm \right) t \tag{50}$$

eq. (48) reads

$$\xi = \mu \mp e \sin \mu$$

when $\mid l \mid < 1$ the above relationship can be inverted for μ and yields

$$\mu = \xi \pm \sum_{q=1}^{\infty} \frac{2J_q(qe)}{q} \sin q\xi$$

where J_q is the Bessel function of order q.

Therefore one sees that an initially sinusoidal profile is subsequently distorted by creation of the higher order harmonics. Explicitely one gets

$$\Pi = W_0 \frac{1}{e} \sum_{q=1}^{\infty} \frac{2J_q(qe)}{q} \sin q\xi$$

It is easily seen that the critical time $t_c^\pm$ corresponds to the value for which $\mid e \mid = 1$. Therefore

$$t_c^\pm = \left| \left(\frac{F}{p_0}\right) \frac{1}{\delta} \frac{1}{2 \mid l \mid \sqrt{g_\pm} \Psi_\pm} \right| \tag{50 a}$$

In many situations encountered in Relativistic Astrophysics and Cosmology one deals with a barotropic fluid [*6*]. This corresponds to a fluid where the rest-mass energy is negligible compared to the internal energy, i. e. the fluid is in the so-called ultrarelativistic regime [*9*].

A barotropic fluid is defined by the following relationships:

$$S = \text{constant}, \qquad \rho = \gamma p, \qquad \gamma = \text{constant}.$$

In this case the critical time is obtained from (50) by inserting the following expression for $\Psi_\pm$

$$\Psi_\pm = (\gamma - 1) [3 + 2\gamma Y - \gamma(5 + 2Y) g_\pm] \tag{51}$$

where $Y = \dfrac{b^2}{\eta}$ is an adimensional parameter measuring the ratio of the magnetic energy to the fluid's energy.

For the sake of physical interpretation it is helpful to consider separately the cases of propagation along and normal to the magnetic field.

In the former case $\theta = 0$ and one finds

$$t_c^+ = \frac{[(1+\gamma)\gamma]^{1/2}}{(\gamma - 1)} \frac{1}{2\,|\,l\,|\,\delta} \left[\frac{(1+Y)^3}{Y}\right]^{1/2} \left| \frac{1 - Y(\gamma - 1)}{3 - (2 + 3\gamma)Y - 2Y^2} \right|$$

while t_c^- goes to infinity.

In the latter case $\theta = \frac{\pi}{2}$ and

$$t_c^+ = \frac{(1+\gamma)^{1/2}}{\gamma - 1} \frac{1}{2\,|\,l\,|\,\delta} [(1+Y)^3(1+\gamma Y)]^{1/2} \frac{1}{|\,(1 - 3\gamma)Y - 2\,|}$$

while t_c^- goes to infinity. This is consistent with the well known results that the slow waves degenerate into Alfven waves and material waves in the cases of propagation along and normal to the magnetic field. Of course in these cases the slow waves do no longer correspond to simple roots of the characteristic equation and therefore they must be handled separately.

6. CONFORMALLY FLAT SPACETIMES

The barotropic fluids of interest in relativistic astrophysics and cosmology have $\gamma = 1$ [*10*] or $\gamma = 3$ [*4*]. In the first case the system is completely exceptional [*2*], in agreement with our previous results which show that t_c goes to infinity for $\gamma = 1$.

The case $\gamma = 3$ corresponds to a radiative fluid.

Astrophysically it occurs in the latest stages of the gravitational collapse of a star to a black hole (where the radiative energy density and pressure are dominating) and in the radiative period of cosmic evolution.

Because of its astrophysical importance and analytical simplicity only the case $\gamma = 3$ will be treated here. It is easily seen that, for $\rho = 3p$, the energy tensor of G. R. M. H. is traceless, $T^\alpha{}_\alpha = 0$.

At this stage it is useful to employ the following Lemma, which holds for any symmetric traceless conservative energy tensor.

LMMEA. — Let $(\mathcal{M}, g_{\alpha\beta})$ be a conformally flat manifold

(52) $$g_{\alpha\beta} = e^{\Psi}\eta_{\alpha\beta}$$

and $T^{\alpha\beta} = T^{\beta\alpha}$ a traceless tensor.

(53) $$T^\alpha{}_\alpha = 0$$

Then

$$\nabla_\alpha T^{\alpha\beta} = 0 \Leftrightarrow \partial_\alpha(\tilde{T}^{\alpha\beta}) = 0$$

with

$$\tilde{T}^{\alpha\beta} = e^{3\Psi}T^{\alpha\beta}$$

Proof.

$$\nabla_\alpha T^{\alpha\beta} = \partial_\alpha T^{\alpha\beta} + \Gamma^\alpha{}_{\alpha\rho} T^{\rho\beta} + \Gamma^\beta{}_{\alpha\rho} T^{\alpha\rho}.$$

From (52) one has

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}[\delta^\alpha{}_\beta \Psi_{,\gamma} + \delta^\alpha{}_\gamma \Psi_{,\beta} - \eta_{\beta\gamma}\eta^{\alpha\mu}\Psi_{,\mu}]$$

$$\Gamma^\alpha{}_{\alpha\rho} = 2\Psi_{,\rho}$$

Hence, by using (53), it follows

$$\partial_\alpha T^{\alpha\beta} + 3\Psi_{,\alpha} T^{\alpha\beta} = 0\cdot$$

From the above Lemma it follows that, in our case, the conservation equations in a conformally flat metric (52) write

$$(54)\quad \partial_\alpha \tilde{T}^{\alpha\beta} = 0 \quad , \quad \tilde{T}^{\alpha\beta} = (\rho + \tilde{p} + |\tilde{b}|^2)\tilde{u}^\alpha \tilde{u}^\beta - \left(\tilde{p} + \frac{1}{2}|\tilde{b}|^2\right)\eta^{\alpha\beta} - \tilde{b}^\alpha \tilde{b}^\beta$$

where

$$(54')\quad \tilde{u}^\alpha = e^{\Psi/2}u^\alpha \quad , \quad \tilde{b}^\alpha = e^{3\Psi/2}b^\alpha \quad , \quad \tilde{\rho} = e^{2\Psi}\rho \quad , \quad \tilde{p} = e^{2\Psi}p.$$

Since the fluid is assumed to be barotropic, S = const. everywhere and the only remaining equations to be considered are Maxwell's:

$$\nabla_\alpha(u^\alpha b^\beta - b^\beta u^\alpha) = 0$$

A straightforward calculation shows that the above equations reduce to

$$(55)\qquad \partial_\alpha(\tilde{u}^\alpha \tilde{b}^\beta - \tilde{u}^\beta \tilde{b}^\alpha) = 0$$

under the transformation (54′).

Therefore, in the case of a radiative fluid, one has a method of generating exact solution of G. R. M. H. in conformally flat spacetimes, starting from exact solution of G. R. M. H. in Minkowski's spacetime.

It follows that, when $\gamma = 3$, the results of the preceeding section on the signal distortion and the critical time can be transferred to the case of an arbitrary conformally flat spacetime by the transformation (54′).

It is well known [*11*] that all the Robertson-Walker models are conformally flat. Also, a suitable model for the universe in the radiative period is provided by the Robertson-Walker spacetime with vanishing spatial curvature. In this case one has for the proper time

$$d\tau^2 = e^\Psi[dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2]$$

where

$$\Psi = 2 \ln a$$

a being the expansion factor of the universe [*10*].

It follows

$$\tau = \int_0^t a(q)dq$$

which gives the relationship between the proper time τ and the conformal coordinate time t.

The results of the preceeding section are then seen to hold also in this case provided that t is interpreted as the conformal coordinate time. In particular the critical time $\tau_c^{\pm}$ is simply given by

$$\tau_c^{\pm} = \int_0^{t_c^{\pm}} a(q)dq$$

where $t_c^{\pm}$ is any of the critical times previously evaluated, computed for $\gamma = 3$.

Detailed applications to more general situations of astrophysical and cosmological importance are under current investigation and will be published elsewhere.

APPENDIX

We give here the explicit expression for the coefficient β of the linear term in the growth equation (24), in the general case

$$\begin{aligned}
\beta = {} & [\mathrm{B}\varphi_\beta - (\mathrm{G} - 2\mathrm{U}^2)b_\beta] \Big\{ b^\alpha \nabla_\alpha \Omega^\beta - u^\alpha \nabla_\alpha \mathrm{B}^\beta + \frac{1}{\eta}(\gamma u^\alpha b^\beta - u^\beta b^\alpha)\nabla_\alpha(\lambda \mathrm{U}^2\mathrm{A}^2) \\
& + \mathrm{B}^\alpha \nabla_\alpha u^\beta - \Omega^\alpha \nabla_\alpha b^\beta + \frac{1}{\eta}[u^\alpha b^\beta \nu \lambda \mathrm{U}^2\mathrm{A}^2 + \gamma \mathrm{B}^\beta u^\alpha + \gamma b^\beta \Omega^\alpha \\
& - u^\beta \mathrm{B}^\alpha - b^\alpha \Omega^\beta - \frac{\gamma+1}{\eta}(\gamma u^\alpha b^\beta - u^\beta b^\alpha)\lambda \mathrm{U}^2\mathrm{A}^2]\partial_\alpha p \Big\} \\
& + \left(\frac{\mathrm{G} - 2\mathrm{U}^2}{\lambda \mathrm{U}}\mathrm{B}b_\beta - \mathrm{U}\varphi_\beta\right)\{ \lambda u^\alpha \nabla_\alpha \Omega^\beta - b^\alpha \nabla_\alpha \mathrm{B}^\beta + (g^{\alpha\beta} - 2u^\alpha u^\beta)b_\rho \nabla^\alpha \mathrm{B}^\rho \\
& - \frac{1}{\eta}(\eta\gamma^{\alpha\beta} + \gamma\,|\,b\,|^2 u^\alpha u^\beta - b^\alpha b^\beta)\partial_\alpha(\lambda \mathrm{U}^2\mathrm{A}^2) + (\lambda \Omega^\alpha + u^\alpha \mu \lambda \mathrm{U}^2\mathrm{A}^2 \\
& - 2u^\alpha b_\rho \mathrm{B}^\rho)\nabla_\alpha u^\beta - \mathrm{B}^\alpha \nabla_\alpha b^\beta + [(g^{\alpha\beta} - 2u^\alpha u^\beta)\mathrm{B}_\rho - 2b_\rho(u^\alpha \Omega^\beta + u^\beta \Omega^\alpha)]\nabla_\alpha b^\rho \\
& - \frac{1}{\eta}[(\gamma+1)\lambda \mathrm{U}^2\mathrm{A}^2\gamma^{\alpha\beta} + (\gamma\,|\,b\,|^2 - \eta)(u^\alpha \Omega^\beta + u^\beta \Omega^\alpha) + |\,b\,|^2 u^\alpha u^\beta \nu \lambda \mathrm{U}^2\mathrm{A}^2 \\
& - 2\gamma u^\alpha u^\beta b_\sigma \mathrm{B}^\sigma - b^\alpha \mathrm{B}^\beta - b^\beta \mathrm{B}^\alpha - \left(\frac{\gamma+1}{\eta}\right)\lambda \mathrm{U}^2\mathrm{A}^2(\eta\gamma^{\alpha\beta} + \gamma\,|\,b\,|^2 u^\alpha u^\beta - b^\alpha b^\beta)]\partial_\alpha p \Big\} \\
& + \frac{\mathrm{A}^2}{r}\Big\{ r\nabla_\alpha \Omega_\alpha + r'_p u^\alpha \partial_\alpha(\lambda \mathrm{U}^2\mathrm{A}^2) + r'_p \lambda \mathrm{U}^2\mathrm{A}^2 \nabla_\alpha u^\alpha + \left(r'_p \Omega^\alpha + r''_{p^2}\lambda \mathrm{U}^2\mathrm{A}^2 u^\alpha\right)\partial_\alpha p \Big\} \\
& + \left[(r\mathrm{T} + c^2 f r'_s)\mathrm{U}^2 - \frac{r'_s}{r}\,|\,b\,|^2\mathrm{G}\right]\Omega^\alpha \partial_\alpha \mathrm{S}
\end{aligned}$$

where

$$\begin{aligned}
\nu & \equiv r'_p f'_p + f r''_p \\
\mathrm{B}^\beta & \equiv \frac{\lambda}{\eta}\{ \mathrm{B}[\mathrm{B}^2 - (\eta + \gamma\,|\,b\,|^2)\mathrm{U}^2](\mathrm{U}u^\beta - \varphi^\beta) - \mathrm{UBA}^2 u^\beta + \mathrm{U}^2(\lambda\gamma \mathrm{U}^2 - \mathrm{B}^2)b^\beta \} \\
\Omega^\beta & \equiv \frac{\lambda}{\eta}\{ \mathrm{U}[\mathrm{B}^2 - (\eta + \gamma\,|\,b\,|^2)\mathrm{U}^2](\mathrm{U}u^\beta - \varphi^\beta) + (\gamma - 1)\mathrm{U}^3\mathrm{B}b^\beta \}.
\end{aligned}$$

REFERENCES

[1] Y. CHOQUET-BRUHAT, *J. Math. pures et appl.*, t. **48**, 1969, p. 117.
[2] A. GRECO, Rend. Acc. Naz. Lincei. *Scienze fisiche*, vol. LII, 1972, p. 507.
[3] A. LICHNEROWICZ, Relativistic Fluid Dynamics. *C. I. M. E.*, Bressanone 7-16 giugno 1970, Ed. Cremonese, Roma 1971, p. 87.
[4] S. WEINBERG, *Astrophys. J.*, t. **168**, 1971, p. 175.
[5] J. L. ANDERSON, *Gen. Rel. Grav.*, t. **7**, 1976, p. 53.
[6] E. P. T. LIANG, *Astrophys. J.*, t. **211**, 1977, p. 361.
[7] A. M. ANILE and A. GRECO, *Internal Report*, University of Palermo, 1977.
[8] G. BOILLAT, *La propagation des ondes*, Gauthier-Villars, Paris, 1965.
[9] S. WEINBERG, *Gravitation and Cosmology*, J. Wiley and Sons, New York, 1972, p. 51.
[10] Ya B. ZELDOVICH and I. NOVIKOV, *Relativistic Astrophysics*, The University of Chicago press, Chicago, 1971, p. 200.
[11] S. W. HAWKING and G. F. R. ELLIS, *The large scale structure of spacetime*, Cambridge University Press, 1973, p. 135.

(Manuscrit reçu le 8 juin 1978)