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Section A :
Physique théorique.

Bound states in completely integrable systems with two types of particles

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ABSTRACT. — Connection between the trajectories in the completely integrable systems with two types of particles and the geodesics on the Hermitian matrices space is established. Thereby the problem of the existence of the bound states is clarified.

1. We consider dynamical system with the Hamiltonian

$$H = \frac{1}{2} p^2 + U(q), \tag{1}$$

where $p = (p_1, \dots p_n)$, $q = (q_1, \dots q_n)$,

$$p^2 = \sum_{j=1}^{n} p_j^2 ,$$

$$U(q) = g^2 \sum_{\substack{1 \leqslant j < i \leqslant n_1 \\ n_1 < j < i \leqslant n}} a^2 \sinh^{-2} [a(q_i - q_j)] - g^2 \sum_{\substack{1 \leqslant i \leqslant n_1 \\ n_1 < j \leqslant n}} a^2 \cosh^{-2} [a(q_i - q_j)] . \tag{2}$$

We shall take that

$$\sum_{j=1}^{n} p_j = 0\,, \qquad \sum_{j=1}^{n} q_j = 0\,. \tag{3}$$

This Hamiltonian describes the interaction of n particles set on a straight line. The system contains n_1 particles of the same sign and $n_2 = n - n_1$ particles of the opposite sign. Each pair of particles with opposite sign is attracted with the potential $-\, g^2a^2 \cosh^{-2}[a(q_i - q_j)]$. At the same time particles with the same sign are repulsed with the potential

$$g^2a^2 \sinh^{-2}[a(q_i - q_j)]\,.$$

This system is completely integrable [1]. The proof follows from the completely integrability of a simpler system with the potential

$$\mathrm{U}(q) = \sum_{1 \leqslant i < j \leqslant n} a^2 \sinh^{-2}[a(q_i - q_j)]\,. \tag{4}$$

The replacement

$$q_j \to q_j + i\frac{\pi}{2a} \qquad n_1 < j \leqslant n \tag{5}$$

transforms potential (4) to potential (2).

The present paper is analogous to paper [2], where systems with potential (4) were considered. In the work mentioned the problem of Hamiltonian's equations integration is reduced to the resolving of algebraic equation of the nth degree. For this purpose free motion along geodesics x_t in the space of Hermitian positively defined matrices $n \times n$ is considered. Then the logarithms of the matrices x_t eigenvalues determine the coordinates of the particles moving in potential (4). Here we prove that for the system with two types of particles the geodesic flow in the space of the Hermitian matrices X_{n_1,n_2} with the signature (n_1, n_2) should be considered. In particular from the forms of geodesics it follows that in systems with the potential (2) in contrast to those with potential (4) the bound states do exist. But they aren't stable in respect to the initial data with the exception of case.

It should be emphasized that for the simplification of the problem we expand the configuration space with the dimension $n - 1$ (see (3)) to the space X_{n_1,n_2} ($\dim \mathrm{X}_{n_1,n_2} = n^2 - 1$). The inverse transition from the expanded phase space to the reduced phase space in dynamical systems with the symmetries is presented in the most general form in [3]. Besides the above mentioned work [2] (see also [4]), the idea of the phase space expansion in the problem of Hamiltonian systems integration has been used in the recent work [5].

2. We shall demonstrate the situation in the simplest case $n_1 = n_2 = 1$. Then from (2) we obtain

$$H = \frac{1}{2} p^2 - g^2 \cosh^{-2} q \qquad (a = 1). \tag{6}$$

Consider an unparted hyperboloid $\mathbb{H}^2 : x_1^2 + x_2^2 - x_3^2 = 1$ (see fig. 1).

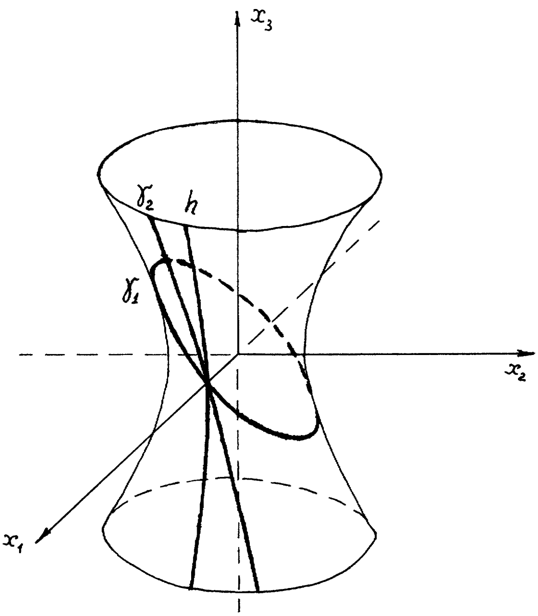

FIG. 1.

Though by analogy with the general case a three-dimensional hyperboloid should have been considered, for simplification we consider $\mathbb{H}^2$ on which the Hamiltonian (6) is also realized.

There is a metric on $\mathbb{H}^2$ invariant relative to its group of motion SO(2, 1). The geodesics of this metric are intersections of $\mathbb{H}^2$ and the planes $\{ A_1x_1 + A_2x_2 + A_3x_3 = 0 \}$. Let's project the motion along the geodesics on the geodesic $h = \mathbb{H}^2 \cap \{ x_2 = 0 \}$ by the horizontal section $\mathbb{H}^2 \cap \{ x_3 = c \}$. Then it is easy to see that we have the Hamiltonian (6), where $\cosh^2 q = x_1^2 + x_2^2$. There are two types of geodesics: closed (γ_1) and unclosed (γ_2). Finite motions of particle on h correspond to closed geodesics and infinite motions correspond to unclosed geodesics.

Only in the case $n_1 = n_2 = 1$ the set of the initial conditions corresponding to the bound states has a complete measure.

3. Now we'll describe the space of the Hermitian matrices X_{n_1,n_2} with

the signature $(n_1, n_2)(n_1 + n_2 = n)$. Let's agree that $n_1 \geqslant n_2$. Each matrix $x \in X_{n_1,n_2}$ can be represented as

$$x = g\sigma g^+ \tag{7}$$

where $g \in SL(n, \mathbb{C})$,

$$\sigma = \begin{pmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{pmatrix} \tag{8}$$

I_{n_1}, I_{n_2} are unite matrices of order n_1 and n_2. Representation (7) isn't single for it is evident that matrices $v \in SU(n_1, n_2) \subset SL(n, \mathbb{C})$ conserve σ:

$$\sigma = v\sigma v^+ . \tag{9}$$

Thus the space under consideration can be identified with homogeneous space $SL(n, \mathbb{C})/SU(n_1, n_2)$. It is a symmetric pseudo-Riemann space. The exact definition and a complete theory of these spaces are represented in [6]. For the complete list of the irreducible symmetric pseudo-Riemann spaces see [7].

Let $\mathfrak{A}$ be a pseudo-Hermitian matrix

$$\mathfrak{A} = \sigma\mathfrak{A}^+\sigma \tag{10}$$

with a zero trace. Then it has a form:

$$\mathfrak{A} = \begin{pmatrix} A & C \\ -C^+ & B \end{pmatrix} \begin{matrix} \}\, n_1 \\ \}\, n_2 \end{matrix} \tag{11}$$

and $A = A^+$, $B = B^+$, $\mathrm{Sp}\, A + \mathrm{Sp}\, B = 0$. The orbit of the one parametric subgroup $\exp\{\mathfrak{A}t\}$ passing through the point $\sigma \in X_{n_1,n_2}$ is the geodesic on X_{n_1,n_2}. Each geodesic X_t on X_{n_1,n_2} can be obtained from the previous by the transformation:

$$x_t = b \exp\{\mathfrak{A}t\}\, \sigma\, (\exp\{\mathfrak{A}t\})^+ b^+ \tag{12}$$

$$b \in SL(n, \mathbb{C}),$$

or in view of (10)

$$x_t = b \exp\{2\mathfrak{A}t\}\, \sigma b^+ \tag{13}$$

It is easy to see that x_t satisfies the equation

$$\frac{d}{dt}(x_t^{-1} \cdot \dot{x}_t + \dot{x}_t x_t^{-1}) = 0 \tag{14}$$

The matrix $\mathfrak{A}$ can be reduced to the canonical form by the transformation from the stationary subgroup $SU(n_1, n_2)$

$$\mathfrak{K} = v\mathfrak{A}v^{-1} \qquad v \in SU(n_1, n_2), \tag{15}$$

where

$$\mathfrak{K} = \begin{pmatrix} \alpha_1 & & & & & & & & \\ & \ddots & & & & & 0 & & \\ & & \alpha_r & & & & & & \\ & & & \beta_1 & & & i\varphi_1 & & \\ & & & & \ddots & & & \ddots & \\ & 0 & & & & \beta_k & & & i\varphi_k \\ & & & i\varphi_1 & & & \beta_1 & & \\ & & & & \ddots & & & \ddots & \\ & & & & & i\varphi_k & & & \beta_k \end{pmatrix} \tag{16}$$

$$n_1 - n_2 \leqslant r \leqslant n\,, \quad 0 \leqslant k \leqslant n_2\,, \quad r + 2k = n\,, \quad \sum_{j=1}^{r} \alpha_j + 2\sum_{j=1}^{k} \beta_j = 0 \tag{17}$$

Correspondingly the matrix $\exp\{2\mathfrak{K}t\}$ has the form:

$$\exp\{2\mathfrak{K}t\} = \begin{pmatrix} e^{2\alpha_1 t} & & & & & & & & \\ & \ddots & & & & & 0 & & \\ & & e^{2\alpha_r t} & & & & & & \\ & & & e^{2\beta_1 t}\cos 2\varphi_1 t & & & ie^{2\beta_1 t}\sin 2\varphi_1 t & & \\ & & & & \ddots & & & \ddots & \\ & & & & & e^{2\beta_k t}\cos 2\varphi_k t & & & ie^{2\beta_k t}\sin 2\varphi_k t \\ & 0 & & ie^{2\beta_1 t}\sin 2\varphi_1 t & & & e^{2\beta_1 t}\cos 2\varphi_1 t & & \\ & & & & \ddots & & & \ddots & \\ & & & & & ie^{2\beta_k t}\sin 2\varphi_k t & & & e^{2\beta_k t}\cos 2\varphi_k t \end{pmatrix} \tag{18}$$

It follows from the form $\mathfrak{K}$ (16) (and $\exp\{2\mathfrak{K}t\}$ (18)) that all the matrices fall into $n_2 + 1$ classes relative to the transformation (15). Thereby we have $n_2 + 1$ classes of the geodesics x_t (13). Each class is characterized by the k « compact » parameteres φ_j and $r + k$ « noncompact » parameters α_j and β_j (see (17)). Observe now, that geodesic x_t is contained in a bounded part of the space X_{n_1,n_2} iff $\alpha_1 = \ldots \alpha_r = \beta_1 = \ldots = \beta_k = 0$. The geodesic in the general case being defined by $n - 1$ parameteres, such « bounded » geodesic is exclusive. It depends only on $k \leqslant n_2$ parameteres.

It should be noted that length s_t along geodesic x_t (13) is defined as:

$$s_t = (\mathrm{sp}\, \mathfrak{A}^2)^{1/2} t = (\mathrm{sp}\, \mathfrak{K}^2)^{1/2} t\,. \tag{19}$$

From (16) we obtain:

$$s_t = \left(\sum_{j=1}^{r} \alpha_j^2 + \sum_{j=1}^{k} 2(\beta_j^2 - \varphi_j^2) \right)^{1/2} t\,. \tag{20}$$

Thereby three cases are possible: 1) $s_t^2 > 0$; 2) $s_t^2 = 0$ (geodesic belongs to isotropic cone), 3) $s_t^2 < 0$.

Now let $x \in X_{n_1,n_2}$. Then by unitary transformation it can be reduced to the diagonal form

$$x = u \exp\{2aq\}\sigma u^+ \tag{21}$$

where $u \in SU(n) \subset SL(n, \mathbb{C})$, $q = \operatorname{diag}(q_1, \dots q_n)$ and a is a parameter.

4. Let x_t be an arbitrary curve in the space X_{n_1,n_2}. We can write the decomposition (21):

$$x_t = u_t \exp\{2aq(t)\}\sigma u_t^+, \qquad u_t \in SU(n). \tag{22}$$

Designate $M = u_t^{-1}\dot{u}_t$ the element of Lie algebra of the group SU(n). Our purpose is to prove the following proposition.

Proposition. — Motion along the geodesic x_t of the space X_{n_1,n_2}

$$x_t = b \exp\{2\mathfrak{A}t\}\sigma b^+ \tag{23}$$

is projected by (22) in trajectory $q(t)$ of Hamiltonian system with potential (2). Matrices b and $\mathfrak{A}$ are defined by initial data $\{q(0), p(0)\}$ in the following way:

$$b = \exp\{aq(0)\} \tag{24}$$

$$\mathfrak{A} = p(0) + \frac{1}{2a}[\operatorname{Ad}(\exp\{-aq(0)\}) - \operatorname{Ad}(\exp\{aq(0)\}\sigma)]M(0) \tag{25}$$

and matrix $M(0) = M(q(t))|_{t=0}$ so as

$$M_{lj}(q) = \begin{cases} d_j & l = j \\ -iga^2 \sinh^{-2}[a(q_l - q_j)] & l, j \gtrless n_1 \\ iga^2 \cosh^{-2}[a(q_l - q_j)] & l \leqslant n_1, \quad j > n_1 \\ & \text{or} \quad l > n_1, \quad j \leqslant n_1 \end{cases} \tag{26}$$

$$d_j = \sum_{l \neq j} M_{jl}$$

Proof. — As has been proved in [*1*] the Hamiltonian system of equations

$$\dot{p}_j = -\frac{\partial U(q)}{\partial q_j}, \qquad \dot{q}_j = p_j \tag{27}$$

with the potential U(q) (2) is equivalent to Lax equation

$$\dot{L}_t = [L, M] \tag{28}$$

with matrix M (26) and matrix

$$L = p(t) - \frac{1}{4a}[\operatorname{Ad}(\exp\{2aq(t)\}\sigma) - \operatorname{Ad}(\exp\{-2aq(t)\}\sigma)]M \tag{29}$$

$$p(t) = \operatorname{diag}(p_1, \dots p_n).$$

Let's compute the expression $x_t^{-1}\dot{x}_t + \dot{x}_t x_t^{-1}$ where the curve $x_t \in X_{n_1,n_2}$ is defined by (22). Taking into account that $M = u_t^{-1}\dot{u}_t$ we'll get

$$x_t^{-1}\dot{x}_t = a \operatorname{Ad}(u_t)\left[2p + \frac{1}{a}\operatorname{Ad}(\exp\{-2aq\}\sigma)M - \frac{1}{a}M\right] \tag{30}$$

$$\dot{x}_t x_t^{-1} = a \operatorname{Ad}(u_t)\left[2p - \frac{1}{a}\operatorname{Ad}(\exp\{\quad 2aq\}\sigma)M + \frac{1}{a}M\right] \tag{31}$$

Then from (29) it follows

$$x_t^{-1}\dot{x}_t + \dot{x}_t x_t^{-1} = 4 \operatorname{Ad}(u_t)L \tag{32}$$

Let's derive for the last equality by

$$\frac{d}{dt}(x_t^{-1}\dot{x}_t + \dot{x}_t x_t^{-1}) = 4 \operatorname{Ad}(u_t)(\dot{L}_t - [L, M]) \tag{33}$$

On the other hand if x_t is the arbitrary geodesic of X_{n_1,n_2} (23) then it satisfies the equation (14). It follows from (14) and (33) that Lax equation (28) is always correct for geodesics. Let $u_t^{-1}\dot{u}_t|_{t=0} = M(0)$ (26). Then the projection $q(t)$ (22) of the geodesic x_t is the trajectory of the original dynamical system. It follows from the equivalence of Lax equation (28) and Hamiltonian ones (27).

If $u_t|_{t=0} = I_n$ then from (22), (23) and (30) we'll get formulae (24) and (25). Proposition is proved.

With the help of this proposition the integration of Hamiltonian system may be reduced to the operations of linear algebra. According to the initial data $(q(0), p(0))$ matrices $\mathfrak{A}$ and b are constructed and geodesic is restored. Then we move along the geodesic from the point $x_t|_{t=0}$ to the arbitrary point x_t. Coordinates $q_j(t)$ of the system at the moment t as it follows from (22) are equal

$$q_j(t) = \frac{1}{2a}\ln|\lambda_j(x_t)| \tag{34}$$

where $\lambda_j(x_t)$ — the eigenvalue of matrix x_t. Momenta may be obtained from the matrix representation (29):

$$p(t) = L(t) + \frac{1}{4a}[\operatorname{Ad}(\exp\{2aq(t)\}\sigma) - \operatorname{Ad}(\exp\{-2aq(t)\}\sigma)]M(t) \tag{35}$$

where

$$L(t) = u_t L(0) u_t^{-1} \tag{36}$$

and u_t is such matrix that $u_t x_t u_t^{-1}$ is the diagonal one.

COROLLARY. — Let E be the energy of our system and $|\mathfrak{A}|$ be the length of the geodesic $x_t|_{t=1}$ which is defined by matrix $\mathfrak{A}$ (23). Then

$$E = \frac{1}{2}|\mathfrak{A}|^2 \tag{37}$$

In fact from (1), (2), (24) and (25) we obtain:

$$\frac{1}{2} | \mathfrak{A} |^2 = \frac{1}{2} \operatorname{Sp} \mathfrak{A}^2 = \mathrm{H}(p(0), q(0)) = \mathrm{E}\,. \tag{38}$$

5. Let's investigate the character of the movement. As it was mentioned before all the geodesics fall into $n_2 + 1$ classes according to the number of « compact » parameters. From the formula of the matrix $\mathfrak{A}$ (25) it follows that in each class there are geodesics x_t (23) which correspond to a set of the initial data. So the movements of our dynamical system fall into $n_2 + 1$ classes. To all appearence the number of the « compact » parameters φ_k $(0 \leqslant k \leqslant n_2)$ of the geodesics defines the number of bound pairs of particles, but we have no rigorous proof of this fact.

At the same time the finite movement of the system corresponds to the compact geodesic. If follows from the continuity of map $x_t \to q(t)$ (22). Observe that these geodesics are not of the general state. Therefore a number of particles run to the infinity with infinitesimal changing of the initial data.

Let's consider the configurational space $\mathfrak{H}$ of our system. Its dimension is $n - 1$ as $\sum_{j=1}^{n} q_j = 0$ (3). Each set of coordinates $(q_1, \ldots q_n)$ corresponds to the point with coordinates $(q_1 - q_2, q_2 - q_3 \ldots q_{n-1} - q_n)$ in the space $\mathfrak{H}$. However the movement does not take place in the whole space. If

$$q_i - q_j = 0 \qquad i, j \gtrless n_1 \tag{39}$$

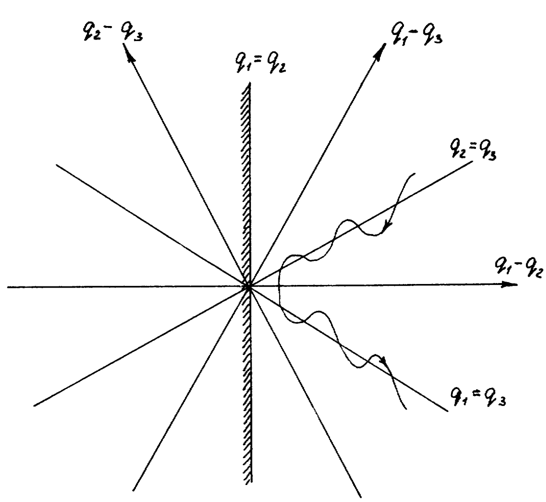

FIG. 2.

then the potential U(q) (2) becomes the infinity. Therefore the trajectories are limited by the hyperplanes (39). On the other side the attractive finite potential is concentrated on the hyperplanes

$$q_i - q_j = 0 \qquad i \leqslant n_1 , \quad j > n_1 \tag{40}$$

Trajectories which are concentrated near such hyperplanes correspond to the bound pairs of particles. The configuration space for three particles where the first two particles have a similar sign is shown in fig. 2.

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