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Coherent states and square integrable representations

by

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ABSTRACT. — The purpose of this paper is to put in evidence the close connection between the existence of coherent states and the square integrability of an irreducible unitary representation of a Lie group. It is shown that, in the case of a reductive group, an irreducible unitary representation with discrete kernel admits a system of coherent states if and only if it belongs to the relative discrete series, and that, in such a situation, it is in fact a coherent state representation (in a sense to be precised in the body of the paper). Results of the same nature are proved in the solvable case, but only for coherent state representations, and under the additional assumption that the group involved is an extension of a torus by an exponential Lie group.

INTRODUCTION

The theory of group representations, whose important rôle in the development of quantum mechanics is well-known, was recently recognized as an adequate tool for studying the coherent states of a quantum model. One of the fundamental questions one can raise in this respect is that of determining when a given irreducible unitary representation of a Lie group, acting as a symmetry group on the underlying quantum model, admits a system of coherent states. Of course, one may ignore a subgroup whose action is trivial and thus assume that the representation in case has discrete kernel.

A very satisfactory answer to this problem was obtained in the nilpotent case. Namely, in [9] one of us proved that if π is an irreducible unitary representation, with discrete kernel, of a simply connected nilpotent Lie group G, and X_π is the orbit under the coadjoint action of G (on the dual vector space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G) associated with π by the Kirillov correspondence, then the following statements are equivalent:

(SCS) π admits a system of coherent states;
(SIR) π is square integrable;
(AV) X_π is an affine variety in $\mathfrak{g}^*$;
(CSR) π is a coherent state representation.

This last assertion requires an explanation. In many significant cases, including those we deal with in the present paper, an irreducible unitary representation π of a Lie group G arises by the geometric quantization procedure from a Hamiltonian G-space X_π (see [*6*], [*14*]). In this process, the symplectic homogeneous space X_π can be regarded as a classical phase space, while the Hilbert space $\mathscr{H}_\pi$, on which the produced representation acts, plays the rôle of the corresponding quantum phase space. From this point of view, it is natural to distinguish, among all irreducible representations which admit coherent states, those representations whose coherent states are based precisely on the associated Hamiltonian G-space X_π. Such representations were called in [*9*] « coherent state representations ». When G is nilpotent, the assignment $\pi \mapsto X_\pi$ is just the Kirillov correspondence between the orbits of G in $\mathfrak{g}^*$ and the equivalence classes of irreducible unitary representations of G.

Our concern in this paper is to investigate to what extent results of the same nature as in the nilpotent case are valid for other classes of Lie groups. It should be said from the beginning that even the progression from nilpotent to solvable, which is a natural step, raises a number of difficulties which we have not succeeded to surpass entirely. First of all, we were not able to relate in a nontrivial way (SCS) to the remainder of the properties. (Note that either (SIR) or (CSR) trivially implies (SCS)). However, for solvable Lie groups which are extensions of tori by exponential Lie groups, we can prove that (CSR) and (SIR) are equivalent. Clearly, any exponential Lie group is of this type. In addition, we shall show that any quasi-algebraic solvable group (i. e. the identity component of an irreducible real algebraic solvable group) is of this type too. As a matter of fact, in the exponential case, the properties (CSR) and (SIR) are also equivalent to the following reasonable substitute for (AV):

(OAV) X_π is open in its affine hull in $\mathfrak{g}^*$.

This last property should be considered as accidental, the equivalence between it and the others being no longer true in general (as can be illustrated by a simple example — the « diamond » group). For an arbitrary solvable Lie group we have only obtained that (SIR) implies a slightly weaker form of (CSR), but there is no evidence that this is the best possible result. Actually we suspect that the converse implication is also true.

Finally we have found, somehow unexpectedly, that all relevant equivalences (i. e. between (SCS), (SIR) and (CSR)) are still valid in the case of a reductive group.

The proofs are generally based on the same technique as in [9] in the solvable case, while in the reductive case we use at an essential point a result by C. C. Moore in [8].

1. PRELIMINARIES

We write down in this section the basic facts concerning coherent states and square integrable representations of locally compact groups.

We use the following notation concerning a locally compact group G. A left Haar measure is denoted μ_G and the modular function is denoted Δ_G. The connected component of the identity is denoted G_0. If π is an irreducible unitary representation of G then $\mathscr{H}_\pi$ denotes its representation space and $[\pi]$ denotes its unitary equivalence class. By G' (resp. G'') we shall denote the quotient $G/Ker\ \pi$ (resp. $G/(Ker\ \pi)_0$) and by π' (resp. π'') the representation of G' (resp. G') to which π factorizes.

1.1. Let π be an irreducible unitary representation of a locally compact group G. We shall say, after [9, § 1], that π admits a system of coherent states if there exist a closed subgroup H of G and a family $\{ P_x \}$ of one-dimensional projections in $\mathscr{H}_\pi$, where x runs over the space of right cosets $X = G/H$, such that the following conditions are satisfied:

(SCS_1) X admits an invariant measure μ_X, that is $\Delta_G(h) = \Delta_H(h)$ for any $h \in H$ and then $\mu_X = \mu_G/\mu_H$;

(SCS_2) $P_{gx} = \pi(g)P_x\pi(g)^{-1}$ for any $g \in G$ and $x \in X$;

(SCS_3) there exists a nonzero vector $\psi \in \mathscr{H}_\pi$ such that

$$\int \langle P_x\psi, \psi \rangle \, d\mu_X(x) < \infty$$

In such a case, the family $\{ P_x \}$ will be called a *π-system of coherent states based on X*. The last two conditions can be restated in a more convenient form (at least for our purposes). Namely, if one chooses a vector φ of norm one in the range of P_{eH}, where e is the unit element of G, then from (SCS_2) we infer:

(SCS'_2) there exists a unitary character λ of H such that $\pi(h)\varphi = \lambda(h)\varphi$ for all $h \in H$.

Furthermore, one has

(CS) $\qquad P_{gH}\psi = \langle \psi, \pi(g)\varphi \rangle \pi(g)\varphi \qquad$ for any $\psi \in \mathscr{H}_\pi$.

These being noted, the property (SCS_3) says that:

(SCS'_3) there exists a nonzero vector $\psi \in \mathscr{H}_\pi$ such that

$$\int | \langle \psi, \pi(g)\varphi \rangle |^2 d\mu_X(gH) < \infty .$$

As it was observed in [9, § 1], (SCS_2) and (SCS_3) can be replaced by (SCS'_2) and (SCS'_3).

If the conditions (SCS_1), (SCS'_2) and (SCS'_3) hold, we shall say that the pair $\{ \varphi ; \lambda \}$ defines a *π-system of coherent states based on $X = G/H$*; the corresponding coherent states are given in fact by the formula (CS).

1.1.1. REMARK. — Assume that $\{ \varphi ; \lambda \}$ defines a π-system of coherent states based on G/H. Then π is equivalent to a subrepresentation of the induced representation *ind* (H, G, λ) (cf. [9, Prop. 1.2]).

1.1.2. PROPOSITION. — *Let $H \subset K$ be two closed subgroups of G and let π be an irreducible unitary representation of G. Assume that there exist a vector $\varphi \in \mathcal{H}_\pi$ with $\| \varphi \| = 1$ and a unitary character λ of K such that $\pi(k)\varphi = \lambda(k)\varphi$ for all $k \in K$, and let λ_H be the restriction of λ to H. Then the following statements are equivalent:*

i) $\{ \varphi ; \lambda_H \}$ *defines a π-system of coherent states based on $X = G/H$;*

ii) $\{ \varphi ; \lambda \}$ *defines a π-system of coherent states based on $Y = G/K$ and K/H admits a finite K-invariant measure.*

Proof. — *i)* $\Rightarrow$ *ii)*. Since $\Delta_H = \Delta_G \mid H$, the homomorphism $\delta : K \to \mathbb{R}_+$, $\delta(k) = \Delta_K(k)^{-1}\Delta_G(k)$ extends the homomorphism $h \mapsto \Delta_K(h)^{-1}\Delta_H(h)$, $h \in H$, so that one may form the relatively invariant measure $\mu = \delta\mu_K/\mu_H$ on K/H. On the other hand, let ρ be a continuous positive function on G such that $\rho(gk) = \delta(k)^{-1}\rho(g)$, for $g \in G$ and $k \in K$. Then $\mu_Y = \rho\mu_G/\mu_K$ is a quasi-invariant measure on Y.

Now every coset $gK \in G/K$ defines a positive measure μ_{gK} on X by the formula

$$\int f d\mu_{gK} = \rho(g)^{-1} \int f(gkH) d\mu(kH) .$$

According to [3, Ch. II, 3.4], μ_X has an integral decomposition of the form

$$\mu_X = \int \mu_{gK} d\mu_Y(gK) .$$

Thus, if we denote

$$f_\psi(gH) = | \langle \psi, \pi(g)\varphi \rangle |^2$$

and take into account the fact that for a suitable $\psi \in H_\pi$

$$\int f_\psi(gH) d\mu_X(gH) < \infty ,$$

we get that

$$\int f_\psi d\mu_{gK} < \infty ,$$

for almost all $gK \in Y$, and further that

$$\int d\mu_Y(gK) \int f_\psi d\mu_{gK} = \int f_\psi d\mu_X .$$

Now

$$\int f_\psi d\mu_{gK} = \rho(g)^{-1} \int f_\psi(gkH) d\mu(kH) = \rho(g)^{-1} \int f_\psi(gH) d\mu(kH),$$

since clearly $f_\psi(gkH) = f_\psi(gH)$. It follows firstly that $\mu(K/H) < \infty$ and secondly that

$$\int f_\psi d\mu_{gK} = \rho(g)^{-1} f_\psi(gH)\mu(K/H),$$

which further imply that $\rho(gk) = \rho(g)$ for all $g \in G$ and $k \in K$. Hence $\Delta_G \mid K = \Delta_K$, in particular $\Delta_K \mid H = \Delta_G \mid H = \Delta_H$ which means that μ is K-invariant, so that we may take $\rho \equiv 1$. It follows that μ_Y is an invariant measure and that

$$\int |\langle \psi, \pi(g)\varphi \rangle|^2 d\mu_Y(gK) = \mu(K/H)^{-1} \int f_\psi(gH) d\mu_X(gH) < \infty$$

which concludes the proof of *i*) $\Rightarrow$ *ii*).

ii) $\Rightarrow$ *i*). By hypothesis, $Y = G/K$ and K/H admit invariant measures. Hence $X = G/H$ admits an invariant measure too. Then, in the above notation, one has

$$\begin{aligned}\int f_\psi d\mu_{gK} &= \int |\langle \psi, \pi(gk)\varphi \rangle|^2 d\mu(kH) \\ &= \int |\langle \psi, \pi(g)\lambda(k)\varphi \rangle|^2 d\mu(kH) = \int |\langle \psi, \pi(g)\varphi \rangle|^2 d\mu(kH) \\ &= f_\psi(gK)\mu(K/H),\end{aligned}$$

so that

$$\int f_\psi(gH) d\mu_X(gH) = \mu(K/H) \int f_\psi(gK) d\mu_Y(gK) < \infty$$

This completes the proof.

1.2. Let Z denote the center of the locally compact group G. An irreducible unitary representation π of G when restricted to Z is a multiple of a well determined unitary character ζ_π of Z. This will be called *the central character* of π.

The following statements concerning an irreducible unitary representation π are known to be equivalent (see for instance [*1*, Prop. 1.2]):

(SIR′) there exist nonzero vectors $\psi, \varphi \in \mathscr{H}_\pi$ such that

$$\int |\langle \psi, \pi(g)\varphi \rangle|^2 d\mu_{G/Z}(gZ) < \infty ;$$

(SIR″) for every $\psi \in \mathscr{H}_\pi$ and φ in a dense linear subspace of $\mathscr{H}_\pi$

$$\int |\langle \psi, \pi(g)\varphi \rangle|^2 d\mu_{G/Z}(gZ) < \infty ;$$

(SIR''') π is equivalent to a subrepresentation of the induced representation *ind* (Z, G, ζ_π).

If one of these conditions holds, π is called a *square integrable representation* or a *member of the relative discrete series of G.*

1.2.1. REMARK. — It is a trivial observation to note that π is square integrable if and only if π admits a system of coherent states based on G/Z. Furthermore if π is square integrable then, for any φ in a dense linear subspace of $\mathcal{H}_\pi$, with $\|\varphi\| = 1$, the pair $\{\varphi ; \zeta_\pi\}$ defines a π-system of coherent states based on G/Z. It is less trivial to prove that a square integrable representation is a coherent state representation (in the sense explained in the introduction).

1.2.2. PROPOSITION. — *Let G be a connected locally compact group, π an irreducible unitary representation of G, Γ a closed, discrete, normal subgroup of G contained in Ker π, and π_Γ the representation of $G_\Gamma = G/\Gamma$ to which π factorizes. Then π is square integrable if and only if π_Γ is square integrable.*

Proof. — First, it is clear that Γ is central and that Z/Γ is contained in the center Z_Γ of G_Γ. Now let $c\Gamma \in Z_\Gamma$. Then, for any $g \in G$, $gcg^{-1}c^{-1} \in \Gamma$. Since the map $g \mapsto gcg^{-1}c^{-1}$ is continuous, G is connected and Γ discrete, it follows that $gcg^{-1}c^{-1} = e$, for any $g \in G$. Therefore $c \in Z$. We conclude thus that $Z_\Gamma = Z/\Gamma$. To complete the proof it remains to notice that

$$\int |\langle \psi, \pi(g)\varphi \rangle|^2 d\mu_{G/Z}(gZ) = \int |\langle \psi, \pi_\Gamma(g_\Gamma)\varphi \rangle|^2 d\mu_{G_\Gamma/Z_\Gamma}(g_\Gamma Z_\Gamma),$$

where g_Γ denotes the right coset $g\Gamma \in G_\Gamma$.

1.2.3. COROLLARY. — *Let π be an irreducible unitary representation of the connected, locally compact group G, and let π', π'' be the corresponding representations of $G' = G/Ker\ \pi$, $G'' = G/(Ker\ \pi)_0$ respectively. Then π' is square integrable if and only if π'' is square integrable.*

Proof. — One applies the above proposition to the subgroup $\Gamma = Ker\ \pi/(Ker\ \pi)_0$ of G''.

2. THE SOLVABLE CASE

Before going to treat the case of a solvable Lie group, let us introduce some notation concerning an arbitrary Lie group G. The Lie algebra of G is denoted $\mathfrak{g}$ and $\mathfrak{g}^*$ stands for the dual vector space of $\mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ is denoted Ad and the coadjoint action is denoted Ad^*. Given a functional $f \in \mathfrak{g}^*$, $G(f)$ denotes the isotropy subgroup of G (acting via Ad^* on $\mathfrak{g}^*$) at f, and $\mathfrak{g}(f)$ denotes its Lie algebra. As it is known,

$\mathfrak{g}(f) = \{ x \in \mathfrak{g} ; \langle f, [x, \mathfrak{g}] \rangle = 0 \}$. The set of all orbits of G acting on $\mathfrak{g}^*$ will be denoted $\mathfrak{g}^*/Ad^*(G)$. If $X \in \mathfrak{g}^*/Ad^*(G)$, we denote

$$\mathfrak{g}[X] = \{ x \in \mathfrak{g} ; \langle f, x \rangle = 0 \text{ for any } f \in X \}.$$

2.1. The following two lemmas are established in the general case of a connected Lie group G, although they will be used only for G solvable.

2.1.1. LEMMA. — *Let $X \in \mathfrak{g}^*/Ad^*(G)$. Then $\mathfrak{g}[X]$ is an ideal of $\mathfrak{g}$ and one has $\mathfrak{g}[X] = \bigcap_{f \in X} ker\,(f \mid \mathfrak{g}(f))$.*

Proof. — If $x \in \mathfrak{g}[X]$ and $f \in G$, then

$$\langle f, Ad(g)x \rangle = \langle Ad^*(g)^{-1}f, x \rangle = 0, \qquad \text{for any } f \in X.$$

Thus $\mathfrak{g}[X]$ is $Ad(G)$-stable, or equivalently it is an ideal. Now clearly $\bigcap_{f \in X} ker\,(f \mid \mathfrak{g}(f))$ is contained in $\bigcap_{f \in X} ker\, f$. Conversely, let $x \in \mathfrak{g}[X]$; $\mathfrak{g}[X]$ being an ideal, $[x, \mathfrak{g}] \subset \mathfrak{g}[X]$, hence $[x, \mathfrak{g}] \subset ker\, f$ for any $f \in X$, or equivalently $x \in \bigcap_{f \in X} \mathfrak{g}(f)$. On the other hand, $x \in \bigcap_{f \in X} ker\, f$. In conclusion $x \in \bigcap_{f \in X} ker\,(f \mid \mathfrak{g}(f))$.

2.1.2. LEMMA. — *For $X \in \mathfrak{g}^*/Ad^*(G)$ the following assertions are equivalent:*

i) *$\mathfrak{g}(f)$ is an ideal for any $f \in X$;*
ii) *$\mathfrak{g}(f)$ is an ideal for some $f \in X$;*
iii) *X is open in its affine hull.*

Proof. — i) $\Rightarrow$ ii). This is obvious.

ii) $\Rightarrow$ iii) Let $g \in G$ and $x \in \mathfrak{g}(f)$. Since $\mathfrak{g}(f)$ is an ideal,

$$Ad(g)^{-1}x - x \in [\mathfrak{g}, \mathfrak{g}(f)],$$

hence

$$\langle Ad^*(g)f - f, x \rangle = \langle f, Ad(g)^{-1}x - x \rangle = 0.$$

Therefore, X is contained in the affine variety $f + \mathfrak{g}(f)^\perp$, where $\mathfrak{g}(f)^\perp = \{ k \in \mathfrak{g}^* ; \langle k, \mathfrak{g}(f) \rangle = 0 \}$. Moreover, X is a submanifold of the same dimension as $f + \mathfrak{g}(f)^\perp$ and thus it is open in this affine variety.

iii) $\Rightarrow$ i) Let $f \in X$. By hypothesis $X \subset f + W$ where W is a linear subspace in $\mathfrak{g}^*$ of the same dimension as X. If $x \in \mathfrak{g}$ and

$$y \in \mathfrak{g}[W] = \{ u \in \mathfrak{g} ; \langle k, u \rangle = 0 \text{ for all } k \in W \},$$

then

$$\langle f, Ad\,(\exp tx)y - y \rangle = \langle Ad^*\,(\exp tx)^{-1}f - f, y \rangle = 0.$$

By taking the derivative in $t = 0$, we get

$$0 = \langle f, ad(x)y \rangle = \langle f, [x, y] \rangle .$$

This proves that $\mathfrak{g}[W] \subset \mathfrak{g}(f)$. Since both these spaces have the same dimension, it follows that $\mathfrak{g}(f) = \mathfrak{g}[W]$. But W does not depend on the choice of f in X. Hence $\mathfrak{g}(f) = \mathfrak{g}(Ad^*(g)f) = Ad(g)(\mathfrak{g}(f))$ for any $g \in G$, which shows that $\mathfrak{g}(f)$ is $Ad(G)$-stable, and thus it is an ideal.

2.2. In the remainder of this section, unless otherwise stated, G will denote a connected and simply connected solvable Lie group.

Let $f \in \mathfrak{g}^*$. Since $G(f)_0$ is simply connected, there exists a unique character χ_f of $G(f)_0$ such that $d\chi_f = 2\pi i f \mid \mathfrak{g}(f)$. We denote $Q_f = (Ker\ \chi_f)_0$. If $X \in \mathfrak{g}^*/Ad^*(G)$, we set $G[X] = \bigcap_{f \in X} Q_f$. Clearly, the Lie algebra of $G[X]$ is $\mathfrak{g}[X]$.

2.2.1. By extending the Auslander-Kostant-Pukanszky procedure, we constructed in [*10*] a series of irreducible unitary representations of G. Without entering into details, we shall recall here this construction.

Let $f \in \mathfrak{g}^*$ and let Γ be a subgroup of $G(f)$ containing $G(f)_0$, such that there exists a unitary character χ of Γ which extends χ_f. For any positive, strongly admissible polarization $\mathfrak{h}$ of G at f and any character χ of Γ with the above property, one constructs by holomorphic induction a unitary representation $\rho(f, \chi, \mathfrak{h})$ of G. This representation is irreducible if and only χ cannot be extended to a subgroup larger than Γ, and its equivalence class does not depend on $\mathfrak{h}$. Furthermore, an irreducible representation of the form $\rho(f, \chi, \mathfrak{h})$ is normal (see [*12*, p. 81] for the definition) if and only if the orbit $X_f = Ad^*(G)f$ is locally closed in $\mathfrak{g}^*$ and $G(f)/\Gamma$ is finite. Conversely, any irreducible normal representation π of G is unitarily equivalent to such a representation $\rho(f, \chi, \mathfrak{h})$, where the orbit X_f is uniquely determined by $[\pi]$; accordingly, it will be alternatively denoted X_π.

Finally we mention that when G is of type I, all the irreducible unitary representations of G are normal, the above construction becomes a bit simpler (since for any $f \in \mathfrak{g}^*$, $\Gamma = G(f)$), and in fact it is just the Auslander-Kostant construction (cf. [*2*]).

2.2.2. LEMMA. — *Let π be an irreducible normal representation of G. Then $(Ker\ \pi)_0 = G[X_\pi]_0$.*

Proof. — The representation π'' of $G'' = G/(Ker\ \pi)_0$ is irreducible and normal too. By what we said above, there exist $f'' \in \mathfrak{g}''^*$, where $\mathfrak{g}''$ is the Lie algebra of G'', a subgroup Γ'' of $G''(f'')$ with $\Gamma''_0 = G''(f'')_0$, a character χ'' of Γ'' and a polarization $\mathfrak{h}''$ of $\mathfrak{g}''$ at f'', such that $[\pi''] = [\rho(f'', \chi'', \mathfrak{h}'')]$. Let $p'' : G \to G''$, $dp'' : \mathfrak{g} \to \mathfrak{g}''$ be the canonical projections. Then, the naturality of the construction in 2.2.1 ensures us that, if we take $f = f'' \circ dp''$, $\Gamma = p''^{-1}(\Gamma'')$, $\chi = \chi'' \circ p''$ and $\mathfrak{h} = (dp'')^{-1}(\mathfrak{h}'')$, then $[\pi] = [\rho(f, \chi, \mathfrak{h})]$. Consequently $X_\pi = Ad^*(G)f$, while of course $X_{\pi''} = Ad^*(G'')f''$. It is not difficult to see that $G[X_\pi] = p''^{-1}(G''[X_{\pi''}])$. In particular, $(Ker\ \pi)_0 \subset G[X_\pi]$, hence $(Ker\ \pi)_0 \subset G[X_\pi]_0$.

To prove the other inclusion, let us consider the quotient group $\bar{G} = G/G[X_\pi]_0$, with Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{g}[X_\pi]$. Now if $[\pi] = [\rho(f, \chi, \mathfrak{h})]$, then clearly f, χ and $\mathfrak{h}$ « factorize » to the entities $\bar{f}$, $\bar{\chi}$, $\bar{\mathfrak{h}}$ corresponding to $\bar{G}$. Moreover, the naturality argument gives us $[\pi] = [\bar{\pi} \circ \bar{p}]$, where $\bar{p} : G \to \bar{G}$ is the canonical projection and $\bar{\pi} = \rho(\bar{f}, \bar{\chi}, \bar{\mathfrak{h}})$. Therefore $Ker\ \pi = \bar{p}^{-1}(Ker\ \bar{\pi})$. In particular, $Ker\ \pi \supset G[X_\pi]_0$, hence $(Ker\ \pi)_0 \supset G[X_\pi]_0$.

2.3. Let π be a square integrable representation of G. As it was proved in [*4*], π is normal and for any $f \in X_\pi$, $G(f)/Z$ is compact; in addition, $G(f)/G(f)_0Z$ is a finite group. Here Z denotes, as usually, the center of G. Let $Y_f = G/G(f)_0Z$. This is a finite covering space of the orbit X_π, and thus a Hamiltonian G-space. In particular it possesses an invariant measure μ; it is given by the invariant volume form which comes from the canonical G-invariant symplectic 2-form on Y_f.

2.3.1. THEOREM. — *Let π be a square integrable representation of the connected and simply connected solvable Lie group G. Then π admits a system of coherent states based on Y_f, where $f \in X_\pi$.*

Proof. — By hypothesis, $[\pi] = [\rho(f, \chi, \mathfrak{h})]$, where χ is a character of a subgroup Γ of $G(f)$ containing $H = G(f)_0Z$, and $\mathfrak{h}$ is a positive strongly admissible polarization of $\mathfrak{g}$ at $f \in X_\pi$. It is easy to see that the restriction of χ to Z is just the central character ζ_π. Thus, if we denote by χ_H and π_H the restriction to H of χ and π respectively, then $\chi_H^{-1} \otimes \pi_H$ is trivial on Z and hence defines a representation of H/Z. Note that H/Z, being isomorphic to $G(f)_0/G(f)_0 \cap Z$ is compact, connected and solvable; therefore it is a torus. Consequently, π_H splits as a direct sum of characters on H whose restriction to Z is ζ_π. If λ is such a character then $P_\lambda = \int \overline{\lambda(h)}\pi(h)d\mu_{H/Z}(hZ)$ ($\mu_{H/Z}$ being a normalized Haar measure on H/Z) is the orthogonal projection on the isotypic subspace of $\mathscr{H}_\pi$ corresponding to λ.

Now let $\varphi \in \mathscr{H}_\pi$ be a nonzero vector such that

$$\int |\langle \psi, \pi(g)\varphi \rangle|^2 d\mu_{G/Z}(gZ) < \infty, \qquad \text{for any } \psi \in \mathscr{H}_\pi .$$

Since $\varphi \neq 0$, there exists a character λ on H such that $\varphi_\lambda = P_\lambda\varphi \neq 0$; there is no loss of generality in assuming $\| \varphi_\lambda \| = 1$. We will show that $\{ \varphi_\lambda ; \lambda \}$ defines a π-system of coherent states based on Y_f.

First it is clear that $\pi(h)\varphi_\lambda = \lambda(h)\varphi_\lambda$ for any $h \in H$. Now let $\psi \in \mathscr{H}_\pi$. Then

$$\langle \psi, \pi(g)\varphi_\lambda \rangle = \int \langle \psi, \overline{\chi(h)}\pi(gh)\varphi \rangle\, d\mu_{H/Z}(hZ),$$

hence

$$|\langle \psi, \pi(g)\varphi_\lambda \rangle|^2 \leq \int |\langle \psi, \pi(gh)\varphi \rangle|^2 d\mu_{H/Z}(hZ)$$

and further

$$\int |\langle \psi, \pi(g)\varphi_\lambda \rangle|^2 d\mu_{G/Z}(gZ) = \int d\mu_{G/Z}(gZ) \int |\langle \psi, \pi(gh)\varphi \rangle|^2 d\mu_{H/Z}(hZ)$$
$$= \int \Delta_{G/Z}(hZ)^{-1} |\langle \psi, \pi(g)\varphi \rangle|^2 d\mu_{G/Z}(gZ).$$

But H/Z being compact, $\Delta_{G/Z}(hZ) = 1$, and thus we finally get

$$\int |\langle \psi, \pi(g)\varphi_\lambda \rangle|^2 d\mu_{G/Z}(gZ) = \int |\langle \psi, \pi(g)\varphi \rangle|^2 d\mu_{(G/Z)}(gZ) < \infty.$$

To conclude the proof it remains only to apply Proposition 1.1.2.

2.3.2. REMARK. — In particular, if π is a square integrable representation with the corresponding orbit X_π simply connected, then π is a coherent state representation, that is π admits a system of coherent states based on X_π.

2.4. We shall restrict ourselves now to the case when G is exponential. As it is well-known G is of type I, hence the Auslander-Kostant construction [2] works and all equivalence classes of irreducible unitary representations can be obtained by their procedure. In addition, every orbit in $\mathfrak{g}^*$ under the coadjoint representation is simply connected, since for any $f \in \mathfrak{g}^*$ the isotropy subgroup $G(f)$ is connected and simply connected.

2.4.1. THEOREM. — *Let G be a connected and simply connected exponential Lie group. The following statements, concerning an irreducible unitary representation π of G, are equivalent:*

(CSR) *π admits a system of coherent states based on X_π;*

(OAV) *X_π is open in its affine hull;*

(SIR) *π' is a member of the relative discrete series of $G' = G/Ker\ \pi$.*

Proof. — (CSR) $\Rightarrow$ (OAV). Fix a pair $\{\varphi; \lambda\}$ defining a π-system of coherent states based on X_π, with $\varphi \in \mathscr{H}_\pi$ and λ a unitary character of $G(f)$, for some $f \in X_\pi$. Recall that π is contained in the induced representation *ind* $(G(f), G, \lambda)$. (Cf. Remark 1.1.1). We want to apply the Mackey subgroup theorem [7, Th. 12.1] to decompose the restriction to $G(f)$ of *ind* $(G(f), G, \lambda)$. To do this we need to prove first that the space of double cosets $G(f)\backslash G/G(f)$ is countably separated or, which amounts to the same thing, that the orbits of $G(f)$ acting on the left on $G/G(f)$ are locally closed. Since $\mathfrak{g}^*$ is a G-module of exponential type, we may apply [3, Ch. I, Th. 3.8 and Remark 3.9] to deduce that $G/G(f)$ is homeomorphic to the orbit $X_\pi \subset \mathfrak{g}^*$. Now the action of $G(f)$ on $\mathfrak{g}^*$ being also of exponential type, its orbits in $\mathfrak{g}^*$, in particular those in X_π, are locally closed. Consequently, Mackey's theorem cited above can be used to obtain the direct integral decomposition

$$ind\,(G(f),\,G,\,\lambda)\,|\,G(f) = \int^{\oplus} ind\,(G(f) \cap g^{-1}G(f)g,\,G(f),\,\lambda_g)d\nu(\dot{g}),$$

where $\lambda_g(h) = \lambda(ghg^{-1})$ for $h \in G(f) \cap g^{-1}G(f)g$, $\dot{g} = G(f)gG(f)$ and ν is an admissible measure on $G(f)\backslash G/G(f)$ (see [7] and also [9, 3.2]).

Now $\pi \,|\, G(f)$ is contained in $ind\,(G(f), G, \lambda) \,|\, G(f)$ and, on the other hand, λ is contained in $\pi \,|\, G(f)$. Therefore, λ is contained in $ind\,(G(f), G, \lambda) \,|\, G(f)$. This further implies (see [9, 3.3]) that λ is contained in

$$ind\,(G(f) \cap g^{-1}G(f)g,\,G(f),\,\lambda_g),$$

for g running over a measurable subset M_λ of strictly positive Haar measure in G.

Let us fix an element $g \in M_\lambda$. We will prove now that $g^{-1}G(f)g = G(f)$. Since $ind\,(G(f) \cap g^{-1}G(f)g, G(f), \lambda_g)$ contains a finite-dimensional representation, by [7, Th. 8.2], the homogeneous space $G(f)/G(f) \cap g^{-1}G(f)g$ admits a finite invariant measure. In view of [*11*, Th. 7.1] this implies that $G(f)/G(f) \cap g^{-1}G(f)g$ is compact. We immediately get that $g^{-1}G(f)g = G(f)$, since $G(f)/G(f) \cap g^{-1}G(f)g$ is diffeomorphic to an euclidian space which cannot be compact unless it reduces to a single point.

Now let $G_\lambda = \{\, g \in G\,;\, g^{-1}G(f)g = G(f) \,\}$. This is clearly a closed subgroup of G. Since $M_\lambda \subset G_\lambda$, G_λ will be of strictly positive Haar measure in G, and thus it must be in fact the whole G. Thus $G(f)$ is a normal subgroup of G, which by Lemma 2.1.2 proves that X_π is open in its affine hull.

(OAV) $\Rightarrow$ (SIR). We begin by a few comments which will put our problem under a more convenient light. First, in view of Lemma 2.1.2, we may consider that the hypothesis consists in assuming $\mathfrak{g}(f)$ to be an ideal, for some $f \in X_\pi$. Next, according to Corollary 1.2.3, we may replace in the conclusion G' by $G'' = G/(Ker\,\pi)_0$, and π' by π''. Further, by Lemma 2.2.2, $(Ker\,\pi)_0 = G[X_\pi]_0$. Finally, in view of [*4*, Th. 3], we have to prove that $G''(f'')/Z''$ is compact, where Z'' denotes the center of G'' and f'' is the functional on the Lie algebra $\mathfrak{g}''$ of G'' to which $f \in X_\pi$ factorizes.

Now $\mathfrak{g}(f)$ being an ideal and $[\mathfrak{g}, \mathfrak{g}(f)]$ being contained in $ker\,(f \,|\, \mathfrak{g}(f))$, it results that $ker\,(f \,|\, \mathfrak{g}(f))$ is an ideal too, hence $\mathfrak{g}[X_\pi] = ker\,(f \,|\, \mathfrak{g}(f))$. By going down to $\mathfrak{g}''$ we obtain $ker\,(f'' \,|\, \mathfrak{g}''(f'')) = 0$. As

$$[\mathfrak{g}'', \mathfrak{g}''(f'')] \subset ker\,(f'' \,|\, \mathfrak{g}''(f'')) = 0,$$

it follows that $\mathfrak{g}''(f'')$ coincides with the center $\mathfrak{z}''$ of $\mathfrak{g}''$. We know that $G(f)$ is connected and simply connected. Hence $G''(f'')$ will be connected too. It follows that $G''(f'') = Z_0''$. On the other hand Z'' is always contained in $G''(f'')$. Thus $G''(f'') = Z''$.

(SIR) $\Rightarrow$ (SCR). In view of Corollary 1.2.3 we may consider that π'' is square integrable. As noticed above, the orbit $X_{\pi''}$ is simply connected. According to Remark 2.3.2, this implies that π'' admits a system of coherent states based on $X_{\pi''}$. Thus, there exist $\varphi \in \mathscr{H}_\pi = \mathscr{H}_{\pi''}$ with $\| \varphi \| = 1$ and a character λ'' on $G''(f'')$ such that $\{ \varphi ; \lambda'' \}$ defines a π''-system of coherent states based on $X_{\pi''} \simeq G/G''(f'')$. It is an easy matter to check that $\{ \varphi ; \lambda \}$ defines a π-system of coherent states based on $X_\pi \simeq G/G(f)$.

2.4.2. REMARK. — The following example illustrates the fact that (SIR) does not necessarily imply (OAV) for non-exponential solvable Lie groups. Let $\mathfrak{g}$ be the solvable Lie algebra spanned by $\{ e_1, e_2, e_3, e_4 \}$ with the nonvanishing brackets: $[e_1, e_2] = e_3$, $[e_1, e_3] = - e_2$, $[e_2, e_3] = e_4$. The corresponding simply connected Lie group is known as the « diamond group » (see [3, Ch. VIII, § 1]). It is a type I solvable Lie group, diffeomorphic to $\mathbb{R}^4$, whose center is $Z = \{ (2\pi n, 0, 0, t);\ n \in \mathbb{Z}, t \in \mathbb{R} \}$. Let $f \in \mathfrak{g}^*$ be the functional: $\langle f, e_1 \rangle = \langle f, e_2 \rangle = \langle f, e_3 \rangle = 0$, $\langle f, e_4 \rangle = 1$. Then $G(f)$ is simply connected and $G(f)/Z$ is compact, hence the representation π associated to the orbit $X = Ad^*(G)f$ is square integrable. However, $\mathfrak{g}(f) = \mathbb{R}e_1 + \mathbb{R}e_4$ is clearly not an ideal.

2.5. In this subsection, G denotes a (not necessarily simply connected) connected solvable Lie group of the following type: it contains a connected and simply connected closed subgroup M which is exponential, such that G/M is a torus. Such a group is known to be of type I. (This is an easy consequence of Mackey's theory of representations of group extensions). Consequently, if π is an irreducible unitary representation of G, it lifts to an irreducible normal representation $\tilde{\pi}$ of the universal convering group $\tilde{G}$ of G. As recorded in 2.2.1, $\tilde{\pi}$ is unitarily equivalent to a representation of the form $\rho(f, \tilde{\chi}, \mathfrak{h})$, where $f \in \mathfrak{g}^*$, $\tilde{\chi}$ is a unitary character of a subgroup $\tilde{\Gamma} \subset \tilde{G}(f)$ of finite index, and $\mathfrak{h}$ is a strongly admissible polarization at f. Now, since $\tilde{\chi}$ when restricted to the center $\tilde{Z}$ of $\tilde{G}$ is exactly the central character $\zeta_{\tilde{\pi}}$, it drops down to a character χ of $\Gamma = \tilde{\Gamma}/L \subset G(f)$, L being the kernel of the canonical epimorphism of $\tilde{G}$ onto G. Then, one can construct the holomorphically induced representation $\rho(f, \chi, \mathfrak{h})$ of G, which clearly is unitarily equivalent to π. We shall denote by Y_π the homogeneous symplectic manifold G/Γ, which is a finite covering space of the orbit $X_\pi = Ad^*(G)f$.

With these preparatives, we may now formulate the converse to Theorem 2.3.1.

2.5.1. THEOREM. — *Let π be an irreducible unitary representation of G which admits a system of coherent states based on Y_π. Then π' is square integrable.*

Proof. — In view of the above discussion, we may assume that π is of the form $\rho(f, \chi, \mathfrak{h})$. Let $M(f) = G(f) \cap M$. We claim that $M(f)$ is connected and simply connected. Indeed, this is a consequence of the following lemma, which will be stated below, after introducing some more notation.

Denote by $\mathfrak{m}$ the Lie algebra of M, by l the restriction of f to $\mathfrak{m}$, by $M(l)$ the isotropy subgroup of M (acting on $\mathfrak{m}^*$) at l, and by $\mathfrak{m}(f)$, $\mathfrak{m}(l)$ the Lie algebras corresponding to $M(f)$, $M(l)$ respectively. Finally, let $\mathfrak{a} = \mathfrak{m} + \{ x \in \mathfrak{g} ; \langle f, [x, \mathfrak{m}] \rangle = 0 \}$, which is an ideal in $\mathfrak{g}$ since $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{m}$, and let $\mathfrak{a}^\perp \subset \mathfrak{g}$ be the space of all linear functionals that vanish on $\mathfrak{a}$.

2.5.2. Lemma. — *The mapping $a \in M(l) \mapsto Ad^*(a)f \in \mathfrak{g}^*$ induces a homeomorphism of $M(l)/M(f)$ onto $f + \mathfrak{a}^\perp \subset \mathfrak{g}^*$.*

This can be proved exactly as Lemma II.1.2 in [2], so that we shall omit the details.

Now M being exponential, $M(l)$ is connected and simply connected. On the other hand, the above lemma shows that $M(l)/M(f)$ is also connected and simply connected. It follows that $M(f)$ is connected (and, being a subgroup of M, simply connected) as claimed before.

By hypothesis, there exists a vector of norm one $\varphi \in \mathcal{H}_\pi$ and a unitary character λ of $\Gamma \subset G(f)$, such that $\{ \varphi ; \lambda \}$ defines a π-system of coherent states based on Y_π. Notice that $M(f) \subset G(f)_0 \subset \Gamma$, and denote by λ' the restriction of λ to $M(f)$.

2.5.3. Lemma. — *The pair $\{ \varphi ; \lambda' \}$ defines a π-system of coherent states based on $G/M(f)$.*

This clearly follows from Proposition 1.1.2, if we are able to prove that $G(f)/M(f)$ is compact. In turn, the compactness of $G(f)/M(f) = G(f)M/M$, will follow from the compactness of G/M, provided that we can prove that $G(f)M$ is closed in G. Now $G(f)M$ is easily seen to be the isotropy subgroup of G, acting on the orbit space $\mathfrak{g}^*/Ad^*(M)$, at the point $Ad^*(M)f \in \mathfrak{g}^*/Ad^*(M)$. We next observe that $\mathfrak{g}$, hence $\mathfrak{g}^*$, is an M-module of exponential type, since $\mathfrak{m}$ is such a module and M acts trivially on $\mathfrak{g}/\mathfrak{m}$. Thus, the fact that $G(f)M$ is closed, and hence Lemma 2.5.3, is a consequence of the following result.

2.5.4. Lemma. — *Let T be a topological group acting on a topological space X, such that for any $x \in X$ the map $a \in T \mapsto ax \in X$ is continuous. Assume that all the points of X are locally closed. Then the isotropy subgroup of T at any $x \in X$ is closed.*

Let us prove this statement. Fix a point $x \in X$ and let $T(x)$ be the corresponding isotropy subgroup. Further, let $T(\overline{\{x\}}) = \{ a \in T ; a\overline{\{x\}} = \overline{\{x\}} \}$. Clearly, $T(x) \subset T(\overline{\{x\}})$. The point is that the other inclusion is also true.

Indeed, $\{x\}$ being the only relatively open one-point subset in $\overline{\{x\}}$, every homeomorphism of $\overline{\{x\}}$ must fix it. Thus

$$T(x) = T(\overline{\{x\}}) = \bigcap_{y \in \{\bar{x}\}} \{ a \in T ; ay \in \overline{\{x\}} \},$$

which shows that it is closed.

We continue now the proof of the theorem. According to Lemma 2.5.3 and Remark 1.1.1, π is equivalent to a subrepresentation of *ind* $(M(f), G, \lambda')$. On the other hand, as noticed before, $\mathfrak{g}^*$ is an M-module of exponential type. Therefore, when viewed as an $M(f)$-module it is also of exponential type.

So, the arguments used in the first part of the proof of Theorem 2.4.1 ((CSR) $\Rightarrow$ (OAV)) work in the case at hand too, and lead us ultimately to the conclusion that $M(f)$ is a normal subgroup of G, or equivalently that $\mathfrak{m}(f)$ is an ideal in $\mathfrak{g}$. Now let $\mathfrak{k} = ker\,(f \mid \mathfrak{m}(f))$. Since $\mathfrak{m}(f)$ is an ideal, $[\mathfrak{g}, \mathfrak{m}(f)] \subset \mathfrak{m}(f)$. On the other hand $[\mathfrak{g}, \mathfrak{m}(f)] \subset [\mathfrak{g}, \mathfrak{g}(f)] \subset ker\ f$. Therefore $\mathfrak{k}$ is an ideal in $\mathfrak{g}$ too. Denote $K = \exp \mathfrak{k}$, $G_1 = G/K$, $M_1 = M/K$, $\mathfrak{g}_1 = \mathfrak{g}/\mathfrak{k}$, $\mathfrak{m}_1 = \mathfrak{m}/\mathfrak{k}$, f_1 the functional on $\mathfrak{g}_1$ induced by f, $\mathfrak{h}_1 = \mathfrak{h}/\mathfrak{k}_{\mathbb{C}}$, $\Gamma_1 = \Gamma/K$, and χ_1 the character on Γ_1 induced by χ. Then $\pi_1 = \rho(f_1, \chi_1, \mathfrak{h}_1)$ is a unitary representation of G_1 which lifts to π. In other words, π drops down to G_1 and yields the representation π_1.

To prove the theorem, it clearly suffices to check that π_1 is square integrable. To this end, let us note that $\mathfrak{m}_1(f_1) = \mathfrak{m}(f)/\mathfrak{k}$ is 1-dimensional and $[\mathfrak{m}_1(f_1), \mathfrak{g}_1] = 0$. Thus, $\mathfrak{m}_1(f_1)$ is contained in the center $\mathfrak{z}_1$ of $\mathfrak{g}_1$, so that $M_1(f_1)$ is contained in the center Z_1 of G_1.

Now $G_1(f_1)/M_1(f_1)$ is isomorphic to $G(f)/M(f)$, which, as we already noticed, is compact. It follows that $G_1(f_1)/Z_1$ is compact, which finishes the proof.

2.5.5. Remark. — Let G be a real algebraic solvable group which is connected with respect to the Zariski topology. Viewing G as a Lie group, let G_0 denote the connected component of the identity with respect to the underlying Lie group topology. We claim that G_0 is a solvable Lie group of the type described at the beginning of this subsection, in particular that Theorem 2.5.1 works for G_0.

Indeed, as it is known, G can be decomposed into a semidirect product of a torus T and a unipotent algebraic normal subgroup U. Now U is a connected and simply connected nilpotent Lie group, and T is an almost direct product of two uniquely defined real subtori T' and T'', such that T' is $\mathbb{R}$-split and T'' is anisotropic over $\mathbb{R}$. These being said, it is easy to see that $M = T'_0 U$ is a connected and simply connected exponential (in fact, completely solvable) normal subgroup of G_0, and that G_0/M is a compact connected Lie group.

3. THE REDUCTIVE CASE

Throughout this section G will denote a connected and simply connected reductive Lie group, with Lie algebra $\mathfrak{g}$.

3.1. The semisimple Lie groups (with finite center) with a non-empty discrete series of irreducible unitary representations have been characterized by Harish-Chandra [5], who also has given a parametrization of the set of all equivalence classes of square integrable representations. His results were extended by Wolf [15] to the case of a reductive group. Finally, W. Schmid [13] has recently proved that all the discrete series representations can be realized in terms of the geometric quantization, as was conjectured by Kostant and Langlands. We shall present here the « orbital picture » of the relatively discrete series, extracted from their results.

As above, G is a connected and simply connected reductive Lie group, with Lie algebra $\mathfrak{g}$ and center Z. The Killing form on $\mathfrak{g}$ permits to identify $\mathfrak{g}$ with $\mathfrak{g}^*$; let $f \mapsto x_f$ be the canonical bijection of $\mathfrak{g}^*$ onto $\mathfrak{g}$. An element $f \in \mathfrak{g}^*$ will be called *elliptic* if the corresponding $x_f \in \mathfrak{g}$ has the following properties: $ad(x_f)$ is semisimple and $G(f)/Z$ is compact. An orbit $X \in \mathfrak{g}^*/Ad^*(G)$ will be called *elliptic* if it consists of elliptic elements, and will be called *regular* if it is of maximal possible dimension. The set of all regular elliptic orbits in $\mathfrak{g}^*$ will be denoted Λ.

The group G has a non-empty relative discrete series if and only if Λ is non-empty. Moreover there is a canonical bijection between Λ and the set $\hat{G}_{disc}$ of all equivalence classes of square integrable representations of G. If $X \in \Lambda$, we denote by $[\pi_X]$ the corresponding class in $\hat{G}_{disc}$, and conversely if $[\pi] \in \hat{G}_{disc}$ we denote by X_π the corresponding orbit in Λ. The irreducible unitary representation π_X associated to an orbit $X \in \Lambda$ can be realized on an L^2-cohomology space of a holomorphic line bundle over X, by a construction which fits into the general framework of quantization.

We mention finally the following fact to be used below. If $X \in \Lambda$ and $f \in X$, then $G(f)$ is a connected Cartan subgroup of G and $G(f)/Z$ is a compact Cartan subgroup of G/Z.

3.2. Here are our results concerning the reductive case.

3.2.1. PROPOSITION. — *Let π be a square integrable representation of the connected and simply connected reductive Lie group G. Then π admits a system of coherent states based on X_π.*

Proof. — Let $f \in X_\pi$. Since $G(f)$ is abelian, there exists a character χ of $G(f)$ extending the central character ζ_π of π. Denoting by $\pi_{G(f)}$ the restriction of π to $G(f)$, we see that $\chi^{-1} \otimes \pi_{G(f)}$ is trivial on Z, hence it becomes a representation of $G(f)/Z$. But this quotient group is a torus. It follows that $\pi_{G(f)}$ splits as a direct sum of unitary characters of $G(f)$. Now pick a character λ occuring in this direct sum decomposition and then a vector φ of norm one in the isotypic subspace of $\mathscr{H}_\pi$ corresponding to λ. Since G is unimodular, the dense subspace of $\mathscr{H}_\pi$ involved in the definition (SIR″) is in fact the whole $\mathscr{H}_\pi$, so that we have

$$\int | \langle \psi, \pi(g)\varphi \rangle |^2 d\mu_{G/Z}(gZ) < \infty$$

for any $\psi \in \mathscr{H}_\pi$. Finally, by making use of Proposition 1.1.2, we get that $\{ \varphi ; \lambda \}$ defines a π-system of coherent states based on X_π.

3.2.2. THEOREM. — *Let π be an irreducible unitary representation of the connected and simply connected reductive Lie group G. If π admits a system of coherent states, then π' is a square integrable representation of $G' = G/Ker\ \pi$.*

Proof. — By Corollary 1.2.3 it is enough to prove that π'' is a square integrable representation of $G'' = G/(Ker\ \pi)_0$. On the other hand, in view of Proposition 1.1.2 we may enlarge, if necessary, the subgroup involved in the system of coherent states of π (whose existence is assumed by hypothesis), to include $(Ker\ \pi)_0$. In this way, π'' will inherit a system of coherent states too. Thus, there will be no loss of generality in assuming from the begining that $Ker\ \pi$ is discrete.

Now G being reductive, it is a direct product $A \times S$ with A abelian and S semisimple. Further, the irreducibility of π implies that $\pi = \alpha \otimes \sigma$, exterior tensor product, where α is a unitary character of A and σ an irreducible unitary representation of S acting on the same Hilbert space as π. We parenthetically add that $A \simeq \mathbb{R}$, since $Ker\ \pi$ is discrete. The same reason ensures us that $Ker\ \sigma$ is discrete too.

We want to prove that σ admits itself a system of coherent states. To this end let us note that if the π-system of coherent states is defined by a pair $\{ \varphi ; \nu \}$, with ν a unitary character of a closed subgroup K, then by using once more Proposition 1.1.2, we may consider that K is the set of all $k \in G$ such that $\pi(k)\varphi \in \mathbb{C}\varphi$. In this case K contains both $Ker\ \pi$ and the center Z of G. Furthermore, we will show that $K = A \times H$, where H denotes the projection of K onto S.

Indeed, we first have $A \times \{ e \} \subset H$ since $\pi(a, e)\varphi = \alpha(a)\varphi \in \mathbb{C}\varphi$, for any $a \in A$. On the other hand, if $h \in H$, there exists $a \in A$ such that $k = (a, h) \in K$ and then

$$\alpha(a)\sigma(h)\varphi = \pi(k)\varphi = \nu(k)\varphi ,$$

hence

$$\pi(e, h)\varphi = \sigma(h)\varphi = \alpha(a)^{-1}\nu(k)\varphi \in \mathbb{C}\varphi .$$

Thus $K = A \times H$; accordingly, $\nu = \alpha \otimes \lambda$, where λ is the restriction of ν to $\{ e \} \times H$. Further, we have for any $\psi \in \mathcal{H}_\pi = \mathcal{H}_\sigma$

$$\int | \langle \psi, \pi(a, s)\varphi \rangle |^2 d\mu_{G/K}((a, s)K) = \int | \langle \psi, \sigma(s)\varphi \rangle |^2 d\mu_{S/H}(sH)$$

which shows that $\{ \varphi ; \lambda \}$ defines a σ-system of coherent states based on S/H.

Let C be the center of S. Since $Z = A \times C$ is contained in K, C will be contained in H. We intend to prove that H/C is compact. This will be done by using Theorem 2 in [8].

We need to introduce some more notations. The group S is a direct product $S_1 \times \ldots \times S_n$ of simple groups and consequently $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_n$ where σ_i is an irreducible representation of S_i. We denote by S^* (resp. S_i^*) the adjoint group of S (resp. S_i) and observe that $S^* = S/C$. Finally let $p : S \to S^*$ and $p_i : S \to S_i^*$ be the canonical projections.

These being settled, we claim that $H/C = p(H)$ is compact. Indeed, assume the contrary. Then there exists $1 \leq j \leq n$ such that $p_j(H)$ is non-compact. In addition, $\sigma(h)\varphi \in \mathbb{C}\varphi$ for any $h \in H$. In view of [8, Th. 2] these two facts imply that $\sigma_i(s_i)\varphi = \varphi$ for any $s_i \in S_i$. Since σ_i is irreducible, it follows that σ_i is the trivial representation of S_i. Hence $S_i \subset Ker\ \sigma$, which contradicts the discretness of $Ker\ \sigma$.

Now, for a suitably normalized invariant measure $\mu_{S/H}$ on S/H, we have

$$\begin{aligned} \int | \langle \psi, \sigma(s)\varphi \rangle |^2 d\mu_{S/C}(sC) &= \int d\mu_{S/H}(sH) \int | \langle \psi, \sigma(sh)\varphi \rangle |^2 d\mu_{H/C}(hC) \\ &= \mu_{H/C}(H/C) \int | \langle \psi, \sigma(s)\varphi \rangle |^2 d\mu_{S/H}(sH) < \infty , \end{aligned}$$

for any $\psi \in \mathcal{H}_\sigma$. Thus, σ is a square integrable representation of S.

To conclude the proof it suffices to notice that

$$\int | \langle \psi, \pi(a, s) \rangle |^2 d\mu_{G/Z}((a, s)Z) = \int | \langle \psi, \sigma(s)\varphi \rangle |^2 d\mu_{S/C}(sC),$$

which means that π is a square integrable representation.

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