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WOLFGANG GRAF

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Section A :
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Differential forms as spinors

by

Wolfgang GRAF

Fachbereich Physik der Universität,
D-7750 Konstanz, Germany

ABSTRACT. — An alternative notion of spinor fields for spin 1/2 on a pseudoriemannian manifold is proposed. Use is made of an algebra which allows the interpretation of spinors as elements of a global minimal Clifford-ideal of differential forms. The minimal coupling to an electromagnetic field is introduced by means of an « U(1)-gauging ». Although local Lorentz transformations play only a secondary role and the usual two-valuedness is completely absent, all results of Dirac's equation in flat space-time with electromagnetic coupling can be regained.

1. INTRODUCTION

It is well known that in a riemannian manifold the laplacian $\square := -(d\delta + \delta d)$ operating on differential forms admits as « square root » the first-order operator $d - \delta$, d being the exterior derivative and δ the generalized divergence.

In 1928 Dirac solved a similar problem in the riemann-flat manifold of special relativity by introducing the differential matrix operator $\gamma^\mu \dfrac{\partial}{\partial x^\mu}$ acting on spinor fields.

That these two problems and their solutions despite their appearance are essentially the same was not recognized until 1960/1961, when E. Kähler showed [1] that at least for flat space-time Dirac's equation

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu} = \varkappa \psi \tag{1.1}$$

[1] Compare also Lichnerowicz (1962, 1964) with regard to the Petiau-Duffin-Kemmer algebra.

can be completely reinterpreted in terms of certain inhomogeneous differential forms ϕ obeying

$$(d - \delta)\phi = \varkappa\phi . \tag{1.2}$$

It is our purpose to develop the corresponding notion of spinor field on a riemannian manifold based entirely on differential forms and to propose it as a conceptually more economical alternative to the usual notion ([2]). Our proposal will rest on the following facts, most of them classical ([3])

a) spinors (for spin 1/2) emerge naturally in the representation theory of Clifford-algebras (whereas the Lorentz group has in addition also tensor and higher spin representations),

b) there is an isomorphism of the linear structures underlying a Clifford-algebra over a vector space V with inner product on the one hand and the exterior algebra over V on the other hand,

c) the Grassmann-algebra over V can be regarded in a natural way as a (reducible) representation module of the Clifford-algebra over V,

d) the finite dimensional irreducible representations of the Clifford-algebra are isomorphic (as modules) to its minimal ideals,

e) there is a unique bijection (implicit in Chevalley (1954), explicitly given by Kähler (1960) and also used by Atiyah (1970)) which for a given vector space V and quadratic form Q maps any Clifford-algebra onto the corresponding Grassmann-algebra (= exterior algebra with inner product induced by Q),

f) this map extends to the corresponding differential operators: Dirac's operator $\gamma^\mu\nabla_\mu$ on the one hand, $d - \delta$ on the other.

The main difference to the usual treatment of spinor fields is our use of vector bundles related to the cotangent bundle instead of principal bundles with structure groups homomorphic to the rotation groups $O(p, q)$. This choice guarantees the naturalness of *f*) and amounts to considerable technical simplifications.

Another peculiarity worth mentioning is the following: our algebras and bundles, being derived from the cotangent space of real manifolds, will be primarily over the reals. Their complexification is then made by « gauging » with the circle group U(1). This has the advantage of automatically introducing minimal interactions in terms of gauge-covariant derivatives.

([2]) In which a spinor field on a riemannian manifold is a cross section of the associated bundle of type id to the spin bundle which is an extension of the bundle of orthonormal frames (comp. Lichnerowicz (1964)).

([3]) Comp. Brauer and Weyl (1935), Chevalley (1954), Rashevskii (1957), Bourbaki (1959).

A very thorough analysis of representations of Clifford-bundles was done by Karrer (1973) and recently slightly generalized from a different point of view by Popovici (1976).

2. SOME ALGEBRAS AND THEIR RELATIONS

Let us start from a real n (finite) dimensional vector space V (in our later applications it will be the cotangent space of a differentiable manifold) and recall some definitions and elementary properties (comp. Chevalley (1954, 1955), Bourbaki (1958, 1959)):

A) The *tensor algebra* T(V) over $\mathbb{R}$ is the $\mathbb{R}$-vector space of the direct sum of the powers $\otimes^p V$ together with the usual associative tensor product $\otimes$ of its elements. Since V is a finite dimensional vector space, it is canonically isomorphic to its image $\otimes^1 V$ in T(V). Therefore we will identify $\otimes^1 V$ with V and define $\otimes^0 V := \mathbb{R}$. This tensor-algebra $T(V) = \left(\bigoplus_{p=0}^{\infty} \otimes^p V, \otimes \right)$ is $\mathbb{Z}$-*graded*: $(\otimes^p V) \otimes (\otimes^q V) \subset \otimes^{p+q} V$ and infinite-dimensional if $n \geqslant 1$.

On T(V) there are two important involutive morphisms (both being linear automorphisms of $\oplus \otimes^p V$):

a) the *main automorphism* α with

$$\alpha(a \otimes b) = \alpha(a) \otimes \alpha(b) \tag{2.1}$$

$$\alpha(a) = a \quad \text{if} \quad a \in \otimes^0 V, \qquad \alpha(a) = -a \quad \text{if} \quad a \in \otimes^1 V, \tag{2.2}$$

b) the *main anti-automorphism* β mapping T(V) to its opposite algebra:

$$\beta(a \otimes b) = \beta(b) \otimes \beta(a), \tag{2.3}$$

with

$$\beta(a) = a \quad \text{if} \quad a \in \otimes^0 V + \otimes^1 V. \tag{2.4}$$

B) The *exterior algebra* $\Lambda(V)$ over the $\mathbb{R}$ vector space V can be defined as the quotient-algebra T(V)/J of T(V) by the two-sided ideal $J \subset T(V)$ generated by the elements of the form $a \otimes a$, where $a \in V$.

As customary, we will denote exterior multiplication by the sign $\wedge$. Since J is homogeneous in the $\mathbb{Z}$-gradation of T(V) also $\Lambda(V)$ is $\mathbb{Z}$-graded: $\Lambda(V) = \oplus \Lambda^p(V)$, with $\Lambda^p(V) \wedge \Lambda^q(V) \subset \Lambda^{p+q}(V)$. As before, we make the identifications $\Lambda^1(V) = V$ and $\Lambda^0(V) = \mathbb{R}$. The subspaces $\Lambda^p(V)$ are $\binom{n}{p}$-dimensional and $\Lambda(V)$ is 2^n-dimensional. For homogeneous elements $a \in \Lambda^p(V)$ and $b \in \Lambda^q(V)$, their exterior product is either commutative or anticommutative

$$a \wedge b = (-1)^{pq} b \wedge a. \tag{2.6}$$

The morphisms α and β of T(V) pass to the quotient $\Lambda(V)$. Denoting them with the same symbols α and β we now have

(2.7) $$\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$$

(2.8) $$\beta(a \wedge b) = \beta(b) \wedge \beta(a)$$

(2.9) if $a \in \Lambda^p(V)$, then $\alpha(a) = (-1)^p a$ and $\beta(a) = (-1)^{\binom{p}{2}} a$.

C) As *Grassmann-algebra* $\Lambda(V, Q)$ we will denote the pair $(\Lambda(V), Q)$ consisting of an exterior algebra $\Lambda(V)$ together with an inner product $(\,,\,)_Q : \Lambda(V) \times \Lambda(V) \to \mathbb{R}$ induced in $\Lambda(V)$ by a quadratic form Q over V as follows ([4]):

i) if $a \in \Lambda^p(V)$ and $b \in \Lambda^q(V)$ with $p \neq q$, then $(a, b)_Q := 0$,

ii) if $a = a_1 \wedge a_2 \wedge \ldots \wedge a_p$ and $b = b_1 \wedge \ldots \wedge b_p$ with $a_i, b_i \in \Lambda^1(V)$, then $(a, b)_Q := \det(B(a_i, b_j))$, where B is the bilinear form associated to Q by

(2.10) $$2B(x, y) := Q(x + y) - Q(x) - Q(y)$$

iii) the case of general $a, b \in \Lambda(V)$ can then be reduced by linearity to *i*) and *ii*).

D) The *Clifford-algebra* C(V, Q) of the real vector space V with quadratic form Q is defined as the quotient algebra T(V)/J′, where the two-sided ideal J′ is generated by elements of the form $a \otimes a - Q(a)\cdot 1$, with $a \in V$. As before, we can and will identify V with its image in C(V, Q). Denoting Clifford-multiplication by the sign $\vee$ ([5]), we have for $a, b \in V$ the familiar relations ([6])

(2.11) $$a \vee b + b \vee a = 2B(a, b),$$

with the bilinear form B as defined in (2.10). The ideal J′ being inhomogeneous of even degree in T(V) induces a $\mathbb{Z}_2$-gradation of the Clifford-algebra, $C(V, Q) = C^+ + C^-$, where C^+ is the image of the elements of even degree in T(V) and C^- is the image of the elements of odd degree in T(V). Since $\alpha(J') = \beta(J') = J'$, the morphisms α and β induce morphisms (designated by the same symbols) in C(V, Q)

(2.12) $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$

(2.13) $\beta(a \vee b) = \beta(b) \vee \beta(a)$

(2.14) $\alpha(a) = \beta(a) = a$ if $a \in \mathbb{R}$, $\quad -\alpha(a) = \beta(a) = a$ if $a \in V$.

In particular, for $a^+ \in C^+$ $\alpha(a^+) = a^+$ and for $a^- \in C^-$ $\alpha(a^-) = -a^-$.

([4]) If there is no danger of confusion, instead of $(a, b)_Q$ we shall also write $a.b$.

([5]) If there is no risk of confusion, we will also write ab instead of $a \vee b$.

([6]) Compare for example Kastler (1961). More suggestively, if $\{e^i\}$ is any basis in V, we have $e^ie^j + e^je^i = 2g^{ij}$, with $g^{ij} := B(e^i, e^j)$.

Remark. — The Clifford-algebra as defined above although closely related is not even abstractly isomorphic to the Clifford-algebra generally used in physics, which is a matrix algebra generated by matrices γ^i such that $\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij}$. Whereas the former is an algebraic structure with a distinguished subspace V, the latter does not pay attention to the particular set of γ^i (in fact, all such sets are equivalent under $\gamma^i \mapsto \pm S\gamma^i S^{-1}$). This distinction can also be seen in their automorphism groups: for non-degenerate Q, for the former it is the $\binom{n}{2}$-parametric rotation group $O(p, q)$, whereas for the latter it is an $(2^n - 1)$ (resp. $(2^n - 2)$)-parametric Lie group for n even (resp. odd).

3. THE KÄHLER-ATIYAH ALGEBRA

In this section we will define a new algebraic structure (7) over $\oplus \Lambda^p(V)$ containing both the Grassmann-algebra $\Lambda(V, Q)$ and the Clifford-algebra $C(V, Q)$ as substructures.

First, for any element X of the dual vector space V* define the *contraction* of an element of T(V) with $X \in V^*$ as the $(V^*, T(V))$-bilinear map $V^* \times T(V) \to T(V)$ of degree -1 with

i) $X \lrcorner 1 = 0$

ii) $X \lrcorner a = X(a)$, if $a \in V \subset T(V)$

iii) $X \lrcorner (a \otimes b) = (X \lrcorner a) \otimes b + \alpha(a) \otimes (X \lrcorner b)$.

(In particular, $X \lrcorner X \lrcorner$ will annihilate any element of T(V).)

Since $X \lrcorner J = J$ and $X \lrcorner J' = J'$ the contraction also passes to the quotients $\Lambda(V)$ and $C(V, Q)$ and to $\Lambda(V, Q)$, and we have

$$X \lrcorner (a \wedge b) = (X \lrcorner a) \wedge b + \alpha(a) \wedge (X \lrcorner b) \quad \text{if} \quad a, b \in \Lambda(V), \Lambda(V, Q) \tag{3.1}$$

$$X \lrcorner (a \vee b) = (X \lrcorner a) \vee b + \alpha(a) \vee (X \lrcorner b) \quad \text{if} \quad a, b \in C(V, Q), \tag{3.2}$$

with

$$X \lrcorner a = 0 \qquad \text{if} \quad a \in \mathbb{R} \subset \Lambda(V), \Lambda(V, Q), C(V, Q) \tag{3.3}$$

and

$$X \lrcorner a = X(a) \quad \text{if} \quad a \in V \subset \Lambda(V), \Lambda(V, Q), C(V, Q).$$

Define the element $\tilde{a} \in V^*$.

Q-adjoint to $a \in V \subset T(V), \Lambda(V), \Lambda(V, Q), C(V, Q)$ to be the linear function

$$\tilde{a} \lrcorner b = B(a, b), \qquad b \in V. \tag{3.4}$$

(7) Already introduced by Kähler (1960) and Atiyah (1970).

For any $a \in \mathrm{V} \subset \Lambda(\mathrm{V}, \mathrm{Q})$ and $b \in \Lambda(\mathrm{V}, \mathrm{Q})$ define their product $a \nabla b$ as

$$a \nabla b := a \wedge b + \tilde{a} \lrcorner b . \tag{3.5}$$

Then

$$a \nabla a = \mathrm{Q}(a) \cdot 1 . \tag{3.6}$$

By the theorem on the universality of Clifford-algebras (comp. Chevalley (1954, 1955), Bourbaki, 1959), the ∇-algebra generated by this relation on the elements of $\Lambda(\mathrm{V}, \mathrm{Q})$ is the Clifford-algebra $\mathrm{C}(\mathrm{V}, \mathrm{Q})$ with $\vee$ replaced by ∇.

Conversely, if for a Clifford-algebra $\mathrm{C}(\mathrm{V}, \mathrm{Q})$ we define the Δ-product of $a \in \mathrm{V} \subset \mathrm{C}(\mathrm{V}, \mathrm{Q})$ with $b \in \mathrm{C}(\mathrm{V}, \mathrm{Q})$ as

$$a \Delta b := a \vee b - \tilde{a} \lrcorner b , \tag{3.7}$$

we get

$$a \Delta a = 0 , \tag{3.8}$$

which is the defining relation of the exterior algebra. Since in the Clifford-algebra $a \vee a = \mathrm{Q}(a)$, this exterior-algebra can be made a Grassmann-algebra.

This correspondence of Clifford- and Grassmann-algebras does not depend on Q being nondegenerate or not, in particular if $\mathrm{Q} = 0$, the Q-adjoint vanishes and $\mathrm{C}(\mathrm{V}, 0) = \Lambda(\mathrm{V}, 0) = \Lambda(\mathrm{V})$.

We now have the following situation:

on the direct sum $\oplus \Lambda^p(\mathrm{V})$ of the linear spaces $\Lambda^p(\mathrm{V})$ we can not only impose the structure of a Grassmann-algebra by means of $\wedge$ and Q, but also the structure of a Clifford-algebra, and any of the two multiplications $\wedge$ and $\vee$ can be reduced to the other [8]. Consequently, we make the following definition:

a *Kähler-Atiyah-algebra* $\mathrm{KA}(\mathrm{V}, \mathrm{Q})$ corresponding to a vector space V with quadratic form Q is the quadruple $(\oplus \Lambda^p(\mathrm{V}), \wedge, \cdot, \vee)$ consisting of the elements $\oplus \Lambda^p(\mathrm{V})$ together with an exterior product $\wedge$, an inner product $\cdot$ induced by Q (comp. 2.C) and a Clifford-product $\vee$, such that $a \vee b = a \wedge b + a \,.\, b$, $a, b \in \Lambda^1(\mathrm{V})$ and the products $\wedge, ., \vee$ being distributive with respect to addition.

Neglecting $\vee$ we have an Grassmann-algebra $\Lambda(\mathrm{V}, \mathrm{Q})$. If moreover we neglect . we have an exterior algebra. And neglecting $\wedge$ and . we have a Clifford-algebra.

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(8) Reminding the reduction of union to intersection in a boolean lattice by means of complementation.

For general elements, Clifford- and exterior product are related as follows (comp. Kähler (1960, 1962)):

$$(3.9)\quad a \vee b = $$
$$= \sum_p \frac{(-1)^{\binom{p}{2}}}{p!} g^{i_1 j_1} \dots g^{i_p j_p} \alpha^p(\overset{*}{e}_{i_1} \lrcorner \dots \overset{*}{e}_{i_p} \lrcorner a) \wedge (\overset{*}{e}_{j_1} \lrcorner \dots \overset{*}{e}_{j_p} \lrcorner b)$$

$$(3.10)\quad a \wedge b = $$
$$= \sum_p \frac{(-1)^{\binom{p}{2}}}{p!} g^{i_1 j_1} \dots g^{i_p j_p} (\overset{*}{e}_{i_1} \lrcorner \dots \overset{*}{e}_{i_p} \lrcorner \alpha^p(a)) \vee (\overset{*}{e}_{j_1} \lrcorner \dots \overset{*}{e}_{j_p} \lrcorner b),$$

where $g^{ik} := \mathrm{B}(e^i, e^k)$, and $\overset{*}{e}_i$ is the dual basis to any basis $e^j \in \Lambda^1(\mathrm{V})$ [9]. In what follows, Kähler's formulas (3.9) and (3.10) will never be used save for the special cases a or $b \in \Lambda^1(\mathrm{V})$:

$$(3.11)\qquad a \in \Lambda^1(\mathrm{V}) : a \vee \psi = a \wedge \psi + \tilde{a} \lrcorner \psi ,$$

$$(3.12)\qquad b \in \Lambda^1(\mathrm{V}) : \phi \vee b = b \wedge \alpha\phi - \tilde{b} \lrcorner \alpha\phi .$$

Moreover, for Q nondegenerate, instead of (3.9) we have the more compact and easier to handle expression

$$(3.13)\quad \phi \vee \psi = \sum_p \frac{1}{p!} g_{i_1 j_1} \dots g_{i_p j_p} (\phi \,.\, e^{i_1} \dots e^{i_p} \beta\psi) e^{j_1} \wedge \dots \wedge e^{j_p} ,$$

in which the p-elements of the product are directly displayed. The factors $e^{i_1} \dots e^{i_p} \beta\psi$ are calculated successively with the aid of (3.11), and g_{ij} denotes the inverse matrix of g^{ij}. Our formula (3.13) is easily demonstrated using the general identity

$$(3.14)\qquad (\beta\psi \vee \phi)_0 = \psi \,.\, \phi$$

between the scalar part of the Clifford-product and the inner product.

4. ALGEBRAIC SPINORS

As the Clifford-algebra C(V, Q) is semisimple (simple for n even) its finite-dimensional irreducible representations are already given by its

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[9] Taken in conjunction, $\vee$ and $\wedge$ are not associative, e. g.

$$e^i \vee (e^j \wedge e^k) - (e^i \vee e^j) \wedge e^k = g^{ik} e^j$$

and . is not even associative by itself, e. g.

$$r \,.\, (e^i \,.\, e^j) - (r \,.\, e^i) \,.\, e^j = r g^{ij}, \qquad r \in \mathbb{R}.$$

minimal ideals; we shall take the left ideals I_L (10). Consequently we define:

The elements of I_L we call *algebraic spinors* corresponding to the minimal left ideal I_L (11).

The decomposition of C(V, Q) into minimal ideals can also be characterized (comp. van der Waerden (1967)) by a *spectral set* $\{ P_i \}$ of $\vee$-idempotent elements of C(V, Q) such that

i) $\Sigma P_i = 1$
ii) $P_i P_j = 0$ for $i \neq j$, $\quad P_i P_j = P_i$ for $i = j$
iii) rank P_i is minimal $\neq 0$,

where rank P is defined as the rank of the $\oplus \Lambda^p(V)$-morphism $\mathscr{P} : \psi \mapsto \psi P$. Then $C(V, Q) = \Sigma I_{L_i}$, $I_{L_i} := C(V, Q)P_i$ and a I_{L_i}-spinor is a $\psi \in C(V, Q)$ such that $\psi P_i = \psi$. Conversely, any I_L-spinor can be characterized by an idempotent P of minimal rank $\neq 0$ with $\psi P = \psi$.

Taking as we have done the minimal ideals themselves as representation modules instead of some vector spaces isomorphic to them, we have to be careful about the notion of equivalence of representations.

It is classical that *i*) any simple algebra with unit is isomorphic to a matrix algebra over a field (theorem of Wedderburn), *ii*) the automorphisms of such an algebra are given by its inner automorphisms $\psi \mapsto S\psi S^{-1}$ (theorem of Noether-Skolem) and *iii*) their finite-dimensional irreducible representations are all equivalent under inner automorphisms.

However, our Clifford-algebra C(V, Q) not being just an algebra (simple for n even) but an algebraic structure essentially consisting of an algebra together with a distinguished subspace V, and our representation spaces I being certain subalgebras of C(V, Q), the above mentioned classical results apply only in part. In particular, the Noether-Skolem-theorem restricts the automorphisms to inner automorphisms $\psi \mapsto s(\psi) := S\psi S^{-1}$ such that $SVS^{-1} \subset V$ — for Q nondegenerate they are isomorphic to the orthogonal group O(Q)! (12) Consequently we define:

two representations I_L and I'_L (not necessarily irreducible) of C(V, Q) are *equivalent*, if there is an automorphism s of C(V, Q) such that $I'_L = s(I_L)$.

Remarks : *i*) this is the only place in our approach to spinors where the (generalized) « Lorentz-group » O(Q) appears. Our ideals I_L are just representations of the Clifford-algebra. They could be made into spinorial representations of O(Q) by postulating $I_L \mapsto SI_L$ under automorphisms s

(10) Recall a left ideal I is a subset of C(V, Q) such that C(V, Q)I $\subset$ I. It is called a minimal ideal if it does not contain any smaller left ideal different from I and zero.

(11) Compare Chevalley (1955), Bourbaki (1959) and Crumeyrolle (1969) for essentially the same definition for the case of a neutral quadratic form Q.

(12) For n odd, the only additional automorphism which appears is $\psi \mapsto \alpha(\psi)$ leading also the full orthogonal group O(Q) as automorphism group.

(compare Chevalley (1955)). Although this would correspond to the usual transformation of spinors, this would contradict our I's being substructures of C(V, Q),

ii) the irreducible representations are now in general not all equivalent ([13]). As the discussion of Kähler's Dirac equation will show, the different equivalence classes also behave physically differently,

iii) by means of the Kähler-Atiyah-algebra KA(V, Q), spinors can now be interpreted as elements of the Grassmann-algebra $\Lambda(V, Q)$. In particular, the Grassmann-algebra itself can be considered as a (in general reducible) representation module of the Clifford-algebra corresponding to the idempotent $P = 1$,

iv) characterizing the representations I_L by their corresponding idempotent elements P will provide a convenient starting point for the globalization done in the next sections.

Before we globalize, a short discussion of the orthogonal group O(Q) will be useful. We suppose Q to be nondegenerate. Then the elements of O(Q) can be characterized by means of the theorem of Cartan-Dieudonné (comp. Chevalley (1954)) as follows:

if for an endomorphism $s : V \to V$.

$Q(s(x)) = Q(x)$, then s is a finite product of *symmetries* s_z with respect to the hyperplane orthogonal to z:

$$x \mapsto s_z(x) := x - 2z(x.z)/(z.z) \quad ([14]).$$

In the language of Clifford-algebras such a symmetry is more compactly expressed as $x \mapsto s_z(x) = -zxz^{-1}$ (compare Rashevskii (1955/1957)).

The four disconnected pieces of the orthogonal group O(Q) (two, if Q is definite) can then conveniently characterized in terms of the element $S \in C(V, Q)$ corresponding up to a sign to the automorphism s by two discrete parameters ε_1 and ε_2 as follows:

a) case $n = p + q$ *even* (where (p, q) is the signature of Q).

For any automorphism s of C(V, Q) there is a $S \in C(V, Q)$ such that

$$\psi \mapsto S\psi S^{-1} = s(\psi) \tag{4.1}$$

and

$$\begin{cases} \alpha(S) = \varepsilon_1 S & \text{with } \varepsilon_1, \varepsilon_2 = \pm 1 \quad \text{if } p, q \neq 0 \\ \beta(S) = \varepsilon_2 S^{-1} & \text{and } \varepsilon_1 = \pm 1, \varepsilon_2 = 1 \quad \text{if } p \text{ or } q = 0 . \end{cases} \tag{4.2}$$

b) case $n = p + q$ *odd*.

([13]) Since for any invertible $T \in C(V, Q)$ also TPT^{-1} is idempotent if P is, there are in general more than countably many equivalence classes $[I_L]$.

([14]) $x \mapsto -s_z(x)$ is the reflection of the vector x on the hyperplane orthogonal to z, z being nonisotropic ($Q(z) \neq 0$).

For any automorphism s of C(V, Q) there is a $S \in C(V, Q)$ such that

$$\psi \mapsto s(\psi) = S\alpha^k(\psi)S^{-1} \tag{4.3}$$

with $k := (1 - \varepsilon_1)/2$ and

$$\left\{ \begin{array}{llll} \alpha(S) = \varepsilon_1 S & \text{with} \quad \varepsilon_1, \varepsilon_2 = \pm 1 & \text{if} & p, q \neq 0 \\ \beta(S) = \varepsilon_2 S^{-1} & \text{and} \quad \varepsilon_1 = \pm 1, \varepsilon_2 = 1 & \text{if} & p \text{ or } q = 0 . \end{array} \right. \tag{4.4}$$

For small n there is even a converse, given by the following

THEOREM. — If $n \leqslant 5$, then for any $S \in C(V, Q)$ obeying (4.2) or (4.4), $s \in O(Q)$.

Proof (sketch): for any invertible $S \in C(V, Q)$ we can put $S = \exp(t\sigma)$, with $t \in \mathbb{R}$ and $\sigma \in C(V, Q)$ normalized to $\sigma \cdot \sigma = 0, \pm 1$. If $\varepsilon_1 = \varepsilon_2 = 1$ then (4.2, 4) imply $\alpha(\sigma) = -\beta(\sigma) = \sigma$. For $n \leqslant 5$ the only solution is $\sigma \in \Lambda^2(V, Q)$, that is, σ is the generator of an infinitesimal rotation $\psi \mapsto [\sigma, \psi]$ and consequently $S = \exp(t\sigma) \in O(Q)$. In the remaining cases $(\varepsilon_1, \varepsilon_2) = (-1, 1), (-1, -1)$ and $(1, -1)$ (the two last cases not occuring if Q is definite) multiply S by the special rotations x, t and xt with $x, t \in V$ and $x^2 = -t^2 = 1$, reducing S to the case $\varepsilon_1 = \varepsilon_2 = 1$ already treated.

5. SOME VECTOR BUNDLES RELATED TO THE COTANGENT BUNDLE

Since the algebraic structures considered above all posess a $\mathbb{R}$-linear structure inherited from the vector space V, for their generalization to manifolds we will use the formalism of vector bundles (with additional algebraic structures) ([15]). Our manifolds M will be real finite (n-)dimensional C^∞-manifolds. Also our bundles, cross sections and maps will be C^∞.

A) The basic bundle, replacing V, will be the *cotangent bundle* τ_M^* of the manifold M. Cross sections $c \in \operatorname{Sec}(\tau_M^*)$ will also be called *1-forms*.

B) Given a cross section $g \in \operatorname{Sec}(\tau_M \times \tau_M)$ such that in each fiber $\pi^{-1}(x)\, g_x$ will be a quadratic form over the cotangent space $T_x(M)^*$, we denote the pair (τ_M^*, g) a *riemannian vector bundle* ([16]).

C) We denote the vector-bundle whose fibers $\Lambda T_x(M)^*$ are exterior algebras over $V = T_x(M)^*$, the *Cartan-bundle* $\Lambda\tau_M^*$ of exterior differential

([15]) Our main reference on vector bundles is Greub, Halperin and Vanstone (1972, 1973). We use their notation.

([16]) Note that this definition is slightly more general than the usual definition of Riemann space as $Q := g_x$ may be degenerate and is a form over cotangent space instead of tangent space.

forms on M. As is well known, on a Cartan-bundle the *exterior derivative d* can be uniquely characterized by the following conditions:

i) $d(a + b) = da + db$

ii) $d(a \wedge b) = da \wedge b + \alpha(a) \wedge db$

iii) $d^2 = 0$

iv) $X \lrcorner (df) = X(f)$,

for any $a, b \in \operatorname{Sec} \Lambda\tau_M^*$, $f \in \operatorname{Sec} \Lambda^0\tau_M^*$ and $X \in \operatorname{Sec} \tau_M$.

In particular, d will be homogeneous of degree $+1$ in the $\mathbb{Z}$-gradation of the ring of cross sections of $\Lambda T_M^* = (\oplus \Lambda^p\tau_M^*, \wedge)$.

D) The pair $(\Lambda\tau_M^*, g)$, where each fiber $(\Lambda T_x(M)^*, g_x)$ is a Grassmann-algebra, we call *Hodge-bundle* on M with metric g.

If for any $x \in M$, $Q := g_x$ is nondegenerate, in addition to d there is the *divergence* δ, formally g-adjoint to d, defined by [17]

$$\delta := \star^{-1} d \star \alpha \tag{5.1}$$

where the operator $\star$ (« Hodge-star ») is defined as the unique linear isomorphism $\star : \Lambda^p\tau_M^* \to \Lambda^{n-p}\tau_M^*$, given by

$$\varphi \wedge \star \psi = (\varphi . \psi)\varepsilon \tag{5.2}$$

for all p-forms $\varphi \in \operatorname{Sec} \Lambda^p\tau_M^*$, where ε is a local orienting n-form with

$$(\varepsilon . \varepsilon)^2 = 1 . \tag{5.3}$$

Whereas $\star$ is a local operator depending on an orientation, δ is independent of the particular orienting n-form and can therefore be immediately globalized also to a non-orientable M. Because $d^2 = \delta^2 = 0$ the laplacian for differential forms $-\square := d\delta + \delta d$ can be written also as a square

$$\square = (d - \delta)^2 . \tag{5.4}$$

E) A vector bundle is called a *Clifford-bundle* $\mathscr{C}(\tau_M^*, g)$ if each fiber is a Clifford-algebra $C(T_x(M)^*, g_x)$ [18].

If g is nondegenerate, there is a particular differential operator $\nabla\!\!\!\!/$ called the *Dirac-operator*, odd in the $\mathbb{Z}_2$-gradation of $C(T_x(M)^*, g_x)$, defined as follows [19].

For any $t^* \in \operatorname{Sec} \tau_M^* \subset \operatorname{Sec} \mathscr{C}(\tau_M^*, g)$ and any $t \in \operatorname{Sec} \tau_M$, consider the bilinear tensorial map of type (1, 1) given by

$$\psi \mapsto t^*\nabla_t\psi , \tag{5.5}$$

---

[17] Compare de Rham (1960).

[18] Essentially the « cotangential Clifford-bundle » of Karrer (1973) and Popovici (1976).

[19] Compare Karrer (1973).

where ψ is any element of Sec $\mathscr{C}(\tau^*_M, g)$ and ∇_t is the riemannian derivative of ψ considered as element of the tensor-bundle, in the direction of t ([20]). Then $\not\nabla$ is defined as the tensorial trace of this map ([21])

$$(5.6) \qquad \not\nabla := \mathrm{Tr}\,(t^*\nabla_t)\,.$$

In terms of a local basis $\{ e^i \}$ of 1-forms and its dual basis $\{ \overset{*}{e}_j \}$ of vector fields, we can also write

$$(5.7) \qquad \not\nabla = e^i \nabla_{\overset{*}{e}_i}\,.$$

In particular, taking a local coordinate basis $\{ dx^\mu \}$ we have

$$(5.8) \qquad \not\nabla = dx^\mu \vee \nabla_{\frac{\partial}{\partial x^\mu}}\,,$$

which is the definition used by Kähler (1960, 1962).

The relation to the Dirac operator of special relativity is as follows. In a connected and simply connected riemann-flat space-time (τ^*_M, g) there exists a global coordinate-basis $\{ dx^\mu \}$ orthonormalized with respect to the usual Lorentz bilinear form. For $n = 4$ there is a representation $dx^\mu \mapsto \gamma^\mu$ of the Clifford-bundle by 4×4 constant matrices and we get Dirac's operator in the usual form

$$(5.9) \qquad \gamma^\mu \frac{\partial}{\partial x^\mu}\,,$$

but acting on the cross sections of the representation of $\mathscr{C}(\tau^*_M, g)$. For an irreducible representation it becomes exactly Dirac's original operator acting on « four-spinors ». The only difference to the usual four-spinors consist in the action of the Lorentz group on them. This point will be discussed in section 8.

F) As *Kähler-Atiyah-bundle* $\mathscr{K}\mathscr{A}(\tau^*_M, g)$ we define the quadruple $(\oplus \Lambda^p \tau^*_M, \wedge, ., \vee)$ consisting of the vector-bundle $\oplus \Lambda^p \tau^*_M$ together with the products $\wedge, ., \vee$ such that the restriction to its fibers are Kähler-Atiyah-algebras $\mathrm{KA}(\mathrm{T}_x(\mathrm{M})^*, g_x)$. Obviously this bundle has the Cartan-, Hodge- and Clifford-bundles as substructures.

The Dirac operator can now be reformulated as follows. Take any local neighbourhood $\mathrm{U} \subset \mathrm{M}$ with coordinate basis $\{ dx^\mu \}$. Then in U, $\not\nabla = dx^\mu \vee \nabla_\mu$ can be written as $dx^\mu \wedge \nabla_\mu + \widetilde{dx^\mu} \lrcorner \nabla_\mu$. Applying this operator to a local form $\psi = \frac{1}{p!} \psi_{\rho_1 \dots \rho_p} dx^{\rho_1} \wedge \dots \wedge dx^{\rho_p}$,

$$dx^\mu \wedge \nabla_\mu \psi = \frac{1}{p!} \nabla_{[\mu} \psi_{\rho_1 \dots \rho_p]} dx^\mu \wedge dx^{\rho_1} \wedge \dots \wedge dx^{\rho_p} = d\psi$$

([20]) Because $\nabla_t \mathrm{J}' \subset \mathrm{J}'$, ∇_t passes to the quotient bundle $\mathscr{C}(\tau^*_M, g)$.

([21]) No intrinsic characterization of $\not\nabla$ seems to be known without using the riemannian derivative explicitly.

and

$$\widetilde{dx^\mu} \lrcorner \nabla_\mu \psi = \frac{1}{(p-1)!} \nabla_\mu \psi^\mu_{\rho_2 \ldots \rho_p} dx^{\rho_2} \wedge \ldots \wedge dx^{\rho_p} = - \delta\psi .$$

The form of the right-hand sides of the two last formulas being independent of any basis, the result globalizes to

$$(5.10) \qquad \nabla\!\!\!\!/ = d - \delta \qquad ([22]).$$

The operator $d + \delta$ is by construction formally g-selfadjoint, that is, with respect to the functional

$$(5.11) \qquad \int_\Omega (\varphi \cdot \psi)\varepsilon = \int_\Omega \varphi \wedge \star \psi$$

for all $\varphi, \psi \in \operatorname{Sec} \mathscr{K}\mathscr{A}(\tau^*_M, g)$ such that $\Omega := (\operatorname{supp} \varphi) \cap (\operatorname{supp} \psi)$ is compact and orientable, oriented with a normalized n-form ε. Is is easily verified that our Dirac operator $d - \delta$ is formally selfadjoint (in the above sense) with respect to

$$(5.12) \qquad \int_\Omega \alpha(\varphi) \wedge \star \psi .$$

The local bilinear form over $\mathscr{K}\mathscr{A}(\tau^*_M, g)$ making $d - \delta$ formally selfadjoint is thus given by

$$(5.13) \qquad H(\varphi, \psi) := \star^{-1}(\alpha(\varphi) \wedge \star \psi)_n .$$

This symmetric bilinear form can also be written in terms of the Clifford-product as

$$(5.14) \qquad H(\varphi, \psi) = (\alpha\beta(\varphi) \vee \psi)_0 .$$

The corresponding quadratic form is then expressible as

$$(5.15) \qquad H(\varphi) := \star^{-1}(\alpha(\varphi) \wedge \star\varphi)_n = (\alpha\beta(\varphi) \vee \psi)_0 = \alpha(\varphi) . \varphi .$$

The differential identity responsible for selfadjointness is

$$(5.16) \qquad d(\alpha(\psi) \wedge \star\varphi + \varphi \wedge \star\alpha(\psi))_{n-1} = (H(\psi, \nabla\!\!\!\!/\varphi) - H(\nabla\!\!\!\!/\psi, \varphi))\varepsilon .$$

For any two solutions of the Kähler-Dirac equation $\nabla\!\!\!\!/\psi = \varkappa\psi$, the right hand side vanishes and we get a conservation law (trivial for $\varphi = \psi$).

REMARK. — Instead of using local orienting n-forms ε, it would have been more profitable to introduce the globally defined « volume element », locally given in a coordinate neighbourhood by

$$(5.17) \qquad \mathrm{vol} := |\det g_{\mu\nu}|^{1/2} dx^1 \wedge \ldots \wedge dx^p .$$

---

[22] Compare Kähler (1960, 1962), Lichnerowicz (1964), Atiyah (1970) and Karrer (1973).

This is properly speaking not a differential form, but a pseudodifferential form which gets a sign from the jacobian determinant upon going from one coordinate patch to another (compare de Rham (1960)). The dimensionality of our algebras and bundles would then be doubled and an additional $\mathbb{Z}_2$-gradation appears.

6. SPIN STRUCTURES AND SPINOR FIELDS

Our task will now be to give an appropriate globalization of an algebraic spinor to a spinor field.

Customarily (comp. the SC-structures of Karrer (1973)) one takes the complex Clifford-bundle induced by the complexification of the basic vector bundle (in our case, the cotangent bundle). It is well known (comp. Chevalley (1954), Bourbaki (1959)) that for nondegenerate Q and even dimension $n = 2r$ or odd dimension $n = 2r + 1$ of V, the complexified Clifford-algebra $\tilde{C}(\tilde{V}, Q)$ has 2^r-dimensional modules as irreducible representations. For this reason a spinor field is then defined as a cross section of a bundle (the spinor bundle) whose fibers are irreducible (meaning 2^r-dimensional) representation modules of $\tilde{C}(\tilde{V}, Q)$. The structure of the algebras then becomes independent of the particular signature of the quadratic form (as long as it is nondegenerate) and the corresponding bundles depend only on global properties of the manifold, existing only under certain conditions.

Our algebraic spinors being real (in the next section we will complexify them by U(1)-gauging), we will use a somewhat different and finer as usual notion of global irreducibleness.

Recall the $\vee$-idempotent elements P of C(V, Q) introduced in section 4.

A *generalized spin structure* is a nontrivial cross section P of the Clifford-bundle $\mathscr{C}(\tau_M^*, g)$ such that P is $\vee$-idempotent and of minimal global rank, where the global of P rank is defined as

$$\text{rank } P := \max_{x \in M} (\text{rank } P_x). \tag{6.1}$$

This definition can be somewhat simplified as follows. Consider first the case that C(V, Q) is a simple algebra (this is the case for n even). Then by the theorem of Wedderburn, as an algebra C(V, Q) is isomorphic to an algebra of (say N $\times$ N) matrices over a field K. A Clifford-idempotent P is then mapped to an idempotent matrix P. For idempotent matrices P there is the identity trace P = rank P (where rank P is the K-rank of the matrix P), as can be seen by using the Jordan-normal form of P. Now the trace and the rank of a matrix M and the scalar part and rank of the corresponding Clifford element m are related as follows: m_0 = trace (M)/N and rank $m = (2^n/\text{N})$ rank M, as can be seen by taking $M = I$. Taking an idempotent P_x we have finally rank $P_x = 2^n(P_x)_0$. If C(V, Q) is not simple it is at most semisimple being the direct sum of two isomorphic simple algebras of dimension 2^{2r} (n being odd $= 2r + 1$). Therefore also in this remaining case rank $P_x = 2^n(P_x)_0$. We want now to show that for M connected, $(P_x)_0$ (and therefore rank P_x) is a constant, by using an infinitesimal argument. Consider a first-order variation $P \mapsto P + p$. The condition that also $P + p$ be idempotent gives $Pp + pP = p$. As P is idempotent this implies $PpP = (1 - P)p(1 - P = 0$, leaving only the terms $Pp(1 - P) + (1 - P)pP$ in the Peirce-decomposition of p with respect to P. Taking the scalar part of $p = Pp(1 - P) + (1 - P)pP$ gives $p_0 = 0$, by means of the cyclic permutability of the factors of the scalar part and $(1 - P)P = P(1 - P) = 0$. Therefore P_0 is a constant for each connected component of our manifold M. In particular, for a connected M rank P_x is a constant, and instead of (6.1) we have the handier relation

(6.1′) $\text{rank } P = \text{rank } P_x = 2^n(P_x)_0 , \qquad x \in M, \quad M \text{ connected.}$

If P is a generalized spin structure such that rank P is moreover locally minimal for almost all $x \in M$, then we speak also of an *elementary spin structure* ([23]).

In general, we will speak simply of a *spin structure*, if only a nonzero idempotent P is given. In particular, $P = 1$ will be called the *trivial spin structure.*

Obviously, the trivial spin structure $P = 1$ will always exist, whereas the existence of an elementary spin structure will in general impose global restrictions on M.

Two spin structures P and P′ are called *equivalent*, if there is an invertible cross section S with $P' = SPS^{-1}$, such that S induces an automorphism $s \in O(g)$. If moreover, for any $x \in M s_x$ belongs to the subgroup $O_1(g_x)$ containing the identity, P and P′ are called *strongly equivalent.*

The following example is physically important (Lorentz-signature !) and mathematically instructive, as it shows that the existence of a metric g with particular signature may in some cases imply the existence of elementary spin structures, without further restrictions on M.

Example. — Consider a time-orientable 4-dimensional Riemann manifold with Lorentz signature (1, − 1, − 1, − 1). On M there will always exist a timelike vector field $\tilde{t}$ with $g_x^{-1}(\tilde{t}_x) = 1$ (comp. Markus (1955)). Consequently there is a nontrivial spin structure P given by $P = (1 + t)/2$. Since P has global minimal rank 8 which is also the local minimal rank, P is an elementary spin structure for (τ_M^*, g). The time-reversed $P' = (1 - t)/2$ is another elementary spin structure, equivalent but not strongly equivalent to P.

([23]) In the complexified version, this would correspond to Karrer's SC-structure.

If moreover M is orientable, with normalized orienting n-form ε, then there are also the elementary spin structures given by $(1 \pm \varepsilon t)/2$, which are weakly equivalent. They are not equivalent to P or to P'.

Thus, for a M with g-signature $(1, -1, -1, -1)$ there are four strongly inequivalent (resp. two weakly inequivalent) classes of spin structures.

Remark. — In a riemannian manifold with the opposite Lorentz-signature $(1, 1, 1, -1)$, an everywhere timelike vector field $\tilde{t}$ does not induce any nontrivial spin structure, in spite of the fact that there are local P's with minimal rank 4. Upon complexification, the role of the two different Lorentz-signatures becomes symmetrical and $\tilde{t}$ induces nontrivial spin structures in both cases, which are however not elementary.

A P-*spinor field* on (τ_M^*, g) corresponding to the spin structure P is defined as a cross section ψ of the Clifford-bundle $\mathscr{C}(\tau_M^*, g)$, such that $\psi P = \psi$.

According to the type of spin structure, we call a P-spinor field simply a spinor field, a trivial spinor field, a non-trivial spinor field, an elementary spinor field or a generalized spinor field.

In addition it is useful to introduce the notion of a $\overset{\circ}{P}$-*local spinor field* as a cross section ψ of the Clifford-bundle, such that on supp $(\psi) \subset M$ there exists an idempotent $\overset{\circ}{P}$ with $\psi\overset{\circ}{P} = \psi$. In particular, any ψ which is non-invertible is a local spinor field.

Regarding the Clifford-bundle $\mathscr{C}(\tau_M^*, g)$ canonically embedded into a Kähler-Atiyah-bundle $\mathscr{K}\mathscr{A}(\tau_M^*, g)$, a P-spinor field ψ can now be interpreted in a natural way as a differential form. This was the main objective of the present investigation.

The arguments for differential forms as spinor fields would perhaps be more convincing if it were possible to use the usual operator $d - \delta$ now acting on P-spinor fields as Dirac operator. In particular, $\nabla\!\!\!/$ should map a P-spinor field to a P-spinor field. This is however in general not the case as we shall now discuss.

Because the riemannian derivative ∇_X is a linear endomorphism of the left ideal generated by an idempotent P if and only if $\nabla_X(\psi P)(1 - P) = 0$, we have the following theorem:

THEOREM. — ∇_X is a linear endomorphism of the left ideal generated by an idempotent P, if and only if $P\nabla_X P = 0$.

Proof: since $P^2 = P$, the product rule gives

$$\nabla_X(\psi P) = \nabla_X((\psi P)P) = \nabla_X(\psi P)P + \psi P\nabla_X P .$$

Sufficiency being evident, necessity follows because $\psi P\nabla_X P = \psi\nabla_X P(1-P)$ should vanish for any ψ (in particular, for $\psi = 1$).

As the Dirac operator $d - \delta$ can be written locally as $e^i\nabla_{\tilde{e}_i}^*$, we have the

COROLLARY. — Kähler's Dirac equation $\not{\nabla}\psi = \varkappa\psi$ has a P-spinorial solution ψ if and only if $P\nabla_X P = 0$.

Proof : By means of our assumptions we can write

$$0 = \not{\nabla}\psi - \varkappa\psi = \not{\nabla}(\psi P) - \varkappa\psi = (\not{\nabla}\psi)P + e^i\psi\nabla_{\tilde{e}_i^*}P - \varkappa\psi = (\not{\nabla}\psi)P - e^i\psi P\nabla_{\tilde{e}_i^*}P(1-P) - \varkappa\psi P.$$

The middle term in the last expression does not lie in the left P-ideal and vanishes exactly if $P\nabla_X P = 0$.

This condition

$$\text{(C)} \qquad P\nabla_X P = 0$$

imposes strong restrictions on the riemannian manifold (τ_M^*, g) as will be seen by its integrability conditions.

For any $X, Y \in \operatorname{Sec} \tau_M$, define the cotensorial riemannian *curvature 2-form* $K(X, Y)$ by the $\vee$-commutator [24]

$$(6.2) \quad (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})\psi := [K(X, Y), \psi], \qquad \psi \in \operatorname{Sec} \mathscr{K}\mathscr{A}(\tau_M^*, g).$$

In local coordinates $\{ dx^\mu \}$

$$(6.3) \qquad K(X, Y) = 1/4 X^\mu Y^\nu R_{\mu\nu\varkappa\lambda} dx^\varkappa \wedge dx^\lambda,$$

where $X = X^\mu \dfrac{\partial}{\partial x^\mu}$ and $Y = Y^\mu \dfrac{\partial}{\partial x^\mu}$.

Differentiating (6.1) with respect to Y, substracting the expression with X, Y interchanged and $P\nabla_Z P(=0)$ with $Z := [X, Y]$, we get using the identity $P(\nabla_X P)P = 0$ the relation $P[K(X, Y), P] = 0$, which can also be written

$$(6.4) \qquad PK(X, Y)(1 - P) = 0.$$

Differentiating (6.4) with respect to $Z_1, \ldots, Z_r$ and using the relations (6.1) and (6.4), we get the necessary conditions

$$(6.5) \qquad P\nabla_{Z_1} \ldots \nabla_{Z_r} K(X, Y)(1 - P) = 0,$$

which must hold for all vector fields $X, Y, Z_1, \ldots, Z_r$.

Example. — Take any elementary canonical spin structure $P = (1 + t)/2$ of a riemannian bundle with signature $(1, -1, -1, -1)$. Since $t^2 = 1$, condition (C) is now $\nabla_X t + t \wedge \nabla_X t = 0$, which holds if and only if the covariant derivative of t vanishes $\nabla_X t = 0$. The integrability conditions can be written $\tilde{t} \lrcorner \nabla_{Z_1} \ldots \nabla_{Z_r} K(X, Y) = 0$. For g the Schwarzschild metric, these conditions cannot be fulfilled. Clearly they hold in a riemann-flat space-time.

[24] Although closely related, K(X, Y) is not identical to the usual curvature 2-form, which is a $0(g)$-Lie-algebra-valued 2-form.

The difficulties connected with condition (C) can be resolved as follows: If we want *a*) not to restrict our riemannian manifolds severely, *b*) not to introduce any additional geometric structures and *c*) to retain $d - \delta$ in the spirit of Kähler as Dirac operator, we are obliged to discard the generalized spin structures in favor of the trivial spin structure $P = 1$, that is, regarding any section of $\mathscr{K}\mathscr{A}(\tau^*_M, g)$ (consequently any differential form) as spinor field ([25]).

Since $P = 1$ may be decomposed into a sum of elementary local spin structures $\overset{\circ}{P}$, this amounts in the general case to consider several irreducible representations (particles) at once, which for strong gravitational fields cannot be decoupled (particle creation).

However, there remains the remote possibility that in the general relativistic setting of coupled Einstein-Kähler-Dirac equations, selfcoupling effects plus boundary conditions trow ψ into a generalized spin structure P such that $P\nabla_X P = 0$ holds.

7. U(1)-GAUGING OF ALGEBRA BUNDLES

It is well known that the homogeneous set $dF = 0$ of Maxwell's equations can be understood in terms of connections on principal U(1)-bundles. In this context it is natural to regard the typical minimal coupling to charged matter fields as induced connections on the corresponding associated bundles ([26]). Our main problem is now to extend our algebraic vector bundles to associated U(1)-bundles in a manner which conserves and extends their algebraic structures. Let us only sketch how this may be done.

The finite-dimensional representations σ_m of U(1) are given by

$$\sigma_m(e^{i\alpha}) = e^{im\alpha}, \qquad m \in \mathbb{Z}. \tag{7.1}$$

Therefore, the U(1)-associated bundles $\Sigma^{(m)}_M$ can be indexed by integers m. In particular, for $m = 0$ we get the bundle of complex valued functions, whereas for $m \neq 0$, their cross sections may be regarded only locally as complex valued functions on M. As the fiber product of two cross sections $\psi^{(m)} \in \mathrm{Sec}\ \Sigma^{(m)}_M$ and $\psi^{(m')} \in \mathrm{Sec}\ \Sigma^{(m')}_M$ lies in $\mathrm{Sec}\ \Sigma^{(m+m')}_M$, $\bigoplus_m \Sigma^{(m)}_M$ is $\mathbb{Z}$-graded with respect to products. Call m the « charge weight » of $\psi^{(m)}$. To any connection on the principal bundle, locally given by ([27])

$$\overset{\circ}{\nabla}_X = X + iX \lrcorner A, \tag{7.2}$$

([25]) In accordance with Karrer (1973), the trivial spin structure $P = 1$ together with the local quadratic form H and the derivative ∇_X could be called a « complete spin structure ».

([26]) For a discussion of U(1)-bundles in the context of magnetic monopoles see Greub and Petry (1975).

([27]) iA is the U(1)-Lie-algebra-valued connection 1-form.

there is a corresponding connection on $\Sigma_M^{(m)}$, locally given by

$$\tilde{\nabla}_X^{(m)} = X + imX \lrcorner A. \tag{7.3}$$

Denote by ι the $\mathbb{C}$-antilinear automorphism $\iota : \Sigma_M^{(m)} \to \Sigma_M^{(-m)}$ given by complex conjugation $\psi \mapsto \bar{\psi}$.

For $\psi \in \mathrm{Sec}\,(\oplus\, \Sigma_M^{(m)})$ there is a hermitean form h, defined by

$$h(\psi, \psi') := \sum_m \overline{\psi^{(m)}}\psi'^{(m)}. \tag{7.4}$$

Between h, $\overset{\circ}{\nabla}_X$ and $\tilde{\nabla}_X^{(m)}$ there holds the identity

$$\overset{\circ}{\nabla}_X h(\psi, \psi') = h(\tilde{\nabla}_X\psi, \psi') + h(\psi, \tilde{\nabla}_X\psi'), \tag{7.5}$$

where $\tilde{\nabla}_X\psi := \sum_m \tilde{\nabla}_X^{(m)}\psi^{(m)}$.

Let us call the $\mathbb{Z}$-graded structure $(\oplus\, \Sigma_M^{(m)}, h, \tilde{\nabla}_X)$ the *generalized hermitean line-bundle* associated to the principal U(1)-bundle on M ([28]).

The U(1)-gauged bundle $\Lambda\tilde{\tau}_M^*$, of exterior forms is then defined as the complex vector bundle

$$\Lambda\tilde{\tau}_M^* := (\oplus\, \Sigma_M^{(m)}) \otimes \Lambda\tau_M^*, \tag{7.6}$$

where $\otimes$ is the Whitney-product of bundles. Moreover, we identify $(\oplus\, \Sigma_M^{(m)}) \otimes \Lambda\tau_M^*$ with $\Lambda\tau_M^* \otimes (\oplus\, \Sigma_M^{(m)})$.

If

$$\Phi_p^{(l)} = \phi^{(l)} \otimes \phi_p \qquad \text{and} \qquad \Psi_q^{(m)} = \psi^{(m)} \otimes \psi_q,$$

the additional structures are extended as follows

$$
\begin{array}{ll}
i) & \Phi_p^{(l)} \wedge \Psi_q^{(m)} := (\phi^{(l)}\psi^{(m)}) \otimes (\phi_p \wedge \psi_q) \\
ii) & \Phi_p^{(l)} \,.\, \Psi_q^{(m)} = (\phi^{(l)}\psi^{(m)}) \otimes (\phi_p \,.\, \psi_q) \\
iii) & \Phi_p^{(l)} \vee \Psi_q^{(m)} := (\phi^{(l)}\psi^{(m)}) \otimes (\phi_p \vee \psi_q) \\
iv) & \tilde{\mathrm{H}}(\Phi_p^{(l)}, \Psi_q^{(m)}) := \begin{cases} 0 , & \text{if } l \neq m \\ h(\phi^{(l)}, \psi^{(m)}) \otimes (\alpha(\phi_p).\psi_q), & \text{if } l = m \end{cases} \\
v) & \iota(\Phi_p^{(l)}) := \overline{\phi^{(l)}} \otimes \phi_q \\
vi) & \star\Phi_p^{(l)} := \phi^{(l)} \otimes \star\phi_p \qquad (\text{if } \varepsilon \text{ in given}) \\
vii) & \nabla_X^{(l)}\Phi_p^{(l)} := (\tilde{\nabla}_X^{(l)}\phi^{(l)}) \otimes \phi_p + \phi^{(l)} \otimes (\nabla_X\phi_p)
\end{array}
$$

([28]) For details on hermitean line bundles see Kostant (1970).

The gauge-covariant exterior derivative D is defined as

$$iix) \qquad D^{(l)}\Phi_p^{(l)} := \mathrm{Tr}\,\big(\overset{*}{x} \wedge \nabla_X^{(l)}\Phi_p^{(l)}\big)\,,$$

where the trace is taken with respect to the entries $\overset{*}{x}$ and X. Analogously we define the gauge-covariant divergence Δ as

$$ix) \qquad \Delta^{(l)}\Phi_p^{(l)} := \mathrm{Tr}\,\big(\tilde{\overset{*}{x}} \lrcorner \nabla_X^{(l)}\Phi_p^{(l)}\big)\,.$$

Define the $(\wedge, ., \vee)$-derivation C (« charge weight ») as

$$x) \qquad C(\Phi_p^{(l)}) = l\Phi_p^{(l)}\,.$$

All these structures then extend to the whole of $\Lambda\tau_M^*$ by $\mathbb{C}$-linearity.

As before, the divergence Δ can be expressed as $\Delta = \star^{-1}D\star\,\alpha$ and is formally selfadjoint with respect to the hermitean form

$$(7.7) \qquad ((\Phi, \Psi))_\Omega := \int_\Omega (\alpha(\Phi)\,.\,\Psi)\varepsilon\,.$$

The corresponding differential identity is

$$(7.8) \quad d(\Psi \wedge \star\,\alpha(\bar{\Phi}) + \alpha(\bar{\Phi}) \wedge \star\,\Psi)^{(0)}_{n-1} = (\alpha\bar{\Phi} \wedge \star\,\tilde{\nabla}\Psi - \alpha\overline{\tilde{\nabla}\Phi} \wedge \star\,\Psi)^{(0)}_n\,,$$

where the gauge-covariant Dirac-operator is defined as

$$(7.9) \qquad \tilde{\nabla} := D - \Delta\,.$$

The complex vector bundle $\Lambda\tau_M^*$ together with the structures thus defined, we call a U(1)-*gauged Kähler-Atiyah-bundle.*

On this new bundle a spin structure is defined exactly as before by an idempotent cross section P, with the difference that it must not necessarily be real and of zero charge-weight. If P is idempotent with C(P) = 0, we call it a *neutral spin structure* with respect to the charge-weight. As for any complex Clifford-algebra over a vector space with dimension even $= 2r$ or odd $= 2r + 1$, there locally always exist neutral P with minimal rank 2^r (compare Chevalley (1954)) with respect to the cross sections with zero charge-weight.

Example. — If in our standard example of section 6 we consider U(1)-gauged bundles, the nontrivial neutral spin structure P given by $(1 + t)/2$ is no longer elementary, as now locally there are idempotents with lowest rank 4 given for example by $P'_x := (1 + t_x)(1 + ie_x^2e_x^3)/4$, where $e_x^1 = t_x$, e_x^2 and e_x^3 are real 1-forms corresponding to an orthonormalized basis $\{ e^i \}$ at x. This situation does not change for the case of the opposite signature $(+ + + -)$, only the 1-form t has to be replaced by the neutral 1-form it.

In the orientable case there are in addition the distinguished neutral nontrivial and nonelementary spin structures of rank 8 given by $P = (1 \pm i\varepsilon)/2$.

REMARK 1. — A gauging like the U(1)-gauging introduced above could be done with any Lie-group whatsoever without impairing the algebraic structure of the Kähler-Atiyah-bundle. A promising candidate is SO(3) and the quaternions.

REMARK 2. — In flat spacetime of signature $(+ + + -)$ a Kähler-Dirac equation completely equivalent to the usual Dirac equation is obtained by choosing a P-spinor field ψ of charge weight 1 corresponding to the neutral spin structure $P = (1 + it)(1 + ie^1e^2)/4$ (for the equivalence, see Kähler (1961)).

Whereas the electromagnetic current (8.4) corresponding to P has positive definite charge density, the nontrivial spin structures $(1 \pm i\varepsilon)/2$ would lead to a vanishing electromagnetic current.

8. REMARKS ON THE ROLE OF THE PSEUDOORTHOGONAL GROUP. THE U(1)-GAUGED LIE DERIVATIVE

We have already seen (section 6) that with regard to derivatives our generalized spinors in general behave differently from the usual spinors. There is another important difference which we want now to discuss.

In our treatment of spinor fields the local transformations $\psi \mapsto S\psi$, where S may be understood as an element of a representation homomorphic to the pseudoorthogonal (« Lorentz »-) group O(Q), are conspicuously absent ([29]). Instead, the associated principal bundles to our vector bundles all have structure groups homomorphic to the full linear group $GL(\mathbb{R}, n)$ inherited from the cotangent bundle. The only place where the pseudoorthogonal group appeared, was in the definition of equivalence classes of left ideals of the Clifford-algebra (resp. bundle).

The only justification of calling our constructions « spinors » (resp. « spinor fields ») rests upon our generalized spinors being irreducible representations of the Clifford-algebra (resp. -bundle) and on the possibility of exactly mirroring Dirac's flat space-time equation including electromagnetic couplings.

However, if our riemannian bundle (τ_M^*, g) admits infinitesimal isometries, there will exist a group of global transformations as relict of the global Lorentz-transformations of the trivial and locally riemann-flat bundle. Thus: if any, global transformations will appear instead of local Lorentz-transformation. Let us now be more precise:

The *Lie derivative* L_X of a differential form ψ with respect to the vector field X is defined as usual by

$$L_X\psi := d(X \lrcorner \psi) + X \lrcorner (d\psi). \tag{8.1}$$

([29]) In fact, in our context they would result in a inconsistency.

Being the anticommutator of two $\wedge$-antiderivations, L_X is a $\wedge$-derivation, that is, the usual product rule holds with respect to the exterior product $\wedge$. L_X commutes with d. By an *infinitesimal isometry* of the Riemann bundle (τ_M^*, g) we will understand a vector field $X \in \tau_M$ (the « Killing vector field »), such that for any two 1-forms $a, b \in \operatorname{Sec}(\tau_M^*)$ the condition

$$\text{(K)} \qquad X(a.b) = (L_X a).b + a.(L_X b)$$

holds (implying that L_X is also a derivation with respect to the inner product .).

Since on the other hand $X(a.b) = (\nabla_X a).b + a.(\nabla_X b)$, (K) is equivalent to the g-skew-symmetry of the tensor field A_X of type (1, 1):

$$\text{(K')} \qquad (A_X a).b + a.(A_X b) = 0, \qquad A_X := L_X - \nabla_X.$$

With respect to a local coordinate basis $\{ dx^\mu \}$ this condition reduces to the familiar Killing-equation in the form

$$\text{(K'')} \qquad (g^{\rho\mu}X^\nu{}_{;\rho} + g^{\rho\nu}X^\mu{}_{;\rho})\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = 0.$$

If X and Y are two Killing vector fields, then also their commutator [X, Y] is a Killing field. Evidently the set of infinitesimal isometries forms a real Lie algebra. It is well known (comp. Kobayashi-Nomizu (1963), Petrov (1964)) that this algebra is at most of dimension $n(n+1)/2$, the maximal dimension being achieved for spaces of constant curvature, in particular for a trivial manifold which is locally riemann-flat.

Since the Dirac operator $\nabla\!\!\!/ = d - \delta$ involves the metric g (supposed now to be nondegenerate), L_X and $\nabla\!\!\!/$ commute if X is an infinitesimal isometry.

For U(1)-gauged fields, the notions of infinitesimal isometry and the corresponding Lie derivative have to be generalized appropriately.

Call the triple (τ_M^*, g, F) a *Riemann-Maxwell manifold*, if the pair (τ_M^*, g) is a riemannian bundle and the real 2-form F on M is closed, $dF = 0$.

An *infinitesimal Riemann-Maxwell isometry* is defined as the pair (X, λ) such that X is an infinitesimal isometry of (τ_M^*, g) (that is, condition (K) holds), and λ is a real function such that

$$\text{(M)} \qquad (X) \lrcorner F = d\lambda.$$

Condition (M) implies $L_X F = d(X \lrcorner F) + X \lrcorner (dF) = d^2\lambda = 0$, and if $F = da$, then $L_X a = d(X \lrcorner a) + X \lrcorner (da) = d(\lambda + \rho)$, $\rho := X \lrcorner a$. In particular, (M) is almost trivially satisfied by the pair $(X, \lambda) = (0, \text{const.})$, which trivially satisfies (K). If iF is the curvature 2-form of a principal U(1)-connection, we define the U(1)-*gauged Lie derivative* $L_{X,\lambda}$ of a U(1)-gauged differential form Φ with respect to (X, λ) by

$$(8.2) \qquad L_{X,\lambda}\Phi := D(X \lrcorner \Phi) + X \lrcorner D\Phi + i\lambda C(\Phi).$$

$L_{X,\lambda}$ commutes with the gauge covariant exterior derivative D, if condition (M) holds.

If moreover (X, λ) is an infinitesimal Riemann-Maxwell isometry, then $L_{X,\lambda}$ also commutes with the U(1)-gauged Dirac-operator $\tilde{\nabla}$. In this case, for any solution Φ of the generalized Kähler-Dirac equation $\tilde{\nabla}\Phi = \varkappa\Phi$ (with real $\varkappa$), also $L_{X,\lambda}\Phi$ is a solution, and the neutral $(n-1)$-form

$$(8.3) \qquad j_{X,\lambda}(\Phi) := (\bar{\Phi} \wedge \star\, L_{X,\lambda}\alpha\Phi + L_{X,\lambda}\alpha\Phi \wedge \star\, \bar{\Phi})^{(0)}_{n-1}$$

is conserved : $dj_{X,\lambda} = 0$. In particular, the almost trivial infinitesimal R-M isometry (0, const.) leads to the *electromagnetic current*

$$(8.4) \qquad j(\Phi) := i(\bar{\Phi} \wedge \star\, \alpha C\Phi + \alpha C\Phi \wedge \star\, \bar{\Phi})^{(0)}_{n-1} \quad [30].$$

As in the case without gauge-coupling, the set of infinitesimal R-M isometries has the structure of a Lie algebra, with Lie bracket now defined as

$$(8.5) \qquad [(X, \lambda), (X', \lambda')] := ([X, X'], X(\lambda') - X'(\lambda)).$$

In particular, the Lie product with (0, const.) always vanishes. By exponential mapping, we get the corresponding global Lie group which replaces $(\text{Poincaré} \times U(1))_{\text{global}}$ of the trivial and riemann-flat manifold of special relativity. In general, only $U(1)_{\text{global}}$ will remain, which corresponds to constant phase transformations of Φ.

9. CONCLUSION

A notion of spinor field for spin 1/2 has been developed based ultimately only on differential forms on a riemannian manifold. The point of departure from the usual notion of spinor field can be traced to our particular choice of representations of the Clifford algebra C(V, Q): our irreducible representations are not only isomorphic to the minimal left ideals of C(V, Q) (as modules), but are identical to them. This has the effect that local Lorentz-transformations are almost completely absent and no vestige of spin-transformations $\psi \mapsto S\psi$ with their typical two-valuedness is present. Moreover, not all irreducible representations (in this sense) are equivalent. The electromagnetic minimal coupling is introduced in a manner conserving all algebraic features and opening new perspectives for the description of

[30] By means of a local base $\{ e^i \}$ and its dual $\{ \mathring{e}_j \}$ this current can also be expressed in terms of Clifford-operations as

$$j(\Phi) = -\, i(\alpha\beta\bar{\Phi}e^i\Phi)^{(0)}_0 \mathring{e}_i \lrcorner\, \varepsilon\,.$$

Obviously this can also be done for the general current of (8.3).

interacting many-particle configurations. In spite of all basic differences, for flat spacetime the usual Dirac equation with electromagnetic coupling can be obtained by selecting appropriate spin structures. Departures are to be expected only for strong gravitational fields. Therefore we propose our notion of spinor field as a viable *alternative* to the usual spinor concept (which is geometrically and conceptually more complicated than ours). The generalized Kähler-Dirac equation $\tilde{\nabla}\Phi = \varkappa\Phi$ will be studied in more detail in a subsequent paper.

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