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Section A :
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Generators for quasi-free completely positive semi-groups

by

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ABSTRACT. — We construct quasi-free completely positive (CP) semi-groups on the CCR-C*-algebra, show that they can be extended, in certain representations, to a dynamical semi-group on the associated von Neumann algebra and determine the infinitesimal generator.

RÉSUMÉ. — Nous construisons les semi-groupes complètement positifs et quasi-libres sur la C*-algèbre de relations de commutation.
Ces sémi-groupes pouvant être étendus à l'algèbre de von Neumann associée à certaines représentations, on détermine le générateur infinitésimal.

1. INTRODUCTION

In the algebraic approach to non-equilibrium statistical mechanics, it is generally assumed that the dynamics of an open system, idealized as a C*- or a von Neumann algebra, is given by means of a one parameter semi-group of completely positive maps on the algebra [*1*] [*10*]. In case this semi-group extends to a group of $*$-automorphisms, the system is called conservative, if not the system is called dissipative.

In this paper we study a particular class of dynamical systems, namely quasi-free boson systems. Our algebra will be the CCR-C* algebra $\overline{\Delta(H, \sigma)}$

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build over a symplectic space (H, σ). [2] [3], while the CP maps will be of « quasi-free » type.

These CP maps where introduced in [4] [5]. The full class of these maps was characterized in [6]. Some further results, concerning extremality, dilation, implementation and relaxation, were obtained in [7].

It is clear that semi-groups of quasi-free CP maps cannot be strongly continuous, as they map Weyl-operators into Weyl-operators. We show however (theorem 4.4 below) that in certain representations, determined by quasi-free states [8], the semi-group may be extended to a so-called dynamical semi-group [9] [10] on the von Neumann algebra generated by the representation.

As an ultraweakly continuous semi-group of normal maps on a von Neumann algebra, there exists a densely defined and closed generator. We obtain this generator explicitly in theorems 4.7, 9 below. Formally it is of the Lindblad type [10].

The characterization of unbounded generators of dynamical semi-groups being far from complete, the results obtained here should contain some information on the structure of these generators.

The paper is organized as follows:

In § 2 we gather some results on symplectic spaces, operators and semi-groups on it. We briefly recall the definition of the CCR-C* algebra $\overline{\Delta(H, \sigma)}$. In § 3 we construct the class of quasi-free CP semi-groups. § 4 shows, the extension of the semi-group to certain associated von Neumann algebras, is possible. Finally the explicit form of the generator is obtained. If moreover we ask for the existence of a separating vector in the representation space, the domain of the generator is fully determined.

For the general theory of semi-groups and their generators we refer to [13] [14] [15]. For a treatment of quasi-free semi-groups on the CAR algebra, see [21].

2. SYMPLECTIC SPACES AND THE CCR ALGEBRA

The one particle space (H, σ) is a real symplectic space, i. e.

i) H is a real linear, possibly infinite dimensional, space.

ii) σ is a real, bilinear antisymmetric and non degenerated form, defined on H.

On H, we define the topology, induced by the family of seminorms $\{ p_\phi \}$,

$$p_\phi(\psi) = | \sigma(\phi, \psi) | .$$

The resulting locally convex space is Hausdorff. We call this topology the s-topology.

Given any continuous operator T: H $\rightarrow$ H, a unique operator T^+ is defined through the formula $\sigma(Tx, y) = \sigma(x, T^+y)$. A complex structure is an operator $J: H \to H$, such that $J^+ = -J$, and $J^2 = -1$.

A symplectic base of H, is a set of vectors $\{ f_i, g_i \}_{i=1,\dots}$ such that

i) $\{ f_i, g_i \}$ generate H (we suppose H separable).

ii) $\sigma(f_i, g_j) = \delta_{ij} \quad \forall i, j$

iii) $\sigma(f_i, f_j) = \sigma(g_i, g_j) = 0 \quad \forall i, j$

We suppose $\{ f_i, g_i \}$ is ordered as follows: $\{ f_1, g_1, f_2, g_2, f_3, \dots \}$ and denoted as $\{ e_1, e_2, e_3, \dots \}$. Then defining

$$J: H \to H$$

by

$$Je_{2k+1} = e_{2k+2} \qquad k = 0, 1, \dots$$

$$Je_{2k+2} = -e_{2k+1} \qquad k = 0, 1, \dots$$

it is easily checked that J extends to a complex structure.

The following formulas hold

$$\forall \phi \in H: \qquad \phi = \sum_k \sigma(\phi, Je_k)e_k \tag{1}$$

$$\forall \psi, \phi \in H: \qquad \sigma(\phi, \psi) = \sum_k \sigma(\phi, Je_k)\sigma(e_k, \psi) \tag{2}$$

DEFINITION 2.1. [8]. — $\mathcal{Q}$ is the set of all operators $Q: H \to H$ s. t.

i) $s_Q(\psi, \phi) \equiv -\sigma(Q\psi, \phi)$ defines a real scalar product on H.

ii) $Q^*Q \geq 1$ where $*$ and $\geq$ are taken with respect to s_Q.

It follows then that $Q^* = -Q = Q^+$, hence Q is bounded for the s_Q-norm topology on H; moreover Q is invertible.

Let $\bar{H}^Q$ denote the completion of H for the s_Q norm topology. The following properties are well known. Suppose H is sequentially s-complete, then $\forall Q \in \mathcal{Q}$, H is s_Q-norm complete. [8] Conversely if H is not sequentially s-complete, then $\bar{H}^Q$ is sequentially s-complete [12], II cor. 29. Henceforth we will suppose H is sequentially s-complete.

DEFINITION 2.2. — A continuous semi-group on H is a 1-parameter family of s-continuous, everywhere defined, operators A_t, $t \in R^+$ s. t.

i) $A_0 = 1$

ii) $A_tA_s = A_{t+s}$

iii) the map $t \to A_t$ is s-continuous.

As H is sequentially s-complete, by the closed graph theorem, $\{ A_t \}$ is a strongly continuous semi-group on H, equipped with the s_Q-norm topology, whenever $Q \in \mathcal{Q}$.

The infinitesimal generator Z of A_t is defined by

$$Z\psi = \lim_{t \to 0} \frac{1}{t}(T_t - 1)\psi \qquad \text{for any} \quad \psi \in H$$

such that the limit exists (in any of the topologies). When H is sequentially s-complete, Z is s & s_Q-norm densely defined. Moreover Z is s_Q-norm closed.

In the sequel we make use of the following properties ((+), (++)).

i) Let A_t be a continuous semi-group in H, Z its generator, then for all $\psi \in \mathscr{D}(Z)$ the map $t \mapsto ZA_t\psi$ is s-continuous. Hence $t \mapsto \sigma(ZA_t\psi, A_t\psi)$ is continuous for all $\psi \in \mathscr{D}(Z)$ [*13*] [*14*].

ii) Let A_t be continuous, then for fixed ψ, and finite s, the set $\{ A_t\psi \mid t \in [0, s] \}$ is contained in a finite dimensional subspace of H.

The CCR C*-algebra $\overline{\Delta(H, \sigma)}$ ([*2*] [*3*]) is the C*-algebra obtained by completing the $*$-algebra $\Delta(H, \sigma)$ generated by the Weyl elements δ_ψ, $\psi \in H$, satisfying

$$\delta_\psi \delta_\phi = e^{-i\sigma(\psi,\phi)}\delta_{\psi+\phi}$$
$$(\delta_\psi)^* = \delta_{-\psi}$$

We refer to [*3*] for the exact definition of the norm with respect to which the completion is to be taken.

We recall that ([*8*]) any $Q \in \mathscr{Q}$ determines a quasi-free state on $\overline{\Delta(H, \sigma)}$ through the formula

$$\omega_Q(\delta_\psi) = e^{1/2\sigma(Q\psi,\psi)} .$$

3. QUASI-FREE CP SEMIGROUPS

Let A be any operator on H.

Denoting $\sigma_A(\psi, \phi) \equiv \sigma(\psi, \phi) - \sigma(A\psi, A\phi)$. It was shown in [*6*], that the map

$$\begin{aligned} \tau : \Delta(H, \sigma) &\to \Delta(H, \sigma) \\ \tau(\delta_\psi) &= \delta_{A\psi} f(\psi) \end{aligned} \tag{3}$$

f being a functional on H, such that $f(0) = 1$, extends to a CP map on $\overline{\Delta(H, \sigma)}$ iff ω, defined by

$$\omega(\delta_\psi) = f(\psi)$$

extends to a state on the C algebra $\overline{\Delta(H, \sigma_A)}$.

Imposing some regularity conditions, the general form of semi-groups, consisting of CP maps of type (3), was exhibited in [7]. In the following we suppose $f(\psi)$ to be the generating functional of a quasi-free state on $\overline{\Delta(H, \sigma_A)}$.

THEOREM 3.1. — Let $\tau_t : \overline{\Delta(H, \sigma)} \to \overline{\Delta(H, \sigma)}$ be a one parameter semi-group of quasi-free CP maps, i. e.

$$\tau_t(\delta_\psi) = \delta_{A_t\psi} f(\psi) \tag{4}$$

such that A_t is continuous and for all ψ the map $t \mapsto f_t(\psi)$ is differentiable.

Suppose $f_t(\psi)$ is the *generating* functional of a quasi-free state on $\overline{\Delta(H, \sigma_{A_t})}$, then it is of the form

$$f_t(\psi) = \exp\,[\sigma(B_t\psi, \psi)] \tag{5}$$

where

$$B_t = \int_0^t A_x^+ Y A_x dx \tag{6}$$

Here Y satisfies:

i) Y is uniquely and everywhere defined

ii) $Y^+ = -Y$ is bounded for all s_Q-norm topologies.

iii) $\sigma(Y\psi, \psi) \leqslant 0 \quad \forall\psi$

iv) $\forall\psi, \phi \in \mathscr{D}(Z)$

$$|\sigma(Z\psi, \phi) + \sigma(\psi, Z\phi)| \leqslant -\frac{1}{2}[\sigma(Y\psi, \psi) + \sigma(Y\phi, \phi)] \tag{7}$$

Conversely, any continuous semi-group $\{A_t\}$ and operator Y with the above properties, define a CP semi-group through the formulas (4) (5) (6).

Proof. — As $t \to f_t(\psi)$ is differentiable for all ψ and as $B_t^+ = -B_t$, we obtain that the map $t \to \sigma(B_t\psi, \phi)$ is differentiable for all ψ, ϕ in H.

On the other hand, as Q is invertible when $Q \in \mathscr{Q}$

$$\forall t \quad \exists B_t^Q \text{ s. t.} \qquad B_t = QB_t^Q$$

then $B_t^Q = B_t^{Q*}$ where the adjoint is taken w. r. t. s_Q. As B_t^Q is everywhere defined, B_t^Q is bounded for the s_Q-norm topology. We have that

$$\lim_{t\to 0} \frac{1}{t}[\sigma(QB_t^Q\psi, \phi) - \sigma(Q\psi, \phi)] = -\lim_{t\to 0}\frac{1}{t}[s_Q(B_t^Q\psi, \phi) - s_Q(\psi, \phi)]$$

exists for all ψ and ϕ in H. Thus there is an s_Q-bounded operator Y^Q such that

$$\frac{d}{dt} s_Q(B_t^Q\psi, \phi)\Big|_{t=0} = s_Q(Y^Q\psi, \phi)$$

Defining $Y = QY^Q$ we obtain *i)* and *ii)*. Using Prop. 4.2 in [7] we obtain

$$f_t(\psi) = \exp\left[\int_0^t dx\left(\frac{d}{dt'}, f_{t'}(A_x\psi)\Big|_{t'=0}\right)\right]$$
$$= \exp\left[\int_0^t dx\, \sigma(YA_x\psi, A_x\psi)\right].$$

To show *iii)* we note that $f_t(\psi)$ is the generating functional of a state on a C*-algebra and that as such $|f_t(\psi)| = |\omega_t(\delta_\psi)| \leqslant 1$ hence, for all t, $\sigma(B_t\psi, \psi) \leqslant 0$. On the other hand, $\tau_0 = 1$, and we have $\sigma(B_0\psi, \psi) = 0$ thus

$$\frac{d}{dt}\sigma(B_t\psi, \psi)\Big|_{t=0} \leqslant 0,$$

which is by definition $\sigma(Y\psi, \psi) \leqslant 0$.

Finally we express that

$$f_t(\psi) = \exp\left[\int_0^t dx\, \sigma(YA_x\psi, A_x\psi)\right]$$

defines a state on $\overline{\Delta(H, \sigma_{A_t})}$. That $f_t(\psi)$ generates a state, implies ([8])

$$|\sigma_{A_t}(\psi, \phi)| \leqslant -\frac{1}{2}[\sigma(B_t\psi, \psi) + \sigma(B_t\phi, \phi)]\ ;$$

Noting once more that the equality is reached for $t = 0$, we derive for $\psi, \phi \in \mathscr{D}(Z)$

$$|\sigma(Z\psi, \phi) + \sigma(\psi, Z\phi)| \leqslant -\frac{1}{2}[\sigma(Y\psi, \psi) + \sigma(Y\phi, \phi)]$$

Conversely, suppose $\{A_t\}$ is a continuous semi-group on H, and Y is an operator on H enjoying properties *i*), *ii*), *iii*) and *iv*).

Taking ψ, ϕ in $\mathscr{D}(Z)$, by *iv*) we obtain for all $x \geqslant 0$

$$-\sigma(ZA_x\psi, A_x\phi) - \sigma(A_x\psi, ZA_x\phi) \leqslant -\frac{1}{2}[\sigma(YA_x\psi, A_x\psi) + \sigma(YA_x\phi, A_x\phi)] \quad (8)$$

The left and right hand sides of the inequality are integrable on any bounded interval by property (+); integrating (8) yields

$$-\int_0^s [\sigma(ZA_x\psi, A_x\phi)dx + \sigma(A_x\psi, ZA_x\phi)]dx \leqslant -\frac{1}{2}\int_0^s [\sigma(YA_x\psi, A_x\psi) + \sigma(YA_x\phi, A_x\phi)]dx$$

Hence

$$\sigma(\psi, \phi) - \sigma(A_t\psi, A_t\phi) \leqslant -\frac{1}{2}[\sigma(B_t\psi, \psi) + \sigma(B_t\phi, \phi)]$$

using the other inequality we arrive at (7). This together with *i*) *ii*) and *iii*) imply that $f_t(\psi)$ defines a state on $\Delta(H, \sigma_{A_t})$. That τ_t defines a semi-group on $\overline{\Delta(H, \sigma)}$ follows now from [7] prop. 4.2. ■

EXAMPLE 3.2. — [4] [16]. — Let $H = R^2$

$$\sigma((x, y), (x', y')) = \frac{1}{2}(xy' - yx')$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A_t z = e^{-\lambda t}z \qquad \lambda > 0 \quad t \geqslant 0 \quad z \in R^2$$

$$f_t(z) = \exp\left[-\frac{\theta}{4}(1 - e^{-2\lambda t})\|z\|^2\right], \qquad \theta \geqslant 1$$

then $f_t(z)$ is of the form (6) with

$$Y = -\frac{\lambda\theta}{2}J$$

At the risk of being confusing we now introduce

DEFINITION 3.3. — Any semi-group $\{\tau_t\}$ as in theorem 3.1 will be called a quasi-free CP semi-group.

4. GENERATORS OF QUASI-FREE CP SEMIGROUPS

DEFINITION 4.1 [10]. — A one parameter semi-group $\{\tau_t\}$ on a von Neumann algebra $\mathcal{M}$ is called a dynamical semi-group whenever

i) $\forall t$ τ_t is a CP map $\mathcal{M} \to \mathcal{M}$
ii) $\tau_t(1) = 1$ for any t
iii) $\tau_0 = 1$
iv) $\forall t$ τ_t is an ultraweakly continuous map
v) $t \to \tau_t(x)$ is ultraweakly continuous.

It follows that $\{\tau_t\}$ is a contraction semi-group.

If $\{\tau_t\}$ is a dynamical semi-group, there exists an ultraweakly dense set in $\mathcal{M}$, called $\mathcal{D}(\mathcal{L})$ such that for all $x \in \mathcal{D}(\mathcal{L})$ the

$$\lim_{t\to 0} \frac{\tau_t(x) - x}{t} \quad \text{exists in ultraweak (u. w.)-sense.}$$

The limit is called $\mathcal{L}(x)$; it is again an element of $\mathcal{M}$ $\mathcal{L}$ is called the generator of $\{\tau_t\}$. It can be shown that $\mathcal{L}$ is uw-closed [13] [15].

Using the methods of [11] [13] [15] the following can be shown.

THEOREM 4.2. — Let τ_t be a dynamical semi-group on $\mathcal{M}$. Then

1. $\forall x \in \mathcal{M}$, $\forall s$, the element $x_s = \int_0^s \tau_t(x)dt$ is well defined and $x_s \in \mathcal{D}(\mathcal{L})$
Moreover $\tau_s(x) - x = \mathcal{L}(x_s)$
2. Let $\mathcal{C}$ be an u. w. dense set, $\mathcal{C} \subseteq \mathcal{D}(\mathcal{L})$, s. t.

$$\tau_t(\mathcal{C}) \subseteq \mathcal{C}, \qquad \text{then } \mathcal{C} \text{ is a core for } \mathcal{L}. \quad \blacksquare$$

Let ω_Q be any quasi-free state on $\overline{\Delta(H, \sigma)}$, and τ_t a quasi-free CP semi-group, then $\forall t$, $\tau_t^*(\omega_Q)$ is again quasi-free and

$$\tau_t^* \cdot \omega_Q = \omega_{A_t^+ QA_t + B_t}$$

We now introduce our main hypothesis in order to ensure condition *iv*) of definition 4.1 is satisfied when τ_t is considered in a quasi-free representation of the CCR.

DEFINITION 4.3. — A quasi-free state ω_Q is said to be approximately τ_t-invariant, iff $\forall t \geqslant 0$

$$Q - A_t^+ Q A_t - B_t \tag{9}$$

is an s_Q-trace class operator.

THEOREM 4.4. — Let ω_Q a quasi-free state, τ_t a quasi-free semi-group on $\overline{\Delta(H, \sigma)}$. If H is sequentially s-complete, and ω_Q is approximately τ_t invariant then τ_t extends uniquely to a dynamical semi-group on

$$\Pi_{\omega_Q}(\overline{\Delta(H, \sigma)})'' \equiv \mathcal{M}.$$

Proof (For convenience we denote $\Pi_{\omega_Q}(x)$ as x. — ω_Q and $\tau_t^*(\omega_Q)$ are quasi equivalent. Indeed, as

$$H = \bar{H}^Q = \bar{H}^{A_t^+ Q A_t + B_t}$$

both ω_Q and $\tau_t^*(\omega_Q)$ are factor states [8] and the condition ensures that they are quasi-equivalent [17].

We now show that for any $\omega \in \mathcal{M}_*^+$, the positive part in the predual of $\mathcal{M}$, $\tau_t^*(\omega)$ is again in $\mathcal{M}_*^+$.

Denote by τ_χ the gauge automorphism $\overline{\Delta(H, \sigma)} \to \overline{\Delta(H, \sigma)}$ defined by $\tau_\chi(x) = \delta_\chi x \delta_{-\chi}$, $\chi \in H$ and, given $\omega \in \mathcal{M}_*^+$, define the state ω^χ

$$\omega^\chi = \omega_0 \tau_\chi$$

then $\tau_t^*(\omega_Q^\chi) = (\tau_t^*(\omega_Q))^{A_t^+ \chi}$.

Hence, $\forall \chi \in H$, the state $\tau_t^*(\omega_Q^\chi)$ is quasi-equivalent to ω_Q.

For a general state $\omega \in \mathcal{M}_*^+$, there is a sequence ρ_n, where ρ_n is a finite linear combination of states of the form ω_Q^χ such that $\rho_n \to \omega$ in norm [18].

On the other hand, $\forall t$, the map $\tau_t^* : \overline{\Delta(H, \sigma)}^* \to \overline{\Delta(H, \sigma)}^*$ as the dual of a normalized positive map on a C*-algebra, is norm continuous (in fact $\| \tau_t^* \| = 1$)

Thus we obtain $\tau_t^*(\rho_n) \to \tau_t^*(\omega)$ in norm and as $\mathcal{M}_*$ is norm closed $\tau_t^*(\omega) \in \mathcal{M}_*^+$. This is nothing but saying that for any t, the map $\tau_t : \Pi_\omega(\overline{\Delta(H, \sigma)}) \to \Pi_\omega(\overline{\Delta(H, \sigma)})$ is ultraweakly continuous.

This implies that $\forall t$, τ_t can be extended to an uw-continuous map $\bar{\tau}_t : \mathcal{M} \to \mathcal{M}$. Using a Kaplansky-type approximation, we show that $\bar{\tau}_t$ is positive.

In the same way we have $\bar{\tau}_t \otimes \mathbb{1}_n : \mathcal{M} \otimes \mathcal{M}_n \to \mathcal{M} \otimes \mathcal{M}_n$ are positive, such that $\bar{\tau}_t$ is CP.

Remains to show *v*) of definition 4.1, i. e. for all ω in $\mathcal{M}_*$ and for all x

in $\mathscr{M}$ the map $t \mapsto \omega(\tau_t(x))$ is continuous. This is clearly true for $\omega = \omega_Q \cdot \tau_\chi$ and

$$x = \sum_{i=1}^{n} \lambda_i \delta \psi_i$$

Using once more the norm density of the linear combinations of the states $\omega_Q \cdot \tau_\chi$ in $\mathscr{M}_*^+$, we obtain the continuity of

$$t \mapsto \omega(\tau_t(x))$$

for all $\omega \in \mathscr{M}_*$, and $x = \sum_{i=1}^{n} \lambda_i \delta \psi_i$

Finally, for general $x \in \Pi(\overline{\Delta(H, \sigma)})$, there is a sequence x_n, $x_n \in \Pi(\overline{\Delta(H, \sigma)})$

hence

$$x_n \to x \text{ in norm}$$

$$\begin{aligned} | \omega(\tau_t(x)) - \omega(\tau_t(x_n)) | &= | (\omega_0 \tau_t)(x - x_n) | \\ &\leqslant || x - x_n || < \varepsilon \qquad \text{for } n \text{ large enough.} \end{aligned}$$

By the uniform convergence $t \mapsto \omega(\tau_t(x))$ is continuous.

Then, using the method of [*19*], we obtain the continuity for all x in $\mathscr{M}$. ■

Let ω_Q be any quasi-free state on $\overline{\Delta(H, \sigma)}$; for all ψ and ϕ in H, the map

$$\lambda \in \mathbb{R} \mapsto \omega_Q(\delta_{\lambda\psi + \phi})$$

is infinitely differentiable. Hence, $\forall \psi$, there exists a selfadjoint operator $B_Q(\psi)$ on $\mathscr{H}_{\omega_Q}$, the GNS space for ω_Q such that

$$\Pi_{\omega_Q}(\delta_{\lambda\psi}) = e^{i\lambda B_Q(\psi)},$$

(Remark that $\mathscr{H}_{\omega_Q}$ is separable).

Moreover $\forall \psi, \phi, \eta \in H$ we have

i) $$\Pi_{\omega_Q}(\delta_\psi)\Omega_{\omega_Q} \in \mathscr{D}(B_Q(\phi))$$

here Ω_{ω_Q} denotes the cyclic vector in the GNS space.

ii) $$B_Q(\psi)\Pi_{\omega_Q}(\delta_\phi)\Omega_{\omega_Q} \in (B_Q(\eta)).$$

If $\{ e_k \}$ is a symplectic base for H, we will denote $B_Q(e_k)$ by B_k. From now on we drop all indices refering to ω_Q.

The following equalities are easily verified:

$$\begin{aligned} &\langle \Pi(\delta_\phi)\Omega, B_k[B_l, \Pi(\delta_\psi)]\Pi(\delta_\eta)\Omega \rangle \\ &\qquad = 2[\sigma(e_k, \psi + \phi + \eta) + i\sigma(Qe_k, -\phi + \psi + \eta)] \\ &\qquad \cdot \sigma(e_l, \psi) \langle \pi(\delta_\phi\Omega, \pi(\delta_\psi)\pi(\delta_\eta)\Omega \rangle \end{aligned} \tag{10}$$

$$\begin{aligned} &\langle \Pi(\delta_\phi)\Omega, [B_k, \Pi(\delta_\psi)]B_l\Pi(\delta_\eta)\Omega \rangle \\ &\qquad = 2[-\sigma(e_l, \psi + \phi + \eta) + i\sigma(Qe_l, -\phi + \psi + \eta)] \\ &\qquad \cdot \sigma(e_k, \psi) \langle \pi(\delta_\phi)\Omega, \pi(\delta_\psi)\pi(\delta_\eta)\Omega \rangle \end{aligned} \tag{11}$$

LEMMA 4.5. — Let τ_t and ω_Q as in theorem 4.4 Π as above. Then $\forall k, l \in \mathbb{N}$ and $\forall \psi, \phi \in H$, and for all finite s the elements

$$\int_0^s \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \tag{12}$$

$$\int_0^s \Pi(\tau_t(\delta_\psi))B_l\Pi(\delta_\phi)\Omega dt \tag{13}$$

are well defined and belong to respectively $\mathscr{D}(B_kB_l)$ and $\mathscr{D}(B_k)$.

Proof (for (12)). — It is easily checked that the maps

$$t \to \| \Pi(\tau_t(\delta\psi))\Pi(\delta\phi)\Omega \|$$

and

$$t \to \| B_l\Pi(\tau_t(\delta\psi))\Pi(\delta\phi)\Omega \|$$

are continuous; thus the elements

$$\int_0^s \Pi(\tau_t(\delta\psi))\Pi(\delta\phi)\Omega dt$$

and

$$\int_0 B_l\Pi(\tau_t(\delta\psi))\Pi(\delta\phi)\Omega dt$$

exist in Bochner sense.

By the continuity of $t \mapsto \tau_t$, $\int_0^s \Pi(\tau_t(\delta_\psi))dt$ exists as an operator (in ultraweak sense).

Let $\xi \in \mathscr{D}(B_l)$, then

$$\begin{aligned}\langle B_l\xi, \int_0^s \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \rangle &= \int_0^s \langle B_l\xi, \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega \rangle dt \\ &= \int_0^s \langle \xi, B_l\Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega \rangle dt \\ &= \langle \xi, \int_0^s B_l\Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \rangle\end{aligned}$$

Thus, since $B_l = B_l^*$, $\int_0^s \Pi(\tau_t(\delta_\phi))\Pi(\delta_\phi)\Omega dt \in \mathscr{D}(B_l)$. In the same way we prove that

$$\int_0^s B_l\Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \text{ is in } \mathscr{D}(B_k). \quad \blacksquare$$

PROPOSITION 4.6. — Let H be finite dimensional, τ_t a CP quasi-free semigroup; ω_Q a quasi-free state on $\overline{\Delta(H, \sigma)}$.

Then there exist real sequences

$$\{ a_{lk} \} \qquad k, l = 1, \ldots, 2N$$
$$\{ b_{lk} \} \qquad k, l = 1, \ldots, 2N$$

such that $\forall x \in \mathscr{D}(\mathscr{L})$ and all ξ, η in a dense set $\mathscr{D}$, one has

$$\begin{aligned} \langle \xi, \mathscr{L}(x)\eta \rangle = -\frac{i}{4}\sum_{k,l} a_{lk}[\langle B_lB_k\xi, x\eta \rangle - \langle B_k\xi, xB_l\eta \rangle] \\ + a_{kl}[\langle B_k\xi, xB_l\eta \rangle - \langle \xi, xB_kB_l\eta \rangle] \\ -\frac{1}{4}\sum_{k,l} b_{lk}[\langle B_lB_k\xi, x\eta \rangle - \langle B_k\xi, xB_l\eta \rangle] \\ - b_{kl}[\langle B_k\xi, xB_l\eta \rangle - \langle \xi, xB_kB_l\eta \rangle] \end{aligned} \qquad (12)$$

Moreover there is a core $\mathscr{C}$ for $\mathscr{L}$, such that $\forall x \in \mathscr{C}$ and $\forall n \in \mathscr{D}$

$$\begin{aligned} \mathscr{L}(x)\eta = -\frac{i}{4}\sum_{k,l}(a_{kl}B_k[B_l, x]\eta + a_{kl}[B_k, x]B_l\eta) \\ -\frac{1}{4}\sum_{k,l}(b_{kl}B_k[B_l, x]\eta - b_{kl}[B_k, x]B_l\eta). \end{aligned} \qquad (13)$$

Proof. — Let $\mathscr{D}$ be the linear span of $\{ \Pi(\delta_\psi)\Omega \mid \psi \in H \}$ in $\mathscr{H}$. Denote by $\mathscr{C}$ the linear span of $\left\{ \int_0^s \Pi(\tau_t(\delta_\psi))dt \mid s < \infty,\ \psi \in H \right\}$ in $\mathscr{M}$.

Then, by theorem 4.2, $\mathscr{C}$ is a core for $\mathscr{L}$.

Then define (e_k being a symplectic base)

$$a_{kl} = \sigma(Je_k, Z^+J_{el})$$
$$b_{kl} = \sigma(Je_k, YJ_{el})$$

By theorem 4.2 weknow

$$\mathscr{L}\left(\int_0^s \Pi(\tau_t(\delta_\psi))dt\right) = \Pi(\tau_s(\delta_\psi)) - \Pi(\delta_\psi)$$

on the other hand, the function

$$t \mapsto \frac{d}{dt}\langle \Pi(\delta_\phi)\Omega, \Pi(\tau_t(\delta_\psi))\Pi(\delta_\eta)\Omega \rangle$$

being continuous,

$$\langle \Pi(\delta_\phi)\Omega, \Pi(\tau_s(\delta_\psi) - \delta_\psi)\Pi(\delta_\eta)\Omega \rangle = \int_0^s \frac{d}{dt}[\langle \Pi(\delta_\phi)\Omega, \Pi(\tau_t(\delta_\psi))\Pi(\delta_\eta)\Omega \rangle]dt.$$

For notational convenience, we put $\phi = \eta = 0$.

$$\left\langle \Omega, \mathscr{L}\left[\left[\int_0^s \Pi(\tau_t(\delta_\psi))dt\right]\Omega\right\rangle\right.$$

$$= \int_0^s dt\omega_Q(\tau_t(\delta_\psi))[-\sigma(ZA_t\psi, QA_t\psi) + \sigma(YA_t\psi, A_t\psi)]$$

which by (1) equals

$$= \int_0^s dt\omega_Q(\tau_t(\delta_\psi))\sum_{k,l}[\sigma(A_t\psi, Je_l)\sigma(Ze_l, Je_k)\sigma(Qe_k, A_t\psi) - \sigma(A_t\psi, Je_l)\sigma(e_l, YJe_k)\sigma(e_k, A_t\psi)]$$

substituting (Je_l) for (e_l) and $(-e_l)$ for (Je_l)

$$= -\int_0^s dt\omega_Q(\tau_t(\delta_\psi))\sum_{k,l}[\sigma(A_t\psi, e_l)\sigma(Ze_l, Je_k)\sigma(Qe_k, A_t\psi) - \sigma(A_t\psi, e_l)\sigma(Je_l, YJe_k)\sigma(e_k, A_t\psi)]$$

$$\begin{aligned} = -\frac{i}{2}\int_0^s dt\omega_Q(\tau_t(\delta_\psi))\sum_{k,l} a_{lk}[&\sigma(e_l, A_t\psi)\sigma(e_k, A_t\psi) \\ &+ i\sigma(Qe_k, A_t\psi)\sigma(e_l, A_t\psi)] \\ &+ a_{kl}[-\sigma(e_k, A_t\psi)\sigma(e_l, A_t\psi) \\ &+ i\sigma(e_k, A_t\psi)\sigma(Qe_l, A_t\psi)]] \\ -\frac{1}{2}\int_0^s dt\omega_Q(\tau_t(\delta_\psi))\sum_{k,l} b_{kl}[&\sigma(e_l, A_t\psi)\sigma(e_k, A_t\psi) \\ &+ i\sigma(Qe_k, A_t\psi)\sigma(e_l, A_t\psi)] \\ &+ b_{lk}[+\sigma(e_k, A_t\psi)\sigma(e_l, A_t\psi) \\ &- i\sigma(e_k, A_t\psi)\sigma(Qe_l, A_t\psi)] \end{aligned}$$

by (10) (11) this equals

$$\begin{aligned} = -\int_0^s dt\Big[\frac{i}{4}\sum_{k,l}&[a_{lk}\langle \Omega, B_k[B_l, \tau_t(\delta_\psi)]\Omega\rangle + a_{kl}\langle \Omega, [B_k, \tau_t(\delta_\psi)]B_l\Omega\rangle] \\ -\frac{1}{4}\sum_{kl}&[b_{lk}\langle \Omega, B_k[B_l, \tau_t(\delta_\psi)]\Omega\rangle - b_{kl}\langle \Omega, [B_k, \tau_t(\delta_\psi)]B_l\Omega\rangle]\Big] \end{aligned} \tag{14}$$

as all terms in the sum are integrable, we obtain by making use of lemma 4.5.

$$\begin{aligned} = -\frac{i}{4}\sum_{kl}\Big[& a_{lk}\left\langle \Omega, B_k\left[B_l, \int_0^s \tau_t(\delta_\psi)dt\right]\Omega\right\rangle \\ &+ a_{kl}\left\langle \Omega, \left[B_k, \int_0^s \tau_t(\delta\psi)dt\right]B_l\Omega\right\rangle\Big] \\ -\frac{1}{4}\sum_{kl}\Big[& b_{lk}\left\langle \Omega, B_k\left[B_l, \int_0^s \tau_t(\delta_\psi)dt\right]\Omega\right\rangle \\ &- b_{kl}\left\langle \Omega, \left[B_k, \int_0^s \tau_t(\delta_\psi)dt\right]B_l\Omega\right\rangle\Big] \end{aligned} \tag{15}$$

which is (13).

For general x in $\mathscr{D}(\mathscr{L})$, we proceed as follows $\exists x_\alpha \in \mathscr{C}$ such that $x_\alpha \to x$ u. w. and $\mathscr{L}(x_\alpha) \to \mathscr{L}(x)$ u. w. thus, for $(\xi, \eta) \in \mathscr{D}$:

$$\langle \xi, \mathscr{L}(x)\eta \rangle = \lim_\alpha [\langle \xi; \mathscr{L}(x_\alpha)\eta \rangle]$$

$$= \lim_\alpha \left[\frac{-i}{4} \sum_{kl} a_{lk}[\langle B_l B_k \xi, x_\alpha \eta \rangle - \langle B_k \xi, x_\alpha B_l \eta \rangle + \ldots]\right]$$

since there isn't but a finite number of terms, this equals

$$= -\frac{i}{4} \sum_{kl} a_{lk} \langle B_l B_k \xi, x\eta \rangle - \langle B_k \xi, x B_l \eta \rangle + \ldots$$

which is (12). ■

NOTATION 4.7. — For fixed x in $\mathscr{D}(\mathscr{L})$, the right hand side in (12) defines a bilinear form on $\mathscr{D}$. We will denote it as

$$\phi_x(\xi, \eta)$$

Let $(H_n)_{n\in N}$ be an increasing and absorbing net of finite dimensional regular symplectic subspaces of H. Then we define $\mathscr{M}_F \subseteq \mathscr{M}$ as

$$\mathscr{M}_F \equiv \bigcup_{n\in N} \Pi(\overline{\Delta(H_n, \sigma)})''$$

PROPOSITION 4.8. — Let H be infinite dimensional and sequentially s-complete, ω_Q and τ_t as in theorem 4.4.

Then there exist real infinite sequences $\{a_{lk}\}$, $\{b_{ek}\}$ such that $\forall x \in \mathscr{D}(\mathscr{L}) \cap \mathscr{M}_F$ and all ξ, in a dense set $\mathscr{D}$ one has

$$\begin{aligned} \langle \xi, \mathscr{L}(x)\eta \rangle = \frac{i}{4} \sum_{k,l} & [a_{lk}[\langle B_l B_k \xi, x\eta \rangle - \langle B_k \xi, x B_l \eta \rangle] \\ & + a_{kl}[\langle B_k \xi, x B_l \eta \rangle - \langle \xi, x B_k B_l \eta \rangle]] \\ & - \frac{1}{4} \sum_{k,l} [b_{lk}[\langle B_l B_k \xi, x\eta \rangle - \langle B_k \xi, x B_l \eta \rangle] \\ & - b_{lk}[\langle B_k \xi, x B_l \eta \rangle - \langle \xi, x B_k B_l \eta \rangle]] \qquad (16) \\ & \equiv \phi_x(\xi, \eta) \end{aligned}$$

Moreover there is a core $\mathscr{C}$ for $\mathscr{L}$, such that $\forall x \in \mathscr{C}$, a formula similar to (13) holds.

Proof. — Define $\mathscr{D}$ as in proportion 4.7, and $\mathscr{C}$ as the linear span of $\left\{ \int_0^s \Pi(\tau_t(\delta_\psi))dt \mid s < \infty, \psi \in \mathscr{D}(z) \right\}$. Again by theorem 4.2 $\mathscr{C}$ is a core for $\mathscr{L}$.

Choose a symplectic base in $\{e_k\}$ in $\mathscr{D}(Z^+)$ and define

$$a_{kl} = \sigma(Je_k, Z^+Je_l)$$
$$b_{kl} = \sigma(Je_k, YJe_l)$$

If $\psi \in \mathscr{D}(Z)$, then by property (+) stated in § 2, the map

$$t \mapsto \frac{d}{dt}\omega_Q(\tau_t(\delta_\psi)) = \omega_Q(\tau_t(\delta_\psi))[-\sigma(ZA_t\psi, QA_t\psi) + \sigma(YA_t\psi, A_t\psi)]$$

is continuous.

The proof is then a mere extension of the method in 4.7. Indeed, by the continuity, we arrive at a formula similar to (14). The elements $\int_0^s \Pi(\tau_t(\delta_\psi))dt$ belong to $\mathscr{M}_F$, by property (+ +) stated in § 2.

Hence we obtain (15), and thus (13).

For general $x \in \mathscr{D}(\mathscr{L})$, there is a net $\{x_\alpha\} \in \mathscr{C}$ such that $x_\alpha \to x$ and $\mathscr{L}(x_\alpha) \to \mathscr{L}(x)$ u. w.

Thus for $\xi, \eta \in \mathscr{D}$

$$\begin{aligned}\langle \xi, \mathscr{L}(x)\eta \rangle &= \lim_\alpha \langle \xi, \mathscr{L}(x_\alpha)\eta \rangle \\ &= \lim_\alpha \phi_{x_\alpha}(\xi, \eta)\end{aligned} \tag{17}$$

If moreover $x \in \mathscr{M}_F$, then for l sufficiently large we obtain e. g.,

$$\langle B_lB_k\xi, x\eta \rangle - \langle B_k\xi, xB_l\eta \rangle = 0\,.$$

Since any term in (17) is convergent to a term which eventually vanishes, we obtain (16). ■

Formula (16) is of the form

$$\begin{aligned}\langle \xi, \mathscr{L}(x)\eta \rangle &= \lim_{N\to\infty} \sum_{k,l=1} -\frac{i}{4}[a_{lk} \langle B_lB_k\xi, x\eta \rangle - \langle B_k\xi, xB_l\eta \rangle] \\ &\quad + a_{kl} \ldots + \ldots \\ &\equiv \lim_{N\to\infty} \phi_x^N(\xi, \eta)\end{aligned}$$

We remark however that in general the bilinear forms ϕ_x^N are not associated to a generator of a quasi-free CP semi-group on $\overline{\Pi(\Delta(H_N, \sigma))}''$, the reason being that when Z generates a semi-group in H, and P_N is the projection onto H_N, P_NZP_N need not generate a semi-group on H_N, except when H_N is a reducing subspace.

Remark also that if $\sum_{k,l} (a_{lk} + a_{kl})B_kB_l$ can be given a sense as a self-adjoint operator, then by re-summing the series, we recover the Lindblad form of the generator [*10*] [*22*].

If we impose the condition that $\mathscr{M}$ has a separating vector, then we can determine $\mathscr{D}(\mathscr{L})$, in the same way as was done in [11] for spatial derivations.

THEOREM 4.9. — Let H, ω_Q, τ_t, $\mathscr{D}$ as above. Suppose moreover that ω_Q extends faithfully to $\mathscr{M}$.

An element x in $\mathscr{M}_F$ belongs to $\mathscr{D}(\mathscr{L})$ iff the bilinear form

$$\phi_x(\cdot, \cdot) : \mathscr{D} \times \mathscr{D} \to \mathrm{C}$$

has a bounded extension to $\mathscr{H} \times \mathscr{H}$.

Proof. — If $x \in \mathscr{D}(\mathscr{L}) \cap \mathscr{M}_F$, then by proposition 4.7 and 4.9, there is an element $\mathscr{L}(x)$ in $\mathscr{M}$ such that $\phi_x(\xi, \eta) = \langle \xi, \mathscr{L}(x)\eta \rangle$. Hence ϕ_x has a bounded extension.

Let $\phi_x(\cdot, \cdot)$ have a bounded extension, then there is a bounded operator B_x such that for all $\xi, \eta \in \mathscr{D}$

$$\langle \xi, B_x\eta \rangle = \phi_x(\xi, \eta)$$

As $\mathscr{D}(\mathscr{L}) \cap \mathscr{M}_F$ is u. w. dense in $\mathscr{M}$, there is a net x_α in this set such that

$$x_\alpha \to x \text{ u. w.}$$

then $\phi_{x_\alpha}(\xi, \eta) \to \phi_x(\xi, \eta)$ for all $\xi, \eta \in \mathscr{D}$ or $\langle \xi, \mathscr{L}(x_\alpha)\eta \rangle \to \langle \xi, B_x\eta \rangle$.

As both $\mathscr{L}(x_\alpha)$ and B_x are bounded we have that $\mathscr{L}(x_\alpha) \to B_x$ weakly, hence $B_x \in \mathscr{M}$.

In case $\mathscr{M}$ has a separating vector, then the weak and ultraweak topologies coincide [20]. Thus we constructed a net x_α in $\mathscr{D}(\mathscr{L})$

$$x_\alpha \to x \text{ u. w.}$$

and

$$\mathscr{L}(x_\alpha) \to B_x \text{ u. w.}$$

as $\mathscr{L}$ is u. w. closed, we obtain $x \in \mathscr{D}(\mathscr{L})$ and $\mathscr{L}(x) = B_x$. ∎

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