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Remarks on the use of the stable tangent bundle in the differential geometry and in the unified field theory (*)

by

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SUMMARY. — A method of generating the stable tangent bundle of a differentiable manifold from its differentiable structure is indicated, which may be used to describe higher dimensional unified field theories and to overcome the difficulty of the high dimensionality. This idea is exemplified by the Einstein-Mayer-Cartan theory.

Also, some applications of the stable tangent bundle in the differential geometry are considered. These include interpretations of affine, projective and conformal connections and of the so called f-structures, with complemented frames, where f is a (1, 1)-tensor field with $f^3 = \pm f$.

It is known that the general aim of the unified field theory was not achieved. However, it gave interesting developments and it is still worthy of attention.

A largely developed type of unified theories are the so-called five-dimensional theories, which are based on the consideration of a five-dimensional physical universe. (Actually, there are also higher dimensional theories.)

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One of the important difficulties of such theories is the lack of a physical interpretation for the fifth dimension.

However, by an examination of the mentioned theories we may see that they are actually using only four point coordinates but five vector coordinates, and that for the five vector coordinates there are physical interpretations. Hence, if we are able to develop the theory by using a naturally defined vector bundle with 5-dimensional fibre on a four dimensional manifold, the previous difficulty disappears.

Then, the simplest idea is to take the stable tangent bundle $T(M) \oplus \theta$, where θ is the trivial line bundle $M \times R$. This bundle might be introduced either formally or as the restriction of the usual tangent bundle of the manifold $M \times R$ to $M \times \{0\}$. But, we think that such definitions are unsatisfactory for the philosophy of the unified theories because the first gives no unification and the second makes an indirect use of a fifth point coordinate.

Hence, we have to answer the question of whether it is possible to generate the bundle $T(M) \oplus \theta$ from the manifold structure of M only, and whether we may define for this bundle the operations which allow the development of the unified theories.

In this note, we shall answer the previous questions in the affirmative for any bundle of the form $T(M) \oplus \theta^h$. Also, we shall use this opportunity in order to develop some purely geometrical applications of the stable tangent bundle $T(M) \oplus \theta^h$ ($h \geq 1$).

The whole development is in the C^∞-category, which, as a matter of fact, is necessary for the intended definition of the stable tangent bundle only. But the fields of the geometric objects used by the unified theories may be of an arbitrary class C^k.

1. THE STABLE TANGENT BUNDLE OF A DIFFERENTIABLE MANIFOLD

If M^n is an n-dimensional differentiable manifold and T(M) is the usual tangent bundle of M, then the *stable tangent bundle* is defined as $T(M) \oplus \theta^h$, where θ^h denotes the trivial vector bundle $M \times R^h$ on M. This bundle is used in some problems of differential topology, and here we want to use it in the differential geometry and in the unified field theories. In order to answer to the questions listed in the introduction, we shall define the stable tangent bundle by the help of some general schema.

Let us consider the manifold M above and let A be an associative and commutative algebra over the real field R.

Then, a function $f : M \to A$ will be called *differentiable* iff, for every R-linear map $\phi : A \to R$, the composed function $\phi \circ f : M \to R$ is differentiable. Clearly, if A is either the real or the complex field this is the same with the classical notion of differentiability. Also, the introduced notion has a *local character*, i. e. it depends on the germs of f only.

Now, if $x \in M$, any operator v_x which sends the germs at x of the A-valued differentiable functions into A and has the properties

$$1) \qquad v_x(\alpha f + \beta g) = \alpha v_x(f) + \beta v_x(g) \qquad (\alpha, \beta \in R),$$

$$2) \qquad v_x(fg) = v_x(f).g(x) + f(x).v_x(g),$$

will be called an *A-tangent vector* to M at x. The set of such vectors defines the *A-tangent space* $T_x(M ; A)$, and this is a linear space over R.

Next, if for every $x \in M$ we have some v_x and if for every differentiable function $f : M \to A$ the function $\phi(x) = v_x(f)$ is also differentiable, we shall say that $v = \{ v_x \}$ is an A-vector field on M and we shall denote $\phi = v(f)$. Hence, we see that the A-vector fields are operators on the A-valued differentiable functions which satisfy the properties corresponding to 1) and 2) above. Also, it is clear that we may define in the classical manner the bracket of two A-vector fields and that the set of these fields defines a Lie algebra over R.

Further, we may define the A-tangent covectors as elements of the dual space $T_x^*(M ; A)$, and, next, using the definition of the general tensors as R-multilinear functions on tangent vectors and covectors, we may define all the spaces of the A-*tensors* on M, A-tensor fields and the usual operations with them.

Particularly, we may define *R-valued A-differential forms* and the exterior differential calculus.

Moreover, we may also construct an exterior differential calculus with *A-valued A-differential forms.* The exterior derivative will be defined by the usual global formula, which is possible because we have the bracket operation. To see that the usual properties hold, it suffices to consider the global proofs encountered in the classical case. (See, for instance, [9]).

Especially, it follows that we may associate to M and A the *de Rham cohomology spaces* $R^i(M ; A)$ (not to be confused with the cohomology of M with coefficients in A), constructed in the usual way but with A-valued forms. These spaces are invariant by diffeomorphisms of M.

Finally, it is important that one may also obtain the absolute differential calculus for A-tensors.

Namely, an *A-linear connection* will be an operator ∇, which associates to every A-tangent vector v_x $(x \in M)$ and every germ w of an A-field around x a new A-vector $s_x = \nabla_v w$ in such a way that

$$1) \qquad \nabla_{(v^1 + v^2)} w = \nabla_{v^1} w + \nabla_{v^2} w ; \qquad \nabla_v(w_1 + w_2) = \nabla_v w_1 + \nabla_v w_2 ;$$

$$2) \qquad \nabla_{\alpha v} w = \alpha \nabla_v w ; \qquad \nabla_v(fw) = (v_x f)w + f(x)\nabla_v w ;$$

where $\alpha \in R$ and f is the germ of a real valued differentiable function around x. Moreover, we shall ask that whenever we have the A-fields v, w defined on some open subset $U \subseteq M$, the values of $\nabla_v w$ define again an A-field on U. Then, it follows just like in the classical case (see for instance [7]) that the connection is defined by its action on global A-fields. ∇ will be called the *covariant derivative* and it may be extended in the classical manner to all the A-tensor fields. Moreover, because we have the bracket operation, we may define, again in the classical manner, the *torsion* and the *curvature* of ∇ [8] and derive the *Bianchi identities*.

Hereafter, we shall apply the previous definitions to the R-algebra generated over R by the basis $(1, a_1, \ldots, a_h)$, whose multiplication table is defined by

$$(1.1) \qquad 1.a_i = a_i.1 = a_i,\ 1.1 = 1,\ a_i.a_j = 0 \qquad (i, j = 1, \ldots, h).$$

We shall denote this algebra by $\mathcal{R}(h)$. Clearly $\mathcal{R}(0) = R$ and $\mathcal{R}(1)$ is the so-called algebra of the *dual numbers*. If

$$(1.2) \qquad x = x_0 + \sum_{i=1}^{h} x_i a_i,\ y = y_0 + \sum_{i=1}^{h} y_i a_i$$

(where $x_0, x_i, y_0, y_i \in R$) are elements in $\mathcal{R}(h)$, we have

$$(1.3) \qquad xy = x_0 y_0 + \sum_{i=1}^{h} (x_0 y_i + x_i y_0) a_i .$$

Then, it is simple to see that a function $f : M \to \mathcal{R}(h)$ is differentiable iff all its real components are such and if we put

$$(1.4) \qquad f = f_0 + \sum_{i=1}^{h} f_i a_i ,$$

we see just like for the classical tangent vectors that $T_x(M ; \mathcal{R}(h))$ is a finite-dimensional real linear space.

Namely, if t^α $(\alpha = 1, \ldots, n)$ are local coordinates at $x \in M$, we have by [7, p. 21]

$$f(x') = f(x) + \sum_{\alpha=1}^{n} (t'^\alpha - t^\alpha) g_\alpha(x'),$$

where x has the local coordinates t^α, x' has the local coordinates t'^α, and

$$g_\alpha(x') = g_\alpha^0(x') + \sum_{i=1}^{h} g_\alpha^i(x') a_i ,$$

where
$$g_\alpha^0(x) = (\partial f_0/\partial t^\alpha)_x,\ g_\alpha^i(x) = (\partial f_i/\partial t^\alpha)_x .$$

It follows that for any $\mathscr{R}(h)$-vector v at x we have

$$vf = \sum_{\alpha=1}^{n}(vt^\alpha)(\partial f/\partial t^\alpha) + \sum_{i=1}^{h}(va_i)f_i ,$$

where the derivatives of f are taken componentwise and all the functions have to be evaluated at x.

Hence, v is defined by the values in $\mathscr{R}(h)$ of vt^α and of va_i. Since, by (1.1), we get easily that we must have $va_i = \sum_{j=1}^{h'} \alpha_i^j a_j$, it follows that *a natural basis* of $T_x(M;\mathscr{R}(h))$ over the reals consists of the operators T_α, $a_i T_\alpha$ and $P_{ij}(M)$, where

$$(1.5) \qquad T_\alpha f = \partial f_0/\partial t^\alpha + \sum_{i=1}^{h}(\partial f_i/\partial t^\alpha)a_i,\ P_{ij}(M)f = f_i a_j$$

whence the dimension of $T_x(M;\mathscr{R}(h))$ is $n + nh + h^2$.

Also, we see that

$$(1.6) \qquad T(M;\mathscr{R}(h)) = \bigcup_{x\in X} T_x(M;\mathscr{R}(h))$$

is a vector bundle on M, namely $\underbrace{T(M) \oplus \ldots \oplus T(M)}_{(h+1)\ \text{times}} \oplus \theta^{h^2}$ where θ^{h^2} denotes the trivial bundle $M \times R^{h^2}$.

Now, let us consider the stable tangent bundle $T(M) \oplus \theta^h$, which we shall denote thereafter by $T(M;R(h))$. It is clear that this is a subbundle of $T(M,\mathscr{R}(h))$, which is generated by the local bases T_α and $P_i(M) = P_{ii}(M)$. Moreover, we shall see immediately that this subbundle is closed with respect to all the considered operations and constructions for $\mathscr{R}(h)$-tangent vectors. The elements of $T(M;R(h))$ will be called *R(h)-tangent vectors* and all the corresponding operations will get the label R(h) instead of $\mathscr{R}(h)$.

By the previous development, $T(M;\mathscr{R}(h))$ and also $T(M : R(h))$ are generated by the manifold M thereby achieving the goal which we proposed.

Thus, every R(h)-vector (field) has a unique representation of the form

$$(1.7) \qquad v = X + \sum_{i=1}^{h} \xi^i P_i ,$$

where X is a classical tangent vector (field) which we'll call the *projection* of v. We also see that the classical vectors are R(h)-vectors too. The following formulas may be easily established.

$$(1.8)\qquad \begin{aligned} vf &= Xf_0 + \sum_{i=1}^{h} (Xf_i + \xi^i f_i)a_i\,, \\ [v, w] &= [X, Y] + \sum_{i=1}^{h} (X\eta^i - Y\xi^i)P_i, \end{aligned}$$

where v and f are as above and $w = Y + \sum_{i=1}^{h} \eta^i P_i$. Also, if $\phi : M \to N$ is a differentiable mapping we have

$$(1.9)\qquad \phi_* v = \phi_* X + \sum_{i=1}^{h} \xi^i P_i(N)\,.$$

Next, the R(h)-tensor calculus and exterior calculus may be developed and we only mention the following formulas

$$(1.10)\qquad \begin{aligned} \Phi \wedge \Psi &= \Phi_0 \wedge \Psi_0 + \sum_{i=1}^{h} (\Phi_0 \wedge \Psi_i + \Phi_i \wedge \Psi_0)a_i\,, \\ d(\Phi \wedge \Psi) &= d\Phi \wedge \Psi + (-1)^{\deg\Phi}\Phi \wedge d\Psi\,, \\ d\Phi &= d\Phi_0 + \sum_{i=1}^{h} (d\Phi_i + (-1)^{\deg\Phi}\Phi_i \wedge \theta^i)a_i\,, \end{aligned}$$

where (dt^α, θ^i) are the dual cobases of (T_α, P_i) and the rest of the notation is obvious.

Let us mention that, while we have of course $d^2 = 0$, we have no Poincaré type lemma. In fact, we have $d\theta^i = 0$, but there is no $U \subseteq M$ and no function $f : U \to R(h)$ such that $\theta^i = df$, since the contrary assumption leads to a contradiction.

As for the corresponding cohomology spaces we have

PROPOSITION. — *If* $H^i(M, R)$ *are the classical real cohomology spaces of the manifold, there is an injection* $\alpha : H^i(M, R) \to R^i(M; R(h))$ *and a surjection* $\beta : R^i(M,; R(h)) \to H^i(M, R)$ *such that* $\beta \circ \alpha = \mathrm{id}.$, *whence the exact sequences*

$$\begin{aligned} 0 \to H^i(M, R) &\xrightarrow{\alpha} R^i(M; R(h)) \to \operatorname{coker}\alpha \to 0 \\ 0 \to \ker\beta &\to R^i(M; R(h)) \xrightarrow{\beta} H^i(M, R) \to 0 \end{aligned}$$

split.

In fact, β is defined by restricting the arguments of the representative form of a class of $R^i(M; R(h))$ to usual vector fields, and α is obtained by extending trivially the classical differential forms on M to $R(h)$-forms such that the extended form be zero whenever an argument is P_i. (Of course, $R^i(M; R(h))$ is not to be confused with $R^i(M; \mathscr{R}(h))$.

Finally, we may consider $R(h)$-linear connections ∇, for which covariant derivatives, curvature and torsion are available. Then, the projection of the restriction of ∇ to usual vector fields on T(M) defines the *induced connection*, which is a usual linear connection ∇'. Similarly, the projections of $\nabla_X P_i$ ($i = 1, \ldots, h$) define h classical tensor fields of type (1,1) on M. Also, if we project $\nabla_{P_i} X$ on the θ^h-component of the stable tangent bundle we get h new tensor fields of type (1,1) and from $\nabla_{P_i} P_j$ we get h^2 scalar functions. Conversely, the $R(h)$-connection ∇ may be reconstructed from the previously mentioned objects.

If we define $DT_\alpha(v) = \nabla_v T_\alpha$, $DP_i(v) = \nabla_v P_i$, we get the local expression of the connection ∇ under the form

$$
(1.11) \qquad
\begin{aligned}
DT_\alpha &= \sum_{\beta=1}^{n} \omega_\alpha^\beta T_\beta + \sum_{j=1}^{h} \varpi_\alpha^j P_j, \\
DP_i &= \sum_{\beta=1}^{n} \pi_i^\beta T_\beta + \sum_{j=1}^{h} \nu_i^j P_j,
\end{aligned}
$$

where ω_α^β, ϖ_α^j, π_i^β, ν_i^j are scalar valued $R(h)$-Pfaff forms, which form a matrix ω defined on coordinate neighbourhoods of M and with the classical transformation law

$$
(1.12) \qquad \omega' = g\omega g^{-1} + dg.g^{-1}, \qquad g = \begin{pmatrix} \partial t'^\alpha / \partial t^\beta & 0 \\ 0 & \delta_i^j \end{pmatrix}.
$$

As in the classical theory, it follows that the curvature of ∇ is given by the local matrices of scalar valued $R(h)$-two-forms

$$
(1.13) \qquad \Omega = d\omega - \omega \wedge \omega,
$$

the Bianchi identities are $D\Omega = 0$, where D is the *covariant-exterior derivative* and the Chern-Weil construction of the characteristic classes may be mimiced.

Also, the torsion of ∇ may be expressed by the following local forms

$$
(1.14) \qquad
\begin{aligned}
\tau^\alpha(v, w) &= \sum_{\beta=1}^{n} (\omega_\beta^\alpha(v) w^\beta - \omega_\beta^\alpha(w) v^\beta) + \sum_{i=1}^{h} (\pi_i^\alpha(v)\eta^i - \pi_i^\alpha(w)\xi^i), \\
\tau^i(v, w) &= \sum_{\beta=1}^{n} (\varpi_\beta^i(v) w^\beta - \varpi_\beta^i(w) v^\beta) + \sum_{j=1}^{h} (\nu_j^i(v)\eta^j - \nu_j^i(w)\xi^j),
\end{aligned}
$$

where
$$v = \sum_{\alpha=1}^{n} v^\alpha T_\alpha + \sum_{i=1}^{h} \xi^i P_i, \; w = \sum_{\alpha=1}^{n} w^\alpha T_\alpha + \sum_{i=1}^{h} \eta^i P_i \,.$$

It is interesting to point out that the class of the R(h)-connections contains many classical connections as particular cases.

First, if we ask $\nabla_{P_i} v = 0$ ($i = 1, \ldots, h$), we get a usual linear connection on the vector bundle T(M ; R(h)) and if we ask moreover that $\nabla_X P_i = 0$ ($i = 1, \ldots, h$), we get a usual linear connection on T(M), which is the induced connection ∇'.

Next, let us consider the set of the 1-dimensional subspaces of the local fibres of T(M ; R(h)). This set obviously defines a locally trivial bundle with fibre the $(n + h - 1)$-dimensional real projective space, and we may consider the principal bundle of the corresponding projective frames, which we shall denote by P(M ; R(h)). Moreover, the local bases $\{ T_\alpha, P_i \}$ define local cross-sections of P(M ; R(h)).

It follows that an infinitesimal connection on P(M ; R(h)) may be expressed, by the help of the previous local cross-sections, under the form (1.11), where the matrix ω consists of classical Pfaff forms and it satisfies the law (1.12) and the *normalization condition*

$$v_1^1 = 0 \,. \tag{1.15}$$

It is natural to call such a connection a $(n + h - 1)$*-dimensional projective connection* on M. Particularly, for $h = 0$ we get connections on the space of the tangent directions of M, for $h = 1$ we get E. Cartan's *projective connections* on M, and for $h = 2$, and replacing (1.15) by the system of conditions

$$\begin{gathered} \omega_\alpha^\beta + \omega_\beta^\alpha = 0, \qquad \pi_1^\alpha = -\varpi_\alpha^1, \qquad \pi_2^\alpha = -\varpi_\alpha^2 \,, \\ v_1^1 = v_2^2 = 0, \qquad v_1^2 = -v_2^1 \,, \end{gathered} \tag{1.16}$$

we get E. Cartan's *conformal connections* on M.

It is clear now that all the mentioned infinitesimal connections may be identified with R(h)-connections. Namely, the $(n + h - 1)$-dimensional projective connections are R(h)-connections ∇ such that

$$\nabla_{P_i} v = 0 \,, \qquad \mathrm{pr}_{P_1}(\nabla_X P_1) = 0 \quad (i = 1, \ldots, h) \tag{1.17}$$

and, for $h = 1$, we have the usual projective connections. For the conformal connections we must have $h = 2$, (1.17) and (1.16). Essentially, such interpretations of the projective and conformal connections go back to R. König (1920) and J. A. Schouten (1924). By the present interpretation we also get a natural definition of the torsion of these connections.

Let us finally remark that the fibres of the previously introduced projective bundle have a special structure. Namely, they have a distinguished

$(n-1)$-plane defined by the points T_α and other h distinguished points P_i, which enables us to consider different principal subbundles of $P(M ; R(h))$. For instance, if we take the bundle of the projective frames which have n vertices in the distinguished plane, its infinitesimal connections will be $R(h)$-connections ∇ which satisfy (1.17) and

$$\text{(1.18)} \qquad \mathrm{pr}_{\theta^h}(\nabla_X Y) = 0,$$

for any two vector fields X and Y on M. If, moreover, $h = 1$ these are just the *affine connections* on M [*8*].

Before ending this section let us still make the following remark. If we intend to use the stable tangent bundle in geometry only, without any relation to unified field theories, it suffices to define it simply as the restriction of $T(M \times R^h)$ to $M \times \{0\}$. We have a natural action of the group R^h on $M \times R^h$ by translations on the last coordinates and the $R(h)$-vector fields may be identified with invariant cross-sections of $T(M \times R^h)$. All the operations considered in this (and the next) section are then induced by the corresponding operations on $T(M \times R^h)$.

2. STABLE ALMOST COMPLEX AND ALMOST PRODUCT STRUCTURES

In view of the definitions of Section 1, we may define on $T(M ; R(h))$ structures which are similar to the classical structures studied in differential geometry. Usually, such structures define some corresponding structures on T(M) and the first may be used for an easier description of the latest. Our aim in this section is to consider from this viewpoint the stable almost complex and stable almost product structures which seem to give more interesting results.

DÉFINITION. — Let F be a differentiable field of endomorphisms of the fibres of $T(M ; R(h))$, and suppose that

$$\text{(2.1)} \qquad F^2 = \varepsilon I,$$

where I is the identity transformation. Then, if $\varepsilon = -1$, F is said to be a *stable almost complex* (s. a. c.) structure on M and if $\varepsilon = +1$, F is a *stable almost product* (s. a. p.) structure on M.

In both cases, F is clearly non-degenerate and for an s. a. c. structure we must necessarily consider $n + h$ even.

PROPOSITION. — *An s. a. c. (s. a. p.) structure* F *on* M *may be identified with the* G*-structure defined on* M *by the following system of tensor fields:* i) *a tensor field f of type* (1, 1), ii) *h vector fields ξ_i and h covector fields η^i*

globally defined on M *and* h^2 *scalar functions* Φ_i^j $(i, j = 1, \ldots, h)$, *such that the following relations hold*

$$(2.2)\qquad \begin{aligned} &\varepsilon f^2 = \mathrm{I} - \sum_{i=1}^{h} \eta^i \otimes \xi_i, \qquad \eta^i \circ f = -\sum_{j=1}^{h} \Phi_j^i \eta^j, \\ &f(\xi_i) = -\varepsilon \sum_{j=1}^{h} \Phi_i^j \xi_j, \qquad \eta_j(\xi^i) = \delta_i^j - \varepsilon \sum_{k=1}^{h} \Phi_i^k \Phi_k^j. \end{aligned}$$

Proof. — In fact, if we have F we shall define

$$(2.3)\qquad \begin{aligned} &f(\mathrm{X}) = \mathrm{pr}_{\mathrm{T(M)}}\mathrm{F}(\mathrm{X}), \qquad \eta^i(\mathrm{X}) = \text{coefficient } \mathrm{pr}_{\mathrm{P}_i}\mathrm{F}(\mathrm{X}), \\ &\xi_i = \varepsilon \mathrm{pr}_{\mathrm{T(M)}}\mathrm{F}(\mathrm{P}_i), \qquad \Phi_i^j = \text{coefficient } \mathrm{pr}_{\mathrm{P}_j}\mathrm{P}_i, \end{aligned}$$

where the notation is like in Section 1. Then (2.2) holds because it is just a transcription of (2.1).

Conversely, by (2.3) we may reconstruct F and get (2.1).

The proposition suggests us to consider *distinguished* s. a. c. (p.) (d. s. a. c. (p.)) structures, defined by the supplementary condition that F sends all the R(h) vector fields P_i into T(M). This means $\Phi_i^j = 0$ and (2.2) become

$$(2.4)\qquad \varepsilon f^2 = \mathrm{I} - \sum_{i=1}^{h} \eta^i \otimes \xi_i, \ \eta^i \circ f = 0, \ f(\xi_i) = 0, \ \eta^j(\xi_j) = \delta_j^i.$$

It follows that a d. s. a. c. structure is the same thing as an f-structure $(f^3 + f = 0)$ with complementary frames and, in the case $h = 1$, with an almost contact structure on M. For $\varepsilon = 1$ we have $f^3 - f = 0$ and, in order to consider both cases simultaneously, we shall understand thereafter by an (f, ε)-*structure* a structure with $f^3 - \varepsilon f = 0$. (The difference between the two cases $\varepsilon = \pm 1$ is however essential in many respects.)

For further considerations, we need also R(h)-*Riemann metrics*, i. e. R(h)-tensor fields g of type (0, 2) symmetrical, non-degenerate and positive definite. The simplest manner of obtaining such metrics is the following: we start with a classical Riemann metric γ on M and define

$$(2.5)\qquad g(v, w) = \gamma(\mathrm{X}, \mathrm{Y}) + \sum_{i=1}^{h} \alpha^i \beta^i,$$

where

$$(2.6)\qquad v = \mathrm{X} + \sum_{i=1}^{h} \alpha^i \mathrm{P}_i, \qquad w = \mathrm{Y} + \sum_{i=1}^{h} \beta^i \mathrm{P}_i.$$

In this case we'll call g the *trivial extension* of γ.

Naturally if we have a s. a. c. (p.) structure F and any R(h)-metric g they are to be called *compatible* iff

$$g(Fv, Fw) = g(v, w), \tag{2.7}$$

and in this case we call (g, F) a *metric s. a. c. (p.) structure.*

Now, if we suppose that F is distinguished and g is the trivial extension of some Riemann metric γ, (2.7) may be expressed by the help of the corresponding (f, ε)-structure. Namely, we have

$$Fv = fX + \sum_{i=1}^{h} \eta^i(X)P_i + \varepsilon \sum_{i=1}^{h} \alpha^i \xi_i, \tag{2.8}$$

for v of (2.6), and a similar expression for Fw, whence the compatibility relation (2.7) becomes

$$\begin{gathered} \gamma(fX, fY) = \gamma(X, Y) - \sum_{i=1}^{h} \eta^i(X)\eta^i(Y), \\ \gamma(fX, \xi_i) = 0, \qquad \gamma(\xi_i, \xi_j) = \delta_{ij}. \end{gathered} \tag{2.9}$$

It is easy to see that the first condition (2.9) implies the other two conditions and, in view of known definitions, we see that (g, F) may be identified with a *metric (f, ε)-structure with complementary frames* and, for $h = 1$, $\varepsilon = -1$, with a *metric almost contact structure.*

Let us remark that (2.8) may also be written as

$$F = f' + \varepsilon \sum_{i=1}^{h} \theta^i \otimes \xi_i + \sum_{i=1}^{h} \eta'^i \otimes P_i, \tag{2.8'}$$

where f', η'^i are the trivial extensions of f and η^i respectively and θ^i are the 1-forms of the cobases of Section 1.

In the case $\varepsilon = -1$, there is one more interesting object, namely the *fundamental form* Ω which is a scalar R(h)-two-form defined by

$$\Omega(v, w) = g(v, Fw), \tag{2.10}$$

or, in view of the formulas (2.5)-(2.9)

$$\Omega(v, w) = \gamma(X, fY) - \sum_{i=1}^{h} (\beta^i \eta^i(X) - \alpha^i \eta^i(Y)). \tag{2.11}$$

The term $\gamma(X, fY) = \Theta(X, Y)$ defines a classical two-form, called the fundamental form of the f-structure. (If $\varepsilon = 1$, Ω is a symmetric tensor and we don't use it.)

In view of Section 1, we may consider F-structures such that $d\Omega = 0$, i. e. *stable almost Kähler structures.* If we remark from (2.11) that

$$\Omega = \Theta' + \sum_{i=1}^{h} \theta^i \wedge \eta'^i, \tag{2.12}$$

where Θ' is the trivial extension of Θ, it is easy to see that $d\Omega = 0$ is equivalent to $d\Theta = 0$ and $d\eta^i = 0$. In this case, we'll say that the respective f-structure is *closed* and if, moreover, $h = 1$, that it is a *metric cosymplectic structure.* If we replace $d\Omega = 0$ by the weaker condition $d\Omega = 0$ (mod. $\theta^i = 0$), we have the equivalent condition $d\Theta = 0$ and the corresponding f-structure is sometimes called a K-structure. Finally, if we consider $h = 1$ and the more special condition

$$d\Omega = -\,\theta \wedge \Omega, \tag{2.13}$$

we get the equivalent condition $\Theta = d\eta$, i. e. we have just a contact *metric structure* on M.

Further, and coming back to the consideration of both cases $\varepsilon = \pm 1$, since there is a generalized bracket, we may define the R(h)-Nijenhuis tensor

$$N(v, w) = F^2[v, w] + [Fv, Fw] - F[Fv, w] - F[v, Fw], \tag{2.14}$$

and if $N(v, w) = 0$ we'll say that F defines a *formally integrable structure.* Using the previous formulas, one sees that $N(v, w) = 0$ is equivalent to the following set of relations

$$\begin{gathered} N^f(X, Y) - \varepsilon \sum_{i=1}^{h} d\eta^i(X, Y)\xi_i = 0, \\ (L_{fX}\eta^i)(Y) - (L_{fY}\eta^i)(X) = 0, \\ L_{\xi_i} f = 0, \quad L_{\xi_i}\eta^j = 0, \qquad [\xi_i, \xi_j] = 0, \end{gathered} \tag{2.15}$$

where N_f is the classical Nijenhuis tensor of f and L denotes the Lie derivative. By the same calculations like in [*1*, p. 50], we can see that the first relation (2.15) implies all these relations and this first relation characterizes the so-called *normal f-structures.* It follows that the normality of f is equivalent with the formal integrability of F.

Now, before proceeding with the F-structures let us consider an arbitrary Riemann metric g on the stable tangent bundle T(M ; R(h)). Then, it follows, by just the same calculations like in the classical case [*8*], that there is a unique R(h)-connection ∇ which has no torsion and is such that $\nabla g = 0$. This will be the R(h)-*Levi-Civita connection* of (M, g). It is easy to see that if g is the trivial extension of a usual Riemann metric γ, the R(h)-Levi-Civita connection of g will be the trivial extension of the Levi-Civita connection of γ.

Consider a metric d. s. a. c. (p.) structure (F, g) on M and let ∇ be the R(h)-Levi-Civita connection of g. Then, we may consider a lot of conditions which are used to delimitate important classes of almost Hermitian manifolds (see, for instance, [6]) and thus introduce corresponding classes of f-structures.

Such are, for instance, the conditions

(2.16) $\nabla_v F = 0$ (the R(h)-Kähler condition),

(2.17) $(\nabla_v F)(w) + (\nabla_w F)(v) = 0$ (the R(h)-nearly Kähler condition),

(2.18) $(\nabla_v F)(w) + (\nabla_{Fv} F)(Fw) = 0$ (the R(h)-quasi-Kähler condition),

etc., which are respectively equivalent to

(2.16′) $\nabla_X f = 0, \quad \nabla_X \xi_i = 0 \qquad (i = 1, \ldots, h),$

(2.17′) $(\nabla_X f)(Y) + (\nabla_Y f)(X) = 0, \qquad \nabla_X \xi_i = 0,$

$$(\nabla_X f)(Y) + (\nabla_{fX} f)(fY) + \varepsilon \sum_{i=1}^{h} \eta^i(Y) \nabla_{fX} \xi_i = 0,$$

(2.18′) $(\nabla_X \eta^i)(Y) - \eta^i(\nabla_{fX} fY) = 0,$

$$\nabla_X \xi_i - f \nabla_{fX} \xi_i = 0, \qquad \eta^j(\nabla_{fX} \xi_i) = 0, \qquad \nabla_{\xi_i} \xi_j = 0,$$

These conditions define special classes of (f, ε)-structures which probably deserve more attention. For instance, for $\varepsilon = -1$, we may see like in the classical theory that the manifolds satisfying (2.16), i. e. (2.16′), are characterized by the normality condition (2.15) together with the closedness of the forms η^i and Θ. In the compact case, these forms are harmonic and, since clearly $\eta^i \wedge \ldots \wedge \eta^h \wedge \Theta^{(n-h)/2}$ is a volume element, we get

(2.19) $b^1(M) \geq h, \quad b^2(M) \geq \binom{h}{2} + 1, \quad b^3(M) \geq \binom{h}{3} + h, \ldots,$

where $b^i(M)$ are the Betti numbers of the manifold M. Particularly, for a compact manifold which has a metric cosymplectic normal structure all the Betti numbers are nonvanishing.

We see from the discussion in this section that the consideration of the stable structures gives a unified viewpoint on interesting classes of usual structures on a differentiable manifold.

3. COMMENTS ON UNIFIED FIELD THEORIES

We do not intend here to give an essentially new unified field theory nor to discuss the physical significance of such theories. As we said in the introduction, our aim is simply to provide a mathematical argument in favor of the *five-dimensional unified field theories* and we shall discuss this in the case of an interesting example, the Einstein-Mayer-Cartan theory.

One of the main physical difficulties of the previously mentioned unified theories is just their five-dimensional character. Actually, under the usual interpretations, this implies a five-dimensional physical universe, which is in contradiction with the generally accepted views about the physical reality.

As a matter of fact, this difficulty arises only if we want to have five coordinates for the *points* of the universe, because we have no interpretation for the fifth coordinate. But if we are interested in *five-dimensional vectors over a four-dimensional manifold*, such an interpretation exists. For instance, following E. Cartan [*4*], we may attach to a particle a vector of the form $m_0\bar{t} + e\bar{v}$, where $\bar{t}$ is the usual unit tangent vector to the trajectory of the particle, $\bar{v}$ points into the fifth dimension, m_0 is the rest-mass and e is the charge of this particle.

Hence, if we are able to develop the mathematics of the five-dimensional unified theories using some kind of five-dimensional vectors over four-dimensional manifolds, the mentioned difficulty disappears. Such a mathematical development may be obtained using for vectors the elements of the stable tangent bundle $T(M^4 ; R(1))$. And, moreover, we have a development where we need in no way five punctual coordinates.

One more thing. Most of the relativistic and unified theories consider that the fundamental geometric structure of the universe is defined by a metric. Hence, if we want it to be a pseudo-Riemann metric on the four-dimensional universe we are not able to derive a unitary theory. In this note, we shall agree with an argument of E. Cartan [*3*] showing that it is sensible to consider that the fundamental geometric structure of the universe is given by a connection. Namely, the mentioned argument says that the fundamental physical fact is that any two sufficiently near observers are able to locate the reference frame of each other. This may be interpreted geometrically by the existence of a connection. And now, since our vectors are in $T(M^4 ; R(1))$ it is natural to consider an $R(1)$-connection ∇.

Thus, our main hypothesis is that *the physical universe is a four-dimensional differentiable manifold* M^4 *endowed with an* $R(1)$*-linear connection* ∇, *which satisfies some field equations.*

In the sequel, we shall show that this hypothesis is consistent with the unified theory of Einstein-Mayer in the form given by E. Cartan [*5*]. In fact, this is an older theory but it allows a good illustration of the main idea and we are not interested in other aspects of unified theories.

To get the Einstein-Mayer-Cartan field equations, we begin by asking for ∇ the following conditions (where the notation is like in the previous sections) :

1) $\nabla_P v = 0$ for every $R(1)$-vector field v $(P = P_1)$; intuitively this means that there is no displacement along P;

2) the universe M^4 has a pseudo-Riemannian metric γ whose trivial extension g to T(M, R(1)) is invariant by ∇, i. e. $\nabla g = 0$;

3) the torsion $T_\nabla(X, Y)$ is collinear with P for every two classical vector fields X, Y on M;

4) for every vector field X on M, $\nabla_X X$ is also a vector field.

In order to get the consequences of these hypotheses, we shall write the operator ∇ under the form

$$\begin{aligned} \nabla_X Y &= \nabla'_X Y + h(X, Y)P, \\ \nabla_X P &= f(X) + k(X)P, \end{aligned} \tag{3.1}$$

which together with 1) defines ∇. In (3.1), ∇' is the induced connection on M, h is a tensor field of type (0, 2) and f a tensor field of type (1, 1), and k a 1-form on M.

Now, it is easy to see that 4) is equivalent with the skew-symmetry of the tensor h, 3) is equivalent with the fact that ∇' is without torsion, which, in view of 2) shows that ∇' is the Levi-Civita connection of γ, and we also have

$$k(X) = 0, \quad \gamma(fX, Y) = -h(X, Y) \tag{3.2}$$

and

$$T(X, Y) = 2h(X, Y)P, \quad T(X, P) = f(X), \tag{3.3}$$

where T denotes the torsion of the connection ∇. The relations (3.2) show that k and f are well defined by the other elements of (3.1) and we shall have 16 *field potentials* namely the components of the tensors γ and h.

Hence, h is a quadratic skew-symmetric tensor field related to the torsion of ∇ and, in the considered theory it is accepted that h is just the electromagnetic field.

Further, in order to define the field equations the following tensor fields related to the curvature of ∇ are introduced :

$$\bar{R}(X, Y)Z = R'(X, Y)Z + h(Y, Z)f(X) - h(X, Z)f(Y), \tag{3.4}$$

where R' is the usual curvature of the induced connection ∇' and $\bar{R}$ is the projection on T(M) of the curvature of ∇, and

$$A(X, Y, Z) = (\nabla'_X h)(Y, Z) - (\nabla'_Y h)(X, Z), \tag{3.5}$$

which is the projection of the curvature of ∇ onto P.

By the usual contractions, we may associate to $\bar{R}$ a *Ricci tensor* $\bar{\rho}$ and a scalar curvature $\bar{r}$, which allow us to introduce

$$R^* = \bar{\rho} - (\bar{r}/2 + \| h \|^2/4)\gamma. \tag{3.6}$$

Also, we shall consider the vector field B defined by contraction from $\nabla_X f$ [4].

Now, following [4], the field equations are

$$(3.7) \qquad R^* = T, \quad B = \tau,$$

where T is the momentum-energy tensor field and τ is the vector field related to the electricity.

Hence, all the equations of [4] are available without asking for an embedding of M as a totally geodesic submanifold of a five-dimensional manifold like in the original exposition, and this is just what we wanted to obtain.

Let us make one more remark. Because of the hypothesis $\nabla_P v = 0$ we could simply ask in the previous theory that ∇ be a usual connection on the vector bundle T(M, R(1)) rather than an R(1)-connection. But, by considering it as an R(1)-connection we have the torsion and thus obtained a schema where the electromagnetic field is related to the torsion, while the gravitational field is related to the curvature of a connection. On the other hand, it is possible to renounce to the hypothesis $\nabla_P v = 0$ and consider, if so desired, theories with more field potentials.

Finally, we shall remark that the previous manner of introducing five-dimensional theories may also be applied for the Kaluza-type theories (see, for instance, [*5*, *10*]).

To obtain such theories, we shall start from a different main hypothesis, namely, that *the physical universe is a four-dimensional differentiable manifold, endowed with an* R(1)-*pseudo-Riemann metric g*.

Then, we have the R(1)-Levi-Civita connection of g and we have, both, enough field potentials and enough analytical machinery to develop unified theories.

For instance, most authors are asking a condition which is equivalent to $g(P, P) = 1$. Also, there is a canonical 1-form p defined by

$$p(u) = g(u, P),$$

whose restriction to T(M) defines a classical 1-form a on M. Consider next $\tilde{g}$ defined by

$$\tilde{g}(u, v) = g(u, v) - p(u)p(v)$$

and the metric γ induced by $\tilde{g}$ on T(M). Then g is defined by γ and a i. e. by a configuration on M alone and (γ, a) may be considered as the basic field potentials [*5*]. Let us remark that the contravariant vector field A defined on M by $g(X, A) = a(X)$ gives the intersection line of T(M) with the hyperplane orthogonal to P with respect to g.

An ample study by Thiry [*10*] studies the integrability of the field equations in a Kaluza type theory, which, in our notation, does not use the condition $g(P, P) = 1$. As for the field equations themselves, in view of a result of E. Cartan [*2*], they must be the Einstein 5-dimensional equations

$$(3.8) \qquad R - (\rho/2)g = 0,$$

where R is the Ricci tensor and ρ the scalar curvature of the R(1)-Levi-Civita connection of g.

But of course, our proposed schema does not give any answer to the difficulties related to the integration of the field equations (which led, for instance, to the essentially five-dimensional theory of [5]), but this is beyond the aim of our note.

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