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On a class of non linear Schrödinger equations. III. Special theories in dimensions 1, 2 and 3

by

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ABSTRACT. — This is the third of a series of papers where we study a class of non linear Schrödinger equations of the form

$$i\frac{du}{dt} = (-\Delta + m)u + f(u)$$

in $\mathbb{R}^n$, where m is a real constant and f a complex valued non linear function. Here we consider the case of space dimensions $n = 1, 2$ and 3. Under suitable assumptions on f, we prove the existence of global solutions of the initial value problem. These solutions are continuous functions of the time with values in $H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We also study the scattering theory for the pair of equations that consists of the previous one and of the equation

$$i\frac{du}{dt} = (-\Delta + m)u\,.$$

In particular, we prove the existence of the wave operators and asymptotic completeness for a class of repulsive interactions. The assumptions made on f cover the case of a single power $f(u) = \lambda\,|\,u\,|^{p-1}u$ with various restrictions on p and λ. For the existence of global solutions, we require $p \geqslant 2$ and in addition $p < 5$ if $n = 3$ and $\lambda > 0$, $p < 1 + 4/n$ if $\lambda < 0$. For asymptotic completeness, we require $\lambda \geqslant 0$ and $p > 1 + 4/n$, and in addition $p < 5$ if $n = 3$.

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0. INTRODUCTION

This is the third of a series of papers where we study a class of non linear equations related to the Schrödinger equation. The equations we consider are of the following type

$$i\frac{du}{dt} = H_0 u + f(u) \tag{0.1}$$

where $H_0 = -\Delta + m$, Δ is the Laplace operator in $\mathbb{R}^n$, m is a real constant and f is a non linear complex valued function.

In the first two papers we have developed a general theory for the equation (0.1) in space dimension $n \geqslant 2$, concentrating our attention on two problems:

(1) Existence and uniqueness of solutions of the Cauchy problem.

(2) Asymptotic behaviour in time of the solutions, and in particular scattering theory for the pair of equations consisting of (0.1) and of the free linear Schrödinger equation

$$i\frac{du}{dt} = H_0 u\,. \tag{0.2}$$

In [*1*], under suitable assumptions on the interaction and for initial data in the Sobolev space $H^1(\mathbb{R}^n)$, we have proved the existence and uniqueness of a global solution of the Cauchy problem for the equation (0.1). In [*2*] for suitable interactions and for suitable initial data, we have proved the existence of solutions of the Cauchy problem at infinite initial time and thereby the existence of the wave operators, and established asymptotic completeness for a class of repulsive interactions. The whole treatment makes extensive use of three conservation laws: the conservation of the L^2-norm, the conservation of the energy and a third conservation law which we called pseudoconformal conservation law.

The general theory developed in [*1*] and [*2*] is satisfactory in several respects. First all dimensions $n \geqslant 2$ can be treated in a unified way. Second the spaces where one proves the existence of global solutions are large and natural in the sense that they are the largest spaces where the relevant conservation laws make sense. These spaces are $H^1(\mathbb{R}^n)$, corresponding to the conservation of the L^2-norm and of the energy, and another space called Σ (see (1.6)), corresponding to all three conservation laws. However this general theory exhibits the following complication: the space where one first solves the local Cauchy problem at finite time is different, in fact larger, than the natural space ($H^1(\mathbb{R}^n)$) mentioned above. As a consequence, while the solution remains in $H^1(\mathbb{R}^n)$ for all times if the initial data belong to $H^1(\mathbb{R}^n)$, the continuity properties of the solution appear naturally in terms of a weaker topology, and continuity in $H^1(\mathbb{R}^n)$ is recovered by an additional compactness argument. Another feature of this theory is that the assumptions made on the interaction term $f(u)$ include (for $n \geqslant 3$) an unnatural coupled restriction between the behaviour of $f(u)$ for large and small values of u.

In this paper we develop a special theory for $n = 2, 3$ in which the global Cauchy problem for the equation (0.1) is solved in $H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. In such a theory the first complication mentioned above disappears and the solution comes out naturally as a continuous function of the time with values in $H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Furthermore for $n = 3$, the unnatural restriction which relates the behaviour of $f(u)$ for large and small values of u is removed. In contrast to the general theory however this special theory does not extend immediately to higher dimensions (*). The present paper contains also a treatment of the case of dimension $n = 1$. This case is especially simple since one can solve the Cauchy problem in $H^1(\mathbb{R})$. In order to save space, this theory will be presented together with those for $n = 2, 3$. The assumptions on f that ensure the solvability of the global Cauchy problem cover the case of a single power $f(u) = \lambda |u|^{p-1}u$ where $p \geqslant 2$, and in addition $p < 5$ if $n = 3$ and $\lambda > 0$, $p < 1 + 4/n$ if $\lambda < 0$. Existence of the wave operators is ensured if in addition $p > (n + 1 + \sqrt{2n+1})/n$ and asymptotic completeness if in addition $\lambda \geqslant 0$ and $p > 1 + 4/n$.

There is a certain amount of freedom in the choice of the spaces where to look for solutions of the equation (0.1), and the choice made here seems to be one of the simplest. On the other hand for $n = 1$ one can develop various theories in spaces larger than $H^1(\mathbb{R})$, similar to the general theories of [*1*] and [*2*]. An example of such a theory with basic space $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is briefly described in the Appendix.

This paper is largely self-contained as far as the results are concerned. However proofs will be shortened or even suppressed whenever they are similar to those contained in [*1*] and [*2*], to which we send back the reader for details and for a number of side remarks.

The paper is organized as follows. Section 1 contains some notation, definitions, preliminary results, and in particular the list of assumptions on the interaction term f. Section 2 is devoted to the proof of existence and uniqueness of the solution of the Cauchy problem in a small time interval. Section 3 contains the statement of the conservation laws and an outline of their derivation. Section 4 contains the proof of existence of global solutions of the Cauchy problem. Section 5 is devoted to the study of the Cauchy problem with initial time in a neighbourhood of infinity

(*) The case $n = 3$ has been also considered by LIN and STRAUSS, who obtain results which partly overlap with those of the present paper [*3*]. The existence problem for $n = 2$ and 3 has also been studied in [*4*].

and to the extension of the three conservation laws to infinite times. Section 6 is devoted to the study of the asymptotic behaviour of solutions and in particular to the proof that all solutions are dispersive for repulsive interactions. Section 7 contains a brief description of the continuity properties of the solutions as functions of the initial time and of the initial data. In section 8 we collect the results from sections 5, 6 and 7 concerning the wave operators. In the Appendix we describe briefly an alternative theory for $n = 1$ with basic space $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

1. PRELIMINARIES

In this section, we introduce the main notation and definitions that will be used in this paper. Most of this material is common to this paper and to [*1*] and [*2*] and it is already contained in sections 1 of [*1*] and [*2*] to which we refer for more details. In particular, all the notation is the same as in [*1*] and [*2*] with the notable exception of the spaces X, $\mathscr{X}.(.)$ and $Y.(.)$ which are slightly different.

We denote by n the number of space dimensions, which in this paper can be 1, 2, 3, by $\| \, . \, \|_q$ the norm in $L^q \equiv L^q(\mathbb{R}^n)$, $1 \leqslant q \leqslant \infty$, except for $q = 2$ where the subscript 2 will be omitted. Pairs of conjugate indices are written as q and $\bar{q}$ with $2 \leqslant q \leqslant \infty$ and $q^{-1} + \bar{q}^{-1} = 1$. $H^1(\mathbb{R}^n)$ will denote the usual Sobolev space with norm defined by

$$\| v \|_{H^1}^2 = \| v \|^2 + \| \nabla v \|^2 . \tag{1.1}$$

We shall use extensively the following Sobolev inequalities:

$$\| v \|_q \leqslant a_q \| \nabla v \|^{1-\eta} \| v \|^\eta \tag{1.2}$$

which is valid for any q such that

$$\begin{cases} 2 \leqslant q \leqslant \infty & \text{if} \quad n = 1 \\ 2 \leqslant q < 2n/(n-2) & \text{if} \quad n = 2, 3 . \end{cases} \tag{1.3}$$

with η defined by

$$1/q = 1/2 - (1 - \eta)/n \tag{1.4}$$

(so that $1/2 \leqslant \eta \leqslant 1$ if $n = 1$, and $0 < \eta \leqslant 1$ if $n = 2, 3$), and

$$\| v \|_\infty \leqslant b_q \| \nabla v \|_q^{n/q} \| v \|_q^{1-n/q} \tag{1.5}$$

which we will use for any $q > n$ and satisfying (1.3). Actually (1.2) also holds (and will be used in section 4) for $n = 3$ and $q = 6$ (i. e. $\eta = 0$).

The free evolution, associated with the equation (0.1), is defined by

$$U(t) = \exp(-itH_0) .$$

Its relevant properties can be read in the following lemma (cf. lemma 1.2 of [*1*]).

LEMMA 1.1. — For any $q \geqslant 2$, for any $t \neq 0$, $U(t)$ is a bounded operator from $L^{\bar{q}}$ to L^q and the map $t \to U(t)$ is strongly continuous. Moreover for all $t \in \mathbb{R}\backslash\{0\}$ one has

$$\| U(t)v \|_q \leqslant (4\pi |t|)^{n/q - n/2} \| v \|_{\bar{q}}$$

for all $v \in L^{\bar{q}}$ (for $q = 2$, $U(t)$ is unitary and strongly continuous for all $t \in \mathbb{R}$).

We shall also need the Hilbert space Σ that consists of those $v \in H^1$ such that $xv \in L^2$ with the norm defined by

$$\| v \|_\Sigma^2 = \| v \|^2 + \| \nabla v \|^2 + \| xv \|^2 . \tag{1.6}$$

The free evolution maps Σ into Σ. Furthermore, for all $v \in \Sigma$,

$$xU(-t)v = U(-t)(x + 2it\nabla)v .$$

The space Σ satisfies the following property:

LEMMA 1.2. — (Cf. lemma 1.3 of [2]). Let $v \in \Sigma$ and let q satisfy (1.3). Then for all $t \in \mathbb{R}$, $U(t)v \in L^q$ and $U(t)v$ satisfies the estimate

$$\| U(t)v \|_q \leqslant \tilde{a}_q(1 + |t|)^{\eta - 1} \| v \|_\Sigma \tag{1.7}$$

where η is defined by (1.4) and the $\tilde{a}_q$'s are constants depending only on n and q.

For any interval I of the real line $\mathbb{R}$, for any Banach space $\mathscr{B}$, we denote by $\mathscr{C}(I, \mathscr{B})$ (respectively $\mathscr{C}_b(I, \mathscr{B})$) the space of continuous (respectively bounded continuous) functions from I to $\mathscr{B}$. $\mathscr{C}_b(I, \mathscr{B})$ is a Banach space when equipped with the uniform topology and coincides with $\mathscr{C}(I, \mathscr{B})$ for compact I. For any interval I of the real line $\mathbb{R}$, we denote by $\bar{I}$ the closure of I in $\bar{\mathbb{R}}$, where $\bar{\mathbb{R}}$ is the compactification of $\mathbb{R}$ with two points $+\infty$ and $-\infty$ (i. e. $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$) with the obvious topology.

We shall need the Banach space $X = H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with the norm given by

$$\| v \|_X = \operatorname{Max} (\| v \|_{H^1}, \| v \|_\infty) . \tag{1.8}$$

For any interval $I \subset \mathbb{R}$, we define the following spaces:

$\mathscr{X}(I) = \mathscr{C}(I, X)$

$\mathscr{X}_a(I) = \mathscr{C}_b(I, H^1) \cap \mathscr{C}(I, L^\infty)$

$\mathscr{X}_b(I) = \mathscr{C}_b(I, X)$

$$\mathscr{X}_0(I) = \Big\{ v \in \mathscr{X}_b(I) : \sup_{t \in I} \operatorname{Max} [\| v(t) \|_{H^1}, (1 + |t|)^{1-\varepsilon} \operatorname{Max} (\| v(t) \|_r, \| v(t) \|_\infty)] \equiv |v|_{0I} < \infty \Big\}$$

for some r such that $r \geqslant 2$ for $n = 1, 2, 3$ and $r < 6$ for $n = 3$, with ε defined by $1/r = 1/2 - (1 - \varepsilon)/n$, so that $1/2 \leqslant \varepsilon \leqslant 1$ for $n = 1$ and $0 < \varepsilon \leqslant 1$ for $n = 2, 3$.

$Y(I) = \{ v : v \in \mathscr{X}(I) \text{ and } v(t) = U(t - s)v(s) \text{ for all } s \text{ and } t \text{ in } I \}$.

$Y_i(I) = \mathscr{X}_i(I) \cap Y(I) \quad i = b, 0.$

$\mathscr{X}_b(\mathrm{I})$ and $\mathscr{X}_0(\mathrm{I})$ are Banach spaces with respective norms

$$|v|_{\mathrm{I}} = \operatorname*{Sup}_{t\in\mathrm{I}} \| v(t) \|_{\mathrm{X}}$$

and $|v|_{0\mathrm{I}}$ defined above, $\mathscr{X}(\mathrm{I})$ and $\mathscr{X}_a(\mathrm{I})$ are Fréchet spaces when equipped with the topology of the uniform convergence on the compact subsets of I. As a consequence of the Sobolev inequality (1.2), for $n = 1$, $\mathrm{X} = \mathrm{H}^1$ and $\mathscr{X}_a = \mathscr{X}_b$. Furthermore $\mathrm{Y}(\mathrm{I})$ is equal to $\mathrm{Y}_b(\mathrm{I})$ and isomorphic to H^1 for all I.

If I and J are two intervals of $\mathbb{R}$, we shall use the notation

$$|\phi|_{\mathrm{I,J}} \equiv \operatorname*{Sup}_{s\in\mathrm{I}} |\phi(s)|_{\mathrm{J}} \tag{1.9}$$

$$|\phi|_{\mathrm{I,0J}} \equiv \operatorname*{Sup}_{s\in\mathrm{I}} |\phi(s)|_{0\mathrm{J}} \tag{1.10}$$

for the norms in the Banach spaces $\mathscr{C}_b(\mathrm{I}, \mathrm{Y}_b(\mathrm{J}))$ and $\mathscr{C}_b(\mathrm{I}, \mathrm{Y}_0(\mathrm{J}))$ respectively.

We stress again that the spaces defined above are different from those denoted by the same symbols in [*1*] *and* [*2*]. We have kept the same notation because they play the same role as the previous ones. Note also that the relation between X and $\mathscr{X}$ and $\mathscr{X}_b$ on the one hand and between $\mathscr{X}$. and Y. on the other hand are the same as previously.

As in [*1*] and [*2*] we make the convention that the letter C, possibly with subscript, shall denote a real non negative constant depending only on the dimension of the space n and on f, but independent of any time, interval, or other functions appearing in the same equation. Constants C without subscript may vary from equation to equation, constants C_i with the same subscript i are the same in all equations where they appear.

The assumptions made on f will be taken from the following list:

(H1 *a*) f is a twice continuously differentiable function from $\mathbb{C}$ to $\mathbb{C}$, with $f(0) = 0$ and $f'(0) = 0$, where f' stands for $\partial f/\partial z$ or $\partial f/\partial \bar{z}$.

(H2 *a*) If $n = 2, 3$: there exists a real number p_2 with $2 \leqslant p_2 < (n+2)/(n-2)$ such that for all $z \in \mathbb{C}$

$$|f'(z)| \leqslant \mathrm{C}(|z| + |z|^{p_2 - 1}). \tag{1.11}$$

Under assumption (H1 *a*), assumption (H2 *a*) restricts only the behaviour of $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ at large $|z|$.

(H2 *b*) f satisfies the estimate

$$|f''(z)| \leqslant \mathrm{C}|z|^{p_1 - 2} \quad \text{for} \quad |z| \leqslant 1, \tag{1.12}$$

where $f''(z)$ stands for any of the second derivatives with respect to z and $\bar{z}$, and p_1 satisfies

$$p_1 - 1 > \mathrm{Max}\,(1/(1-\varepsilon), (4-2\varepsilon)/n) \tag{1.13}$$

with the same ε as in the definition of the spaces $\mathscr{X}_0(\mathrm{I})$.

(H3) For all $z \in \mathbb{C}$, $f(\bar{z}) = \overline{f(z)}$. For all $z \in \mathbb{C}$ and all $\omega \in \mathbb{C}$ with $|\omega| = 1$, $f(\omega z) = \omega f(z)$.

If f is continuous, it follows from (H3) that there exists a (unique) real function $V(z) = V(|z|)$ with $V(0) = 0$ such that

$$f(z) = \partial V(z)/\partial \bar{z}. \tag{1.14}$$

(H4) f is continuous and satisfies (H3). Furthermore, there exists a real number p_3 and a constant C_3 such that

$$1 \leqslant p_3 < 1 + 4/n \tag{1.15}$$

and, for all $\rho \geqslant 0$,

$$V(\rho) \geqslant - C_3(\rho^2 + \rho^{p_3+1}). \tag{1.16}$$

(H5) f is continuous and satisfies (H3),and, for all $z \in \mathbb{C}$, $V(z) \geqslant 0$ and $W(z) \leqslant 0$, where

$$\begin{cases} W(\rho) = (n+2)V(\rho) - (n/2)\rho V'(\rho) & \text{for all} \quad \rho \geqslant 0 \\ W(z) = W(|z|) & \text{for all} \quad z \in \mathbb{C}. \end{cases} \tag{1.17}$$

Remark 1.1. — The assumption $f'(0) = 0$ in (H1 *a*) is unnecessary for $n = 1$ as will be clear in the subsequent estimates. In any case, if (H3) holds, a linear term in f can be included in the free evolution H_0, so that this condition does not restrict the class of admissible f's.

Remark 1.2. — For $n = 2$ the condition (H2 *a*) can be weakened to requiring only that f be bounded for large values of z by a power series with suitably restricted coefficients.

Remark 1.3. — The optimal value of ε in (H2 *b*), namely that giving the weakest restriction on p_1, is $\varepsilon = (3 - \sqrt{2n+1})/2$. For this ε the restriction on p_1 becomes

$$p_1 > (n + 1 + \sqrt{2n+1})/n. \tag{1.18}$$

As in [*1*] and [*2*], in the proof of the conservation laws, we shall need to regularize the basic equations. For this purpose we shall use functions h and g from $\mathbb{R}^n$ to $\mathbb{R}$ satisfying the following assumptions:

(h1) h is even, positive, $h \in L^1$ and $\|h\|_1 = 1$.

(g1 *a*) $g \in \mathscr{C}^1$, $0 \leqslant g \leqslant 1$ and $|\nabla g| \leqslant 1$.

For any h and g satisfying (h1) and (g1 *a*) respectively, we introduce a subscript ν which can represent either the pair (h, g) or g or the empty set. Then the regularized interaction is defined by

$$[f_\nu(v)](x) = \begin{cases} \{ h * [g f(h * v)] \}(x) & \text{if} \quad \nu = (h, g) \\ g(x) f(v(x)) & \text{if} \quad \nu = g \\ f(v(x)) & \text{if} \quad \nu = \varnothing \end{cases} \tag{1.19}$$

and, correspondingly, if f satisfies (H3) and is continuous,

$$[\mathrm{V}_\nu(v)](x) = \begin{cases} g(x)[\mathrm{V}(h * v)](x) & \text{if} \quad \nu = (h, g) \\ g(x)\mathrm{V}(v(x)) & \text{if} \quad \nu = g \\ \mathrm{V}(v(x)) & \text{if} \quad \nu = \varnothing . \end{cases} \tag{1.20}$$

2. EXISTENCE OF LOCAL SOLUTIONS

In this section, we recast the Cauchy problem for the equation (0.1) in the form of the integral equation

$$u(t) = \mathrm{U}(t - t_0)u_0 - i \int_{t_0}^{t} d\tau\, \mathrm{U}(t - \tau) f(u(\tau)) \tag{2.1}$$

and we prove the local existence and uniqueness of solutions by a fixed point technique.

Let $t_1, t_2 \in \mathbb{R}$ and let $v(t)$ be a family of complex valued functions defined on $\mathbb{R}^n$ and depending on a parameter $t \in \mathbb{R}$. We define the operators $\mathrm{G}_\nu(t_1, t_2)$ by the formula

$$[\mathrm{G}_\nu(t_1, t_2)v](t) = - i \int_{t_1}^{t_2} d\tau\, \mathrm{U}(t - \tau) f_\nu(v(\tau)) \tag{2.2}$$

where the f_ν are defined by (1.19).

We first derive some properties of these operators. For this purpose, we introduce the following notation. If f satisfies assumption (H1 a), we define a continuous non decreasing function $\mathrm{R}(\rho)$ from $\mathbb{R}^+$ to $\mathbb{R}^+$ by:

$$\mathrm{R}(\rho) = \operatorname*{Sup}_{|z| \leqslant \rho} \operatorname{Max} \left(| \partial^2 f / \partial z^2 |, | \partial^2 f / \partial \bar{z}^2 |, | \partial^2 f / \partial z \partial \bar{z} | \right). \tag{2.3}$$

Clearly

$$| f(z_1) - f(z_2) | \leqslant 4 \operatorname*{Max}_{i=1,2} \{ | z_i | \mathrm{R}(| z_i |) \} | z_1 - z_2 | \tag{2.4}$$

and

$$| f'(z_1) - f'(z_2) | \leqslant 2 \operatorname*{Max}_{i=1,2} \mathrm{R}(| z_i |) | z_1 - z_2 | \tag{2.5}$$

for all z_1 and $z_2 \in \mathbb{C}$, where f' stands for $\partial f / \partial z$ or $\partial f / \partial \bar{z}$.

LEMMA 2.1. — Let f satisfy (H1 a), let h satisfy (h1), let g satisfy (g1 a). Then, for any ν, for any interval I, the maps $(t_1, t_2, v) \to \mathrm{G}_\nu(t_1, t_2)v$ are continuous from $\mathrm{I} \times \mathrm{I} \times \mathscr{X}(\mathrm{I})$ to $\mathrm{Y}_b(\mathbb{R})$. Moreover for any $t_1, t_2 \in \mathrm{I}$, $t_1 \leqslant t_2$, for any compact interval L such that $[t_1, t_2] \subset \mathrm{L} \subset \mathrm{I}$, for any $t \in \mathbb{R}$, for any $v_1, v_2 \in \mathscr{X}(\mathrm{I})$, the G_ν's satisfy the following estimates:

$$\| [\mathrm{G}_\nu(t_1, t_2)v_1](t) - [\mathrm{G}_\nu(t_1, t_2)v_2](t) \|_{\mathrm{H}^1} \leqslant \mathrm{C} \operatorname*{Max}_{i} \{ | v_i |_{\mathrm{L}} \mathrm{R}(| v_i |_{\mathrm{L}}) \} | v_1 - v_2 |_{\mathrm{L}} | t_1 - t_2 | \tag{2.6}$$

and

$$\begin{aligned}
&\| [G_\nu(t_1, t_2)v_1](t) - [G_\nu(t_1, t_2)v_2](t) \|_\infty \\
&\qquad \leqslant C \operatorname*{Max}_i \{ | v_i |_L R(| v_i |_L) \} | v_1 - v_2 |_L \\
&\qquad \times \begin{cases} [| t_1 - t |^{1/n} + | t_2 - t |^{1/n}] & \text{if } \ t \in [t_1, t_2] \\ | | t_1 - t |^{1/n} - | t_2 - t |^{1/n} | & \text{if } \ t \notin [t_1, t_2] . \end{cases}
\end{aligned} \tag{2.7}$$

Proof. — The proof is similar to that of lemma 2.1 of [*1*], with however different estimates for the quantity

$$\Psi = [G_\nu(t_1, t_2)v_1](t) - [G_\nu(t_1, t_2)v_2](t) . \tag{2.8}$$

It follows from (2.4) and (2.5) that for all ν, for all v_1 and $v_2 \in X$ and for all q satisfying (1.3), f_ν satisfies the following estimates:

$$\| f_\nu(v_1) - f_\nu(v_2) \|_{\bar{q}} \leqslant C \operatorname{Max} R(\| v_i \|_\infty) \operatorname*{Max}_i \| v_i \|_l \, \| v_1 - v_2 \| \tag{2.9}$$

and

$$\begin{aligned}
&\| \nabla f_\nu(v_1) - \nabla f_\nu(v_2) \|_{\bar{q}} \leqslant C \operatorname{Max} R(\| v_i \|_\infty) \\
&\qquad \times \{ \operatorname*{Max}_i \| v_i \|_l (\| v_1 - v_2 \| + \| \nabla v_1 - \nabla v_2 \|) \\
&\qquad + \operatorname*{Max}_i \| \nabla v_i \| \, \| v_1 - v_2 \|_l \}
\end{aligned} \tag{2.10}$$

where $l = n/(1 - \eta)$ and η is defined by (1.4).

Now application of lemma 1.1 yields immediately

$$\begin{aligned}
\| \Psi \|_q \leqslant C \int_{t_1}^{t_2} d\tau \, | t - \tau |^{\eta - 1} \operatorname*{Max}_i R(\| v_i(\tau) \|_\infty) \\
\times \operatorname*{Max}_i \| v_i(\tau) \|_l \, \| v_1(\tau) - v_2(\tau) \|
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\| \nabla \Psi \|_q \leqslant C \int_{t_1}^{t_2} d\tau \, | t - \tau |^{\eta - 1} \operatorname*{Max}_i R(\| v_i(\tau) \|_\infty) \\
&\qquad \times \{ \operatorname*{Max}_i \| v_i(\tau) \|_l (\| v_1(\tau) - v_2(\tau) \| + \| \nabla v_1(\tau) - \nabla v_2(\tau) \|) \\
&\qquad + \operatorname*{Max}_i \| \nabla v_i(\tau) \| \, \| v_1(\tau) - v_2(\tau) \|_l \} .
\end{aligned} \tag{2.12}$$

By taking $q = 2$ in (2.11) and (2.12), using the definition of $\mathscr{X}_b(L)$ and integrating over the variable τ, we obtain (2.6). For $n = 1$, (2.7) follows from (2.6) by the use of (1.5) with $q = 2$. For $n = 2$ or 3, one estimates $\| \Psi \|_\infty$ by the use of the Sobolev inequality (1.5) with q satisfying (1.3) and in addition $q > n$ (or equivalently $\eta < 2 - n/2$), and one uses the estimates (2.11) and (2.12). The estimate (2.7) is obtained for the special choice $\eta = 1/n$.

From lemma 2.1, one derives immediately the following corollary.

COROLLARY 2.1. — With the same notation and assumptions as in lemma 2.1, the G_ν's satisfy the following estimate:

$$\begin{aligned} \| [G_\nu(t_1, t_2)v_1](t) - [G_\nu(t_1, t_2)v_2](t) \|_X \\ \leqslant C_0 \operatorname{Max} (| t_1 - t_2 |, | t_1 - t_2 |^{1/n}) \\ \times \operatorname{Max}_i \{ | v_i |_L R(| v_i |_L) \} | v_1 - v_2 |_L \quad (2.13) \end{aligned}$$

for all $t \in \mathbb{R}$ and all v_1 and v_2 in $\mathscr{X}(I)$.

As in [1], we rewrite the equation 2.1 and its regularized version in terms of G_ν. We define, for all ν,

$$[F_\nu(t_0)v](t) = [G_\nu(t_0, t)v](t) \quad (2.14)$$

and

$$[A_\nu(t_0, u_0)v](t) = \begin{cases} [F_\nu(t_0)v](t) + U(t - t_0)u_0 & \text{if } \nu = g \text{ or } \nu = \varnothing \quad (2.15) \\ [F_\nu(t_0)v](t) + U(t - t_0)(h * u_0) & \text{if } \nu = (h, g) \quad (2.16) \end{cases}$$

where h and g satisfy $(h1)$ and $(g1\ a)$ respectively. Then the equation (2.1) becomes

$$A(t_0, u_0)v = v \quad (2.17)$$

and the corresponding regularized equations are defined by

$$A_\nu(t_0, u_0)v = v\,. \quad (2.18)$$

We now state some elementary properties of the equation (2.18).

LEMMA 2.2. — Let f satisfy (H1 a), let h satisfy $(h1)$ and g satisfy $(g1\ a)$. Let I and J be two intervals of $\mathbb{R}$, $I \subset J$, let $t_0 \in I$, let $u_0 \in X$ be such that the function $t \to U(t - t_0)u_0$ belongs to Y(J) and let $u \in \mathscr{X}(I)$ be a solution of the equation (2.18). Then:

(1) The function

$$\phi(u) : s \to [\phi(u)](s) \equiv U(. - s)u(s) \quad (2.19)$$

belongs to $\mathscr{C}(I, Y(J))$ and satisfies for all s and $s' \in I$ the relation

$$[\phi(u)](s) - [\phi(u)](s') = G_\nu(s', s)u\,. \quad (2.20)$$

Furthermore, if $[\phi(u)](s) \in Y_b(J)$ for some $s \in I$, then

$$\phi(u) \in \mathscr{C}(I, y_b(J))\,.$$

(2) For any $s \in I$, u satisfies the equation

$$A_\nu(s, u(s))u = u\,. \quad (2.21)$$

Proof. — Identical with that of lemma 2.2 of [1].

We now study the existence and uniqueness of local solutions of the equation (2.18).

PROPOSITION 2.1. — Let f satisfy (H1 a), let h satisfy $(h1)$, let g satisfy $(g1\ a)$.

Then for any $\rho > 0$, there exists $T_0(\rho) > 0$, depending only on ρ and f (but independent of ν), such that for any $t_0 \in \mathbb{R}$ and for any $u_0 \in X$ such that $U(. - t_0)u_0 \in B(I, \rho)$, where $I = [t_0 - T_0(\rho), t_0 + T_0(\rho)]$ and $B(I, \rho)$ is the ball of radius ρ in $\mathscr{X}_b(I)$, the equation (2.18) has a unique solution in $\mathscr{X}_b(I)$. This solution belongs to $B(I, 2\rho)$.

Proof. — It follows from corollary 2.1 that if we define $T_0(\rho)$ by

$$4C_0 \operatorname{Max}\,[T_0(\rho), T_0(\rho)^{1/n}]\rho R(2\rho) = 1 \tag{2.22}$$

we ensure that

$$| A_\nu(t_0, u_0)v_1 - A_\nu(t_0, u_0)v_2 |_I \equiv | F_\nu(t_0)v_1 - F_\nu(t_0)v_2 |_I \leqslant \frac{1}{2} | v_1 - v_2 |_I \tag{2.23}$$

for all v_1, v_2 in $B(I, 2\rho)$.

From there on, the proof is identical with those of propositions 2.1, 2.2 and 2.3 of [*1*].

3. CONSERVATION LAWS

In this section, we state the conservation laws for the L^2-norm and the energy, and the pseudoconformal conservation law. We then briefly outline their derivation. The energy function E_g is defined by

$$E_g(v) = \| \nabla v \|^2 + \int gV(v)dx \tag{3.1}$$

for all $v \in X$ and all g satisfying ($g1$ a).

PROPOSITION 3.1. — Let f satisfy (H1 a, 3), let g satisfy ($g1$ a), let J be an interval of $\mathbb{R}$, let $t_0 \in J$, let $u_0 \in X$ and let $u \in \mathscr{X}(J)$ be solution of the equation (2.18) with $\nu = g$. Then for all s and t in J, u satisfies the equalities

$$\| u(t) \| = \| u(s) \| \tag{3.2}$$

$$E_g(u(t)) = E_g(u(s)) . \tag{3.3}$$

If in addition $x.\nabla g \in L^\infty$ and $xu_0 \in L^2$, then $u \in \mathscr{C}(J, \Sigma)$ and for all s and $t \in J$, u satisfies the equality

$$\begin{aligned} \| xU(-t)u(t) \|^2 + 4t^2 \int gV(u(t))dx &= \| xU(-s)u(s) \|^2 + 4s^2 \int gV(u(s))dx \\ &+ \int_s^t 4\tau d\tau \left\{ \int gW(u(\tau))dx + \int (x.\nabla g)V(u(\tau))dx \right\}. \end{aligned} \tag{3.4}$$

Remark 3.1. — If u is a solution of (2.18) in $\mathscr{X}(J)$, then by lemma 2.1, $U(. - t_0)u_0 \in Y(J)$.

Remark 3.2. — The statement $xu_0 \in L^2$ is equivalent to $u_0 \in \Sigma$ if already $u_0 \in X$.

Sketch of the proof. — The proof is similar to the proof of the same conservation laws in [*1*] and [*2*]. One first proves them for the regularized equation, and then one removes the regularization by a limiting procedure.

One first shows (cf. proposition 3.2 of [*1*]) that if, for some interval I containing t_0, $u_h \in \mathscr{X}(I)$ is a solution of the equation (2.18) with $v = (h, g)$, $h \in \mathscr{S}$ and satisfying ($h1$), then for all non negative integer l, $u_h \in \mathscr{C}^1(I, H^l)$ and u_h satisfies the equation (3.3) of [*1*]. If in addition g has compact support and $xu_0 \in L^2$, then (cf. proposition 3.2 of [*2*]) also $xu_h \in \mathscr{C}^1(I, H^l)$ and u_h satisfies the equation (3.1) of [*2*]. Here H^l is the usual Sobolev space. The proof is almost identical with those of propositions 3.2 of [*1*] and 3.2 of [*2*], with the only difference that $\bar{r}$ and q are replaced by 2 and 1 respectively in the estimates (3.5) of [*1*] and (3.5) of [*2*]. Such a u_h is sufficiently regular so that one can derive a regularized version of the conservation laws, as expressed by propositions 3.3 of [*1*] and 3.3 of [*2*]. The proof is a direct computation starting from the regularized differential equation and it is identical with those of propositions 3.3 of [*1*] and 3.3 of [*2*].

Let now $u \in \mathscr{X}(J)$ be the solution of the equation (2.18) mentioned in proposition 3.1. In order to prove the proposition, it is sufficient to prove it for s and t in a small neighbourhood of t_1, where t_1 is an arbitrary point in J (and under the assumption that $xu(t_1) \in L^2$ for the proof of (3.4)) (cf. the proof of proposition 3.4 of [*1*] and 3.4 of [*2*]). In such a small interval, the solution u can be defined by the contraction method of proposition 2.1, and furthermore the regularized equation (2.18) with $v = (h, g)$ also has a unique solution u_h which can be defined by the same method. It is therefore sufficient to derive the conservation laws for u from those for u_h by a limiting argument on h in the situation of proposition 2.1. For this purpose, one picks a fixed function $h \in \mathscr{S}$, h satisfying ($h1$) and one defines a sequence $\{ h_j \}$, $j = 1, 2, \ldots$, by

$$h_j(x) = j^n h(jx).$$

With the same notation as in proposition 2.1, one proves that u_{h_j} tends to u in $\mathscr{C}(I, L^2)$ when $j \to \infty$. The proof is similar to that of proposition 3.1 of [*1*]. It rests on the facts that

$$A_{h_j,g}(t_0, u_0)v \to A_g(t_0, u_0)v$$

in $\mathscr{C}(I, L^2)$ when $j \to \infty$ for fixed $v \in \mathscr{X}(I)$, and that

$$\sup_{t \in I} \| [A_{h_j,g}(t_0, u_0)v_1](t) - [A_{h_j,g}(t_0, u_0)v_2](t) \| \leqslant \frac{1}{2} \sup_{t \in I} \| v_1(t) - v_2(t) \|$$

for all v_1 and v_2 in $B(I, 2\rho)$, uniformly in j. These two facts are easily proved by the same methods as in the proofs of lemma 2.1 and proposition 2.1.

Now for fixed $t \in \mathrm{I}$, the convergence of $u_{h_j}(t)$ to $u(t)$ in L^2 implies that

(1) $$\lim_{j\to\infty} \| u_{h_j}(t) \| = \| u(t) \| ,$$

thereby proving (3.2).

(2) $$\lim_{j\to\infty} \| \nabla u_{h_j}(t) \|^2 = \mathrm{E}_g(u_0) - \int g\mathrm{V}(u(t))dx ,$$

since $\int g\mathrm{V}(v)dx$ is a continuous function of v in the L^2-topology for v in a bounded set of X. By the same compactness argument as in the proof of proposition 3.4 of [*1*], this implies (3.3).

(3) $$\begin{aligned}\lim_{j\to\infty} \| x\mathrm{U}(-t)u_{h_j}(t) \|^2 &= \| x\mathrm{U}(-t_0)u_0 \|^2 \\ &+ 4t_0^2 \int g\mathrm{V}(u_0)dx - 4t^2 \int g\mathrm{V}(u(t))dx \\ &+ \int_{t_0}^{t} 4\tau d\tau \left\{ \int g\mathrm{W}(u(\tau))dx + \int (x\,.\,\nabla g)\mathrm{V}(u(\tau))dx \right\}.\end{aligned}$$

The proof of this fact is similar to that of the corresponding fact in the proof of proposition 3.4 of [*2*], using the continuity property of V mentioned above and the analogous property of W. The only difference is that $\bar{r}$ and q are replaced by 2 and 1 respectively in the estimates (3.24) and (3.27) of [*2*]. By the same compactness argument as in the proof of proposition 3.4 of [*2*], this implies that $u \in \mathscr{C}(\mathrm{I}, \Sigma)$ and that u satisfies (3.4) for all s and t in I, provided g has compact support. In order to remove this last restriction, one finally uses a limiting argument on g, which is of the same type as that in proposition 3.5 of [*2*].

4. EXISTENCE OF GLOBAL SOLUTIONS

In this section, we prove the existence of global solutions of the equation (2.1). This result is obtained by establishing an *a priori* estimate on the X-norm of the solution. The H^1-norm is controlled by the conservation of the L^2-norm and of the energy, while the control of the L^∞-norm comes directly from the equation (2.1).

THEOREM 4.1. — Let f satisfy (H1 a, 2 a, 3, 4), let g satisfy (g1 a), let J be an interval of $\mathbb{R}$, let $t_0 \in \mathrm{J}$ and let $u_0 \in \mathrm{X}$ be such that $\mathrm{U}(.\ - t_0)u_0 \in \mathrm{Y}(\mathrm{J})$. Then:

(1) The equation (2.18) with $v = g$ has a unique solution in $\mathscr{X}(\mathrm{J})$. This solution belongs to $\mathscr{X}_a(\mathrm{J})$.

(2) For all s and $t \in \mathrm{J}$, u satisfies the equalities (3.2) and (3.3).

(3) If in addition $x.\nabla g \in L^\infty$ and $xu_0 \in L^2$ (or equivalently $u_0 \in \Sigma$), then $u \in \mathscr{C}(J, \Sigma)$ and for all s and $t \in J$, u satisfies the equality (3.4).

Proof. — It is sufficient to prove part (1) since once part (1) is proved, parts (2) and (3) are restatements of proposition 3.1. Let I be a bounded open interval, the closure of which is contained in J, let $t_0 \in I$ and let $u \in \mathscr{X}(I)$ be solution of the equation (2.18) with $v = g$. By a standard argument, it is sufficient to obtain an *a priori* estimate for $\| U(t-s)u(s) \|_X$ for all s and $t \in I$.

By assumption (H4), lemma 3.2 of [*1*] and corollary 3.2 of [*1*], the conservation laws (3.2) and (3.3) imply the uniform estimate

$$\| U(t-s)u(s) \|_{H^1} \leqslant a M_0(\| u_0 \|_{H^1}) \tag{4.1}$$

for all s and $t \in I$, where M_0 is a continuous increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $M_0(0) = 0$, polynomially bounded for $n = 2$ or 3, depending only on f and n (see theorem 3.1 of [*1*]). This settles the case $n = 1$, where $X = H^1$.

In order to complete the proof for $n = 2$ and 3, we now derive an *a priori* estimate for $\| U(t-s)u(s) \|_\infty$ with $s, t \in I$. From (2.18) and (2.20) it follows that for all $s, t \in I$

$$\| U(t-s)u(s) \|_\infty \leqslant \| U(t-t_0)u_0 \|_\infty + \| [G_g(t_0, s)u](t) \|_\infty . \tag{4.2}$$

By the Sobolev inequality (1.5) the second term in the r. h. s. of (4.2) can be estimated as

$$\| [G_g(t_0, s)u](t) \|_\infty \leqslant b_q \| [\nabla G_g(t_0, s)u](t) \|_q^{n/q} \; \| [G_g(t_0, s)u](t) \|_q^{1-n/q} \tag{4.3}$$

with $n < q < 2n/(n-2)$. On the other hand, from assumptions (H1 *a*) and (H2 *a*), it follows that for all $z \in \mathbb{C}$

$$| f(z) | \leqslant C(| z |^2 + | z |^{p_2}) . \tag{4.4}$$

By Hölder's inequality, by (1.11), (4.4) and lemma 1.1, one has

$$\| [G_g(t_0, s)u](t) \|_q \leqslant C \left| \int_{t_0}^{s} d\tau \, | t - \tau |^{\eta - 1} \; \{ \| u(\tau) \|_{2\bar{q}}^2 + \| u(\tau) \|_{p_2 \bar{q}}^{p_2} \} \right| \tag{4.5}$$

and

$$\| [\nabla G_g(t_0, s)u](t) \|_q \leqslant C \left| \int_{t_0}^{s} d\tau \, | t - \tau |^{\eta - 1} \times \| \nabla u(\tau) \| \{ \| u(\tau) \|_l + \| u(\tau) \|_{(p_2 - 1)l}^{p_2 - 1} \} \right| \tag{4.6}$$

with η defined by (1.4) and $l = n/(1 - \eta)$.

For $n = 2$, all the norms in the r. h. s. of (4.5) and (4.6) are estimated in terms of the H^1-norm via the Sobolev inequality (1.2). Collecting the previous estimates, one finds after an elementary computation, for all s and t in I,

$$\| U(t-s)u(s) \|_\infty \leqslant \| U(t-t_0)u_0 \|_\infty + C \mid t_0 - s \mid^\eta \{ M_0(\| u_0 \|_{H^1})^2 + M_0(\| u_0 \|_{H^1})^{p_2} \} \quad (4.7)$$

for arbitrary q, $2 < q < \infty$ (the constant C depends on q). This completes the proof for $n = 2$.

For $n = 3$, since $p_2 < (n+2)/(n-2) = 5$ by assumption (H2 a), we can find a q such that

$$n = 3 < q < 2n/(n-2) = 6$$

and

$$p_2 < 1 + 2(1-\eta) + 2/(1+2\eta). \quad (4.8)$$

We fix such a q. From (4.8) it follows that $p_2\bar{q} < 6$, and therefore, by the Sobolev inequality (1.2), by (4.1) and (4.5), we obtain

$$\| [G_g(t_0, s)u](t) \|_q \leqslant C \mid t_0 - s \mid^\eta \{ M_0(\| u_0 \|_{H^1})^2 + M_0(\| u_0 \|_{H^1})^{p_2} \}. \quad (4.9)$$

Similarly, since $3 < l \leqslant 6$, the expression $\| u(\tau) \|_l$ in (4.6) can be estimated in terms of $\| u(\tau) \|_{H^1}$. The same estimate holds for $\| u(\tau) \|_{(p_2-1)l}$, provided $(p_2 - 1)l \leqslant 6$. If $(p_2 - 1)l > 6$, one obtains, by the Sobolev inequality (1.2),

$$\| u(\tau) \|_{(p_2-1)l}^{p_2-1} \leqslant C \| u(\tau) \|_{H^1}^{2(1-\eta)} \| u(\tau) \|_\infty^{p_2-1-2(1-\eta)}. \quad (4.10)$$

Let now L be a compact subinterval of I containing t_0 and let

$$\mu(L) = \operatorname*{Sup}_{s,t\in L} \| U(t-s)u(s) \|_\infty . \quad (4.11)$$

From the previous estimates we obtain

$$\mu(L) \leqslant \operatorname*{Sup}_{t\in L} \| U(t-t_0)u_0 \|_\infty + M \mid L \mid^\eta (1 + \mu(L))^\alpha \quad (4.12)$$

where M depends only on $\| u_0 \|_{H^1}$, $\mid L \mid$ is the length of L and

$$\alpha = (p_2 - 1 - 6/l)3/q = [p_2 - 1 - 2(1-\eta)](1+2\eta)/2 .$$

The condition (4.8) is equivalent to the condition $\alpha < 1$, and therefore implies the required *a priori* estimate for the case $n = 3$. This completes the proof.

5. THE CAUCHY PROBLEM AT INFINITY

In this section, we analyze the integral equation

$$u(t) = U(t)\tilde{u}_0 - i \int_{t_0}^{t} d\tau \, U(t-\tau) f(u(\tau)) \quad (5.1)$$

for t_0 in a neighbourhood of infinity in $\bar{\mathbb{R}}$ and we prove the existence of a unique solution in $\mathscr{X}_0([\mathrm{T}, \infty))$ for T sufficiently large by a fixed point technique. We also prove the existence of the limit of $\mathrm{U}(-t)u(t)$ when t tends to infinity and we extend the conservation laws of the L^2-norm, of the energy, and the pseudoconformal conservation law up to infinite time. Similar results holds in a neighbourhood of $-\infty$. For simplicity, we consider only the original equation (5.1) without regularization and without cut-off. We recall that the quantities f, V and similarly G, F, A and W for $v = \emptyset$ are written without subscript (cf. (1.19, 1.20)).

We first collect some preliminary estimates which give a meaning to the expression (2.2), including the case where t_1 or t_2 is infinite. If f satisfies assumptions (H1 *a*) and (H2 *b*), we define a continuous non decreasing function $\mathrm{R}_0(\rho)$ from $\mathbb{R}^+$ to $\mathbb{R}^+$ by:

$$\mathrm{R}_0(\rho) = \operatorname*{Sup}_{|z| \leqslant \rho} |z|^{2-p_1} \operatorname{Max}\left(|\partial^2 f/\partial z^2|, |\partial^2 f/\partial \bar{z}^2|, |\partial^2 f/\partial z \partial \bar{z}|\right) \tag{5.2}$$

Clearly

$$|f(z_1) - f(z_2)| \leqslant 4 \operatorname*{Max}_i \left\{ |z_i|^{p_1 - 1} \mathrm{R}_0(|z_i|) \right\} |z_1 - z_2| \tag{5.3}$$

and

$$|f'(z_1) - f'(z_2)| \leqslant 2 \operatorname*{Max}_i \left\{ |z_i|^{p_1 - 2} \mathrm{R}_0(|z_i|) \right\} |z_1 - z_2| \tag{5.4}$$

for all z_1 and $z_2 \in \mathbb{C}$, where f' stands for $\partial f/\partial z$ or $\partial f/\partial \bar{z}$.

LEMMA 5.1. — Let f satisfy (H1 *a*, 2 *b*). Then for any interval $\mathrm{I} \subset \mathbb{R}$, the map $(t_1, t_2, v) \to \mathrm{G}(t_1, t_2)v$ is continuous from $\bar{\mathrm{I}} \times \bar{\mathrm{I}} \times \mathscr{X}_0(\mathrm{I})$ to $\mathrm{Y}_0(\mathbb{R})$. Moreover, for any $t_1, t_2 \in \bar{\mathrm{I}} (t_1 \leqslant t_2)$, for any interval L such that $[t_1, t_2] \subset \mathrm{L} \subset \mathrm{I}$, for any $t \in \mathbb{R}$ and for any $v_1, v_2 \in \mathscr{X}_0(\mathrm{I})$, $\mathrm{G}(t_1, t_2)$ satisfies the following estimates:

If $0 \leqslant t_1 \leqslant t_2$:

$$\begin{aligned} \| [\mathrm{G}(t_1, t_2)v_1](t) - [\mathrm{G}(t_1, t_2)v_2](t) \|_{\mathrm{H}^1} & \\ \leqslant \mathrm{C}[(1 + t_1)^{1-(p_1-1)(1-\varepsilon)} - (1 + t_2)^{1-(p_1-1)(1-\varepsilon)}] & \\ \times \operatorname*{Max}_i [|v_i|_{0\mathrm{L}}^{p_1-1} \mathrm{R}_0(|v_i|_{0\mathrm{L}})] |v_1 - v_2|_{0\mathrm{L}} & \end{aligned} \tag{5.5}$$

$$\begin{aligned} \| [\mathrm{G}(t_1, t_2)v_1](t) - [\mathrm{G}(t_1, t_2)v_2](t) \|_r & \\ \leqslant \mathrm{C}(1 + \operatorname{Max}(t_1, |t|))^{\varepsilon - 1} (1 + t_1)^{\gamma} \varphi_\varepsilon((1 + t_2)/(1 + t_1)) & \\ \times \operatorname*{Max}_i \left[|v_i|_{0\mathrm{L}}^{p_1-1} \mathrm{R}_0(|v_i|_{0\mathrm{L}}) \right] |v_1 - v_2|_{0\mathrm{L}} & \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \| [\mathrm{G}(t_1, t_2)v_1](t) - [\mathrm{G}(t_1, t_2)v_2](t) \|_\infty & \\ \leqslant \mathrm{C}(1 + \operatorname{Max}(t_1, |t|))^{\varepsilon - 1} (1 + t_1)^{\gamma} \varphi((1 + t_2)/(1 + t_1)) & \\ \times \operatorname*{Max}_i \left[|v_i|_{0\mathrm{L}}^{p_1-1} \mathrm{R}_0(|v_i|_{0\mathrm{L}}) \right] |v_1 - v_2|_{0\mathrm{L}} & \end{aligned} \tag{5.7}$$

where

$$\gamma = \operatorname{Max}\left[-\frac{n}{2}(p_1 - 1) + 2 - \varepsilon, 1 - (p_1 - 1)(1 - \varepsilon) \right] \tag{5.8}$$

$$\varphi(\sigma) = \operatorname*{Sup}_{1 \leqslant \mu \leqslant \sigma} \int_1^\sigma d\tau \, |1 - \tau/\mu|^{\eta - 1} \tau^{\gamma - 1} \tag{5.9}$$

with

$$\eta = \mathrm{Min}\,(\varepsilon, 1 - n/4) \tag{5.10}$$

and φ_ε is the function defined by (5.9) with η replaced by ε. (Note that $\gamma < 0$ by assumption (H2 *b*)).

If $t_1 \leqslant t_2 \leqslant 0$, one obtains similar estimates with t_1 replaced by $-t_2$ and t_2 replaced by $-t_1$.

If $t_1 \leqslant 0 \leqslant t_2$, one obtains estimates of the same type by combining the previous ones for the intervals $[t_1, 0]$ and $[0, t_2]$.

Remark 5.1. — It is proved in lemma 1.4 of [2] that the function $\varphi(\sigma)$ is increasing, η-Hölder continuous and bounded.

Proof of lemma 5.1. — The proof is similar to that of lemma 2.1 of [2]. We have to obtain suitable estimates for the H^1, L^r and L^∞-norms of the quantity

$$\Psi \equiv [\mathrm{G}(t_1, t_2)v_1](t) - [\mathrm{G}(t_1, t_2)v_2](t) = -i \int_{t_1}^{t_2} d\tau\, \mathrm{U}(t-\tau)(f(v_1(\tau)) - f(v_2(\tau)))\,. \tag{5.11}$$

Now

$$\|\Psi\| \leqslant \mathrm{C} \int_{t_1}^{t_2} d\tau\, \mathop{\mathrm{Max}}_i \left\{ \|v_i(\tau)\|_\infty^{p_1-1} \mathrm{R}_0(\|v_i(\tau)\|_\infty) \right\} \|v_1(\tau) - v_2(\tau)\|$$

by (5.3) and lemma 1.1,

$$\ldots \leqslant \mathrm{C} \int_{t_1}^{t_2} d\tau\,(1 + |\tau|)^{(p_1-1)(\varepsilon-1)} \times \mathop{\mathrm{Max}}_i \left\{ |v_i|_{0\mathrm{L}}^{p_1-1} \mathrm{R}_0(|v_i|_{0\mathrm{L}}) \right\} |v_1 - v_2|_{0\mathrm{L}} \tag{5.12}$$

by the definition of $\mathscr{X}_0(\mathrm{L})$ (see section 1). Similarly

$$\begin{aligned}\|\nabla\Psi\| \leqslant \mathrm{C} \int_{t_1}^{t_2} d\tau \Big\{ &\mathop{\mathrm{Max}}_i \left[\|v_i(\tau)\|_\infty^{p_1-1} \mathrm{R}_0(\|v_i(\tau)\|_\infty)\right] \\ &\times \|\nabla v_1(\tau) - \nabla v_2(\tau)\| + \mathop{\mathrm{Max}}_i \left[\|v_i(\tau)\|_\infty^{p_1-2} \mathrm{R}_0(\|v_i(\tau)\|_\infty)\right] \\ &\times \mathop{\mathrm{Max}}_i (\|\nabla v_i(\tau)\|) \|v_1(\tau) - v_2(\tau)\|_\infty \Big\}\end{aligned}$$

by (5.3), (5.4) and lemma 1.1,

$$\ldots \leqslant \mathrm{C} \int_{t_1}^{t_2} d\tau(1 + |\tau|)^{(p_1-1)(\varepsilon-1)} \times \mathop{\mathrm{Max}}_i \left\{ |v_i|_{0\mathrm{L}}^{p_1-1} \mathrm{R}_0(|v_i|_{0\mathrm{L}}) \right\} |v_1 - v_2|_{0\mathrm{L}}\,. \tag{5.13}$$

The estimate (5.5) follows from (5.12) and (5.13) by integration in the variable τ.

On the other hand

$$\|\Psi\|_r \leqslant \mathrm{C} \int_{t_1}^{t_2} d\tau\, |t - \tau|^{\varepsilon-1} \times \left\| \mathop{\mathrm{Max}}_i \left\{ |v_i(\tau)|^{p_1-1} \mathrm{R}_0(|v_i(\tau)|) \right\} (v_1(\tau) - v_2(\tau)) \right\|_{\bar{r}}$$

by (5.3) and lemma 1.1,

$$\dots \leqslant C \int_{t_1}^{t_2} d\tau \, | t - \tau |^{\varepsilon - 1} \operatorname*{Max}_i \| v_i(\tau) \|_{p_1\bar{r}}^{p_1 - 1} \times \operatorname*{Max}_i R_0(\| v_i(\tau) \|_\infty) \| v_1(\tau) - v_2(\tau) \|_{p_1\bar{r}}$$

by Hölder's inequality,

$$\dots \leqslant C \int_{t_1}^{t_2} d\tau \, | t - \tau |^{\varepsilon - 1}(1 + | \tau |)^{\gamma - 1} \times \operatorname*{Max}_i \{ | v_i |_{0L}^{p_1 - 1} R_0(| v_i |_{0L}) \} | v_1 - v_2 |_{0L} \quad (5.14)$$

where γ is defined by (5.8). In (5.14), we have estimated $\| v_i \|_{p_1\bar{r}}$ by interpolation between $\| v_i \|$ and $\| v_i \|_r$ if $p_1\bar{r} \leqslant r$ and between $\| v_i \|_r$ and $\| v_i \|_\infty$ if $p_1\bar{r} \geqslant r$. The exponent of $(1 + | \tau |)$ that comes out naturally from this estimate is

$$\operatorname{Max}\left[-\frac{n}{2}(p_1 - 1) + 1 - \varepsilon, \, - p_1(1 - \varepsilon) \right] \leqslant \gamma - 1 .$$

The estimate (5.6) now follows from (5.14) and from lemma 1.5 of [*2*]. Finally, by the Sobolev inequality (1.5), we have

$$\| \Psi \|_\infty \leqslant b_q \| \nabla \Psi \|_q^{n/q} \| \Psi \|_q^{1 - n/q} \quad (5.15)$$

with q satisfying (1.3) and in addition $q > n$, or equivalently $\eta < 2 - n/2$. By the same method as above, we obtain

$$\begin{aligned} \| \nabla \Psi \|_q \leqslant C \int_{t_1}^{t_2} d\tau \, | t - \tau |^{\eta - 1} \{ & \operatorname*{Max}_i \| v_i(\tau) \|_{(p_1 - 1)l}^{p_1 - 1} \\ & \times \operatorname*{Max}_i R_0(\| v_i(\tau) \|_\infty) \| \nabla v_1(\tau) - \nabla v_2(\tau) \| \\ & + \operatorname*{Max}_i \| \nabla v_i(\tau) \| \operatorname*{Max}_i \| v_i(\tau) \|_{(p_1 - 1)l}^{p_1 - 2} \\ & \times \operatorname*{Max}_i R_0(\| v_i(\tau) \|_\infty) \| v_1(\tau) - v_2(\tau) \|_{(p_1 - 1)l} \} \end{aligned} \quad (5.16)$$

where η is defined by (1.4) and $l = n/(1 - \eta)$,

$$\dots \leqslant C \int_{t_1}^{t_2} d\tau \, | t - \tau |^{\eta - 1}(1 + | \tau |)^{\beta - 1} \times \operatorname*{Max}_i \{ | v_i |_{0L}^{p_1 - 1} R_0(| v_i |_{0L}) \} | v_1 - v_2 |_{0L} \quad (5.17)$$

where

$$\beta = \operatorname{Max}\left[-\frac{n}{2}(p_1 - 1) + 2 - \eta \, , 1 - (p_1 - 1)(1 - \varepsilon) \right] . \quad (5.18)$$

Similarly, one obtains the same estimate (5.17) for $\| \Psi \|_q$, and by (5.15), this estimate holds also for $\| \Psi \|_\infty$. In order to obtain the t dependence stated in the lemma, we need to take $\eta \leqslant \varepsilon$ (or equivalently $q \geqslant r$). This implies $\beta \geqslant \gamma$. It is then natural to impose that $\beta = \gamma$. This is equivalent to

$$\eta \geqslant \operatorname{Min}\left[\varepsilon, 1 - (p_1 - 1)\left(\frac{n}{2} - 1 + \varepsilon \right) \right] \quad (5.19)$$

which is satisfied for all p_1 satisfying (1.13) provided

$$\eta \geqslant \text{Min}\,[\varepsilon, 2 - n/(2(1-\varepsilon))]. \tag{5.20}$$

One can easily find η satisfying this condition in addition to $\eta \leqslant \varepsilon$ and $\eta < 2 - n/2$. For definiteness, we choose $\eta = \text{Min}\,(\varepsilon, 1 - n/4)$. Then we obtain

$$\|\Psi\|_\infty \leqslant C \int_{t_1}^{t_2} d\tau\, |t-\tau|^{\eta-1}(1+|\tau|)^{\gamma-1} \times \underset{i}{\text{Max}}\, \{ |v_i|_{0L}^{p_1-1} R_0(|v_i|_{0L}) \}\, |v_1 - v_2|_{0L} \tag{5.21}$$

which by lemma 1.5 of [2] implies the estimate (5.7).

In the case $n = 1$, the L^∞-norm of Ψ can be estimated in a much more direct way:

$$\begin{aligned} \|\Psi\|_\infty &\leqslant C \int_{t_1}^{t_2} d\tau\, |t-\tau|^{-1/2} \\ &\qquad \| \underset{i}{\text{Max}}\, \{ |v_i(\tau)|^{p_1-1} R_0(|v_i(\tau)|) \} (v_1(\tau) - v_2(\tau)) \|_1 \\ &\leqslant C \int_{t_1}^{t_2} d\tau\, |t-\tau|^{-1/2} (1+|\tau|)^{\max[-p_1(1-\varepsilon), 1-p_1/2]} \\ &\quad \times \underset{i}{\text{Max}}\, \{ |v_i|_{0L}^{p_1-1} R_0(|v_i|_{0L}) \}\, |v_1 - v_2|_{0L}. \end{aligned} \tag{5.22}$$

This yields an estimate slightly different from (actually slightly better than) that stated in the lemma.

From remark 5.1 and lemma 5.1, one derives immediately the following corollary:

COROLLARY 5.1. — With the same assumptions and notation as in lemma 5.1, for $0 \leqslant t_1 \leqslant t_2 \leqslant \infty$, $G(t_1, t_2)$ satisfies the following estimate:

$$\begin{aligned} &|G(t_1, t_2)v_1 - G(t_1, t_2)v_2|_{0L} \\ &\qquad \leqslant C_1(1+t_1)^\gamma\, \underset{i}{\text{Max}}\, \{ |v_i|_{0L}^{p_1-1} R_0(|v_i|_{0L}) \}\, |v_1 - v_2|_{0L}. \end{aligned} \tag{5.23}$$

We now come back to the equation (5.1), which we rewrite as

$$\tilde{A}(t_0, \tilde{u}_0)v = v \tag{5.24}$$

where the operator $\tilde{A}(t_0, \tilde{u}_0)$ is defined by

$$[\tilde{A}(t_0, \tilde{u}_0)v](t) = U(t)\tilde{u}_0 + [F(t_0)v](t) \tag{5.25}$$

and $F(t_0)$ is defined by (2.14). If t_0 is finite, $\tilde{A}(t_0, .)$ is related to $A(t_0, .)$, defined by (2.15), as follows:

$$\tilde{A}(t_0, \tilde{u}_0) = A(t_0, U(t_0)\tilde{u}_0). \tag{5.26}$$

We first consider the limit of $U(-s)u(s)$ as s tends to infinity.

PROPOSITION 5.1. — Let f satisfy (H1 a, 2 b), let I and J be two intervals of $\mathbb{R}$ (possibly unbounded), $I \subset J$. Let $t_0 \in \bar{I}$, let $\tilde{u}_0 \in H^1$ be such that $U(.)\tilde{u}_0 \in Y_0(J)$ and let $u \in \mathscr{X}_0(I)$ be a solution of the equation (5.24). Then:

(1) The function $\phi(u)$ defined by (2.19) belongs to $\mathscr{C}_b(I, Y_0(J))$ and can be extended by continuity to a function in $\mathscr{C}_b(\bar{I}, Y_0(J))$ still denoted by $\phi(u)$. For all $s, s' \in \bar{I}$, $\phi(u)$ satisfies the relation

$$[\phi(u)](s) - [\phi(u)](s') = G(s', s)u. \tag{5.28}$$

(2) For all $s \in I$, u satisfies the equation

$$\tilde{A}(s, U(-s)u(s))u = u. \tag{5.29}$$

Let now $I = [T, \infty)$ for some $T \in \mathbb{R}$.

(3) There exists $u_+ \in H^1$ such that $U(.)u_+(= [\phi(u)](\infty)) \in Y_0(J)$, and u satisfies the equation

$$\tilde{A}(\infty, u_+)u = u. \tag{5.30}$$

If in addition $\tilde{u}_0 \in X$, then $u_+ \in X$. If $t_0 = \infty$, then $u_+ = \tilde{u}_0$.

(4) $[\phi(u)](s) - U(.)u_+$ belongs to $Y_0(\mathbb{R})$ for all $s \in \bar{I}$ and tends to zero in $Y_0(\mathbb{R})$ when $s \to \infty$. Moreover, for all $s \geqslant 0$, $s \in I$, u satisfies the estimates

$$\| U(t - s)u(s) - U(t)u_+ \|_{H^1} \leqslant C(1 + s)^{1-(p_1-1)(1-\varepsilon)} | u |_{0I}^{p_1} R_0(| u |_{0I}) \tag{5.31}$$

and

$$\begin{aligned} &\| U(t - s)u(s) - U(t)u_+ \|_r + \| U(t - s)u(s) - U(t)u_+ \|_\infty \\ &\qquad \leqslant C(1 + \operatorname{Max}(s, | t |))^{\varepsilon-1}(1 + s)^\gamma | u |_{0I}^{p_1} R_0(| u |_{0I}) \end{aligned} \tag{5.32}$$

where γ is defined by (5.8).

(5) If in addition f satisfies (H3), then for all $t \in I$, u satisfies the relations

$$\| u(t) \| = \| u_+ \| \tag{5.33}$$

$$E(u(t)) = \| \nabla u_+ \|^2. \tag{5.34}$$

Sketch of proof. — The proof of parts (1) to (4) is almost identical with that of proposition 2.1 of [2]. Part (5) follows from part (4), from proposition 3.1 and from the fact that $\int V(u(s))dx$ tends to zero when s tends to infinity.

We next study the Cauchy problem at infinity.

PROPOSITION 5.2. — Let f satisfy (H1 *a*, 2 *b*). For any $\rho > 0$, there exists $T_1(\rho) > 0$, depending only on ρ and f, such that for any $t_0 \in \bar{I}$ and for any $\tilde{u}_0 \in H^1$ such that $U(.)\tilde{u}_0 \in B_0(I, \rho)$, where $I = [T_1(\rho), \infty)$ and $B_0(I, \rho)$ is the ball of radius ρ in $\mathscr{X}_0(I)$, the equation (5.24) has a unique solution u in $\mathscr{X}_0(I)$. This solution belongs to $B_0(I, 2\rho)$. For fixed $\tilde{u}_0$, the map $t_0 \to u$ is continuous from $\bar{I}$ to $B_0(I, 2\rho)$.

Proof. — It follows from corollary 5.1 that if we define $T_1(\rho)$ by

$$2C_1(1 + T_1(\rho))^{\gamma}(2\rho)^{p_1-1}R_0(2\rho) = 1\,, \tag{5.35}$$

then we ensure that

$$| \tilde{A}(t_0, \tilde{u}_0)v_1 - \tilde{A}(t_0, \tilde{u}_0)v_2 |_{0I} \equiv | F(t_0)v_1 - F(t_0)v_2 |_{0I} \leqslant \frac{1}{2} | v_1 - v_2 |_{0I} \tag{5.36}$$

for all $t_0 \in \bar{I}$ and all v_1 and v_2 in $B_0(I, 2\rho)$. From there on, the proof is identical with those of propositions 2.2, 2.3 and 2.4 of [2].

Remark 5.2. — There is some flexibility in the choice of the spaces where to solve the Cauchy problem at infinity. One may for instance replace $\mathscr{X}_0(I)$ by the more general space:

$$\mathscr{X}'_0(I) = \{ v \in \mathscr{X}_b(I) : \operatorname*{Sup}_{t \in I} \{ \operatorname{Max} [\| v(t) \|_{H^1}, (1 + | t |)^{1-\varepsilon} \| v(t) \|_r, (1 + | t |)^{\delta} \| v(t) \|_\infty] \} < \infty \} \tag{5.37}$$

for some δ, $0 < \delta < 1$. The assumption (H2 b) has then to be replaced by a suitable condition involving p_1, δ and ε. The weakest condition, namely the smallest lower bound thereby obtained for p_1, corresponds to $\delta = (\sqrt{2n + 1} - 1)/2 \equiv \delta_0$ and $\delta_1 \leqslant 1 - \varepsilon \leqslant \delta_0$ for some suitable δ_1 depending on n. The range of values of r corresponding to the last condition is

$$(n + 1 + \sqrt{2n + 1})/n \leqslant r \leqslant 2(n + 1 + \sqrt{2n + 1})/n \quad \text{for} \quad n = 1 \text{ or } 2\,,$$

$$1 + \sqrt{7} \leqslant r \leqslant 2(4 + \sqrt{7})/3 \quad \text{for} \quad n = 3\,.$$

Since however the lower bound on p_1 itself, as given in remark 1.3, is not improved by the more general choice (5.37), we have used the simpler space where $\delta = 1 - \varepsilon$.

By strengthening the assumptions on f and $\tilde{u}_0$, we can extend the pseudo-conformal conservation law to infinite times.

PROPOSITION 5.3. — Let f satisfy (H1 a, 2 b, 3) with $p_1 > 1 + 4/n$. Let $T \in \mathbb{R}$, $I = [T, \infty)$, $t_0 \in \bar{I}$, $\tilde{u}_0 \in \Sigma$, and let $u \in \mathscr{X}_0(I)$ be solution of the equation (5.24). Then

(1) $u \in \mathscr{C}(I, \Sigma)$.

(2) Let u_+ be defined as in part (3) of proposition 5.1. Then $u_+ \in \Sigma$.

(3) For all $t \in I$, $u(t)$ satisfies the relation

$$\| xU(- t)u(t) \|^2 + 4t^2 \int V(u(t))dx = \| xu_+ \|^2 - \int_t^{\infty} 4\tau d\tau \int W(u(\tau))dx\,, \tag{5.38}$$

where the integral in the last term is absolutely convergent.

(4) $U(-s)u(s)$ tends to u_+ strongly in Σ when $s \to \infty$.

Proof. — The proof is almost identical with that of proposition 4.1 of [2] and will be omitted.

We now collect the main information obtained in this and the previous section. For this purpose, it is convenient to introduce the following notation. Let $\rho > 0$ and let $T_1(\rho)$ be defined by (5.35). We define

$$I(\rho) = [T_1(\rho), \infty) \tag{5.39}$$

$$K(\rho) = \{ (t_0, \tilde{u}_0) : t_0 \in \bar{I}(\rho),\ U(.)\tilde{u}_0 \in Y_0(I(\rho)) \text{ and } | U(.)\tilde{u}_0 |_{0I(\rho)} \leqslant \rho \} \tag{5.40}$$

and

$$K = \bigcup_{\rho > 0} K(\rho). \tag{5.41}$$

THEOREM 5.1. — Let f satisfy (H1 a, 2 a, 2 b, 3, 4), let $\rho > 0$, let $U(.)\tilde{u}_0 \in Y_0(J)$ where $J = [T, \infty)$ and $-\infty \leqslant T \leqslant 0$, and let $(t_0, \tilde{u}_0) \in K(\rho)$. Then:

(1) The equation (5.24) has a unique solution $u \in \mathscr{X}_0(\mathbb{R}^+)$, which can be uniquely continued to a solution in $\mathscr{X}_a(J)$.

(2) There exists $u_+ \in X$ such that $U(-s)u(s)$ tends to u_+ when $s \to \infty$ in the sense of proposition 5.1 (4), and in particular in X. If $t_0 = \infty$, then $u_+ = \tilde{u}_0$.

(3) For all $t \in J$, $u(t)$ satisfies the relations:

$$\| u(t) \| = \| u_+ \| \tag{5.33}$$

$$E(u(t)) = \| \nabla u_+ \|^2 . \tag{5.34}$$

(4) u is uniformly bounded in $\mathscr{X}_0(\mathbb{R}^+)$ and $\phi(u)$ (defined by (2.19)) is uniformly bounded in $\mathscr{C}_b(\mathbb{R}^+, Y_0(J))$ if $(t_0, \tilde{u}_0) \in K(\rho)$ for some fixed ρ and if $U(.)\tilde{u}_0$ remains in a bounded set of $Y_0(J)$.

(5) If in addition $\tilde{u}_0 \in \Sigma$ and $p_1 > 1 + 4/n$, then $u_+ \in \Sigma$, $u \in \mathscr{C}(J, \Sigma)$ and $U(-s)u(s)$ tends to u_+ strongly in Σ when $s \to \infty$. Furthermore, for all $t \in J$, $u(t)$ satisfies the relation (5.38).

Sketch of proof. — Parts (1), (2), (3) and (5) are repetitions of previous results. Part (4) is trivial for $n = 1$. For $n = 2$ or 3, the uniform boundeness of u follows from proposition 5.2 and from the estimates derived in the proof of theorem 4.1. The uniform boundeness of $\phi(u)$ follows from that of u through (5.28) and lemma 5.1.

6. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS

In this section, we study the asymptotic behaviour in time of the solutions of the equation (5.24). As in the previous paper [2], an important role is played by the pseudoconformal conservation law derived in section 3. The main result is that for repulsive interactions in the sense of assumption (H5), all the solutions of the equation (5.24) in $\mathscr{X}(\mathbb{R})$ are purely dispersive, namely they lie in $\mathscr{X}_0(\mathbb{R})$ (see proposition 6.1).

The first result connects the time decay of various norms of the solutions of (5.24).

LEMMA 6.1. — Let $n = 2$ or 3, and let f satisfy (H1 a, 2 a) and the condition

$$| f'(z) | \leqslant c \, | z |^{p_1 - 1} \qquad \text{for} \qquad | z | \leqslant 1 \tag{6.1}$$

with $p_1 > 1 + 2/n$. (Here f' stands for $\partial f/\partial z$ or $\partial f/\partial \bar{z}$). Let δ satisfy $0 \leqslant \delta < 1$ and

$$p_1 \geqslant 1 + 2(1 + \delta)/n \,. \tag{6.2}$$

Let $t_0 \in \mathbb{R}$ and let $\tilde{u}_0 \in \mathrm{X}$ be such that $\mathrm{U}(.)\tilde{u}_0 \in \mathrm{Y}(\mathbb{R})$ and that

$$\mathrm{M}_0 \equiv \sup_{t \in \mathbb{R}} (1 + | t |)^{\delta} \, \| \mathrm{U}(t)\tilde{u}_0 \|_\infty < \infty \,. \tag{6.3}$$

Let $u \in \mathscr{X}_a(\mathbb{R})$ be solution of the equation (5.24), let

$$\mathrm{M}' = \sup_{t \in \mathbb{R}} \| \nabla u(t) \| \tag{6.4}$$

(M′ is finite by the definition of $\mathscr{X}_a(\mathbb{R})$, see section 1), and let u satisfy in addition

$$\mathrm{M}_\eta \equiv \sup_{t \in \mathbb{R}} (1 + | t |)^{1-\eta} \, \| u(t) \|_q < \infty \tag{6.5}$$

for all q such that $2 \leqslant q < 2n/(n-2)$ and with η defined by (1.4).

Then there is a constant M depending only on f, M_0, M′ and the set $\{ \mathrm{M}_\eta \}$, such that

$$\sup_{s,t \in \mathbb{R}} (1 + | t |)^{\delta} \, \| \mathrm{U}(t - s)u(s) \|_\infty \leqslant \mathrm{M} < \infty \,. \tag{6.6}$$

Proof. — In the proof that follows, we make the convention (already used in the statement of the lemma) that M without subscript or superscript denotes a generic constant that may depend on f, M_0, M′, and the set $\{ \mathrm{M}_\eta \}$, but not otherwise on $\tilde{u}_0$ or u, and not on t_0.

From the equation (5.24) and from (6.3) it follows that

$$\| \mathrm{U}(t - s)u(s) \|_\infty \leqslant \mathrm{M}_0(1 + | t |)^{-\delta} + \| [\mathrm{G}(t_0, s)u](t) \|_\infty \tag{6.7}$$

for all $t \in \mathbb{R}$. By the Sobolev inequality (1.5), the second term in the r. h. s. of (6.7) can be estimated as

$$\| [\mathrm{G}(t_0, s)u](t) \|_\infty \leqslant b_q \, \| [\nabla \mathrm{G}(t_0, s)u](t) \|_q^{n/q} \, \| [\mathrm{G}(t_0, s)u](t) \|_q^{1 - n/q} \tag{6.8}$$

with $n < q < 2n/(n-2)$, or equivalently $0 < \eta < 2 - n/2$. On the other hand, from assumption (H2 a) and from (6.1), it follows that for all $z \in \mathbb{C}$

$$| f(z) | \leqslant \mathrm{C}(| z |^{p_1} + | z |^{p_2}) \tag{6.9}$$

and

$$| f'(z) | \leqslant C(| z |^{p_1 - 1} + | z |^{p_2 - 1}). \tag{6.10}$$

By Hölder's inequality, by (6.9), (6.10) and lemma 1.1, we obtain:

$$\| [G(t_0, s)u](t) \|_q \leqslant C \left| \int_{t_0}^{s} d\tau \, | t - \tau |^{\eta - 1} \sum_j \| u(\tau) \|_{p_j \bar{q}}^{p_j} \right| \tag{6.11}$$

and

$$\| [\nabla G(t_0, s)u](t) \|_q \leqslant C \left| \int_{t_0}^{s} d\tau \, | t - \tau |^{\eta - 1} \| \nabla u(\tau) \| \sum_j \| u(\tau) \|_{(p_j - 1)l}^{p_j - 1} \right| \tag{6.12}$$

where $l = n/(1 - \eta)$. The condition $p_1 > 1 + 2/n$ implies that $p_j \bar{q} > 2$ and $(p_j - 1)l > 2$ for $j = 1, 2$.

For $n = 2$, both $p_j \bar{q}$ and $(p_j - 1)l$ lie in the range of values of q covered by the condition (6.5), i. e. $2 \leqslant q < \infty$. For $n = 3$, because of assumption (H2 a), q can be chosen in such a way that $p_2 \bar{q} < 2n/(n - 2) = 6$ by choosing η sufficiently small, namely $\eta < (5 - p_2)/2$. From now on, we impose this restriction on η. One can then estimate (6.11) for $n = 2$ or 3 by direct use of (6.5). One obtains

$$\| [G(t_0, s)u](t) \|_q \leqslant M \left| \int_{t_0}^{s} d\tau \, | t - \tau |^{\eta - 1} \sum_j (1 + | \tau |)^{1 - \eta - (p_j - 1)n/2} \right|. \tag{6.13}$$

In order to estimate (6.12) however, one needs to consider two cases separately.

First case. — $(p_2 - 1)l < 2n/(n - 2)$. This covers completely the case $n = 2$ and can be achieved for $n = 3$ if $p_2 < 3$ by choosing η sufficiently small. One can then estimate also (6.12) by direct use of (6.5) and (6.4) and obtain

$$\| [\nabla G_0(t_0, s)u](t) \|_q \leqslant M \left| \int_{t_0}^{s} d\tau \, | t - \tau |^{\eta - 1} \times \sum_j (1 + | \tau |)^{1 - \eta - (p_j - 1)n/2} \right|. \tag{6.14}$$

Using (6.8), (6.13), (6.14), the fact that $p_1 \leqslant p_2$ and the elementary estimate

$$\int_{-\infty}^{\infty} d\tau \, | t - \tau |^{\eta - 1} (1 + | \tau |)^{\beta - 1} \leqslant C(1 + | t |)^{\max(\eta - 1, \eta + \beta - 1)} \tag{6.15}$$

which holds for $\beta \neq 0$, $\beta < 1 - \eta$, one obtains

$$\| [G(t_0, s)u](t) \|_\infty \leqslant M(1 + | t |)^{\max(\eta - 1, 1 - (p_1 - 1)n/2)}. \tag{6.16}$$

(Here the condition $\beta < 1 - \eta$ follows from $p_1 > 1 + 2/n$ while the condition $\beta \neq 0$ can always be achieved by a slight change of η if necessary).

We now impose on η the additional restriction $\eta \leqslant 1 - \delta$. Then (6.6) follows from (6.7), (6.16) and (6.2).

Second case. — $(p_2 - 1)l > 2n/(n - 2)$. This case occurs only for $n = 3$ and cannot be avoided if $p_2 \geqslant 3$. One can always assume however that $(p_1 - 1)l < 2n/(n - 2) = 6$ without introducing additional restrictions on δ, by replacing p_1 by $1 + 4/n = 7/3$ if necessary and by taking η sufficiently small. The term with $j = 1$ in (6.12) can then be treated as in the first case. In order to estimate the term with $j = 2$, we use the inequality

$$\| u(\tau) \|_{(p_2-1)l}^{p_2-1} \leqslant \| u(\tau) \|_q^{q/2-1} \| u(\tau) \|_\infty^{p_2-q/2} \tag{6.17}$$

which implies

$$\| [\nabla G(t_0, s)u](t) \|_q \leqslant M \left| \int_{t_0}^{s} d\tau \, | t - \tau |^{\eta-1} \{ (1 + | \tau |)^{1-\eta-(p_1-1)n/2} + (1 + | \tau |)^{(1-\eta)(1-q/2)} \| u(\tau) \|_\infty^{p_2-q/2} \} \right| . \tag{6.18}$$

Let now

$$\mu(T) = \operatorname*{Sup}_{|s|,|t| \leqslant T} (1 + | t |)^\delta \| U(t - s)u(s) \|_\infty . \tag{6.19}$$

Combining (6.7), (6.8), (6.13), (6.18) and (6.15), we obtain for any $T \geqslant | t_0 |$

$$\mu(T) \leqslant M_0 + M \{ \operatorname*{Sup}_{|t| \leqslant T} (1 + | t |)^{\alpha_1} + \operatorname*{Sup}_{|t| \leqslant T} (1 + | t |)^{\alpha_2} \mu(T)^{\alpha_3} \} \tag{6.20}$$

where

$$\alpha_1 = \delta + \operatorname{Max} (\eta - 1, 1 - (p_1 - 1)n/2) \tag{6.21}$$

$$\alpha_2 = \delta + (n/q) \operatorname{Max} (\eta - 1, 1 - (q/2)(1 - \eta) - (p_2 - q/2)\delta) + (1 - n/q) \operatorname{Max} (\eta - 1, 1 - (p_1 - 1)n/2) \tag{6.22}$$

$$\alpha_3 = n(p_2/q - 1/2) . \tag{6.23}$$

We now show that by taking η sufficiently small, one can ensure at the same time that $\alpha_1 \leqslant 0$, $\alpha_2 \leqslant 0$ and $\alpha_3 < 1$. The last condition is equivalent to $p_2 < (n + 2)/(n - 2 + 2\eta) = 5(1 + 2\eta)$ and can be easily satisfied because of assumption (H2 *a*). We impose in addition that $\eta \leqslant 1 - \delta$. Then $\alpha_1 \leqslant 0$ follows from (6.2). In order to ensure that $\alpha_2 \leqslant 0$, it is sufficient that in addition

$$1 - (q/2)(1 - \eta) - (p_2 - q/2)\delta \leqslant - \delta \tag{6.24}$$

Under the assumption $(p_2 - 1)l > 2n/(n - 2) = 6$, the condition (6.24) is easily seen to be satisfied for η sufficiently small. The estimate (6.20) then becomes

$$\mu(T) \leqslant M_0 + M[1 + \mu(T)^{\alpha_3}] \tag{6.25}$$

with $0 \leqslant \alpha_3 < 1$. This implies that $\mu(T)$ is bounded uniformly in T and completes the proof in the second case.

Remark 6.1. — The various conditions imposed on η in the course of the proof are clearly compatible, since all of them are upper bounds on η.

Remark 6.2. — Assumption (H1 *a*) in lemma 6.1 is unnecessarily strong. It would be sufficient to assume instead f to be $\mathscr{C}^1$ with $f(0) = 0$. This is the reason why we have written explicitly the assumption $p_1 > 1 + 2/n$, although for $n = 3$ the assumption (H1 *a*) already implies $p_1 \geqslant 2 > 1 + 2/3$.

We now concentrate on the case of repulsive interactions.

PROPOSITION 6.1. — Let f satisfy (H1 *a*, 2 *a*, 3, 5). Let $t_0 \in \mathbb{R}$, let $\tilde{u}_0 \in \Sigma$ be such that $U(.)\tilde{u}_0 \in Y(\mathbb{R})$ and let u be the solution of the equation (5.24) in $\mathscr{X}_a(\mathbb{R})$ (see theorem 4.1). Then:

(1) $u \in \mathscr{C}(\mathbb{R}, \Sigma)$. For all q satisfying (1.3) and for all s and $t \in \mathbb{R}$, u satisfies the estimate

$$\| U(t-s)u(s) \|_q \leqslant \tilde{a}_q(1+|t|)^{\eta-1}[\| xu(0) \|^2 + \| u(0) \|^2 + E(u(0))]^{1/2} \quad (6.26)$$

where η is defined by (1.4) and $\tilde{a}_q$ is the constant that occurs in (1.7).

(2) Let $n=1$. Then for any $\varepsilon(1/2 \leqslant \varepsilon \leqslant 1)$, $u \in \mathscr{X}_0(\mathbb{R})$, and $\phi(u) \in \mathscr{C}_b(\mathbb{R}, Y_0(\mathbb{R}))$ with

$$| u |_{0\mathbb{R}} \leqslant | \phi(u) |_{\mathbb{R},0\mathbb{R}} \leqslant C[\| xu(0) \|^2 + \| u(0) \|^2 + E(u(0))]^{1/2}. \quad (6.27)$$

(3) Let $n = 2$ or 3, let f satisfy (6.1) with $p_1 \geqslant 1 + (4 - 2\varepsilon)/n$ and let $U(.)\tilde{u}_0 \in Y_0(\mathbb{R})$. Then $u \in \mathscr{X}_0(\mathbb{R})$, $\phi(u) \in \mathscr{C}_b(\mathbb{R}, Y_0(\mathbb{R}))$ and both $| u |_{0\mathbb{R}}$ and $| \phi(u) |_{\mathbb{R},0\mathbb{R}}$ are estimated in terms of $\| u(0) \|_\Sigma$ and $| U(.)\tilde{u}_0 |_{0\mathbb{R}}$ uniformly with respect to t_0.

(4) Let in addition f satisfy (H2 *b*) with $p_1 > 1 + 4/n$, and let u_+ be defined as in proposition 5.1 (3). Then $u_+ \in \Sigma$ and for all $t \in \mathbb{R}$, u satisfies (5.38).

Proof. — The proof of part (1) is almost identical with that of part (1) of proposition 4.3 of [*2*]. Part (2) is an immediate consequence of part (1). Part (3) follows from part (1) and lemma 6.1. Part (4) is a partial repetition of proposition 5.3.

Remark 6.3. — There is some overlap between the various assumptions made on the behaviour of f near the origin. For instance the condition (6.1) with $p_1 \geqslant 1 + (4 - 2\varepsilon)/n$ is contained in (H2 *b*). On the other hand, the condition $p_1 > 1 + 4/n$ is closely related to assumption (H5) (see remark 1.2 of [*2*]).

7. CONTINUITY WITH RESPECT TO INITIAL DATA

In this section, we describe briefly the continuity properties of the solutions of the equation (5.24) with respect to the initial time and initial data. These properties are very similar to those obtained in [*1*] and [*2*]. Indeed, the continuity in a neighbourhood of a given solution depends only on the general structure of the theory, while uniformity of the continuity is obtained whenever one can find a uniform bound on $\phi(u)$. Here we state typical results without proofs and send back the reader to sections 4 of [*1*] and 5 of [*2*] for details.

We shall use systematically the following notation and convention. For any interval J of $\mathbb{R}$, we define the map $\tilde{\omega} : v \to v(0)$ from $\mathrm{Y(J)}$ to H^1, where $v(0)$ is defined as $v(0) = \mathrm{U}(-s)v(s)$ for some (any) $s \in \mathrm{J}$. Clearly $\tilde{\omega}(v) \in \mathrm{X}$ if $0 \in \mathrm{J}$. The map $\tilde{\omega}$ is one to one, and the inverse map is defined by

$$\tilde{\omega}^{-1}(v_0) = \mathrm{U}(.)v_0 . \tag{7.1}$$

In all subsequent propositions, it will be assumed that $\tilde{u}_0 \in \tilde{\omega}(\mathrm{Y}_b(\mathrm{J}))$ or that $\tilde{u}_0 \in \tilde{\omega}(\mathrm{Y}_0(\mathrm{J}))$ for some J. It will then be understood, except in remark 7.2, that continuity properties with respect to $\tilde{u}_0$ are always expressed in terms of the (Banach space) topology image under $\tilde{\omega}$ of that of $\mathrm{Y}_b(\mathrm{J})$ or of that of $\mathrm{Y}_0(\mathrm{J})$, and that joint continuity with respect to $(t_0, \tilde{u}_0)$ is expressed in terms of the product of the natural topology for t_0 and of the previous topology for $\tilde{u}_0$.

Remark 7.1. — If $n = 1$, then for any J, $\tilde{\omega}$ is an isometry from $\mathrm{Y}_b(\mathrm{J})$ onto H^1. Furthermore, by lemma 1.2, Σ is continuously embedded in $\tilde{\omega}(\mathrm{Y}_0(\mathbb{R}))$. This implies obvious simplifications in the following propositions.

PROPOSITION 7.1. — Let f satisfy (H1 *a*). Let I be a bounded interval and J an interval containing I. Let $(t_0, \tilde{u}_0) \in \mathrm{I} \times \tilde{\omega}(\mathrm{Y}_b(\mathrm{J}))$, let $u \in \mathscr{X}_b(\mathrm{I})$ be solution of the equation (5.24) and let $\phi(u)$ be defined by (2.19). Then there exists a neighbourhood $\mathscr{U}$ of $(t_0, \tilde{u}_0)$ in $\mathrm{I} \times \tilde{\omega}(\mathrm{Y}_b(\mathrm{J}))$ such that for all $(t'_0, \tilde{u}'_0) \in \mathscr{U}$, the equation $\tilde{\mathrm{A}}(t'_0, \tilde{u}'_0)v = v$ has a (unique) solution u' in $\mathscr{X}_b(\mathrm{I})$. Furthermore, the map $(t'_0, \tilde{u}'_0) \to \phi(u)$ is continuous from $\mathscr{U}$ to $\mathscr{C}_b(\mathrm{I}, \mathrm{Y}_b(\mathrm{J}))$. If in addition f satisfies (H2 *a*, 3, 4) then the continuity is uniform for t_0 in a compact subset of I and $\tilde{u}_0$ in a bounded set of $\tilde{\omega}(\mathrm{Y}_b(\mathrm{J}))$.

PROPOSITION 7.2. — Let f satisfy (H1 *a*, 2 *b*). Let I be a closed interval and J an interval containing I. Let $(t_0, \tilde{u}_0) \in \bar{\mathrm{I}} \times \tilde{\omega}(\mathrm{Y}_0(\mathrm{J}))$ and let $u \in \mathscr{X}_0(\mathrm{I})$ be solution of the equation (5.24). Then there exists a neighbourhood $\mathscr{U}$ of $(t_0, \tilde{u}_0)$ in $\bar{\mathrm{I}} \times \tilde{\omega}(\mathrm{Y}_0(\mathrm{J}))$ such that for all $(t'_0, \tilde{u}'_0) \in \mathscr{U}$, the equation $\tilde{\mathrm{A}}(t'_0, \tilde{u}'_0)v = v$ has a (unique) solution u' in $\mathscr{X}_0(\mathrm{I})$. Furthermore, the map $(t'_0, \tilde{u}'_0) \to \phi(u')$ is continuous from $\mathscr{U}$ to $\mathscr{C}_b(\bar{\mathrm{I}}, \mathrm{Y}_0(\mathrm{J}))$.

Remark 7.2. — If in proposition 7.2 one adds the assumptions that f satisfies (H3) and that $p_1 > 1 + 4/n$, then one can prove in addition that for fixed $t \in \bar{\mathrm{I}}$, the map $(t'_0, \tilde{u}'_0) \to u'(t)$ is continuous from

$$\mathscr{U} \cap (\bar{\mathrm{I}} \times (\tilde{\omega}(\mathrm{Y}_0(\mathrm{J})) \cap \Sigma))$$

into Σ. Here the topology on u'_0 is the Banach space topology induced by the sup of the norms in $\tilde{\omega}(Y_0(\mathbb{R}))$ and in Σ, and the topology on $u'(t)$ is the natural topology of Σ.

PROPOSITION 7.3. — Let f satisfy (H1 a, 2 a, 2 b, 3, 4), let $J = [T, \infty)$ with $-\infty \leqslant T \leqslant 0$. For any $(t_0, \tilde{u}_0) \in K \cap (\bar{\mathbb{R}}^+ \times \tilde{\omega}(Y_0(J)))$, let $u \in \mathscr{X}_0(\mathbb{R}^+)$ be the solution of the equation (5.24) described in theorem 5.1. Then the map $(t_0, \tilde{u}_0) \to \phi(u)$ is continuous from $K \cap (\bar{\mathbb{R}}^+ \times \tilde{\omega}(Y_0(J)))$ to $\mathscr{C}_b(\bar{\mathbb{R}}^+, Y_0(J))$. The continuity is uniform for $(t_0, \tilde{u}_0) \in \bigcup_{\rho' \leqslant \rho} K(\rho')$ for fixed ρ and for $\tilde{u}_0$ in a bounded set of $\tilde{\omega}(Y_0(J))$.

PROPOSITION 7.4. — Let f satisfy (H1 a, 2 a, 2 b, 3, 5) with $p_1 > 1 + 4/n$. For any $(t_0, \tilde{u}_0) \in \bar{\mathbb{R}} \times (\Sigma \cap \tilde{\omega}(Y_0(\mathbb{R})))$, let $u \in \mathscr{X}_0(\mathbb{R})$ be the solution of the equation (5.24) described in proposition 6.1. Then the map $(t_0, \tilde{u}_0) \to \phi(u)$ is continuous from $\bar{\mathbb{R}} \times (\Sigma \cap \tilde{\omega}(Y_0(\mathbb{R})))$ into $\mathscr{C}_b(\bar{\mathbb{R}}, Y_0(\mathbb{R}))$. The continuity is uniform in $(t_0, \tilde{u}_0)$ for $\tilde{u}_0$ in a bounded set of Σ and in a bounded set of $\tilde{\omega}(Y_0(\mathbb{R}))$.

8. WAVE OPERATORS AND ASYMPTOTIC COMPLETENESS

In this section, we state the implications of the results of sections 5, 6 and 7 to the theory of scattering. We recall that the wave operator Ω_+ is defined as the map $u_+ \to u(0)$ where $u(t)$ is the solution of the equation

$$\tilde{A}(\infty, u_+)v = v . \tag{8.1}$$

The wave operator Ω_- is defined similarly.

The wave operators (and possibly their inverses) will be considered as acting either in the Banach space $\tilde{\omega}(Y_0(\mathbb{R}))$ or in $\tilde{\omega}(Y_0(\mathbb{R})) \cap \Sigma$. The latter space will be equipped either with the topology induced by $\tilde{\omega}(y_0(\mathbb{R}))$ or with the « natural » Banach space topology associated with the maximum of the norms in $\tilde{\omega}(Y_0(\mathbb{R}))$ and in Σ.

PROPOSITION 8.1. — Let f satisfy (H1 a, 2 a, 2 b, 3, 4). Then the wave operators $\Omega_\pm$ map $\tilde{\omega}(Y_0(\mathbb{R}))$ continuously into itself. They are bounded and uniformly continuous on the bounded sets of $\tilde{\omega}(Y_0(\mathbb{R}))$. If in addition $p_1 > 1 + 4/n$, then the wave operators map $\Sigma \cap \tilde{\omega}(Y_0(\mathbb{R}))$ (equipped with its natural topology) continuously into itself.

For repulsive interactions, the results of section 6 imply in addition asymptotic completeness.

PROPOSITION 8.2. — Let f satisfy (H1 a, 2 a, 2 b, 3, 5) with $p_1 > 1 + 4/n$. Then the wave operators $\Omega_\pm$ are bijections of $\Sigma \cap \tilde{\omega}(Y_0(\mathbb{R}))$ into itself. Both $\Omega_\pm$ and $\Omega_\pm^{-1}$ are bounded on the bounded sets of $\Sigma \cap \tilde{\omega}(Y_0(\mathbb{R}))$ and are continuous in the sense of the topology induced on $\Sigma \cap \tilde{\omega}(Y_0(\mathbb{R}))$

by $\tilde{\omega}(Y_0(\mathbb{R}))$, uniformly on the bounded sets of $\Sigma \cap \tilde{\omega}(Y_0(\mathbb{R}))$. Furthermore $\Omega_\pm$ and $\Omega_\pm^{-1}$ are continuous from $\Sigma \cap \tilde{\omega}(Y_0(\mathbb{R}))$ (equipped with its natural topology) into itself.

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APPENDIX

In this appendix, we sketch an alternative theory in dimension $n = 1$, which is similar to that developed in [1] and [2] in the sense that the basic space $X = L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is larger than $H^1(\mathbb{R})$. Most of the methods and results used in [1] and [2] or in this paper carry over without change to this case with the exception of the treatment of the local Cauchy problem, both at finite and infinite initial times. This requires minor modifications, which we present below without proof.

In the assumptions made on f, the condition (H1 a) is replaced by the condition (H1) of [1], namely:

(H1) f is a continuously differentiable function from $\mathbb{C}$ to $\mathbb{C}$ and $f(0) = 0$.

The basic spaces are $X = L^2 \cap L^\infty$, $\mathscr{X}_{(b)}(I) = \mathscr{C}_{(b)}(I, X)$ and

$$\mathscr{X}_0(I) = \{ v \in \mathscr{X}_b(I) \ : \ \operatorname*{Sup}_{t \in I} \{ \operatorname{Max} \| v(t) \| , \ (1 + |t|)^{1-\varepsilon} \operatorname{Max} (\| v(t) \|_r, \| v(t) \|_\infty) \} \equiv | v |_{0I} < \infty \}$$

for some r, $2 < r \leqslant \infty$, and with ε defined by $1/r = 1/2 - (1 - \varepsilon)$.

We first state the result concerning the Cauchy problem at finite initial time for the equation (2.1) or more precisely for the equation (2.18).

PROPOSITION A.1. — Let f satisfy (H1) and the estimate (6.1) with $p_1 \geqslant 2$, let h satisfy ($h1$), let g satisfy ($g1$ a). Then the same conclusions as in proposition 2.1 hold.

The conservation laws for the L^2-norm and for the energy, and the pseudoconformal conservation law are derived in the same way as in section 3. The first two laws yield the existence of global solutions in the same way as in [1], provided f satisfies (H1, 3, 4) and the estimate (6.1) with $p_1 \geqslant 2$.

The solution of the Cauchy problem at infinite initial time is described in the following proposition.

PROPOSITION A.2. — Let f satisfy (H1) and the estimate (6.1) with

$$p_1 - 1 > \operatorname{Max} (\varepsilon/(1 - \varepsilon), 4 - 2\varepsilon) \tag{A.1}$$

with the same ε as in the definition of $\mathscr{X}_0(.)$. Then the same conclusions as in proposition 5.2 hold.

We remark that the weakest restriction on p_1 imposed by the condition (A.1) is $p_1 > (3 + \sqrt{17})/2$, corresponding to $\varepsilon = (7 - \sqrt{17})/4$. This restriction is weaker than that given by (1.18) for $n = 1$.

For repulsive interactions (in the sense of assumption (H5)), all solutions of the equation (2.18) are purely dispersive and lie in $\mathscr{X}_0(\mathbb{R})$. The proof is the same as in section 6.

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