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Non semisimple gauge models: II. Renormalization

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ABSTRACT — In this paper we discuss the renormalizability, within the B. P. H. Z. scheme, of the non-semi-simple gauge models whose classical limit has been fully exposed in paper I. Within our regularization independent approach renormalizability follows from the compensability, by a suitable choice of the radiative corrections to the Lagrangian, of the breakings which can affect the Slavnov identity. These breakings are « *a priori* » controlled by the Lowenstein and Lam Quantum Action Principle and by a system of supplementary renormalization conditions prescribing the equations of motion of the Faddeev-Popov $\bar{c}$ fields and the super-renormalizability of the couplings of those $\bar{c}$'s corresponding to the abelian factors of the gauge group. The supplementary conditions induce algebraic and dimensional (power counting) constraints which insure the compensability of all the breakings of maximum dimension except, of course, the Adler-Bardeen anomaly whose absence we assume. Concerning the remaining soft breakings, we shall show that the Callan-Symanzik equation, in a very general context, also excludes their presence if the hard breakings are absent.

1. INTRODUCTION

In this paper we discuss the renormalization, within the B. P. H. Z. scheme [*1*], of the gauge models [*2*] whose classical limit (tree approximation)

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has been fully described in a previous paper, hereafter referred to as I [3]. Here we will focus on the technical aspects and refer the reader to the introduction of paper I for a survey of the general problems posed by gauge theories.

Our renormalization method is a non trivial generalization of the one employed by Becchi-Rouet-Stora (B. R. S. [4] [5]) in the analysis of purely abelian and semi-simple gauge models. It hinges on the systematic study of the admissible breaking terms of the Slavnov identity (S. I.) [6] as given by the Quantum Action Principle (Q. A. P.) which is known to hold within the chosen renormalization framework [7]. The recent progress of Clark and Lowenstein [8] on the validity of the Q. A. P. for theories with massless particles allows an immediate extension of our results also to models of this kind.

The B. R. S. purely algebraic treatment of the breakings of the S. I. is inadequate for our purposes since here the gauge group is not semi-simple (1) and the class of the admissible breaking terms is not *a priori* restricted on the basis of discrete symmetries as it happens e. g. with charge conjugation in the U(1) H. K. model [5].

A possible way out of these difficulties is exhibited by models with a particularly simple form of the gauge function, namely Feynman's choice, where to each abelian invariant component of the gauge group there corresponds a linear Ward Identity (W. I.), essentially identical to that of Quantum Electrodynamics (Q. E. D.). Within this scheme the abelian Faddeev-Popov ($\Phi.\Pi$) fields [10] are redundant since they turn out to be free, but, when the gauge symmetry is spontaneously broken, we find artificial infrared (U. R.) divergencies which may be avoided by adopting the 't Hooft gauge [11].

In the 't Hooft class of gauges the situation is, of course, different: the abelian $\Phi.\Pi.$ fields are in general no longer free, and we have a global Slavnov identity for the theory. There remains, however a memory of the old status, true in the Feynman gauge: the abelian $\Phi.\Pi.$ fields have, at least in the tree approximation super-renormalizable couplings, and hence they are free at high momenta.

The essential improvement of our method with respect to the B. R. S. one, is to introduce the renormalization condition that the couplings of the abelian $\Phi.\Pi.$ fields remain super-renormalizable to all orders of perturbation theory. Taking into account this condition in a power counting analysis of the breakings, we shall be able to exclude all the terms of maximum dimensions escaping the B. R. S. algebraic treatment, except, of course the Adler-Bardeen [12] anomaly (A. B. A.).

The absence of the lower dimensionality (soft) anomalies will be discussed

(1) The situation here is similar to that shown in I when discussing the classical stability.

by means of the Callan-Symanzik [13] (C. S.) equation, somewhat in analogy with what is done by B. R. S. in Ref. [14].

In Section 2 we recall a number of results obtained in paper I and illustrate the main ideas which will guide our approach to the renormalization problem: i. e. to the discussion of the breakings of the S. I.

Section 3 contains the detailed study of the external fields dependent and of the maximum dimensionality (hard) anomalies. We shall show that, apart from the A. B. A., all other breakings in this class are compensable.

In Section 4 we complete our treatment of the renormalizability of the S. I., excluding the presence of the soft anomalies with arguments based on the Callan-Symanzik equation.

In the concluding Section, besides a summary of the results, we comment upon their extension to theories with massless particles and illustrate the non-renormalization properties of some parameters of these models.

Appendices A and B contain the more technical aspects of the proofs of statements in the text.

2. GENERAL STRATEGY

In this section we recall, for the convenience of the reader, some results and notations appearing in paper I [3], and illustrate the general strategy toward renormalization.

The quantized fields are

$$\{ \psi \} \equiv \{ \mathscr{A}_\mu^\alpha, \varphi_i, c^\alpha, \bar{c}^\alpha \} \tag{1}$$

for which we introduce the sources

$$\{ \mathrm{J} \} \equiv \{ \mathrm{J}_\mu^\alpha, \mathrm{J}_i, \bar{\xi}^\alpha, \xi^\alpha \} \tag{2}$$

The set of Greek indices will be split, when convenient, into a « semi-simple » subset $\alpha_{\mathrm{S}}, \beta_{\mathrm{S}}, \ldots, \mathrm{S} = 1, \ldots, \mathrm{N_S}$ and an « abelian » one $\alpha_{\mathrm{A}}, \beta_{\mathrm{A}}, \ldots$ $\mathrm{A} = 1, \ldots, \mathrm{N_A}$ corresponding to the semi-simple and abelian components of the Lie algebra $\mathscr{G}$ associated with the gauge group.

The Lagrangian involves also the external fields

$$\{ \eta^\alpha \} \equiv \{ \gamma_\mu^{\alpha_{\mathrm{S}}}, \gamma_i, \zeta^{\alpha_{\mathrm{S}}} \} \tag{3}$$

The $\Phi.\Pi.$ charge [10], defined by the $\mathrm{Q}_{\phi\Pi}$ operator according to

$$\begin{aligned}
&\mathrm{Q}_{\phi\Pi}\mathscr{A}_\mu^\alpha = \mathrm{Q}_{\phi\Pi}\varphi_i = 0\,; \quad \mathrm{Q}_{\phi\Pi}\bar{c}^\alpha = -\,\bar{c}^\alpha\,; \quad \mathrm{Q}_{\phi\Pi}c^\alpha = c^\alpha\,;\\
&\mathrm{Q}_{\phi\Pi}\mathrm{J}_\mu^\alpha = \mathrm{Q}_{\phi\Pi}\mathrm{J}_i = 0\,; \quad \mathrm{Q}_{\phi\Pi}\bar{\xi}^\alpha = -\,\bar{\xi}^\alpha\,; \quad \mathrm{Q}_{\phi\Pi}\xi^\alpha = \xi^\alpha\,;\\
&\mathrm{Q}_{\phi\Pi}\gamma_\mu^{\alpha_{\mathrm{S}}} = \gamma_\mu^{\alpha_{\mathrm{S}}}\,; \quad \mathrm{Q}_{\phi\Pi}\gamma_i = \gamma_i\,; \quad \mathrm{Q}_{\phi\Pi}\zeta^{\alpha_{\mathrm{S}}} = 2\zeta^{\alpha_{\mathrm{S}}}
\end{aligned} \tag{4}$$

is conserved.

To both quantized and external fields a naive dimension is assigned

$$\begin{aligned} &\dim \mathscr{A}^\alpha_\mu = \dim c^\alpha = \dim \bar{c}^\alpha = 1\,, \\ &\dim \gamma^{\alpha s}_\mu = \dim \zeta^{\alpha s} = 2\,, \quad \dim \gamma_i = 3 - \dim \varphi_i \end{aligned} \tag{5}$$

where dim φ_i is either 1 or 3/2 according to the boson or fermion character of the matter field.

The classical action

$$\Gamma^{(Cl)}(\psi, \eta) = \int dx\, \mathscr{L}^{Cl}(\psi, \eta)(x) = \int dx\, \mathscr{L}^{Cl}(\mathscr{A}_\mu, \varphi)(x) + \iint dx dy\, c_\alpha(x) K^{\alpha\beta}(x, y) \bar{c}_\beta(y) + \int dx\, \eta^\alpha(x) P^{(0)}_\alpha(x) \tag{6}$$

is the most general dimension four local functional satisfying the identity

$$\int dx \left(\frac{\delta\Gamma}{\delta\varphi_i(x)} \frac{\delta\Gamma}{\delta\gamma_i(x)} + \frac{\delta\Gamma}{\delta\mathscr{A}^{\alpha s}_\mu(x)} \frac{\delta\Gamma}{\delta\gamma^\mu_{\alpha s}(x)} + \frac{\delta\Gamma}{\delta\mathscr{A}^{\alpha A}_\mu(x)} \partial_\mu \bar{c}^{\alpha A}(x) + \left(\partial_\mu \mathscr{A}^\mu_\alpha(x) + \rho_{\alpha i}\varphi_i(x)\right) \frac{\delta\Gamma}{\delta c_\alpha(x)} + \frac{\delta\Gamma}{\delta\bar{c}^{\alpha s}(x)} \frac{\delta\Gamma}{\delta\zeta_{\alpha s}(x)} \right) = 0 \tag{7}$$

exhibiting the Slavnov [6] invariance of the theory, and the mass degeneracy condition for the ghost particles discussed in paper I.

In terms of the connected tree approximation Green functional $Z^{Cl}_c(J, \eta)$ (the Legendre inverse transform of $\Gamma^{Cl}(\psi, \eta)$) Eq. (7) writes

$$\mathscr{S} Z^{Cl}_c(J, \eta) \equiv \int dx \left\{ J_i(x) \frac{\delta}{\delta\gamma_i(x)} + J^{\alpha s}_\mu(x) \frac{\delta}{\delta\gamma^{\alpha s}_\mu(x)} + J^{\alpha A}_\mu(x) \partial^\mu \frac{\delta}{\delta\xi^{\alpha A}(x)} - \xi^{\alpha s}(x) \frac{\delta}{\delta\zeta_{\alpha s}(x)} - \bar{\xi}^\alpha(x) \left(\partial^\mu \frac{\delta}{\delta J^\mu_\alpha(x)} + \rho_{\alpha i} \frac{\delta}{\delta J_i(x)} \right) \right\} Z^{Cl}_c(J, \eta) = 0\,. \tag{8}$$

A quantized extension of the model is defined by means of an effective Lagrangian

$$\mathscr{L}_{\text{eff}}(\psi, \eta)(x) = \mathscr{L}_{\text{eff}}(\varphi, \mathscr{A}_\mu)(x) + \left(c_\alpha K^{\alpha\beta}_{\text{eff}} \bar{c}_\beta\right)(x) + \eta_\alpha(x) P^\alpha(x) \tag{9}$$

equipped with an appropriate set of substraction prescriptions, which for the sake of definiteness, we shall assume to be those of the B. P. H. Z. scheme with substraction index equal to four (in the Zimmermann's notation the Lagrangian is an $\mathscr{N}_4$ [15] operator).

The Lagrangian is a formal power series in $\hbar$

$$\mathscr{L}_{\text{eff}}(\psi, \eta) = \sum_{n=0}^{\infty} \hbar^n \mathscr{L}^{(n)}_{\text{eff}}(\psi, \eta) \tag{10}$$

The coefficient of the zeroth power in Eq. (10) is the classical Lagrangian of Eq. (6), hence in particular

$$\bar{P}^{(0)}_{\alpha_S} = \frac{1}{2} f^{\beta\gamma}_{\alpha_S} \bar{c}_\beta \bar{c}_\gamma \quad (11\,a)$$

$$P^{\mu(0)}_{\alpha_S} = \partial^\mu \bar{c}_{\alpha_S} + f^{\beta\gamma}_{\alpha_S} \mathscr{A}^\mu_\gamma \bar{c}_\beta \quad (11\,b)$$

$$P^{(0)}_i = t^\alpha_{ij}(\varphi_j + q_j)\bar{c}_\alpha \quad (11\,c)$$

where $f^{\beta\gamma}_\alpha$ and t^α_{ij} are the structure constants and the representations of the Lie algebra $\mathscr{G}$.

The main purpose of this paper is to show that the Slavnov identity, Eq. (8), for the connected functional $Z_c(J, \eta)$ or equivalently Eq. (7) for $\Gamma(\psi, \eta)$, can be solved in terms of the coefficients of $\mathscr{L}_{\text{eff}}(\psi, \eta)$ to all orders in $\hbar$.

In practice this means that we will study the violations induced by an arbitrary renormalization process in the S. I. and in the other functional equations characterizing the theory, without looking, from the start, for a « clever » regularization procedure preserving the relevant identities order by order in $\hbar$. According to this crude point of view, the admissible breakings are « *a priori* » controlled, within the Zimmermann formalism, by the Q. A. P. here employed in the weak form given by Stora in [*4*]. In this formulation the Q. A. P. asserts that, given an identity, verified at the tree approximation and exhibiting the invariance of the action under a transformation of the fields into local operators (N-products [*1*] [*15*] in Zimmermann's notation), an arbitrary renormalization procedure preserving power counting introduces quantum breakings which are also local operators. Furthermore the N product index of these breakings, summarizing their power counting properties, is bounded by the dimension of the naive variation of an arbitrary renormalizable Lagrangian.

Now the renormalizability of the theory is proved if the system of equations expressing the vanishing, order by order, of the breakings can be solved, in the sense of formal power series, in terms of the parameters of the Lagrangian. It is an essential point, for our method, that the quantum corrections to the breakings to order n in $\hbar$ depend on the parameters of $\mathscr{L}_{\text{eff}}$ up to order $n - 1$.

As shown by B. R. S. (Ref. [*4*]) the analysis of the breakings to the S. I. is simplified by selecting *a priori* those effective Lagrangians (or, more precisely, the term $c_\alpha K^{\alpha\beta}_{\text{eff}} \bar{c}_\beta$ in Eq. (9)) whose related connected functional $Z_c(J, \eta)$ satisfies the equation

$$\mathscr{S}^2 Z_c(J, \eta) \equiv - \int dx\, \bar{\xi}_\alpha(x) \Bigg(\delta^\alpha_{\alpha_S} \partial_\mu \frac{\delta}{\delta \gamma_{\alpha_{S\mu}}(x)} + \delta^\alpha_{\alpha_A} \Box \frac{\delta}{\delta \xi_{\alpha_A}(x)} + \rho^\alpha_i \frac{\delta}{\delta \gamma_i(x)} \Bigg) Z_c(J, \eta) = 0 \quad (12)$$

which is a necessary condition for the validity of Eq. (8). From this equation it follows a set of consistency conditions, which, in the semi-simple case, allow a purely algebraic treatment of the breakings.

In terms of the vertex functional $\Gamma(\psi, \eta)$, Eq. (12) becomes

$$(\mathscr{S}^2\Gamma)(\psi, \eta) \equiv \int dx\left(\delta^{\alpha\alpha_S}\partial_\mu \frac{\delta}{\delta\gamma_\mu^{\alpha_S}(x)}\Gamma + \delta^{\alpha\alpha_A}\,\Box\,\bar{c}_{\alpha_A}(x) + \rho_i^\alpha \frac{\delta\Gamma}{\delta\gamma_i(x)}\right)\frac{\delta\Gamma}{\delta c^\alpha(x)}(\psi, \eta) = 0 \quad (13)$$

which turns out to be equivalent to the linear relations

$$\frac{\delta\Gamma}{\delta c_\alpha(x)} = \delta^{\alpha\beta_S}\partial_\mu \frac{\delta\Gamma}{\delta\gamma_\mu^{\beta_S}(x)} + \delta^{\alpha\beta_A}\,\Box\,\bar{c}_{\beta_A}(x) + \rho_i^\alpha \frac{\delta\Gamma}{\delta\gamma_i(x)} + \delta^\alpha_{\alpha_A}\mu^{\alpha_A\beta_A}\bar{c}_{\beta_A}(x) \quad (14)$$

or

$$D^\alpha(x)\Gamma \equiv \left[\frac{\delta}{\delta c_\alpha(x)} - \delta^{\alpha\beta_S}\partial_\mu \frac{\delta}{\delta\gamma_\mu^{\beta_S}(x)} - \rho_i^\alpha \frac{\delta}{\delta\gamma_i(x)}\right]\Gamma = \delta^\alpha_{\alpha_A}\mu^{\alpha_A\beta_A}\bar{c}_{\beta_A}(x) + \delta^{\alpha\beta_A}\,\Box\,\bar{c}_{\beta_A}(x). \quad (15)$$

Indeed, these equations, which are nothing but the equations of motion of the $\bar{c}$ fields, give in the tree approximation the most general solution of Eq. (13); consequently, if they are renormalizable, they also give the general solution of Eqs. (12) (13) to all orders. The above mentioned mass degeneracy condition for the ghost particles [3] is written

$$\mu^{\alpha_A\beta_A}\rho_{\beta_A i} = 0. \quad (16)$$

As pointed out in the Introduction, in the non semi-simple case the algebraic conditions are no longer sufficient and must be enriched with the prescription that the abelian $\bar{c}$ fields (i. e. those corresponding to the abelian factor of $\mathscr{G}$) have only super-renormalizable couplings except for the external field vertex $\gamma_i t_{ij}^{\alpha_A}\varphi_j\bar{c}_{\alpha_A}$. This prescription is, of course, verified in the classical limit. Indeed, by substituting Eqs. (11 *b*) (11 *c*) into Eq. (14) we have an explicit solution for the term $c_\alpha K^{\alpha\beta}\bar{c}_\beta$ in $\mathscr{L}^{Cl}$ where the couplings of the abelian $\bar{c}$ fields are all super-renormalizable.

The problem is to implement this super-renormalizability condition to all orders of perturbation theory. This can be done introducing a set $\{\mathscr{C}_{\alpha_A}\}$ of external fields with dimension 2 and Φ.Π. charge -1 and prescribing, by means of the functional equation

$$\mathscr{E}_{\alpha_A}(x)\Gamma(\psi, \eta, \mathscr{C}) \equiv \frac{\delta}{\delta\bar{c}_{\alpha_A}(x)}\Gamma - \frac{\delta}{\delta\mathscr{C}_{\alpha_A}(x)}\Gamma = Y_{\alpha_A}(x), \quad (17)$$

that the sources of the $\mathscr{C}_{\alpha_A}$ and of the $\bar{c}_{\alpha_A}$ fields coincide to all orders. Here $Y_{\alpha_A}(x)$ is a local, dimension three functional, linear in the fields, expect for the term $\gamma_i t_{ij}^{\alpha_A}\varphi_j$ which cannot be avoided being the only dimension four coupling with the $\bar{c}_{\alpha_A}$ fields present at the classical level. In the tree approximation the natural choice is

$$Y^{cl\alpha_A}(x) = -(\delta^{\alpha_A\beta}\Box + \mu^{\beta}_{\alpha_A})c_\beta(x) + \gamma_i(x)t^{\alpha_A}_{ij}\varphi_j(x) \tag{18}$$

which corresponds to the classical vertex functional

$$\Gamma^{cl}(\psi, \eta, \mathscr{C}) = \Gamma^{cl}(\psi, \eta) + \int dx\big(c_\alpha(x)\rho^{\alpha}_{i}t^{\gamma_A}_{ij}\varphi_j(x)\mathscr{C}_{\gamma_A}(x)\big) \tag{19}$$

but, keeping renormalization in mind, we are forced to choose the more general expression

$$Y_{\alpha_A}(x) = Y^{cl}_{\alpha_A}(x) - y^{\beta}_{\alpha_A}\Box c_\beta(x) - Z_{\alpha_A i}\rho^{\beta}_{i}c_\beta(x) + Z_{\alpha_A i}\gamma_i(x) + y^{\beta}_{\alpha_A}\partial^\mu\gamma_{\mu\beta}(x) \tag{20}$$

where the parameters $Z_{\alpha_A i}$, $y^{\beta}_{\alpha_A}$ remain to be determined.

Of course Eq. (17) must be satisfied together with Eq. (15) which, however, in the presence of the $\mathscr{C}_{\alpha_A}$ fields must be modified to the form

$$D^\alpha(x)\Gamma(\psi, \eta, \mathscr{C}) = \delta^{\alpha\beta_A}\Box\bar{c}_{\beta_A}(x) + \mu^{\alpha\beta_A}\bar{c}_{\beta_A}(x) - \rho^{\alpha}_{i}t^{\gamma_A}_{ij}\varphi_j(x)\mathscr{C}_{\gamma_A}(x) \equiv X^\alpha(x) \tag{21}$$

Our program is to discuss first the renormalizability of Eqs. (17) (21) which will be used as supplementary conditions to the S. I. thereby limiting ourselves to the study of theories in which the $\mathscr{C}_{\alpha_A}$ fields can be introduced so as to satisfy them (2).

Exploiting the consequences of these supplementary conditions we shall be able to reabsorb all the highest dimensionality breakings, except the A. B. A. which could only be eliminated in view of an hitherto unproved non-renormalization theorem.

For what concerns the remaining soft anomalies, we shall show that the Callan-Symanzik equation, in a very general context, excludes their presence.

3. RENORMALIZATION OF THE SLAVNOV IDENTITY - HARD ANOMALIES

In this Section we begin to analyze the breakings to the S. I. $\mathscr{S}Z_c(J, \eta) = 0$, and following the general strategy we will first discuss the renormalizability of Eqs. (17) (21) after the introduction in the theory of the set of external fields $\mathscr{C}_{\alpha_A}$ mentioned in the previous Section. We shall take advantage of this opportunity to work out in detail an example of our approach to renormalization which may serve for future reference.

(2) There might be the doubt that Eqs. (17), (21) essentially restrict the class of quantum extensions of the model under consideration. That this is not the case follows from the stability proof of the classical action given in I.

According to the Q. A. P. [7] illustrated in Section 2, Eqs. (17) (21) admit breakings which are N_3-products. We have thus, in general,

$$D_\alpha(x)\Gamma(\psi, \eta, \mathscr{C}) = (\Sigma_\alpha(x)\Gamma)_3(\psi, \eta, \mathscr{C}) + X_\alpha(x) \tag{22 a}$$

$$\mathscr{E}_{\alpha_A}(x)\Gamma(\psi, \eta, \mathscr{C}) = (\Sigma'_{\alpha_A}(x)\Gamma)_3(\psi, \eta, \mathscr{C}) + Y_{\alpha_A}(x) \tag{22 b}$$

where the r. h. s. means the insertion of the vertices $N_3[\Sigma_\alpha(x)]$ and $N_3[\Sigma'_{\alpha_A}(x)]$ into the appropriate Green functions.

Since the vertices Σ_α and Σ'_{α_A} are polynomials in the fields and their derivatives, whose coefficients are formal power series in $\hbar$, i. e.

$$\Sigma_\alpha(x) = \sum_{m\geq n} \hbar^m \Sigma_\alpha^{(m)}(x) \tag{23 a}$$

$$\Sigma'_{\alpha_A}(x) = \sum_{m\geqslant n} \hbar^m \Sigma_{\alpha_A}^{(m)}(x) \tag{23 b}$$

the equations $\Sigma'_{\alpha_A}(x) = \Sigma_\alpha(x) = 0$, ensuring the validity of Eqs. (17) (21) to all orders of perturbation theory, can be solved if for any $n \geqslant 1$ the equations

$$\Sigma_\alpha^{(n)}(x) = 0 \tag{24 a}$$

$$\Sigma_{\alpha_A}^{\prime(n)}(x) = 0 \tag{24 b}$$

are solvable in terms of the coefficients of $\mathscr{L}_{\text{eff}}^{(n)}$.

To prove the solvability of Eqs. (24 *a*) and (24 *b*) remark that [4]:

$$\Sigma_\alpha^{(n)}(x) = D_\alpha(x) \int dy\, \mathscr{L}_{\text{eff}}^{(n)}(\psi, \eta, \mathscr{C})(y) + Q_\alpha^{(n)}(x, \mathscr{L}_{\text{eff}}) \tag{25 a}$$

$$\Sigma_{\alpha_A}^{\prime(n)}(x) = \mathscr{E}_{\alpha_A}(x) \int dy\, \mathscr{L}_{\text{eff}}^{(n)}(\psi, \eta, \mathscr{C})(y) + Q_{\alpha_A}^{\prime(n)}(x, \mathscr{L}_{\text{eff}}) \tag{25 b}$$

where $Q_\alpha^{(n)}(x, \mathscr{L}_{\text{eff}})$ and $Q_{\alpha_A}^{\prime(n)}(x, \mathscr{L}_{\text{eff}})$ sum up the quantum corrections implied by the Q. A. P. and consequently depend on $\mathscr{L}_{\text{eff}}^{(\nu)}$ only for $\nu < n$; clearly we have to show that:

$$Q_\alpha^{(n)}(x, \mathscr{L}_{\text{eff}}) = D_\alpha(x)\hat{\mathscr{L}}(\mathscr{L}_{\text{eff}}) \tag{26 a}$$

and

$$Q_{\alpha_A}^{\prime(n)}(x, \mathscr{L}_{\text{eff}}) = \mathscr{E}_{\alpha_A}(x)\hat{\mathscr{L}}(\mathscr{L}_{\text{eff}}) \tag{26 b}$$

for the same local functional $\hat{\mathscr{L}}(\mathscr{L}_{\text{eff}})$ of dimension four.

Now the coefficients of the fields $\mathscr{C}_{\alpha_A}$ in $Q_\alpha^{(n)}(x)$ have at most dimension one and null $\Phi.\Pi.$ charge; it follows that the possible $\bar{\mathscr{C}}_{\alpha_A}$ dependent terms in $Q_\alpha^{(n)}(x)$, whose general structure is $K^{\alpha,\alpha_A}(\Phi)\bar{\mathscr{C}}_{\alpha_A}$, can be written in the form (26 *a*) with the corresponding term in $\hat{\mathscr{L}}$ given by $c_\alpha K^{\alpha\alpha_A}(\Phi)\bar{\mathscr{C}}_{\alpha_A}$. In much the same way the contributions $K_{ij}^{\alpha_A}\gamma_i\varphi_j$ and $K_\gamma^{\alpha_A\beta_S}\gamma_{\beta_S}^\mu\mathscr{A}_\mu^\gamma$ in $Q'_{\alpha_A}(x)$ can be compensated by introducing in $\hat{\mathscr{L}}$ the couplings $\bar{c}_{\alpha_A}K_{ij}^{\alpha_A}\gamma_i\varphi_j$

and $\bar{c}_{\alpha_A} K_{\gamma}^{\alpha_A \beta_S} \gamma_{\beta_S}^{\mu} \mathscr{A}_{\mu}^{\gamma}$. Now since in the inhomogeneous term $Y_{\alpha_A}(x)$ shown in Eq. (20) the coefficients of the monomials γ_i and $\partial_\mu \gamma_{\beta_S}^{\mu}$ are left undetermined, we are free to transfer these monomials from $Q'_{\alpha_A}(x)$ to $Y_{\alpha_A}(x)$.

Once these adjustements are done we remain with:

$$Q^{(n)\alpha}(x) = \int dy\, H^{\alpha\beta}(x, y; \Phi)\bar{c}_\beta(y) + \Gamma^{\alpha\beta,\gamma\delta}(c_\beta \bar{c}_\gamma \bar{c}_\delta)(x) \quad (27\,a)$$

$$Q'^{(n)\alpha_A}(x) = \int dy\, H'^{\alpha_A\beta}(x, y; \Phi)c_\beta(y) + \Gamma'^{\alpha_A\beta,\gamma\delta}(\bar{c}_\beta c_\gamma c_\delta)(x) + L^{\gamma_S\alpha_A\beta}(\zeta_{\gamma_S}\bar{c}_\beta)(x) \quad (27\,b)$$

where the kernels $H^{\alpha\beta}$ and $H'^{\alpha_A\beta}$ depend only on the φ_i and $\mathscr{A}_\mu^\alpha$ fields.

Due to their Φ.Π. character the operator $D_\alpha(x)$ and $\mathscr{E}_{\alpha_A}(x)$ anticommute among themselves and with each other. This leads, picking out the order $n \geqslant 1$ in Eqs. (22 *a*) (22 *b*), to the following consistency conditions for the n-th order quantum corrections:

$$D_\beta(y)Q_\alpha^{(n)}(x) + D_\alpha(x)Q_\beta^{(n)}(y) = 0 \quad (28\,a)$$

$$\mathscr{E}_{\alpha_A}(x)Q'^{(n)}_{\beta_A}(y) + \mathscr{E}_{\beta_A}(y)Q'^{(n)}_{\alpha_A}(x) = 0 \quad (28\,b)$$

$$\mathscr{E}_{\beta_A}(y)Q_\alpha^{(n)}(x) + D_\alpha(x)Q'^{(n)}_{\beta_A}(y) = 0 \quad (28\,c)$$

Substituting into Eqs. (28 *a*) (28 *b*) the expressions (27 *a*) (27 *b*) yield that $\Gamma^{\alpha\beta,\gamma\delta}$, $\Gamma'^{\alpha_A\beta_A,\gamma\delta}$ are antisymmetric in the first two indices and that $L^{\gamma_S\alpha_A\beta_A}$ is antisymmetric in the indices α_A, β_A. Notice that the antisymmetry of $\Gamma'^{\alpha_A\beta,\gamma\delta}$ and $L^{\gamma_S\alpha_A\beta}$ can be maintained for a generic value of β as a definition.

The mixed integrability condition Eq. (28 *c*) yields:

$$H'^{\alpha\beta_A}(x, y) + H^{\alpha\beta_A}(y, x) = 0 \quad (29\,a)$$

$$\Gamma^{\alpha\beta,\delta\beta_A} + \Gamma'^{\beta_A\delta,\alpha\beta} = 0\,. \quad (29\,b)$$

It thus follows from the consistency conditions (28 *a*) (28 *b*) and (28 *c*) that the quantum correction terms $Q_{\alpha_A}(x)$ and $Q'_{\alpha_A}(x)$ can be given the form (26 *a*) and (26 *b*) with the choice

$$\hat{\mathscr{L}} = \iint dxdy\, c_\alpha(x)H^{\alpha\beta}(x, y)\bar{c}_\beta(y) + \int dx \frac{1}{2}\Gamma^{\alpha\beta,\gamma\delta}c_\alpha(x)c_\beta(x)\bar{c}_\gamma(x)\bar{c}_\delta(x) + \int dx \frac{1}{2} L^{\gamma_S\alpha_A\beta}\zeta_{\gamma_S}(x)\bar{c}_{\alpha_A}(x)\bar{c}_\beta(x)\,. \quad (30)$$

This proves the compensability of the breakings $\Sigma_\alpha(x)$, $\Sigma'_{\alpha_A}(x)$ and hence the renormalizability of the supplementary conditions (17) (21).

We have thus shown that Eqs. (17) (21) and consequently the equation $(\mathscr{S}^2\Gamma)(\psi, \eta) = 0$ can be solved to all orders of perturbation theory by a suitable choice of the term $c_\alpha K^{\alpha\beta}\bar{c}_\beta$ in Eq. (9) and of the bilinear couplings

$\gamma_i\varphi_j\bar{c}_{\alpha_A}$, $\gamma_\mu^{\alpha s}\mathscr{A}_\beta^\mu\bar{c}_{\gamma_A}$ of the abelian $\bar{c}$'s, the remaining η dependent part of $\mathscr{L}_{\text{eff}}$ being arbitrary.

Hence we restrict from now on the effective Lagrangians under consideration by imposing this choice for the abelian $\bar{c}$'s couplings. Such a requirement gives the necessary complement to the B. R. S. algebraic treatment of the highest dimensionality breakings to the S. I. by fixing the asymptotic U. V. behaviour of the vertices with a $\bar{c}_{\alpha_A}$ external leg. For instance:

$$\lim_{p\to\infty} \left.\frac{\delta^3\Gamma(\psi,\eta)}{\delta\bar{\gamma}_\mu^{\alpha s}(p)\delta\tilde{\mathscr{A}}_\beta^\mu(-p)\delta\bar{c}_{\gamma_A}(0)}\right|_{\psi=\eta=0} = 0 \tag{31 a}$$

$$\lim_{p\to\infty} \left.\frac{\delta^3\Gamma(\psi,\eta)}{\delta\bar{\zeta}_{\alpha_S}(p)\delta\bar{\bar{c}}_\beta(-p)\delta\bar{c}_{\gamma_A}(0)}\right|_{\psi=\eta=0} = 0 \tag{31 b}$$

$$\lim_{p\to\infty} \left.\frac{\delta^3\Gamma(\psi,\eta)}{\delta\bar{\gamma}_i(p)\delta\varphi_j(-p)\delta\bar{c}_{\alpha_A}(0)}\right|_{\psi=\eta=0} = 0 \tag{31 c}$$

We begin now the main subject of our study i. e. the discussion of the renormalization of the Slavnov identity, analyzing, as a first step, the breakings by means of the consistency conditions which follow from Eq. (21).

In this exposition we shall repeat without the details of the calculations the essential steps of the B. R. S. [4] method, treating at same length those aspects for which the algebraic conditions do not provide enough information, due to the non semi-simple character of the gauge group.

The breakings of the S. I. are defined in terms of the vertex functional $\Gamma(\psi, \eta)$ by

$$(\mathscr{S}\Gamma)(\psi,\eta) = (\Delta\Gamma)_5(\psi,\eta) \tag{32}$$

According to the Q. A. P. the symbol in the r. h. s. of Eq. (32) means the insertion of the vertex $\int dx\, \mathrm{N}_5[\Delta(x)]$ carrying the $\Phi.\Pi.$ charge -1. Defining the functional operator

$$\mathscr{S}^{(\mathrm{P})} \equiv \int dx \Big[\mathrm{P}_i(x)\frac{\delta}{\delta\varphi_i(x)} + \mathrm{P}_\mu^{\alpha s}(x)\frac{\delta}{\delta\mathscr{A}_\mu^{\alpha s}(x)} + \partial_\mu\bar{c}_{\alpha_A}(x)\frac{\delta}{\mathscr{A}_{\mu\alpha_A}(x)} + \bar{\mathrm{P}}^{\alpha s}(x)\frac{\delta}{\delta\bar{c}^{\alpha s}(x)} + \big(\partial^\mu\mathscr{A}_\mu^\alpha(x) + \rho_i^\alpha\varphi_i(x)\big)\frac{\delta}{\delta c^\alpha(x)} \tag{33}$$

the vertex Δ is related to the effective Lagrangian by

$$\Delta(x) = \mathscr{S}^{(\mathrm{P})}\mathscr{L}_{\text{eff}}(\psi,\eta) + \hbar \mathrm{Q}(x, \mathscr{L}_{\text{eff}}, \hbar) \tag{34}$$

where $\mathrm{Q}(x, \mathscr{L}_{\text{eff}}, \hbar)$ sums up the radiative corrections; $\Delta(x)$ can also be written as a formal power series in $\hbar$ as

$$\Delta(x) = \sum_{m\geq n}^{\infty} \hbar^m \Delta^{(m)}(x) \tag{35}$$

Picking out of Eq. (34) the n-th order terms and selecting the contributions from $\mathcal{L}^{(n)}{}_{\text{eff}}(\psi, \eta)$, the n-th order coefficients $\Delta^{(n)}(x)$ can be computed according to

$$\Delta^{(n)}(x) = [\mathcal{I}^{P^{(n)}}\mathcal{L}^{(0)}_{\text{eff}}(\psi, \eta) + \mathcal{I}^{P^{(0)}}\mathcal{L}^{(n)}_{\text{eff}}(\psi, \eta)](x) + R^{(n)}(\mathcal{L}_{\text{eff}}, x) \quad (36)$$

where $\mathcal{I}^{P^{(n)}}$ and $\mathcal{I}^{P^{(0)}}$ are the n-th and zeroth order terms in the $\hbar$ power series of $\mathcal{I}^{(P)}$ given in Eq. (33) and $R^{(n)}(\mathcal{L}_{\text{eff}}, x)$ depends on the coefficients of $\mathcal{L}_{\text{eff}}$ up to the order $n - 1$. The gauge parameters ρ_i^α are absent since they do not contribute for $n > 0$.

The renormalizability of the S. I. is proved by showing that the equation

$$\Delta^{(n)}(x) = 0 \quad (37)$$

is solvable for any $n \geqslant 1$. In the course of the proof we shall first eliminate from $\Delta^{(n)}(x)$ the terms dependent on the external fields $\eta_a(x)$, and then discuss the remaining, $\eta_a(x)$ independent, part.

a) External field dependent breakings.

Writing explicitely the dependence of $\Delta^{(n)}(x)$ on the $\eta_a(x)$ fields in the form:

$$\Delta^{(n)}(x) = \Delta_0^{(n)}(x) + \gamma_i(x)\Delta_i^{(n)}(x) + \gamma_\mu^{\alpha s}(x)\Delta_{\alpha s}^{(n)\mu}(x) + \zeta^{\alpha s}(x)\Delta_{\alpha s}^{(n)}(x) = \Delta_0^{(n)}(x) + \eta_a(x)\Delta_a^{(n)}(x) \quad (38)$$

we shall discuss the solvability, in terms of the coefficients in $P_a^{(n)}$, of the system

$$\Delta_a^{(n)}(x) = 0 \quad (39)$$

Recall that, in order not to violate the asymptotic behaviours (Eqs. (31 *a*, *b*, *c*)), only the coefficients of the terms involving the semi-simple $\bar{c}$ fields, of $\bar{c}_{\alpha_A}$ and $\partial_\mu\bar{c}_{\alpha_A}$ in $P_a^{(n)}$ are at our disposal to solve Eq. (39), the other $\bar{c}$ couplings being already determined so as to satisfy Eqs. (17) (21). At this point the coefficients of $\mathcal{L}^{(n)}_{\text{eff}}(\psi)$ are still undetermined and will be used to solve the equation $\Delta_0^{(n)}(x) = 0$.

The consistency conditions obtained by B. R. S. [*4*] from Eq. (21) read

$$\mathcal{I}^{\Delta^{(n)}}\bar{P}_{\alpha s}^{(0)}(x) + \mathcal{I}^{P^{(0)}}\Delta_{\alpha s}^{(n)}(x) = 0 \quad (40\,a)$$

$$\mathcal{I}^{\Delta^{(n)}}P_{\mu\alpha s}^{(0)}(x) - \mathcal{I}^{P^{(0)}}\Delta_{\mu\alpha s}^{(n)}(x) = 0 \quad (40\,b)$$

$$\mathcal{I}^{\Delta^{(n)}}P_i^{(0)}(x) - \mathcal{I}^{P^{(0)}}\Delta_i^{(n)}(x) = 0 \quad (40\,c)$$

in terms of the operator

$$\mathcal{I}^{\Delta^{(n)}} \equiv \int dx \left[\Delta_i^{(n)}(x)\frac{\delta}{\delta\varphi_i(x)} + \Delta_\mu^{(n)\alpha s}(x)\frac{\delta}{\delta\mathcal{A}_\mu^{\alpha s}(x)} - \Delta_{\alpha s}^{(n)}(x)\frac{\delta}{\delta\bar{c}_{\alpha s}(x)} \right] \quad (41)$$

The relevant properties of the breakings, ensuing from these consistency conditions, can be summarized in the following relations, whose rather involved deduction is performed at the end of this subsection:

$$\Delta^{(n)}_{\alpha s}(x) = -\left(\mathscr{I}^{\Pi^{(n)}}\bar{\mathrm{P}}^{(0)}_{\alpha s}(x) + \mathscr{I}^{\mathrm{P}^{(0)}}\bar{\Pi}^{(n)}_{\alpha s}(x)\right) \tag{42 a}$$

$$\Delta^{(n)}_{\mu\alpha s}(x) = \mathscr{I}^{\Pi^{(n)}}\mathrm{P}^{(0)}_{\mu\alpha s}(x) + \mathscr{I}^{\mathrm{P}^{(0)}}\Pi^{(n)}_{\mu\alpha s} \tag{42 b}$$

$$\Delta^{(n)}_{i}(x) = \mathscr{I}^{\Pi^{(n)}}\mathrm{P}^{(0)}_{i}(x) + \mathscr{I}^{\mathrm{P}^{(0)}}\Pi^{(n)}_{i}(x) \tag{42 c}$$

where

$$\mathscr{I}^{\Pi^{(n)}} = \int dx \left[\Pi^{(n)}_i(x)\frac{\delta}{\delta\varphi_i(x)} + \Pi^{(n)\alpha s}_{\mu}(x)\frac{\delta}{\delta\mathscr{A}^{\alpha s}_{\mu}(x)} + \bar{\Pi}^{(n)}_{\alpha s}\frac{\delta}{\delta\bar{c}_{\alpha s}(x)}\right]$$

and the field polynomials $\Pi^{(n)}_a(x)$ satisfy

$$\frac{\delta^2\Pi^{(n)}_j(x)}{\delta\tilde{\bar{c}}_{\alpha_A}(p)\delta\tilde{\varphi}_i(-p)} = \frac{\delta^2\Pi^{(n)\alpha s}_{\mu}(x)}{\delta\tilde{\bar{c}}_{\alpha_A}(p)\delta\tilde{\mathscr{A}}_{\nu\beta}(-p)} = 0. \tag{43}$$

From this results we can easily show that the $\eta_a(x)$ dependent breakings are compensable within the allowed class of effective Lagrangians. In fact a comparison of Eq. (42) with Eq. (36) shows that the corrections to be introduced in the n-th order $\eta_a(x)$ field couplings in order to solve Eq. (37) are just given by the $\Pi^{(n)}_a$ polynomials in Eq. (42). Furthermore the condition (43) guarantees that this compensation does not modify the n-th order couplings $\Gamma_{\gamma_i\varphi_j\bar{c}_{\alpha_A}}$, $\Gamma_{\gamma^{\alpha s}_{\mu}\mathscr{A}^{\mu}_{\beta}\bar{c}_{\gamma_A}}$, thus preserving the prescribed power counting behaviours. More explicitely from Eqs. (36) (42) we see that the contribution of the radiative corrections to $\Delta^{(n)}_a(x)$ has the form

$$\mathrm{R}^{(n)}_a(x) = \mathscr{I}^{\mathscr{P}^{(n)}}\mathrm{P}^{(0)}_a(x) + \mathscr{I}^{\mathrm{P}^{(0)}}\mathscr{P}^{(n)}_a(x) \tag{44}$$

where the field polynomials $\mathscr{P}^{(n)}_a(x) = \Pi^{(n)}_a(x) - \mathrm{P}^{(n)}_a(x)$ depend on the coefficients of $\mathscr{L}_{\text{eff}}$ up to order $n-1$. From Eq. (43) we get that the n-th order coefficients of the previously mentioned couplings, which were already determined to preserve the prescribed power counting behaviours, compensate the corresponding contributions coming from radiative corrections. Then, the obvious solution $\mathrm{P}^{(n)}_a = -\mathscr{P}^{(n)}_a$ (equivalent to $\Pi^{(n)}_a = 0$) of the system $\Delta^{(n)}_a = 0$ does not break the super-renormalizability of the abelian $\bar{c}$ couplings.

Thus, to conclude our discussion of the external field dependent breakings, it remains to show that Eq. (42) can be satisfied together with Eq. (43).

In order to arrive at Eqs. (42 a, b, c) we write explicitely the field dependence in $\Delta^{(n)}_a(x)$:

$$\Delta^{(n)}_{\alpha s}(x) = -\frac{1}{6}\Gamma^{\beta\gamma\delta}_{\alpha s}(\bar{c}_\beta\bar{c}_\gamma\bar{c}_\delta)(x) \tag{45 a}$$

$$\Delta^{(n)}_{\mu\alpha s}(x) = \frac{1}{2}\Theta^{\beta\gamma}_{\alpha s\delta}(\mathscr{A}^{\delta}_{\mu}\bar{c}_\beta\bar{c}_\gamma)(x) + \frac{1}{2}\Sigma^{\beta,\gamma}_{\alpha s}(\bar{c}_\beta\partial_\mu\bar{c}_\gamma)(x) \tag{45 b}$$

$$\Delta^{(n)}_{i}(x) = \frac{1}{2}\Theta^{\alpha\beta}_{ij}(\varphi_j\bar{c}_\alpha\bar{c}_\beta)(x) + \frac{1}{2}\Sigma^{\alpha\beta}_{i}(\bar{c}_\alpha\bar{c}_\beta)(x) \tag{45 c}$$

Omitting the calculations, which can be performed by the co-homological methods of B. R. S. (Appendices A, B in ref. [9]), we give the general solutions of Eqs. (40 *a*, *b*, *c*) in terms of the coefficients appearing in Eqs. (45 *a*, *b*, *c*):

$$-\Gamma^{\alpha\beta\gamma}_{\delta\mathbf{s}} = f^{\alpha\lambda}_{\delta\mathbf{s}}\hat{\Gamma}^{\beta\gamma}_{\lambda} + f^{\beta\lambda}_{\delta\mathbf{s}}\hat{\Gamma}^{\gamma\alpha}_{\lambda} + f^{\gamma\lambda}_{\delta\mathbf{s}}\hat{\Gamma}^{\alpha\beta}_{\lambda} - f^{\alpha\beta}_{\lambda}\hat{\Gamma}^{\lambda\gamma}_{\delta\mathbf{s}} - f^{\beta\gamma}_{\lambda}\hat{\Gamma}^{\lambda\alpha}_{\delta\mathbf{s}} - f^{\gamma\alpha}_{\lambda}\hat{\Gamma}^{\lambda\beta}_{\delta\mathbf{s}} \quad (46\,a)$$

$$\Theta^{\beta\gamma}_{\alpha\mathbf{s}\delta} - \hat{\Gamma}^{\beta\gamma}_{\lambda\mathbf{s}}f^{\lambda\mathbf{s}}_{\alpha\mathbf{s}\delta} - \delta^{\beta}_{\beta'_{\mathrm{A}}}\delta^{\gamma}_{\gamma'_{\mathrm{A}}}\delta_{\delta\lambda\mathbf{s}}\Theta^{\beta'_{\mathrm{A}}\gamma'_{\mathrm{A}}}_{\natural\alpha\mathbf{s}\lambda\mathbf{s}}$$
$$= -[f^{\beta}, \hat{\Theta}^{\gamma}]_{\alpha\mathbf{s}\delta} + [f^{\gamma}, \hat{\Theta}^{\beta}]_{\alpha\mathbf{s}\delta} + f^{\beta\gamma}_{\lambda\mathbf{s}}\Theta^{\lambda\mathbf{s}}_{\alpha\mathbf{s}\delta} \quad (46\,b)$$

$$\Sigma^{\beta,\gamma}_{\alpha\mathbf{s}} + 2(\hat{\Theta}^{\beta\gamma}_{\alpha\mathbf{s}} - \hat{\Gamma}^{\beta\gamma}_{\alpha\mathbf{s}}) - \delta^{\beta}_{\beta'_{\mathrm{A}}}\delta^{\gamma}_{\gamma'_{\mathrm{A}}}\Sigma^{\beta'_{\mathrm{A}},\gamma'_{\mathrm{A}}}_{\natural\alpha\mathbf{s}} = -2[f^{\beta}, \hat{\Sigma}]^{\gamma}_{\alpha\mathbf{s}} \quad (46\,c)$$

$$\Theta^{\alpha\beta}_{ij} - \hat{\Gamma}^{\alpha\beta}_{\gamma}t^{\gamma}_{ij} - \delta^{\alpha}_{\alpha'_{\mathrm{A}}}\delta^{\beta}_{\beta'_{\mathrm{A}}}\Theta^{\alpha'_{\mathrm{A}}\beta'_{\mathrm{A}}}_{\natural ij} = [t^{\alpha}, \hat{\Theta}^{\beta}]_{ij} - [t^{\beta}, \hat{\Theta}^{\alpha}]_{ij} - f^{\alpha\beta}_{\gamma}\hat{\Theta}^{\gamma}_{ij} \quad (46\,d)$$

$$\Sigma^{\alpha\beta}_{i} - \Theta^{\alpha\beta}_{ij}q_j = -t^{\alpha}_{ij}(\hat{\Sigma}^{\beta} - \hat{\Theta}^{\beta}q)_j + t^{\beta}_{ij}(\hat{\Sigma}^{\alpha} - \hat{\Theta}^{\alpha}q)_j + f^{\alpha\beta}_{\gamma}(\hat{\Sigma}^{\gamma} - \hat{\Theta}^{\gamma}q)_i \quad (46\,e)$$

The same solutions in the functional form are written:

$$\Delta^{(n)}_{\alpha\mathbf{s}}(x) = -\left(\mathscr{I}^{\Pi^{(n)}}\bar{\mathrm{P}}^{(0)}_{\alpha\mathbf{s}}(x) + \mathscr{I}^{\mathrm{P}^{(0)}}\bar{\Pi}^{(n)}(x)\right) \quad (47\,a)$$

$$\Delta^{(n)}_{\mu\alpha\mathbf{s}}(x) = \left(\mathscr{I}^{\Pi^{(n)}}\bar{p}^{(0)}_{\mu\alpha\mathbf{s}}(x) + \mathscr{I}^{\mathrm{P}^{(0)}}\bar{\pi}^{(n)}_{\mu\alpha\mathbf{s}}(x)\right)$$
$$+ \frac{1}{2}\Theta^{\beta_{\mathrm{A}}\gamma_{\mathrm{A}}}_{\natural\alpha\mathbf{s}\lambda\mathbf{s}}\mathscr{A}^{\lambda\mathbf{s}}_{\mu}(x)\bar{c}_{\beta_{\mathrm{A}}}(x)\bar{c}_{\gamma_{\mathrm{A}}}(x) + \frac{1}{2}\Sigma^{\beta_{\mathrm{A}},\gamma\mathbf{s}}_{\natural\alpha\mathbf{s}}\bar{c}_{\beta_{\mathrm{A}}}(x)\partial_{\mu}\bar{c}_{\gamma\mathbf{s}}(x) \quad (47\,b)$$

$$\Delta^{(n)}_{i}(x) = \mathscr{I}^{\Pi^{(n)}}\mathrm{P}^{(0)}_{i}(x) + \mathscr{I}^{\mathrm{P}^{(0)}}\Pi^{(n)}_{i}(x) + \Theta^{\alpha_{\mathrm{A}}\beta_{\mathrm{A}}}_{\natural ij}q_j\bar{c}_{\alpha_{\mathrm{A}}}(x)\bar{c}_{\beta_{\mathrm{A}}}(x) \quad (47\,c)$$

where

$$\bar{\Pi}^{(n)}_{\alpha\mathbf{s}}(x) = \frac{1}{2}\hat{\Gamma}^{\beta\gamma}_{\alpha\mathbf{s}}\bar{c}_{\beta}(x)\bar{c}_{\gamma}(x) \quad (48\,a)$$

$$\Pi^{(n)}_{\mu\alpha\mathbf{s}}(x) = \hat{\Theta}^{\beta}_{\alpha\mathbf{s}\delta}\mathscr{A}^{\delta}_{\mu}(x)\bar{c}_{\beta}(x) + \hat{\Sigma}^{\beta}_{\alpha\mathbf{s}}\partial_{\mu}\bar{c}_{\beta}(x) \quad (48\,b)$$

$$\Pi^{(n)}_{i}(x) = \hat{\Theta}^{\alpha}_{ij}\varphi_j(x)\bar{c}_{\alpha}(x) + \hat{\Sigma}^{\alpha}_{i}\bar{c}_{\alpha}(x) \quad (48\,c)$$

The coefficients $\Theta^{\alpha_{\mathrm{A}}\beta_{\mathrm{A}}}_{\natural ij}$, $\Theta^{\beta_{\mathrm{A}}\gamma_{\mathrm{A}}}_{\natural\alpha\mathbf{s}\lambda\mathbf{s}}$, $\Sigma^{\beta_{\mathrm{A}},\gamma\mathbf{s}}_{\natural\alpha\mathbf{s}}$ in the preceding formulae obey the equations

$$[f^{\alpha}, \Theta^{\beta_{\mathrm{A}}\gamma_{\mathrm{A}}}_{\natural}]_{\alpha\mathbf{s}\lambda\mathbf{s}} = [f^{\alpha}, \Sigma^{\beta_{\mathrm{A}}}_{\natural}]^{\gamma\mathbf{s}}_{\alpha\mathbf{s}} = [t^{\alpha}, \Theta^{\alpha_{\mathrm{A}}\beta_{\mathrm{A}}}_{\natural}]_{ij} = 0 \quad (49)$$

and the coefficients $\hat{\Gamma}^{\alpha\beta_{\mathrm{A}}}_{\gamma\mathbf{s}}$, $\hat{\Theta}^{\beta_{\mathrm{A}}}_{\alpha\mathbf{s}\gamma}$, $\hat{\Theta}^{\alpha_{\mathrm{A}}}_{ij}$ are given by (see B. R. S., Appendix B)

$$\hat{\Gamma}^{\alpha\beta_{\mathrm{A}}}_{\gamma\mathbf{s}} = c_1 f^{\lambda\mathbf{s}}_{\gamma\mathbf{s}\delta}\Gamma^{\delta\alpha\beta_{\mathrm{A}}}_{\lambda\mathbf{s}} + \hat{\hat{\Gamma}}^{\alpha\beta_{\mathrm{A}}}_{\gamma\mathbf{s}} \quad (50\,a)$$

$$\hat{\Theta}^{\beta_{\mathrm{A}}}_{\alpha\mathbf{s}\gamma\mathbf{s}} = c_2 f^{\sigma}_{\alpha\mathbf{s}\delta}\left(\Theta^{\delta\beta_{\mathrm{A}}}_{\sigma\gamma\mathbf{s}} + (\hat{\Gamma}f)^{\delta\beta_{\mathrm{A}}}_{\sigma\gamma\mathbf{s}}\right) + \hat{\hat{\Theta}}^{\beta_{\mathrm{A}}}_{\alpha\mathbf{s}\gamma\mathbf{s}} \quad (50\,b)$$

$$\hat{\Theta}^{\alpha_{\mathrm{A}}}_{ij} = c_3[t^{\beta}, \Theta^{\beta\alpha_{\mathrm{A}}} + (\hat{\Gamma}t)^{\beta\alpha_{\mathrm{A}}}]_{ij} + \hat{\hat{\Theta}}^{\alpha_{\mathrm{A}}}_{ij}. \quad (50\,c)$$

Analogous expressions hold for $\hat{\Sigma}^{\alpha}_{\beta\mathbf{s}}$ and $\hat{\Sigma}^{\alpha}_{i}$. The factors c_1, c_2 and c_3 are proportional to the values of the quadratic Casimir operator of the gauge group in the adjoint and matter field representations respectively and $\hat{\hat{\Gamma}}^{\alpha\beta}_{\gamma\mathbf{s}}$, $\hat{\hat{\Theta}}^{\beta}_{\alpha\mathbf{s}\gamma}$, $\hat{\hat{\Theta}}^{\alpha}_{ij}$ are solutions of the homogeneous equations corresponding to Eqs. (46). For instance, the general solution of the homogeneous version of Eq. (46 d), obtained by equating to zero its l. h. s., is, for the abelian index

$$\hat{\hat{\Theta}}_{ij}^{\alpha_A} = [t^{\alpha_A}, [t^\beta, A^\beta]]_{ij} \tag{51}$$

where A_{ij}^β is an arbitrary matrix.

The above rather tedious display of formulae explicitely shows the algebraic consequences of the consistency conditions, evidentiating the non compensable contributions to the breakings which have still to be discussed at the light of the prescribed power counting rules. It is indeed clear from Eqs. (47), that Eqs. (42) hold true provided the coefficients $\Theta_{\alpha_S\lambda_S}^{\beta_A\gamma_A}$, $\Theta_{ij}^{\alpha_A\beta_A}$, $\Sigma_{\alpha_S}^{\beta_A,\gamma_S}$ in Eqs. (46) (49) vanish. For this purpose it is sufficient that in Eqs. (45)

$$\Sigma_{\alpha_S}^{\beta_A,\gamma_S} = 0 \tag{52 a}$$

$$\Theta_{\alpha_S\lambda_S}^{\beta_A\gamma_A} = 0 \tag{52 b}$$

$$\Theta_{ij}^{\alpha_A\beta_A} = 0 . \tag{52 c}$$

On the other hand we see that Eqs. (43) can be satisfied if there exists a choice of the coefficients $\hat{\hat{\Gamma}}_{\gamma_S}^{\alpha\beta_A}$, $\hat{\hat{\Theta}}_{\alpha_S\gamma_S}^{\beta_A}$, $\hat{\hat{\Theta}}_{ij}^{\alpha_A}$ in Eqs. (50), for which $\Gamma_{\gamma_S}^{\alpha\beta_A}$, $\hat{\Theta}_{\alpha_S\gamma_S}^{\beta_A}$, $\hat{\Theta}_{ij}^{\alpha_A}$ vanish. With the obviously possible choice

$$\hat{\hat{\Gamma}}_{\gamma_S}^{\alpha\beta_A} = 0 = \hat{\hat{\Theta}}_{\alpha_S\gamma_S}^{\beta_A} \tag{53}$$

the vanishing of $\hat{\Gamma}_{\gamma_S}^{\alpha\beta_A}$ and $\hat{\Theta}_{\alpha_S\gamma_S}^{\beta_A}$ follows immediately from the relations

$$\Gamma_{\lambda_S}^{\alpha\gamma\beta_A} = 0 \tag{54 a}$$

$$\Theta_{\gamma\lambda_S}^{\alpha\beta_A} = 0 \tag{54 b}$$

which are proved, together with Eqs. (52), in Appendix A, making use of the super-renormalizability of the abelian $\bar{c}$ couplings. The possibility of setting to zero the remaining term $\hat{\Theta}_{ij}^{\alpha_A}$, with $\hat{\hat{\Theta}}_{ij}^{\alpha_A}$ as in Eq (51), in ensured if the coefficients $\Theta_{ij}^{\alpha\gamma_A}$ have the following structure

$$\Theta_{ij}^{\alpha\gamma_A} = [t^{\gamma_A}, B^\alpha]_{ij} \tag{55}$$

for some suitable B^α matrix. This is also proved in Appendix A.

So ends the analysis of the renormalizability of the external field couplings and we shall pretend, from now on, that the n-th order breakings do not depend on the external fields. With this proviso we now discuss the remaining $\eta_a(x)$ independent breakings.

b) External field independent breakings.

After B. R. S., the consistency condition for the $\eta_a(x)$ independent n-th order breaking $\Delta_0^{(n)}$ is

$$\mathscr{I}^{P(0)}\Delta_0^{(n)} = 0 \tag{56}$$

and the compensability condition for the same is

$$\Delta_0^{(n)} = \mathscr{I}^{P(0)}\hat{\Delta}_0^{(n)} \tag{57}$$

Our task, to which the rest of this paper is devoted, is to prove Eq. (57) in the hypothesis that the A. B. A. is absent.

The contributions to $\Delta_0^{(n)}$ are classified according to the Φ.Π. field content as follows:

$$\Delta_0^{(n)} = \int dx [W^\alpha(\varphi, \mathscr{A}_\mu)\bar{c}_\alpha + W^{\alpha,\beta\gamma}(\varphi, \mathscr{A}_\mu)c_\alpha\bar{c}_\beta\bar{c}_\gamma + W^{\alpha\beta,\gamma\delta\lambda}c_\alpha c_\beta\bar{c}_\gamma\bar{c}_\delta\bar{c}_\lambda](x) \quad (58)$$

where $W^\alpha(\varphi, \mathscr{A}_\mu)$, $W^{\alpha,\beta\gamma}(\varphi, \mathscr{A}_\mu)$ depend upon the fields φ_i, $\mathscr{A}_\mu^\beta$ and their derivatives and have maximum naive dimension four and two respectively.

Considering first, as usual, the equation $(\mathscr{I}^{P^{(0)}})^2\Delta_0^{(n)} = 0$ we get

$$W^{\alpha\beta,\gamma\delta\lambda} = 0 \quad (59\,a)$$

$$(W^{\alpha,\beta\gamma}(\varphi, \mathscr{A}_\mu)\bar{c}_\beta)(x) = M^{\alpha\beta,\gamma}(\mathscr{M}\bar{c})_\beta(x) \quad (59\,b)$$

with $M^{\alpha\beta,\gamma} = M^{\beta\alpha,\gamma}$ and, as in paper I, $(\mathscr{M}\bar{c})_\beta = (\partial_\mu P_\beta^{(0)\mu} + \rho_{\beta i}P_i^{(0)})$. Some more information about the coefficient $M^{\alpha\beta,\gamma}$ comes from power counting and the super-renormalizability of the abelian $\bar{c}$ couplings. Indeed, from the definition of Δ we obtain

$$\lim_{p\to\infty} \frac{1}{p^2} \frac{\delta^3(\Delta\Gamma)(\psi)}{\delta\bar{c}_\alpha(-p)\delta\bar{\bar{c}}_\beta(p)\delta\bar{c}_{\gamma_A}(0)}\bigg|_{\psi=0} = -M^{\alpha\beta,\gamma_A} + O(\hbar^{n+1}) \quad (60)$$

and, with a power counting analysis similar to the one carried out in Appendix A for the external fields dependent breakings, we get

$$\lim_{p\to\infty} \frac{1}{p^2} \frac{\delta(\mathscr{S}\Gamma)(\psi, \eta)}{\delta\bar{c}_\alpha(p)\delta\bar{\bar{c}}_\beta(-p)\delta\bar{c}_{\gamma_A}(0)}\bigg|_{\psi=\eta=0} = 0 \quad (61)$$

from which the vanishing of $M^{\alpha\beta,\gamma_A}$ follows immediately. Now the consistency condition Eq. (56) writes

$$\iint dxdy\left[\frac{\delta W^\alpha(x)}{\delta\varphi_i(y)}P_i(y) + \frac{\delta W^\alpha(x)}{\delta\mathscr{A}_\mu^{\beta_S}(y)}P_\mu^{\beta_S}(y) + \frac{\delta W^\alpha(x)}{\delta\mathscr{A}_\mu^{\beta_A}(y)}\partial_\mu\bar{c}^{\beta_A}(y)\right]\bar{c}_\alpha(x) + \int dx\, W^{\alpha_S}(x)\bar{P}_{\alpha_S}(x)$$
$$+ \int dx\left[-(\partial_\mu\mathscr{A}_\alpha^\mu(x) + \rho_{\alpha i}\varphi_i(x))\bar{c}_{\gamma_S}(x) + c_\alpha(x)\bar{P}_{\gamma_S}(x)\right]M^{\alpha\beta,\gamma_S}(\mathscr{M}\bar{c}_\beta)(x) = 0 \quad (62)$$

from which, separating the c_α dependent terms, we find $M^{\alpha\beta,\gamma_S} = 0$. Hence the supplementary conditions (17) and (21) reduce the admissible breaking to the simple form:

$$\Delta_0^{(n)} = \int dx\, W^\alpha(x)\bar{c}_\alpha(x) \quad (63)$$

which has still to satisfy the consistency condition (56).

Let us now recall that P_i, $P_\mu^{\beta_S}$, $\partial_\mu\bar{c}^{\beta_A}$ are nothing but the gauge variations of the fields φ_i, $\mathscr{A}_\mu^{\beta_S}$, $\mathscr{A}_\mu^{\beta_A}$ respectively with the infinitesimal gauge parameters $\delta\omega_\alpha$ substituted by $\bar{c}_\alpha$. Writing the gauge variation of the functional $F(\mathscr{A}_\mu, \varphi)$ as $\delta F(\mathscr{A}_\mu, \varphi) = \int dx \frac{\delta F(\mathscr{A}_\mu, \varphi)}{\delta\omega_\beta(x)} \omega_\beta(x)$, we explicitely have

$$P_i^{(0)}(x) = \int \frac{\delta\varphi_i(x)}{\delta\omega_\beta(y)} \bar{c}_\beta(y) dy \tag{64 a}$$

$$P_\mu^{(0)\beta}(x) = \int \frac{\delta\mathscr{A}_\mu^{\beta s}(x)}{\delta\omega_\gamma(y)} \bar{c}_\gamma(y) dy \tag{64 b}$$

$$\partial_\mu \bar{c}^{\beta A}(x) = \int \frac{\delta\mathscr{A}_\mu^{\beta A}(x)}{\delta\omega_\gamma(y)} \bar{c}_\gamma(y) dy . \tag{64 c}$$

Recalling also that

$$\bar{P}_{\alpha s}^{(0)}(x) = \frac{1}{2} f_{\alpha s}^{\beta\gamma} \bar{c}_\beta(x) \bar{c}_\gamma(x) \tag{65}$$

We can write the consistency condition Eq. (56) for the breaking in Eq. (63) as

$$\iint dx dy \frac{\delta W^\alpha(x)}{\delta\omega_\beta(y)} \bar{c}_\beta(y) \bar{c}_\alpha(x) + \int dx\, W^{\alpha s}(x) \frac{1}{2} f_{\alpha s}^{\beta\gamma} \bar{c}_\beta(x) \bar{c}_\gamma(x) = 0 \tag{66}$$

which, upon differentiation with respect to $\bar{c}_\beta(y)$, $\bar{c}_\alpha(x)$ gives for $W^\alpha(x)$ the Wess-Zumino [*16*] consistency condition:

$$\frac{\delta W^\alpha(x)}{\delta\omega_\beta(y)} - \frac{\delta W^\beta(y)}{\delta\omega_\alpha(x)} - f_\gamma^{\alpha\beta} W^\gamma(x) \delta(x - y) = 0 . \tag{67}$$

The discussion of Eq. (67) parallels the one in B. R. S. (Refs. [*4*] [*9*] [*17*]). The starting point is that the general solution of Eq. (67) has the form

$$\Delta^\alpha(x) = \frac{\delta}{\delta\omega_\alpha(x)} \hat{\Delta} + \partial^\mu g_\mu^\alpha(x) + \delta_{\beta_A}^\alpha \Delta_\natural^{\beta A}(x) \tag{68}$$

where the dimension of the functional $\hat{\Delta}$ can be made equal to that of $\Delta^\alpha(x)$, $\partial^\mu g_\mu^\alpha(x)$ is the Adler-Bardeen [*12*] anomaly (A. B. A.), $\Delta_\natural^{\beta A}(x)$ is invariant under global gauge transformations (i. e. those with constant parameters) and is determined up to four divergencies (i. e. terms of the type $\partial_\mu I^\mu(x)$).

Assuming from now on that the A. B. A. is absent to all orders of perturbation theory, we also see that if $\Delta_\natural^{\beta A}(x)$ is likewise absent the breaking $\Delta_0^{(n)}$ is compensable. Indeed, in such a case, from Eq. (68) it follows that

$$\Delta_0^{(n)} = \mathscr{I}^{P(0)} \hat{\Delta} \tag{69}$$

In other words, from Eq. (68) we can conclude that, if the A. B. A. is absent, by a suitable choice of $\mathscr{L}_{\text{eff}}^{(n)}(\psi)$ we can reduce the breaking to the minimal form

$$\Delta_{0\natural}^{(n)} = \int dx\, \Delta_\natural^{\alpha A}(x) \bar{c}_{\alpha_A}(x) \tag{70}$$

with

$$dx\, \Delta_{\natural}^{\alpha \mathrm{A}}(x) \neq 0 \tag{71}$$

and

$$\int dy \frac{\delta \Delta^{\alpha \mathrm{A}}(x)}{\delta \omega_{\beta}(y)} = 0 . \tag{72}$$

In Appendix B we prove that the anomaly $\Delta_{0\natural}^{\alpha \mathrm{A}}$ has no contributions of maximum dimension.

In the following Section we shall complete our investigation of the breakings to the S. I. by showing that $\Delta_{0\natural}^{(n)}(x)$ vanishes if it does not contain terms of maximum dimension.

4. RENORMALIZATION OF THE SLAVNOV IDENTITY - SOFT ANOMALIES

The analysis of the previous Section and of Appendix B points uniquely to the Adler-Bardeen anomaly as the only non compensable breaking of dimension five to the Slavnov identity; here we shall complete the investigation by treating the soft contributions in the hypothesis that the Adler-Bardeen anomaly is absent.

The proof is an extension to the gauge models of the one given by B. R. S. for Lagrangian theories [*14*] with broken symmetries, where the presence of soft anomalies was excluded by means of the Callan-Symanzik [*13*] equation. This extension involves mainly the technique of the proof, while the guiding lines, which we are now going to illustrate, remain unchanged.

We want to show, at the lowest order in $\hbar$ at which they appear, that the breakings to the S. I. must have contributions of maximum dimensionality.

The basic ideas of the proof are the following: let λ be the mass scaling parameter of the theory and b_i's the coefficients of the independent monomials Δ_i's which appear in the breaking Δ. These coefficients are homogeneous functions of λ with degrees $x_i = 5 - \dim(\Delta_i)$ and hence $\lambda \partial_\lambda b_i = x_i b_i$.

According to the Callan-Symanzik equation, the mass scaling transformation applied to the Green functions is equivalent, up to radiative corrections, to the insertion of a soft vertex. From this property there follows a functional differential equation for the breaking Δ. Indeed, defining the mentioned soft insertion by coupling it to an external field Ω we have

$$\Sigma_i \lambda \partial_\lambda b_i \Delta_i = \Sigma_i x_i b_i \Delta_i = \Sigma_i b_i \int dx \frac{\delta}{\delta \Omega(x)} \Delta_i + \mathrm{O}(\hbar \Delta) . \tag{73}$$

Now, since the field Ω has strictly positive dimension, the above equation relates, at the lowest order in $\hbar$, the coefficients b_i of the soft breakings (i. e. the ones with $x_i \neq 0$) with those of terms of higher dimensionality.

Thus, if the terms of dimensionality five are absent in the breaking Δ, all the b_i's are recursively zero.

As already pointed out there are some technical points to be taken care of; in fact the possibility of applying this method depends strictly on the condition that the Ω field does not alter the properties of the breaking by reintroducing contributions of maximum dimension. In particular, the mass scaling operator we are looking for must commute with the functional differential operator $\mathcal{S}$ appearing in the S. I. for the connected Green functional which, taking into account the breaking Δ, writes

$$\mathcal{S}Z_c(J, \eta) \equiv \int dx \left[J_i(x)\frac{\delta}{\delta\gamma_i(x)} + J_\mu^{\alpha s}(x)\frac{\delta}{\delta\gamma_\mu^{\alpha s}(x)} + J_\mu^{\alpha A}(x)\partial^\mu \frac{\delta}{\delta\xi^{\alpha A}(x)} - \xi_{\alpha s}(x)\frac{\delta}{\delta\zeta_{\alpha s}(x)} - \bar{\xi}_\alpha(x)\left(\partial^\mu \frac{\delta}{\delta J_\alpha^\mu(x)} + \rho_i^\alpha \frac{\delta}{\delta J_i(x)}\right)\right] Z_c(J, \eta) = -(\Delta Z_c)(J, \eta) \quad (74)$$

(the Legendre transformation of Eq. (32)). This requires a correction of the naive $\lambda\partial_\lambda$ operator which does not commute with the ρ_i^α gauge parameters in $\mathcal{S}$.

Our program is then the following; first find the appropriate (i. e. symmetric) mass scaling operator and second check that the Ω field does not reintroduce in the theory new anomalies, which either are hard or violate conditions (71), (72). Finally these results are put together to yield the Callan-Symanzik equation which allows a complete treatment of the soft breakings.

The mass scaling operator $\lambda\partial_\lambda$ obeys the commutation rule

$$[\lambda\partial_\lambda, \mathcal{S}] = -\int dx\, \bar{\xi}_\alpha(x)\rho_i^\alpha \frac{\delta}{\delta J_i(x)}, \quad (75)$$

the first step will thus consist in adding to $\lambda\partial_\lambda$ those terms liable to compensate the r. h. s. of Eq. (75). With this in mind, consider the bilocal functional differential operator

$$Q_\rho^{(\varepsilon)} = \hbar^2\Lambda^{\alpha\beta}\int dx \frac{\delta}{\delta\bar{\xi}^\alpha(x+\varepsilon)}\rho_{\beta i}\frac{\delta}{\delta J_i(x)} \quad (76)$$

where $\Lambda^{\alpha\beta}$ is the gauge matrix appearing in the solution of the $\bar{c}$ fields equation of motion (Eq. (12))

$$\Lambda^{\alpha\beta}\left(\delta_\beta^{\lambda s}\partial_\mu \frac{\delta}{\delta\gamma_\mu^{\lambda s}(x)} + \delta_\beta^{\lambda A}\,\square\,\frac{\delta}{\delta\xi^{\lambda A}(x)}\right)Z(J, \eta) + \mu^{\alpha\beta}\frac{\delta}{\delta\xi^\beta(x)}Z(J, \eta) = \frac{i}{\hbar}\bar{\xi}^\alpha(x)Z(J, \eta)$$
$$\equiv M^\alpha(x)Z(J, \eta) + \mu^{\alpha\beta}\frac{\delta}{\delta\xi^\beta(x)}Z(J, \eta) \quad (77)$$

here written in terms of the disconnected Green functional $Z(J, \eta)$.

The operator $Q_\rho^{(\varepsilon)}$ has the following properties:

$$\{ Q_\rho^{(\varepsilon)}, \mathscr{S} \} = \hbar^2 \Lambda^{\alpha\beta} \int dx \left[\frac{\delta}{\delta \bar{\xi}^\alpha(x+\varepsilon)} \rho_{\beta i} \frac{\delta}{\delta \gamma_i(x)} + \rho_i^\beta \frac{\delta}{\delta J_i(x)} \left(\partial_\mu \frac{\delta}{\delta J_{\mu\alpha}(x+\varepsilon)} + \rho_j^\alpha \frac{\delta}{\delta J_j(x+\varepsilon)} \right) \right] \quad (78)$$

$$[Q_\rho^{(\varepsilon)}, \mathscr{S}^2] = [\{ Q_\rho^{(\varepsilon)}, \mathscr{S} \}, \mathscr{S}] = -\hbar^2 \int dx\, M^\alpha(x+\varepsilon) \rho_{\alpha i} \frac{\delta}{\delta J_i(x)} . \quad (79)$$

From Eqs. (77), (79), recalling the ghost mass degeneracy condition $\mu^{\alpha\beta}\rho_{\beta i} = 0$, we can write

$$[Q_\rho^{(\varepsilon)}, \mathscr{S}^2] Z(J, \eta) = -i\hbar \int dx\, \bar{\xi}_\alpha(x+\varepsilon) \rho_i^\alpha \frac{\delta}{\delta J_i(x)} Z(J, \eta) \quad (80)$$

so we immediately deduce that, once the $\varepsilon \to 0$ limit is suitably defined, the operator $\lim_{\varepsilon \to 0} \frac{i}{\hbar} \{ Q_\rho^{(\varepsilon)}, \mathscr{S} \}$ is the appropriate term which should be added to $\lambda \partial_\lambda$ (see Eq. (75)) in order to symmetrize it.

We are going to translate the symmetric mass scaling operator so individuated, into a form useful for the subsequent developments. This is done by introducing, almost for the last time, an external field $r(x)$ of dimensionality one and $\Phi.\Pi.$ neutral and, to first order in $r(x)$ the new disconnected functional (see Eq. (78)):

$$Z^{(\varepsilon)}(J, \eta, r) = \left(1 + i\hbar \int dx\, r(x) \left[\frac{\delta}{\delta \bar{\xi}^\alpha(x+\varepsilon)} \rho_{\beta i} \frac{\delta}{\delta \gamma_i(x)} + \rho_{\beta i} \frac{\delta}{\delta J_i(x)} \left(\partial_\mu \frac{\delta}{\delta J_\mu^\alpha(x+\varepsilon)} + \rho_{\alpha j} \frac{\delta}{\delta J_j(x+\varepsilon)} \right) \Lambda^{\alpha\beta} \right] \right) Z(J, \eta) \quad (81)$$

with corresponding connected generator:

$$\begin{aligned} Z_c^\varepsilon(J, \eta, r) = {} & Z_c(J, \eta) + i\hbar \int dx\, r(x) \Lambda^{\alpha\beta} \left[\frac{\delta}{\delta \bar{\xi}^\alpha(x+\varepsilon)} \rho_{\beta i} \frac{\delta}{\delta \gamma_i(x)} \right. \\ & \left. - \left(\partial_\mu \frac{\delta}{\delta J_\mu^\alpha(x+\varepsilon)} + \rho_{\alpha j} \frac{\delta}{\delta J_j(x+\varepsilon)} \right) \rho_{\beta i} \frac{\delta}{\delta J_i(x)} \right] Z_c(J, \eta) \\ & - \int dx\, r(x) \Lambda^{\alpha\beta} \left[\frac{\delta}{\delta \bar{\xi}^\alpha(x+\varepsilon)} Z_c(J, \eta) \rho_{\beta i} \frac{\delta}{\delta \gamma_i(x)} Z_c(J, \eta) \right. \\ & \left. - \left(\partial_\mu \frac{\delta}{\delta J_\mu^\alpha(x+\varepsilon)} + \rho_{\alpha j} \frac{\delta}{\delta J_j(x+\varepsilon)} \right) Z_c(J, \eta) \rho_{\beta i} \frac{\delta}{\delta J_i(x)} Z_c(J, \eta) \right] . \end{aligned} \quad (82)$$

The Slavnov identity for $Z_c^{(\varepsilon)}(J, \eta, r)$ can be obtained by direct computation to first order in $r(x)$ and up to radiative corrections from Eqs. (74), (82) as:

$$\begin{aligned}\mathscr{S}Z_c^{\varepsilon}(J, \eta, r) = &- \Delta Z_c(J, \eta) + i\hbar \int dx\, r(x) M^{\beta}(x + \varepsilon)\rho_{\beta i} \frac{\delta}{\delta J_i(x)} Z_c(J, \eta) \\ &+ \int dx\, r(x) \left\{ \Lambda^{\alpha\beta} \left[- \frac{\delta}{\delta \bar{\xi}^{\alpha}(x + \varepsilon)} \Delta Z_c(J, \eta)\rho_{\beta i} \frac{\delta}{\delta \gamma_i(x)} Z_c(J, \eta) \right.\right. \\ &+ \frac{\delta}{\delta \bar{\xi}^{\alpha}(x + \varepsilon)} Z_c(J, \eta)\rho_{\beta i} \frac{\delta}{\delta \gamma_i(x)} \Delta Z_c(J, \eta) \\ &+ \left(\partial_{\mu} \frac{\delta}{\delta J_{\mu}^{\alpha}(x + \varepsilon)} + \rho_{\alpha j} \frac{\delta}{\delta J_j(x + \varepsilon)} \right) \Delta Z_c(J, \eta)\rho_{\beta i} \frac{\delta}{\delta J_i(x)} Z_c(J, \eta) \\ &- \left(\partial_{\mu} \frac{\delta}{\delta J_{\mu}^{\alpha}(x + \varepsilon)} + \rho_{\alpha j} \frac{\delta}{\delta J_j(x + \varepsilon)} \right) Z_c(J, \eta)\rho_{\beta i} \frac{\delta}{\delta J_i(x)} \Delta Z_c(J, \eta) \\ &\left.\left. - M_{\alpha}(x + \varepsilon) Z_c(J, \eta)\rho_{\beta i} \frac{\delta}{\delta J_i(x)} Z_c(J, \eta) \right] \right\} + O(\hbar\Delta, r^2). \qquad (83)\end{aligned}$$

The contributions on the r. h. s. of this equation are all regular in the $\varepsilon \to 0$ limit; in particular the second term vanishes by Eq. (77) while the last one can be rewritten, also by Eq. (77), as

$$\int dx\, r(x)\bar{\xi}_{\alpha}(x + \varepsilon)\rho_i^{\alpha} \frac{\delta}{\delta J_i(x)} Z_c(J, \eta) \qquad (84)$$

Defining the connected functional $Z_c(J, \eta, r)$ in the $\varepsilon \to 0$ limiting theory by separating out the singular contributions in the Wilson expansion [*18*] [*1*] of the bilocal operators in the second term on the r. h. s. of Eq. (82), all others being clearly convergent, we can immediately write the S. I. for $Z_c(J, \eta, r)$ as

$$\begin{aligned}\mathscr{S}^{(r)}Z_c(J, \eta, r) \equiv \left(\mathscr{S} - \int dx\, r(x)\bar{\xi}_{\alpha}(x)\rho_i^{\alpha} \frac{\delta}{\delta J_i(x)} \right) Z_c(J, \eta, r) \\ = - (\Delta Z_c)(J, \eta, r) + O(\hbar\Delta, r^2) \qquad (85)\end{aligned}$$

This equation shows that, since the field $r(x)$ is not coupled to a Slavnov invariant vertex, the operator $\mathscr{S}$ has to be replaced by $\mathscr{S}^{(r)}$, and, more important, that the breaking Δ, at its lowest non vanishing order, is not altered by the introduction of $r(x)$.

It follows from the above analysis that the mass scaling operator needed for the construction of the Callan-Symanzik equation of the model is:

$$D_{(\mu)} = \lambda\partial_{\lambda} - \int dx \frac{\delta}{\delta r(x)} \qquad (86)$$

whose commutator with the Slavnov operator $\mathcal{S}^{(r)}$ is straightforwardly computed to be

$$[D_{(\mu)}, \mathcal{S}^{(r)}] = -\int dx\, r(x)\rho_i^{\alpha}\bar{\xi}_{\alpha}(x)\frac{\delta}{\delta J_i(x)} \tag{87}$$

and vanishes at the point $r(x) = 0$.

The second step toward the Callan-Symanzik equation requires characterizing all the soft (less than four dimensional) invariant insertions of the theory and checking that these same insertions do not alter the relevant properties of the breakings. The set of these insertions is in correspondence with the soft Slavnov invariant terms in the action which are compatible with the mass degeneracy condition $\mu^{\alpha\beta}\rho_{\beta i} = 0$. Some elements within this class correspond to invariant vertices while others, typically

$$\mu^{\alpha\beta}\left(\frac{\mathcal{A}_{\nu\alpha}\mathcal{A}_{\beta}^{\nu}}{2} + c_{\alpha}\bar{c}_{\beta}\right)(x) \tag{88}$$

satisfy the invariance requirement only after x integration.

In order to define the soft insertions in the renormalized theory we introduce (for the last time) a set of external scalar fields $\{\Omega(x)\}$ which are coupled, at the tree approximation, with the above mentioned Slavnov invariant terms, thus yielding for the generator of the connected Green functions

$$\mathcal{S}\int dx \frac{\delta}{\delta\Omega(x)} Z_c^{cl}(K, \eta, \Omega) = 0 \tag{89}$$

Now, following the distinction among the insertions, we divide the set $\{\Omega(x)\}$ into subsets $\{\omega(x)\}$, $\{\beta(x)\}$ with the properties

$$\mathcal{S}\frac{\delta}{\delta\omega(x)} Z_c^{cl}(J, \eta, \Omega) = 0 \tag{90 a}$$

$$\mathcal{S}\frac{\delta}{\delta\beta(x)} Z_c^{cl}(J, \eta, \Omega) = \partial_{\mu}B^{\mu}(x). \tag{90 b}$$

An examination of the classical theory leads to the result that the fields $\{\beta(x)\}$ contribute to the classical action a term (³) :

$$\int dx\, \beta_i(x)\lambda_{\alpha\beta}^{i}\left[\frac{\mathcal{A}_{\mu}^{\alpha}\mathcal{A}^{\mu\beta}}{2} + c^{\alpha}\bar{c}^{\beta}\right](x) \tag{91}$$

(³) Obviously terms of the type $\int \beta_i(x)\partial_{\mu}X_i^{\mu}(x)dx$ do satisfy Eq. (90 b), but they correspond to vanishing insertions and hence we will exclude them at least in the tree approximation.

where, in order to preserve the mass degeneracy condition, the matrices $\lambda^i_{\alpha\beta}$ are constrained by

$$\lambda^i_{\alpha\beta} = \lambda^i_{\beta\alpha}, \qquad \lambda^i_{\alpha\beta_S} = 0, \qquad \lambda^i_{\alpha\beta}\rho^\beta_j = 0. \tag{92}$$

Of course, since the β_i field couplings are not Slavnov invariant, their introduction modifies the S. I., which at the tree approximation becomes:

$$\mathcal{S}Z^{cl}_c(J, \eta, \Omega) = \int dx\, \beta_i(x)\partial_\mu B^\mu_i(x) \tag{93}$$

with

$$B^i_\mu(x) = -\lambda^i_{\alpha\beta}\mathcal{A}^\alpha_\mu(x)\bar{c}^\beta(x)\Big|_{\psi(x)=\frac{\delta Z_c}{\delta J(x)}}. \tag{94}$$

In order to define, through the $\{\Omega(x)\}$ fields the soft insertions in the renormalized theory and to verify how the new fields contribute to the breakings, we should analyze this modified S. I. at the higher orders in $\hbar$. Usually this procedure requires a functional translation of Eq. (93) since a field product as in Eq. (94) has no meaning beyond the tree approximation However the present case is much simpler (and, as promised, we can avoid introducing a new external field) since the fields $\lambda^i_{\alpha\beta}\bar{c}^\beta$ are free except for the couplings with the β_i's. As a consequence the breaking due to the β_i fields, defined in Eqs. (93), (94) can be maintained unaltered to all orders of perturbation theory. Thus, extending the $\Phi.\Pi.$ equations of motion (Eqs. (17), (21)) by appropriately modifying the inhomogeneous terms $Y_\alpha(x)$, $X_{\alpha_A}(x)$ to comprehend the new β_i couplings, we can repeat as in the previous Section and without any effect arising from the lack of Slavnov invariance of the classical β_i vertices, the whole analysis of the breakings which are now defined by

$$\mathcal{S}Z_c(J, \eta, \Omega) - \int dx\, \beta_i(x)\partial_\mu B^\mu_i(x) = (\Delta Z_c)(J, \eta, \Omega). \tag{95}$$

It follows that, also in the presence of the Ω fields, the breakings can be reduced to the soft, Ω dependent term:

$$\Delta^{(n)}(\psi, \Omega) = \int dx\, W^{\alpha_A}_{\natural}(\phi, \Omega)(x)\bar{c}_{\alpha_A}(x) \tag{96}$$

where $W^{\alpha_A}_{\natural}(\phi, \Omega)$ is constrained by conditions (71), (72).

We have now gathered all the necessary information to arrive at the Callan-Symanzik equation which we shall write in a short-hand form, sufficient to our purposes. As a first step, let us remark that in the tree approximation, the action of the symmetric mass scaling operator D_μ on the connected Green functional is equivalent to the introduction of a suitable linear combination of the previously defined soft invariant insertions:

$$\left[D_\mu - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right] Z^{cl}_c(J, \eta\Omega, r) = O(r) \tag{97}$$

(where $O(r)$ means terms proportional to r). In the renormalized theory, applying the Lowenstein Q. A. P. [7], we get the modified equation

$$\left[D_\mu - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right] Z_c(J, \eta, \Omega, r) = \hbar[\Xi Z_c]_4(J, \eta, \Omega, r) + O(r) \quad (98)$$

where, in the r. h. s., the hard (N_4) insertion Ξ sums up the radiative corrections (4).

By applying the $\mathscr{S}^{(r)}$ operator to Eq. (98), taking into account Eq. (87) and going to $r(x) \equiv 0$, we get

$$\left[D_\mu - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right] \mathscr{S}^{(r)} Z_c(J, \eta, \Omega, r)\Big|_{r=0} = \hbar\mathscr{S}(\Xi Z_c)(J, \eta, \Omega) \quad (99)$$

Separating $\Xi(J, \eta, \Omega)$ into a Slavnov invariant $\Xi_\natural$ and a non invariant $\Xi_\flat$ part : i. e.

$$\Xi(J, \eta, \Omega) = \Xi_\natural(J, \eta, \Omega) + \Xi_\flat(J, \eta, \Omega) \quad (100)$$

with

$$\mathscr{S}(\Xi_\natural Z_c)(J, \eta, \Omega) = O(\Delta) \quad (101\,a)$$

and

$$\mathscr{S}(\Xi_\flat Z_c)(J, \eta, \Omega) \equiv \Xi'_\flat Z_c(J, \eta, \Omega) + O(\Delta) \quad (101\,b)$$

we have

$$\left[D_\mu - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right] (\Delta Z_c)(J, \eta, \Omega, r)\Big|_{r=0} = \hbar(\Xi'_\flat Z_c)(J, \eta, \Omega) + O(\hbar\Delta)\,. \quad (102)$$

Now, taking into account Eq. (97), we can write the l. h. s. of Eq. (102) in terms of the vertex functional as

$$\begin{aligned}
&\left[D_\mu - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right] (\Delta Z_c)(J, \eta, \Omega, r)\Big|_{r=0} \\
&\quad = \left[D_\mu - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right] (\Delta\Gamma)(\psi, \eta, \Omega, r)\Big|_{\substack{\psi = \frac{\delta Z}{\delta J} \\ r=0}} \\
&\quad + \int dy \frac{\delta}{\delta J_i(y)} \left(D_\mu - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right) Z_c(J, \eta, \Omega, r)\Big|_{r=0} \\
&\quad \cdot \frac{\delta}{\delta\psi_i(y)} (\Delta\Gamma)(\psi, \eta, \Omega, r)\Big|_{\substack{\psi = \frac{\delta Z}{\delta J} \\ r=0}} \\
&\quad = \left[D_\mu - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right] \Delta(\psi, \Omega, r)\Big|_{\substack{\psi = \frac{\delta Z}{\delta J} \\ r=0}} + O(\hbar\Delta)\,. \quad (103)
\end{aligned}$$

(4) In the usual writing of Eq. (98) these terms correspond to a linear combination of derivatives with respect to the coupling and the wave function renormalization constants of the theory.

Manipulating r. h. s. of Eq. (101 *b*) in the same way and recalling that at its lowest non vanishing order Δ is r independent, we can write Eq. (102) directly for the vertices as

$$\left[\lambda\partial_\lambda - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right]\Delta(\psi, \Omega) = \hbar\Xi'_b(\psi, \eta, \Omega) + O(\hbar\Delta). \qquad (104)$$

To reduce this equation to the desired form (Eq. (73)) we must analyze it more closely. Separating in $\Xi_b(\psi, \eta, \Omega)$ the $\eta_a(x)$ fields dependent and independent parts

$$\Xi_b(\psi, \Omega, \eta) = \Xi_b^0(\psi, \Omega) + \gamma_i\Xi_b^i(\psi, \Omega) + \gamma_{\mu\alpha s}\Xi_b^{\mu\alpha s}(\psi, \Omega) + \zeta_{\alpha s}\Xi_b^{\alpha s}(\psi, \Omega). \quad (105)$$

we obtain, at the lowest non vanishing order

$$\begin{aligned}\Xi'_b(\psi, \eta, \Omega) = {} & \mathscr{S}^{(P^0)}\Xi_b^0 + \mathscr{S}^{\Xi_b}\int \mathscr{L}_0 dx \\ & - \gamma_i\big(\mathscr{S}^{(P^0)}\Xi_b^i + \mathscr{S}^{\Xi_b}\overset{\circ}{P}_i\big) - \gamma_{\mu\alpha s}\big(\mathscr{S}^{(P^0)}\Xi_b^{\mu\alpha s} + \mathscr{S}^{\Xi_b}\overset{\circ}{P}{}^{\mu\alpha s}\big) \\ & + \zeta_{\alpha s}\big(\mathscr{S}^{(P^0)}\Xi_b^{\alpha s} + \mathscr{S}^{\Xi}\,\mathring{\bar{P}}{}^{\alpha s}\big)\end{aligned} \qquad (106)$$

where the symbol $\mathscr{S}^{\Xi_b}$ stands, as usual, for the operator

$$\int dx\left(\Xi_b^i(x)\frac{\delta}{\delta\varphi_i(x)} + \Xi_b^{\mu\alpha s}(x)\frac{\delta}{\delta\mathscr{A}_{\mu\alpha s}(x)} - \Xi_b^{\alpha s}(x)\frac{\delta}{\delta\bar{c}_{\alpha s}(x)}\right).$$

Now, since the external fields do not appear on the l. h. s. of Eq. (104), Ξ'_b is also independent of these fields. From the resulting conditions on Ξ_b^i, $\Xi_b^{\mu\alpha s}$, $\Xi_b^{\alpha s}$ (which are completely analogous to Eqs. (35) of paper I), one can show that the η dependent terms of Ξ_b can be inserted in a Slavnov invariant form which obviously should be included in $\Xi_\natural$.

After this Eq. (106) becomes

$$\Xi'_b(\psi, \Omega) = \mathscr{S}^{(P^0)}\Xi_b^0(\psi, \Omega) \qquad (107)$$

and Eq. (104) reduces to

$$\left[\lambda\partial_\lambda - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right]\Delta(\psi, \Omega) = \mathscr{S}^{(P^0)}\Xi_b^0(\psi, \Omega) \qquad (108)$$

which is the desired equation. Indeed, since the non-compensable breakings $\Delta(\psi, \Omega)$ cannot be written in the form $\mathscr{S}^{(P^0)}\hat{\Delta}(\psi, \Omega)$ (otherwise they are clearly compensable) Eq. (108) implies that, at the lowest non vanishing order of Δ, $\mathscr{S}^{(P^0)}\Xi_b^0(\psi, \Omega) = 0$, i. e., $\Xi_b^0(\psi, \Omega) = 0$ and hence

$$\left[\lambda\partial_\lambda - \int dx\, k_a \frac{\delta}{\delta\Omega_a(x)}\right]\Delta(\psi, \Omega) = 0. \qquad (109)$$

which essentially coincides with Eq. (73).

This ends our rather heavy analysis of the renormalization of the S. I. in the absence of the A. B. A. However, it is worthwhile noticing that the arguments exhibited in this Section can be applied also in the presence of A. B. A., excluding the possibility of soft contributions to the anomaly at its lowest non vanishing order. Indeed, as shown by B. R. S. [*14*], the Wess-Zumino consistency conditions [*16*] fix uniquely the structure of A. B. A. preventing it from depending on scalar fields.

5. CONCLUSIONS

The renormalization scheme of the gauge models here discussed and their parametrization, analyzed in Paper I, require the following short comment. In paper I we have shown that the matter fields carry an unstable representation of the abelian factor of the gauge group, i. e. the Slavnov identity is not sufficient to ensure the stability of these representations in the presence of radiative corrections. We also noticed that this problem has no relevance if the parameters of the Lagrangian which determine these representations are not to be renormalized in order to compensate divergencies which could occur outside of the Zimmermann formalism. This is indeed the case since Eqs. (17), (21) or, in other words, the super-renormalizability of the couplings of the abelian $\bar{c}$ fields, guarantee that the vertices, which individuate the mentioned representations, such as $\Gamma_{\gamma_i \varphi_j \bar{c}_{\alpha_A}}$ are not affected by primitive divergencies. In our formalism, in which divergencies never appear explicitely, Eq. (17), (21) must be read as a non renormalization property of the representations of the abelian factor of the gauge group.

Concerning the extension of our method to theories with massless particles we observe that this would be almost straightforward had we at our disposal a renormalization scheme preserving a Q. A. P. in which the breaking terms may be interpreted as local operators with uniformly bounded dimensions (N-products with U. V. power counting index not larger than 5). Such a method has been recently described by Lowenstein and Clark in ref. [*8*].

However the application of our renormalization scheme to theories affected by U. R. problems requires some cautionsness about one more point; in fact in this case it is dangerous to consider vertices with external legs at zero momentum. This is not a serious handicap since, as in Eqs. (A.1), (A.2) we may give to the abelian $\bar{c}$ external leg ($\bar{c}_{\gamma_A}$) a non zero momentum to be held fixed in the limit process. Clearly the asymptotic behaviours of the mentioned vertices are not affected by this choice.

Aside from these rather technical remarks, the problem of the gauge invariance of the model has no new aspects compared to the analysis found in B. R. S. [*9*] to which the interested reader is referred.

Concerning the perturbative unitarity of the S-matrix, whenever asymptotic states are definable, the combinatorial proof given by B. R. S. [*9*] in two special models can also be adapted to our needs by fully exploiting the analysis, given in paper I, of the asymptotic wave equations of the unphysical fields and in particular the assumed constraints on their masses.

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APPENDIX A

In this Appendix we shall show how Eqs. (52), (54), (55) can be deduced from the super-renormalizability of the abelian $\bar{c}$ couplings (Eq. 17).

Let us consider, for example, Eq. (52 *a*). The coefficients $\Sigma_{\alpha_S}^{\beta_A,\gamma_S}$ can be computed in terms of the breaking $(\Delta\Gamma)(\psi, \eta)$ by selecting the n-th order contribution in $\hbar$ from

$$\lim_{p\to\infty} \frac{ip_\mu}{p^2} \frac{\delta^3(\Delta\Gamma)(\psi, \eta)}{\delta\tilde{\gamma}_\mu^{\alpha_S}(p)\delta\tilde{\bar{c}}_{\gamma_S}(-p)\delta\bar{c}_{\beta_A}(0)}\bigg|_{\psi=\eta=0} = \Sigma_{\alpha_S}^{\beta_A,\gamma_S} + O(\hbar^{n+1}) \tag{A.1}$$

where, as usual, the tilde refers to the Fourier transform of the corresponding variable. The l. h. s. of Eq. (A.1) can be evaluated by the very definition of the breakings (Eq. (32)), computing

$$\lim_{p\to\infty} \frac{ip_\mu}{p^2} \frac{\delta^3(\mathscr{S}\Gamma)(\psi, \eta)}{\delta\tilde{\gamma}_\mu^{\alpha_S}(p)\delta\tilde{\bar{c}}_{\gamma_S}(-p)\delta\bar{c}_{\beta_A}(0)}\bigg|_{\psi=\eta=0}$$
$$= \lim_{p\to\infty} \frac{ip_\mu}{p^2}\Big[-\Gamma_{\tilde{\gamma}_\mu^{\alpha_S}(p)\tilde{\varphi}_i(-p)\bar{c}_{\beta_A}(0)}\Gamma_{\tilde{\bar{c}}_{\gamma_S}(-p)\gamma_i(0)}$$
$$-\Gamma_{\tilde{\gamma}_\mu^{\alpha_S}(p)\varphi_i(0)\tilde{\bar{c}}_{\gamma_S}(-p)}\Gamma_{\tilde{\gamma}_i(0)c_{\beta_A}(0)} - \Gamma_{\tilde{\bar{c}}_{\gamma_S}(-p)\gamma_\nu^{\delta_S}(0)}\Gamma_{\tilde{\mathscr{A}}_{\delta_S}^\nu(-p)\tilde{\gamma}_\mu^{\alpha_S}(p)\bar{c}_{\beta_A}(0)}$$
$$+\Gamma_{\tilde{\bar{c}}_{\beta_A}(0)\gamma_\nu^{\delta_S}(0)}\Gamma_{\mathscr{A}_{\delta_S}^\nu(0)\tilde{\gamma}_\mu^{\alpha_S}(p)\tilde{\bar{c}}_{\gamma_S}(-p)} - \Gamma_{\bar{c}_{\delta_S}(0)\tilde{\gamma}_\mu^{\alpha_S}(p)}\Gamma_{\tilde{\zeta}^{\delta_S}(p)\tilde{\bar{c}}_{\gamma_S}(-p)\bar{c}_{\beta_A}(0)}\Big] \tag{A.2}$$

where in the r. h. s. the field subscripts stand for the amputated external legs of the vertices. The fourth term in the r. h. s. of Eq. (A.2) is identically zero by covariance, while the second one vanishes in the limit since the vertex $\Gamma_{\tilde{\gamma}_i(0)\bar{c}_{\beta_A}(0)}$ is obviously constant and, by the usual power counting rules, so is $\Gamma_{\tilde{\gamma}_\mu^{\alpha_S}(p)\varphi_\nu(0)\tilde{\bar{c}}_{\gamma_S}(-p)}$ up to powers of $\log p^2$. The three remaining terms can be evaluated taking advantage of the super-renormalizability of the abelian $\bar{c}$ fields. Indeed, by Eq. (17), we have

$$\Gamma_{\tilde{\gamma}_\mu^{\alpha_S}(p)\tilde{\varphi}_i(-p)\bar{c}_{\beta_A}(0)} = \Gamma_{\tilde{\gamma}_\mu^{\alpha_S}(p)\tilde{\varphi}_i(-p)\bar{\mathscr{C}}_{\beta_A}(0)} \tag{A.3}$$

with U. V. behaviour dominated by $\frac{p^\mu}{p^2}$ and since $\Gamma_{\tilde{\bar{c}}_{\gamma_S}(p)\gamma_i(0)}$ is at most linear in p, the first term on the r. h. s. of Eq. (A.3) is zero in the limit. In much the same way we deduce the vanishing of the contributions from the third and fifth terms. Thus, since the r. h. s. of Eq. (A.2) is zero to all orders in $\hbar$, Eq. (52 *a*) is verified owing to Eq. (A.1).

Along the same lines it is easy to prove Eqs. (52 *b*), (54 *a*), (54 *b*); while computing the coefficient $\Theta_{ij}^{\alpha_A\beta_A}$ in Eq. (52 *e*) we find two terms that survive the power counting analysis, i. e.

$$\lim_{p,q\to\infty} \frac{\delta^4(\mathscr{S}\Gamma)(\psi, \eta)}{\delta\tilde{\gamma}_i(p)\delta\tilde{\varphi}_j(q)\delta\tilde{\bar{c}}_{\alpha_A}(-p-q)\delta\bar{c}_{\beta_A}(0)}\bigg|_{\psi,\eta=0}$$
$$= \lim_{p,q\to\infty}\Big[-\Gamma_{\tilde{\gamma}_i(p)\tilde{\bar{c}}_{\alpha_A}(-p-q)\varphi_k(0)}\Gamma_{\tilde{\gamma}_k(-q)\varphi_j(q)\bar{c}_{\beta_A}(0)}$$
$$+\Gamma_{\tilde{\gamma}_i(p)\tilde{\varphi}_k(-p)\bar{c}_{\beta_A}(0)}\Gamma_{\gamma_k(0)\tilde{\varphi}_j(q)\tilde{\bar{c}}_{\alpha_A}(-p-q)}\Big]. \tag{A.4}$$

However from Eq. (39 *c*) it turns out that this is nothing but the vanishing commutator $[t^{\alpha_A}, t^{\beta_A}]_{ij}$.

It remains to discuss Eq. (53), where the coefficient $\Theta_{ij}^{\alpha_S\gamma_A}$ is computed by:

$$\lim_{p,q\to\infty} \frac{\delta^4(\Delta\Gamma)(\psi, \eta)}{\delta\tilde{\gamma}_i(p)\delta\tilde{\varphi}_j(q)\delta\tilde{\bar{c}}_{\alpha_S}(-p-q)\delta\bar{c}_{\gamma_A}(0)}\bigg|_{\psi=\eta=0} = \Theta_{ij}^{\alpha_S,\gamma_A} + O(\hbar^{n+1}). \tag{A.5}$$

and

$$\lim_{p,q\to\infty} \left. \frac{\delta^4(\mathscr{S}\Gamma)(\psi,\eta)}{\delta\tilde{\gamma}_i(p)\delta\tilde{\varphi}_j(q)\delta\tilde{\bar{c}}_{\alpha_S}(-p-q)\delta\bar{c}_{\gamma_A}(0)}\right|_{\psi=\eta=0} = \lim_{p,q\to\infty} [-\Gamma_{\tilde{\gamma}_i(p)\tilde{\bar{c}}_{\alpha_S}(-p-q)\varphi_k(0)}t^{\gamma}_{kj} + t^{\gamma_A}_{ik}\Gamma_{\tilde{\gamma}_k(p)\tilde{\bar{c}}_{\alpha_S}(-p-q)\varphi_j(0)}] . \tag{A.6}$$

Comparing Eqs. (A.5) and (A.6) we see that the n-th order contributions to the vertex $\Gamma_{\tilde{\gamma}_i(p)\tilde{\bar{c}}_{\alpha_S}(-p-q)\varphi_j(0)}$ which do not commute with all the representatives $t^{\gamma_A}_{ij}$ are constant in the limit $p, q \to \infty$. Choosing for the matrix elements $B^{\alpha_S}_{ij}$ in Eq. (55) the values of these constants we obtain Eq. (55).

APPENDIX B

This Appendix is devoted to the exclusion of the highest dimensionality, external field independent breakings to the Slavnov identity, which, together with the A. B. A. survive the algebraic analysis.

Let us define the n-field test operator

$$X(p_1, a_1 ; \dots ; p_n, a_n) = X(p, a) = \prod_{i=1}^{n} \frac{\delta}{\delta\tilde{\phi}_{a_i}(p_i)} \qquad (B.1)$$

where we have introduced the symbol ϕ_a to indicate both the photons and the matter fields $\{ \phi_a \} \equiv \{ \mathscr{A}^\alpha_\mu, \varphi_i \}$. Unless otherwise stated the momenta p_i $i = 1, \dots, n$ are chosen with vanishing sum and non-exceptional in the Symanzik sense.

The set of the linear combinations of n-field test operators corresponding to given momenta p_i is a linear representation space $S(p)$ of the infinitesimal algebra $\mathscr{G}$ of the gauge group. The action of an element T^{α_A} of $\mathscr{G}_A$ the abelian factor of $\mathscr{G}$ on $S(p)$ is defined by

$$T^{\alpha_A}X(p, a) = - [\mathscr{T}^{\alpha_A}, X(p, a)] = \sum_{ia'_i} t^{\alpha_A}_{a_i,a'_i} X(p_1, a'_1 ; \dots ; p_1, a'_1 ; \dots ; p_n, a'_n) \qquad (B.2)$$

where

$$\mathscr{T}^{\alpha_A} = \int dx\left(\phi_a t^{\alpha_A}_{ab} \frac{\delta}{\delta\phi_b}\right)(x) = \int dx\left(\varphi_i t^{\alpha_A}_{ij} \frac{\delta}{\delta\varphi_j}\right)(x). \qquad (B.3)$$

The complete reducibility of the representation t^{α_A} ensures that this same property holds for T^{α_A}. Thus there exists a basis $\{ X^{(N)}(m_{1_A}, \dots, m_{\alpha_A}, \dots, m_{N_A}, K, p) \} \equiv \{ X^{(N)}(m, K, p) \}$ in $S(p)$ simultaneously diagonalizing the abelian generators T^{α_A}, i. e.:

$$[X^{(N)}(m, K, p), \mathscr{T}^{\alpha_A}] = m_{\alpha_A} X^{(N)}(m, K, p) \qquad (B.4)$$

where K is a degeneration index.

Now selecting the four dimensional part of Eq. (72), yields when $\beta = \beta_A$:

$$\mathscr{T}^{\beta_A}\Delta^{\alpha_A}_{44}(0) = 0 \qquad (B.5)$$

from which we are going to prove that

$$X^{(N)}(m, K, p)\Delta^{\alpha_A}_{44}(0)\,|_{\phi=0} = 0. \qquad (B.6)$$

This is almost immediate if at least one of the m_{α_A}'s is non-vanishing. Indeed let $m_{\beta_A} \neq 0$; we have

$$X^{(N)}(m, K, p)\Delta^{\alpha_A}_{44}(0)\,|_{\phi=0} = m^{-1}_{\beta_A}[X^{(N)}(m, K, p), \mathscr{T}^{\beta_A}]\Delta^{\alpha_A}_{44}(0)\,|_{\phi=0} = 0 \qquad (B.7)$$

where Eqs. (B.3), (B.5) have been used.

The case here all the m_{α_A} vanish can be proved through the power counting method extensively employed in Appendix A. The main point is to show that the polynomial $X^{(N)}(0, p, K)\Delta^{\alpha_A}_{4}(0)\,|_{\phi=0}$ has degree $3 - D_x$ in the components of the momenta p_i, where D_x, the dimension of $X^{(N)}$, is the sum of the naive dimension of the fields ($D_x = \Sigma_i d\Phi_{a_i}$) [5].

---

[5] Notice that D_x is invariant under the action of the generators T^{α_A}.

Assuming for a moment this power counting result which will be proved at the end of this Appendix and writing it as:

$$X^{(N)}(0, p, K)\Delta_{5}^{\alpha A}(0)\,|_{\phi=0} = O(p^{3-D_x}) \tag{B.8}$$

we immediately verify the validity of Eq. (B.6) for vanishing m_{α_A}. Indeed

$$X^{(N)}(0, p, K)\Delta_{54}^{\alpha A}(0)\,|_{\phi=0}$$

contributes to the l. h. s. of Eq. (B.8), considered as a polynomial in p, with its homogeneous part of degree $4 - D_x$, which turns out to vanish.

Now, making use of Eq. (B.6) our goal is soon reached since, for a generic choice of the momenta we can write

$$X^{(N)}(m, K, p)\Delta_{54}^{\alpha A}(0)\,|_{\Phi=0} \propto \Sigma p_i \tag{B.9}$$

for which, recalling that the $X^{(N)}(m, K, p)$'s corresponding to all the possible choices of N, m, K, p span a basis for the whole set of the test operators we conclude that

$$\Delta_{54}^{\alpha A}(x) = \partial^{\mu} I_{\mu}^{\alpha A}(x) \tag{B.10}$$

or

$$\int dx\, \Delta_{54}^{\alpha A}(x) = 0$$

and hence that the five dimensional part of the breaking $\Delta_{50}^{(n)}(\psi)$ is compensable.

It remains to prove Eq. (B.8). We can compute

$$\lim_{p\to\infty} X^{(N)}(0, K, p)\Delta_{5}^{\alpha A}(0)\,|_{\phi=0} + O(\hbar^{n+1}) = \lim_{p\to\infty} X^{(N)}(0, K, p)\frac{\delta}{\delta \bar{c}_{\alpha_A}(0)}(\Delta\Gamma)\Big|_{\psi=\eta=0} \tag{B.11}$$

using the S. I. and the super-renormalizability of the abelian $\bar{c}$ couplings Eq. (17) as

$$\lim_{p\to\infty} X^{(N)}(0, K, p)\frac{\delta}{\delta \bar{c}_{\alpha_A}(0)}(\mathscr{S}\Gamma)\Big|_{\psi=\eta=0} = \lim_{p\to\infty} X^{(N)}(0, K, p)\mathscr{C}^{\alpha_A}\Gamma\,|_{\psi=\eta=0} + O(p^{3-D_x}, \log^{n_c} p) \tag{B.12}$$

where the leading power n_c of log p depends upon the order in $\hbar$. The first term in the r. h. s. of Eq. (B.12) vanishes since

$$X^{(N)}(0, K, p)\mathscr{C}^{\alpha_A}\Gamma\,|_{\psi=\eta=0} = [X^{(N)}(0, K, p), \mathscr{C}^{\alpha_A}]\Gamma\,|_{\psi=\eta=0} \tag{B.13}$$

Thus, comparing Eqs. (B.11) and (B.12), we arrive, for the n-th order contribution in $\hbar$, which is a polynomial in the momenta, at Eq. (B.8).

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