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G. BANDELLONI
C. BECCHI
A. BLASI
R. COLLINA

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Section A :
Physique théorique.

Non semisimple gauge models: I. Classical theory and the properties of ghost states

by

G. BANDELLONI, C. BECCHI (*), A. BLASI and R. COLLINA
Istituto di Scienze Fisiche dell'Università di Genova
e Istituto Nazionale di Fisica Nucleare, Sezione di Genova

ABSTRACT. — In this paper we study a class of non-semisimple gauge models from the point of view of their invariance properties under a set of non linear field transformations (B. R. S. or Slavnov transformations). We first discuss how the Slavnov invariance insures the stability, under small perturbations, of the gauge group and of its representation on the matter field space, thereby individuating a set of stable, Slavnov invariant classical actions. Secondly we analyze the masses of the ghost particles; we see that, contrary to the semisimple case, the Slavnov invariance is no longer sufficient to yield the complete mass degeneracy between the Faddeev-Popov and the longitudinal photons-Goldstone bosons sectors. This mass degeneracy, which is an essential ingredient for gauge invariance, is restored by imposing a special constraint on the parameters of the Lagrangian. The resulting definition of the classical models, i. e. Slavnov invariance plus mass degeneracy, is extendible to the quantum level as shown in a forthcoming paper (II).

1. INTRODUCTION

The gauge models are the result of an historical effort aimed at the construction of renormalizable theories involving vector fields. The contri-

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butions of Yang-Mills [*1*], Feynman [*2*], Faddeev-Popov [*3*], 't Hooft [*4*] [*5*] and many others [*6*] [*7*] [*8*] have provided us with a canonical procedure for the specification of the Lagrangian. Given a gauge group and a set of matter fields which carry a fully reducible representation of it, the model is built by adding to the most general, renormalizable, gauge invariant Lagrangian the well known Faddeev-Popov (Φ.Π.) gauge fixing terms.

The straightforwardness of such a procedure is only apparent since the real problems appear when trying to build a sensible operator theory in a Fock space, for which a necessary prerequisite is to have a quantum extension (renormalization) of the model.

A possible strategy toward renormalization is to use the symmetry properties of the theory as an alternative definition to the historical approach. This point of view is of course meaningful if the symmetry is enough well-behaved to characterize unambiguously the classical models and if it can be maintained to all orders of perturbation theory.

Recently the Slavnov identity (S. I.) [*9*], expressing the invariance of the Lagrangian under a system of non-linear field transformations (Slavnov transformations [*10*]) has proved to be a good definition for theories with semi-simple gauge groups also extendible to the quantum level. Indeed, one can first show, in this case, that the infinitesimal gauge group is stable under small perturbations of the field transformation laws and that its representation on the matter field space is likewise identified up to an equivalence transformation. Secondly, the most general Slavnov invariant Lagrangian is defined up to a field renormalization in terms of the coefficients appearing in the historical model, thus excluding the presence of hidden parameters in the theory (1).

The fulfillment of all these requirements will be summarized by saying that the set of Slavnov invariant Lagrangians is stable.

Classical stability [*11*] [*12*] [*13*] [*14*] does not imply in general the possibility of extending the theory to the quantum level; as an example (and the only one so far known) the Slavnov symmetry can be definitely broken by the occurrence of the Adler-Bardeen anomaly (A. B. A.) [*15*].

The renormalization of the S. I. can be viewed from different angles; a first way is to look for a Slavnov invariant regularization procedure which directly links the renormalizability of the S. I. to the classical stability of the Lagrangian. Indeed, if such a regularization is available and the renormalized Lagrangian including the counter-terms is Slavnov invariant,

(1) This is essential, otherwise these parameters could show up in the form of unexpected divergencies during the process of renormalization. This happens, for example, in the Yukawa model or in scalar Q. E. D. if the quadrilinear couplings for the scalar field are omitted.

the stability properties ensure that all infinities can be compensated by a cut-off dependent renormalization of the fields and the parameters. Among the best known examples of renormalization programs based on this point of view, we mention the Pauli-Villars regularization in Q. E. D. and the dimensional regularization as applied to the gauge theories [16] without fermion fields.

A second approach which, within the B. P. H. Z. [17] framework avoids any specific regularization, is based on the compensation, by suitable finite counter-terms in the Lagrangian, of the breakings which can affect the S. I. according to the renormalized Quantum Action Principle (Q. A. P.) of Lowenstein and Lam [18].

The compensability of the breakings can be investigated by means of two main tools: a power counting analysis and the consistency (integrability) conditions following directly from the structure of the S. I., which can be written as a first order differential equation in terms of the suitable variables.

This point of view has been successfully applied by Becchi-Rouet-Stora (B. R. S.) to renormalizable theories with symmetry breaking [19] and to the renormalization of gauge theories in the case of abelian [20] or semi-simple [10] [11] gauge groups. For a semi-simple gauge field model the consistency conditions for the symmetry breaking are discussed in a purely algebraic fashion and point uniquely to the A. B. A.

Once the S. I. has proved to be renormalizable, there is still a need for a physical interpretation of the theory. In fact the associated Fock-space does not have a positive definite metric, due both to the covariant quantization of the vector fields, as in Q. E. D., and to the presence of the anti-commuting scalar Faddeev-Popov ($\Phi.\Pi.$) fields [3]. Furthermore the Lagrangian of the model is in general not hermitian.

It is, thus, necessary to find, within this indefinite metric Fock space, a subspace with a positive definite norm (physical subspace) where the S-matrix is unitary and independent from the parameters labelling the gauge fixing terms in the Lagrangian. There exists now a systematic approach to the unitarity problem, based on the S. I. and on the peculiar properties (mass degeneracy) of the unphysical states of the models.

As far as gauge invariance is concerned, it can be proved by a direct extension of the method used in massive Q. E. D.

The problem of classifying the local observables, i. e. the local gauge invariant operators, still awaits a global solution, although recently [21], the general guidelines toward such a solution are beginning to clarify.

This paper contains the first part of an analysis of non-semi-simple gauge models where, with the intent of simplifying as much as possible the study of the renormalization process, we shall exclude the presence of massless particles hence considering only models in which all the photons corresponding to the semi-simple factor of the gauge group acquire mass through a Higgs-Kibble [*22*] spontaneous symmetry breaking mechanism. We shall, however, follow a procedure which, in the light of recent developments in the field [*23*], can be directly extended to the massless case.

We study here the definition and the stability properties of the models; the renormalization problem will be fully analyzed in a forthcoming paper.

The stability of the non-semi-simple models suffers two kinds of pathologies as compared to the semi-simple case.

The first of these pathologies lies in the fact that the Slavnov invariance does not identify uniquely the matter field representation of the abelian factor of the gauge group. As a consequence there arises the possibility that the renormalized representation may be inequivalent to the tree-approximation one. It will, however, be shown in the next paper that the abelian representatives are not affected by quantum corrections, so that this first kind of instability has in practice no relevance.

The second pathology comes from the fact that the S. I. is compatible with the introduction of arbitrary mass terms for the abelian photon and Φ.Π. fields.

This phenomenon has been already discussed in the Literature. In particular in the case of the U(1) H. K. model, B. R. S. have shown [*20*] that such a mass term breaks the gauge invariance of the theory. They eliminate it by requiring that the unphysical particles (Goldstone bosons and longitudinal photons) be mass degenerate with the Φ.Π. Notice that in the case of unbroken U(1) gauge symmetry such a mass term is allowed, thus leading to massive Q. E. D.

We shall show that in the general case an analogous mass degeneracy prescription must be imposed, which now turns out to be compatible with the presence of a suitably restricted mass term for the abelian photon and Φ.Π. fields.

In Section 2 we describe the construction of the gauge field models, exhibit their Slavnov symmetry and introduce the necessary ingredients to translate the S. I. into a functional form.

Section 3 is entirely devoted to the study of the stability properties following from the S. I.

In Section 4 we analyze in detail the mass degeneracy condition in the unphysical one-particle sector of the theory and give a heuristic discussion of the relevance of this condition for the gauge invariance of the model.

The concluding Section contains a summary of the results so far obtained and some remarks which provide a bridge toward the renormalization of the gauge field models, which will be the subject of a forthcoming paper.

The more technical aspects of our analysis are treated in Appendices A, B, C.

2. CLASSICAL MODELS AND SLAVNOV INVARIANCE

Let G be a compact, real Lie algebra with $G = S \oplus A$ where S and A are the semi-simple and abelian factors respectively.

The field $\phi(x)$ with components φ_i, $i = 1, \dots, n$, carries an anti-hermitian, fully reducible representation of G according to

$$\delta\varphi_i(x) = t^\alpha_{ij}(\varphi_j(x) + q_j)\omega_\alpha(x), \qquad i, j = 1, \dots, n \quad \alpha = 1, \dots, N \tag{1}$$

while the gauge vector fields $\mathscr{A}^\mu_\alpha(x)$ transform as

$$\delta\mathscr{A}^\mu_\alpha(x) = \partial^\mu\omega_\alpha(x) + f^{\beta\gamma}_\alpha\mathscr{A}^\mu_\gamma(x)\omega_\beta(x) \tag{2}$$

with $\omega^\alpha(x)$ real, differentiable functions of the space-time point x and $f^{\alpha\beta}_\gamma$ the real structure constant of G.

The set of Greek indices $\alpha, \beta, \gamma = 1, \dots, N$ will be split, when convenient into a « semi-simple » subset $\alpha_S, \beta_S, \gamma_S = 1, \dots, N_S$ and an « abelian » subset $\alpha_A, \beta_A, \gamma_A = 1, \dots, N_A$, corresponding to the semi-simple and abelian components of G; in particular $f^{\beta_S\gamma_A}_{\alpha_S} = f^{\beta_S\gamma_S}_{\alpha_A} = f^{\beta_A\gamma_S}_{\alpha_S} = 0$.

A classical Lagrangian invariant under the transformations of Eqs. (1) and (2) is built with the φ_i fields, the covariant antisymmetric tensor

$$\mathscr{G}^{\mu\nu}_\alpha(x) = \partial^\mu\mathscr{A}^\nu_\alpha(x) - \partial^\nu\mathscr{A}^\mu_\alpha(x) - f^{\beta\gamma}_\alpha\mathscr{A}^\mu_\beta(x)\mathscr{A}^\nu_\gamma(x) \tag{3 a}$$

and the covariant derivative

$$D^\mu\varphi_i(x) = \partial^\mu\varphi_i(x) - \mathscr{A}^\mu_\alpha(x)t^\alpha_{ij}(\varphi_j(x) + q_j) \tag{3 b}$$

as [2]

$$\mathscr{L}_{\text{inv}}(\phi, \mathscr{A}_\mu)(x) = -\frac{1}{4}\mathscr{G}^\alpha_{\mu\nu}(x)\mathscr{G}^{\mu\nu}_\alpha(x) + \frac{1}{2}D_\mu\varphi^\dagger_i(x)I_{ij}D^\mu\varphi_j(x) - H(\varphi_i + q_i)(x). \tag{4}$$

The indices $\alpha, \beta, \gamma, \dots$ are raised and lowered by means if a nondegenerate invariant form whose reduction to irreducible components of G defines the coupling constants, the $\mu, \nu = 1, \dots, 4$ indices by the Minkowsky metric tensor and the matrix I_{ij} is a positive definite form anticommuting with the matter field representation. Unless explicitely specified the sum over repeated indices is always understood.

The term $H(\varphi + q)$ in Eq. (4) is an invariant polynomial in the argument $\varphi_i + q_i$ which satisfies

$$\frac{\partial}{\partial\varphi_i}H(\varphi + q)\bigg|_{\phi=0} = 0 \tag{5}$$

(2) For the sake of simplicity we shall here consider only scalar matter fields; the generalization to include fermion fields is straightforward.

and exhibits the spontaneous symmetry breaking mechanism in the direction of the vector q.

The mass matrix m_{ij} of the $\varphi_i(x)$ fields in fact satisfies the eigenvalue equation, obtained from Eqs. (1) and (5)

$$m_{ij} q_j^\alpha = \left. \frac{\partial^2 \mathrm{H}}{\partial \varphi_i \partial \varphi_j} \right|_{\phi=0} q_j^\alpha = 0 \tag{6}$$

where $q_j^\alpha = t_{jk}^\alpha q_k$.

From Eq. (6) we observe that the number of massless Goldstone fields is given by the dimensionality of the orbit of q.

The mass matrix $\mathrm{M}_{\alpha\beta}$ of the gauge vector fields is

$$\mathrm{M}_{\alpha\beta} = q_{\alpha i} q_{\beta i} \,. \tag{7}$$

In the following we shall assume that the orbit of q contains at least the vectors $\{ q^{\alpha s} \}$ i. e. the trivial representation of G is excluded, and hence by Eq. (7) all the gauge fields $\mathscr{A}_{\alpha s}^\mu(x)$ acquire mass by the Higgs-Kibble (H. K.) mechanism. We shall see later on that under this condition and for a generic choice of the parameters in the final Lagrangian all gauge fields can be made massive.

It is well known that the Lagrangian in Eq. (4) is not directly quantizable since it leads to singular field equations. A way out of this difficulty, allowing a correct definition of the propagators, is to introduce the Faddeev-Popov (Φ.Π.) [3] Lagrangian (3)

$$\mathscr{L}(\phi, \mathscr{A}_\mu, c, \bar{c})(x) = \mathscr{L}_{\mathrm{inv}}(\phi, \mathscr{A}_\mu)(x) - \frac{\delta^{\alpha\beta}}{\mathrm{K}} \left\{ \frac{g_\alpha(x) g_\beta(x)}{2} - c_\alpha(x) (\mathscr{M}\bar{c})_\beta(x) \right\} \tag{8}$$

where

$$g_\alpha(x) = \partial^\mu \mathscr{A}_{\mu\alpha}(x) + \rho_{\alpha i} \varphi_i(x) \tag{9 a}$$

$$(\mathscr{M}\bar{c})_\alpha(x) = \int dy \frac{\delta g_\alpha(x)}{\delta \omega_\beta(y)} \bar{c}_\beta(y) = \Box\, \bar{c}_\alpha(x) + f_\alpha^{\beta\gamma} \partial^\mu (\mathscr{A}_{\mu\gamma}(x) \bar{c}_\beta(x)) + \rho_{\alpha i} t_{ij}^\beta (\varphi_j(x) + q_j) \bar{c}_\beta(x) \,. \tag{9 b}$$

The Φ.Π. $c_\alpha(x)$, $\bar{c}_\alpha(x)$ fields obey Fermi statistics and have canonical dimension 1; the components ρ_i^α of the vector ρ^α are the gauge parameters introduced by t'Hooft.

The Lagrangian in Eq. (8) is no longer invariant under the gauge trans-

(3) The Φ.Π. gauge fixing term in Eq. (8), which is often used in the literature, leads to particularly simple form for the propagators. However we shall see in the following that in general the matrix $\frac{1}{k} \delta^{\alpha\beta}$ should be replaced with an arbitrary symmetric positive definite matrix $\Lambda^{\alpha\beta}$ [10]. It is also worthwhile noticing that any choice of the gauge function $g_\alpha(x)$ linear in the fields can be reduced to the form (9 a) by a redefinition of the c_α fields.

formations (Eqs. (1) and (2)), but under the following set of Slavnov [9] transformations:

$$\delta\varphi_i(x) = \delta\lambda \int dy \frac{\delta\varphi_i(x)}{\delta\omega_\alpha(y)} \bar{c}_\alpha(y) = \delta\lambda[t^\alpha_{ij}(\varphi_j(x) + q_j)]\bar{c}_\alpha(x) = \delta\lambda \mathrm{P}_i(x), \quad (10\,a)$$

$$\delta\mathscr{A}^\mu_{\alpha_S}(x) = \delta\lambda \int dy \frac{\delta\mathscr{A}^\mu_{\alpha_S}(x)}{\delta\omega_\beta(y)} \bar{c}_\beta(y) = \delta\lambda[\delta^\beta_{\alpha_S}\partial^\mu + f^{\beta\gamma}_{\alpha_S}\mathscr{A}^\mu_\gamma(x)]\bar{c}_\beta(x) = \delta\lambda\mathrm{P}^\mu_{\alpha_S}(x), \quad (10\,b)$$

$$\delta\mathscr{A}^\mu_{\alpha_A}(x) = \delta\lambda\partial^\mu\bar{c}_{\alpha_A}(x), \quad (10\,c)$$

$$\delta c_\alpha(x) = \delta\lambda g_\alpha(x), \quad (10\,d)$$

$$\delta\bar{c}_{\alpha_S}(x) = \delta\lambda\frac{1}{2} f^{\beta\gamma}_{\alpha_S}\bar{c}_\beta(x)\bar{c}_\gamma(x) = \delta\lambda\bar{\mathrm{P}}_{\alpha_S}(x), \quad (10\,e)$$

$$\delta\bar{c}_{\alpha_A}(x) = 0 \quad (10\,f)$$

where $\delta\lambda$ is an infinitesimal, space-time independent parameter which commutes with the $\varphi_i(x)$, $\mathscr{A}^\mu_\alpha(x)$ fields and anticommutes with the $c_\alpha(x)$, $\bar{c}_\alpha(x)$ fields.

It will turn out to be useful to assign to the fields a $\Phi.\Pi.$ charge $\mathrm{Q}_{\phi\Pi}$ as follows:

$$\mathrm{Q}_{\phi\Pi}c_\alpha = c_\alpha, \qquad \mathrm{Q}_{\phi\Pi}\bar{c}_\alpha = -\bar{c}_\alpha, \quad (11\,a)$$

$$\mathrm{Q}_{\phi\Pi}\varphi_i = 0, \qquad \mathrm{Q}_{\phi\Pi}\mathscr{A}^\mu_\alpha = 0 \quad (11\,b)$$

so that the Lagrangian in Eq. (8) is $\Phi.\Pi.$ neutral.

The above transformations can be summarized in a functional derivative notation. Let

$$\mathscr{I}^{(\mathrm{P})} = \int dx\left[\mathrm{P}_i(x)\frac{\delta}{\delta\varphi_i(x)} + \mathrm{P}^\mu_{\alpha_S}(x)\frac{\delta}{\delta\mathscr{A}^\mu_{\alpha_S}(x)} + (\partial^\mu\bar{c}_{\alpha_A}(x))\frac{\delta}{\delta\mathscr{A}^\mu_{\alpha_A}(x)} + \bar{\mathrm{P}}_{\alpha_S}(x)\frac{\delta}{\delta\bar{c}_{\alpha_S}(x)} + g_\alpha(x)\frac{\delta}{\delta c_\alpha(x)}\right] \quad (12)$$

and

$$\psi(x) \equiv \{\mathscr{A}^\mu_\alpha(x), \phi(x), c_\alpha(x), \bar{c}_\alpha(x)\}; \quad (13)$$

Eqs. (10 *a*, *b*, *c*, *d*, *e*, *f*) take the form

$$\delta\psi(x) = \delta\lambda\mathscr{I}^{(\mathrm{P})}\psi(x). \quad (14)$$

A further gain is acquired by the introduction of a set of external fields

$$\eta(x) = \{\gamma_i(x), \gamma^\mu_{\alpha_S}(x), \zeta_{\alpha_S}(x)\} \quad (15\,a)$$

to which the following $\Phi.\Pi.$ charges and dimensions are assigned

$$\begin{aligned} &\mathrm{Q}_{\phi\Pi}\gamma_i = \gamma_i, \qquad \mathrm{Q}_{\phi\Pi}\gamma^\mu_{\alpha_S} = \gamma^\mu_{\alpha_S}, \qquad \mathrm{Q}_{\phi\Pi}\zeta_{\alpha_S} = 2\zeta_{\alpha_S} \\ &\dim(\gamma_i) = 4 - \dim(\mathrm{P}_i), \qquad \dim(\gamma^\mu_{\alpha_S}) = 2, \qquad \dim(\zeta_{\alpha_S}) = 2. \end{aligned} \quad (15\,b)$$

The external fields $\eta(x)$ are coupled according to the new Lagrangian

$$\mathscr{L}^{(\eta)}(\psi, \eta)(x) = \mathscr{L}(\psi)(x) + (\gamma_i P_i + \gamma^{\mu}_{\alpha s} P^{\alpha s}_{\mu} + \zeta_{\alpha s} \bar{P}^{\alpha s})(x) \tag{16}$$

which is still $\Phi.\Pi$. neutral and invariant under Eq. (12) due to the property

$$\mathscr{S}^{(P)} P_i(x) = \mathscr{S}^{(P)} P^{\mu}_{\alpha s}(x) = \mathscr{S}^{(P)} \bar{P}_{\alpha s}(x) = 0 . \tag{17}$$

Of course the field vector $\gamma(x)$ is assumed to have components only in the non-identity factors which arise from the complete reduction of the representation t^{α}_{ij} into irreducible constituents.

With the aid of the η fields the invariance of the theory under the Slavnov transformations Eq. (12) can be written in terms of the classical action functional

$$\Gamma^{Cl} = \int dx\, \mathscr{L}(\psi, \eta)(x) . \tag{18}$$

in a more compact way:

$$(\mathscr{S}\Gamma^{Cl})(\eta, \psi) \equiv \int dx \left\{ \frac{\delta}{\delta\varphi_i(x)} \Gamma^{Cl} \frac{\delta}{\delta\gamma_i(x)} \Gamma^{Cl} + \frac{\delta}{\delta\mathscr{A}^{\mu}_{\alpha s}(x)} \Gamma^{Cl} \frac{\delta}{\delta\gamma^{\alpha s}_{\mu}(x)} \Gamma^{Cl} \right.$$
$$+ \frac{\delta}{\delta\mathscr{A}^{\mu}_{\alpha A}(x)} \Gamma^{Cl} \partial^{\mu} \bar{c}_{\alpha A}(x)$$
$$\left. + \frac{\delta\Gamma^{Cl}}{\delta c_{\alpha}(x)} (\partial_{\mu}\mathscr{A}^{\mu}_{\alpha}(x) + \rho_{\alpha i}\varphi_i(x)) + \frac{\delta\Gamma^{Cl}}{\delta\bar{c}_{\alpha s}(x)} \frac{\delta\Gamma^{Cl}}{\delta\zeta_{\alpha s}(x)} \right\} \equiv 0 \tag{19}$$

It is also useful to linearize the expression of the Slavnov symmetry of the theory by writing it for the generator of the connected Green functions $Z^{Cl}_c(J, \eta)$. Upon introducing the sources

$$J \equiv \{ J_i, J^{\alpha}_{\mu}, \bar{\xi}^{\alpha}, \xi^{\alpha} \} \tag{20 a}$$

for the fields φ_i, $\mathscr{A}^{\alpha}_{\mu}$, c^{α}, $\bar{c}^{\alpha}$ respectively, with

$$Q_{\phi\Pi} J_i = Q_{\phi\Pi} J^{\alpha}_{\mu} = 0 , \qquad Q_{\phi\Pi} \bar{\xi}^{\alpha} = - \bar{\xi}^{\alpha} , \qquad Q_{\phi\Pi} \xi^{\alpha} = \xi^{\alpha} , \tag{20 b}$$

this generator is defined by

$$Z^{Cl}_c(J, \eta) \Big|_{J = -\frac{\delta\Gamma^{cl}}{\delta\psi}} = \Gamma^{Cl}(\psi, \eta) + \int dx\, J(x)\psi(x) \tag{21}$$

where the subscript is a short-hand notation for

$$\frac{\delta\Gamma^{Cl}}{\delta\varphi_i(x)} = - J_i(x) , \quad \frac{\delta\Gamma^{Cl}}{\delta\mathscr{A}^{\mu}_{\alpha}(x)} = - J^{\alpha}_{\mu}(x) ; \quad \frac{\delta\Gamma^{Cl}}{\delta c_{\alpha}(x)} = \bar{\xi}^{\alpha}(x) , \quad \frac{\delta\Gamma^{Cl}}{\delta\bar{c}_{\alpha}(x)} = \xi^{\alpha}(x) \tag{22 a}$$

and

$$\frac{\delta\Gamma^{Cl}}{\delta\eta(x)} = \frac{\delta Z^{Cl}_c}{\delta\eta(x)} . \tag{22 b}$$

In terms of the functional differential operator

$$\mathcal{S} \equiv \int dx \left\{ J_i(x) \frac{\delta}{\delta \gamma_i(x)} + J_\mu^{\alpha s}(x) \frac{\delta}{\delta \gamma_\mu^{\alpha s}(x)} + J_\mu^{\alpha A}(x) \partial^\mu \frac{\delta}{\delta \xi^{\alpha A}(x)} - \xi^{\alpha s}(x) \frac{\delta}{\delta \zeta^{\alpha s}(x)} - \bar{\xi}_\alpha(x) \left(\partial^\mu \frac{\delta}{\delta J_\alpha^\mu(x)} + \rho_i^\alpha \frac{\delta}{\delta J_i(x)} \right) \right\} \quad (23)$$

the S. I. is now

$$\mathcal{S} Z_c^{Cl}(J, \eta) = 0 . \quad (24)$$

Clearly these functional expressions can be given a meaning also beyond the tree approximation, hence the renormalization program of the models will be based upon the quantum extension of Eqs. (19) (24).

Before ending this section let us remark that Eq. (24) implies:

$$\mathcal{S}^2 Z_c^{Cl}(J, \eta) = - \int dx\, \bar{\xi}_\alpha(x) \left(\delta^{\alpha\alpha_s} \partial^\mu \frac{\delta}{\delta \gamma^{\mu\alpha_s}(x)} + \delta^{\alpha\alpha_A} \Box \frac{\delta}{\delta \xi^{\alpha_A}(x)} + \rho_i^\alpha \frac{\delta}{\delta \gamma_i(x)} \right) Z_c^{Cl}(J, \eta) = 0 , \quad (25)$$

which translated by Eqs. (22) for the vertex functional $\Gamma^{Cl}(\psi, \eta)$ becomes

$$(\mathcal{S}^2 \Gamma^{Cl})(\psi, \eta) = - \int dx \left\{ \partial_\mu \frac{\delta \Gamma^{Cl}}{\delta \gamma_\mu^{\alpha s}(x)} \delta^{\alpha s \alpha} + \Box\, \bar{c}_{\alpha_A}(x) \delta^{\alpha_A \alpha} + \rho_i^\alpha \frac{\delta \Gamma^{Cl}}{\delta \gamma_i(x)} \right\} \frac{\delta \Gamma^{Cl}}{\delta c^\alpha(x)} (\psi, \eta) = 0 . \quad (26)$$

As we shall see in the following the validity of Eq. (26) to all orders of the perturbation expansion fixes the wave equation of the $\bar{c}_\alpha$ fields.

3. STABILITY OF THE CLASSICAL MODELS

Our task in this Section is to check whether any solution of the S. I. in a neighborhood of any Lagrangian built according to the prescriptions of Section 2, can be brought back to this historical form by a suitable linear transformation of the fields, whose general expression is

$$\mathcal{A}_\alpha^\mu \Rightarrow Z_\alpha^\beta \mathcal{A}_\beta^\mu , \quad (27\,a)$$

$$\bar{c}_\alpha \Rightarrow z_\alpha^\beta \bar{c}_\beta , \quad (27\,b)$$

$$\varphi_i \Rightarrow \sigma_{ij}(\varphi_j + p_j) , \quad (27\,c)$$

$$c_\alpha \Rightarrow \bar{Z}_\alpha^\beta c_\beta . \quad (27\,d)$$

To preserve the S. I., modulo a change of the gauge parameters ρ_i^α, we must impose on Eqs. (27) the restrictions

$$\bar{Z}_\beta^\alpha = Z_\beta^\alpha , \quad (28\,a)$$

$$Z_{\beta_A}^{\alpha_A} = z_{\beta_A}^{\alpha_A} , \quad (28\,b)$$

and require that they be implemented by the external fields transformations

$$\gamma^{\alpha s}_{\mu} \Rightarrow Z'^{\alpha s}_{\beta s}\gamma^{\beta s}_{\mu}, \tag{29 a}$$

$$\gamma_i \Rightarrow \sigma^{-1}_{ji}\gamma_j, \tag{29 b}$$

$$\zeta^{\alpha s} \Rightarrow z'^{\alpha s}_{\beta s}\zeta^{\beta s}; \tag{29 c}$$

with

$$Z^{\alpha s}_{\beta s} = (Z'^{-1})^{T_{\alpha s}}_{\beta s}, \tag{30 a}$$

$$z^{\alpha s}_{\beta s} = (z'^{-1})^{T_{\alpha s}}_{\beta s} \tag{30 b}$$

The proof is carried out perturbatively to first order in a « small » quantity ε and is articulated in two steps. First we shall discuss the external field dependent part of the action, thereby having informations on the structure constants of G and its representation on the matter fields space. We shall see that the perturbed structure constants coincide, after a transformation as in Eq. (27 *b*), with the unperturbed ones, while the matter fields representation gives rise to a possible instability which is analyzed in the text. Secondly, we shall be concerned with the external field independent part of the action and its parameters as compared to those of the historical model. This investigation leads to the individuation of a canonical form of the classical action, different from the historical one, which turns out to be stable under perturbations. This last point is also amply commented upon in the text.

According to the above illustrated procedure we write the perturbed action as

$$\Gamma(\psi, \eta) = \Gamma^{\mathrm{Cl}}(\psi) + \varepsilon\Gamma_1(\psi) + \int dx\left(\gamma_i\mathscr{P}_i + \gamma^{\mu}_{\alpha s}\mathscr{P}^{\alpha s}_{\mu} + \zeta_{\alpha s}\bar{\mathscr{P}}^{\alpha s}\right)(x) \tag{31}$$

where

$$\mathscr{P}_i(x) = (T^{\alpha}_{ij}\varphi_j(x) + Q^{\alpha}_i)\bar{c}_{\alpha}(x), \tag{32 a}$$

$$\mathscr{P}^{\mu}_{\alpha s}(x) = \left(q^{\beta}_{\alpha s}\partial^{\mu} + \Theta^{\beta\gamma}_{\alpha s}\mathscr{A}^{\mu}_{\gamma}(x)\right)\bar{c}_{\beta}(x), \tag{32 b}$$

$$\bar{\mathscr{P}}_{\alpha s}(x) = \frac{1}{2}F^{\beta\gamma}_{\alpha s}\bar{c}_{\beta}(x)\bar{c}_{\gamma}(x). \tag{32 c}$$

Eqs. (32 *a*), (32 *b*) and (32 *c*) are first order perturbations of Eqs. (10 *a*), (10 *b*) and (10 *c*) i. e. explicitely

$$Q^{\alpha}_i = t^{\alpha}_{ij}q_j + \varepsilon\bar{Q}^{\alpha}_i, \tag{33 a}$$

$$T^{\alpha}_{ij} = t^{\alpha}_{ij} + \varepsilon\tau^{\alpha}_{ij}, \tag{33 b}$$

$$\Theta^{\beta\gamma}_{\alpha s} = f^{\beta\gamma}_{\alpha s} + \varepsilon\theta^{\beta\gamma}_{\alpha s}, \tag{33 c}$$

$$q^{\beta}_{\alpha s} = \delta^{\beta}_{\alpha s} + \varepsilon\bar{q}^{\beta}_{\alpha s}, \tag{33 d}$$

$$F^{\beta\gamma}_{\alpha s} = f^{\beta\gamma}_{\alpha s} + \varepsilon C^{\beta\gamma}_{\alpha s}. \tag{33 e}$$

The validity of the S. I. for the functional in Eq. (31) can be written as

$$\mathcal{S}^{(\mathcal{P})}\Gamma(\psi, \eta) = 0 \tag{34}$$

where the operator $\mathcal{S}^{(\mathcal{P})}$ is the analog of the one given in Eq. (12) with the substitution $P_i \to \mathcal{P}_i$.

Factoring out the coefficients of the external fields in Eq. (34) gives a set of consistency conditions which are the first order ε expansions of

$$\mathcal{S}^{(\mathcal{P})}\bar{\mathcal{P}}_{\alpha s}(x) = 0\,, \tag{35 a}$$

$$\mathcal{S}^{(\mathcal{P})}\mathcal{P}^{\mu}_{\alpha s}(x) = 0\,, \tag{35 b}$$

$$\mathcal{S}^{(\mathcal{P})}\mathcal{P}_i(x) = 0\,, \tag{35 c}$$

by which we can analyze the behaviour of the perturbed structure constants $F^{\beta\gamma}_{\alpha s}$ and of the representatives $\Theta^{\beta\gamma}_{\alpha s}$, T^{α}_{ij}.

Indeed Eq. (35 *a*) explicitely reads

$$f^{\gamma s\beta}_{\alpha s}C^{\rho\delta}_{\gamma s} + f^{\gamma s\rho}_{\alpha s}C^{\delta\beta}_{\gamma s} + f^{\gamma s\delta}_{\alpha s}C^{\beta\rho}_{\gamma s} + f^{\rho\delta}_{\gamma s}C^{\gamma s\beta}_{\alpha s} + f^{\delta\beta}_{\gamma s}C^{\gamma s\rho}_{\alpha s} + f^{\beta\rho}_{\gamma s}C^{\gamma s\delta}_{\alpha s} = 0 \tag{36}$$

whose general solution is

$$\begin{aligned} C^{\beta\rho}_{\alpha s} &= f^{\beta\gamma s}_{\alpha s}\hat{C}^{\rho}_{\gamma s} - f^{\rho\gamma s}_{\alpha s}\hat{C}^{\beta}_{\gamma s} - f^{\beta\rho}_{\gamma s}\hat{C}^{\gamma s}_{\alpha s}\,, \\ \hat{C}^{\beta}_{\alpha s} &= \frac{1}{f^2} f^{\lambda s}_{\gamma\alpha s}C^{\gamma\beta}_{\lambda s}\,, \qquad f^2 = f^{\beta\gamma}_{\alpha}f^{\alpha}_{\gamma\beta} \end{aligned} \tag{37}$$

or equivalently

$$F^{\beta\gamma}_{\alpha s} = (1 - \varepsilon\hat{C})^{\alpha}_{\alpha'}(1 - \varepsilon\hat{C})^{\beta}_{\beta'}f^{\alpha'\beta'}_{\delta s}(1 + \varepsilon\hat{C})^{\delta s}_{\alpha s}\,. \tag{38}$$

From this equation it is clear that the substitution (27 *b*) with

$$z^{\beta}_{\alpha s} = (1 + \varepsilon\hat{C})^{\beta}_{\alpha s} \tag{39}$$

performed on the classical action restores, to first order in ε, the original structure constants thus ensuring the stability of the semi-simple factor of G. Once this is done, Eq. (35 *b*) yields the system

$$[f^{\beta}, \theta^{\rho}]^{\sigma}_{\alpha s} - [f^{\rho}, \theta^{\beta}]^{\sigma}_{\alpha s} - f^{\beta\rho}_{\gamma s}\theta^{\gamma s\sigma}_{\alpha s} = 0\,, \tag{40 a}$$

$$\theta^{\sigma\gamma s}_{\alpha s}\delta^{\rho}_{\gamma s} + f^{\sigma\gamma s}_{\alpha s}\bar{q}^{\rho}_{\gamma s} + \bar{q}^{\gamma s}_{\alpha s}f^{\rho\sigma}_{\gamma s} = 0\,, \tag{40 b}$$

with general solutions

$$\theta^{\rho\sigma}_{\alpha s} = [f^{\rho}, \hat{\theta}]^{\sigma}_{\alpha s}\,, \tag{41 a}$$

$$\bar{q}^{\rho}_{\alpha s} + \hat{\theta}^{\rho}_{\alpha s} = k\delta^{\rho}_{\alpha s}\,. \tag{41 b}$$

which allow us to write Eqs. (33 *c*), (33 *d*) in the form

$$\Theta^{\beta\gamma}_{\alpha s} = (1 - \varepsilon\hat{\hat{\theta}})^{\delta}_{\alpha s}f^{\beta\eta}_{\delta}(1 + \varepsilon\hat{\hat{\theta}})^{\gamma}_{\eta}\,, \tag{42 a}$$

$$q^{\beta}_{\alpha s} = (1 - \varepsilon\hat{\hat{\theta}})^{\gamma}_{\alpha s}\delta^{\beta}_{\gamma}\,, \tag{42 b}$$

with

$$\hat{\hat{\theta}}^{\delta}_{\alpha s} = \hat{\theta}^{\delta}_{\alpha s} - k\delta^{\delta}_{\alpha s}\,. \tag{43}$$

Here again we see that these perturbations can be reabsorbed by the photon fields renormalization in Eq. (27 *a*) with the choice

$$Z^{\beta}_{\alpha_S} = (1 - \varepsilon\hat{\hat{\theta}})^{\beta}_{\alpha_S} \tag{44}$$

The stability of the adjoint representation carried by these fields is thus proved ; notice that, up to now, the abelian factor of G and its representations have played no role.

The last equations derived from Eq. (35 *c*) are

$$[t^{\alpha}, \tau^{\beta}]_{ij} + [\tau^{\alpha}, t^{\beta}]_{ij} - f^{\alpha\beta}_{\gamma}\tau^{\gamma}_{ij} = 0 \tag{45 a}$$

$$(t^{\alpha}\bar{Q}^{\beta})_i - (t^{\beta}\bar{Q}^{\alpha})_i + (\tau^{\alpha}q^{\beta})_i - (\tau^{\beta}q^{\alpha})_i - f^{\alpha\beta}_{\gamma}\bar{Q}^{\gamma}_i = 0 \tag{45 b}$$

which are solved respectively by

$$\tau^{\alpha}_{ij} = [t^{\alpha}, \hat{\tau}]_{ij} + \delta^{\alpha}_{\beta_A}\tau^{\beta_A}_{\natural ij}\,, \tag{46 a}$$

$$\bar{Q}^{\alpha}_i = t^{\alpha}_{ij}\hat{\hat{Q}}_j + \tau^{\alpha}_{ij}q_j \tag{46 b}$$

where $\tau_{\natural}$ is an arbitrary matrix commuting with the t^{α}'s.

In terms of the expressions (46), Eqs. (33 *a*), (33 *b*) can be written, to first order in ε, as

$$T^{\alpha}_{ij} = (1 - \varepsilon\hat{\tau})_{ik}t^{\alpha}_{kl}(1 + \varepsilon\hat{\tau})_{lj} + \varepsilon\tau^{\beta_A}_{\natural ij}\delta^{\alpha}_{\beta_A} \tag{47}$$

and

$$Q^{\alpha}_i = (1 - \varepsilon\hat{\tau})_{ij}t^{\alpha}_{jk}q_k + \varepsilon t^{\alpha}_{ij}(\hat{\hat{Q}} + \hat{\tau}q)_j + \varepsilon\delta^{\alpha}_{\beta_A}\tau^{\beta_A}_{\natural ij}q_j\,. \tag{48}$$

These last equations are the necessary complement to exhibit the possible pathologies.

Let us first remark that if

$$\tau^{\beta_A}_{\natural ij} = \hat{C}^{\beta_A}_{\gamma_A}t^{\gamma_A}_{ij} \tag{49}$$

then the substitutions (27 *a*), (27 *c*) with

$$\sigma_{ij} = (1 - \varepsilon\hat{\tau})_{ij} \tag{50 a}$$

$$p_i = \hat{\hat{Q}}_i - \varepsilon\hat{\tau}_{ij}q_j \tag{50 b}$$

and

$$Z^{\beta_A}_{\gamma_A} = (1 - \varepsilon\hat{C})^{\beta_A}_{\gamma_A} \tag{50 c}$$

performed on the classical action compensate exactly the remaining perturbations in Eqs. (33). This would imply the stability of the complete algebra G and its representation on the field space. If Eq. (49) is not satisfied we remain with a single possible source of instability in the term $\tau^{\alpha_A}_{\natural}$, related to the abelian factor of G. The analysis of this impediment now proceeds by considering the external field independent part of the action, which likewise must be invariant. Recalling that the instability comes from the φ_i fields representation, we shall only consider the global transformations

$$\delta\varphi_i(x) = T^{\alpha}_{ij}(\varphi_j(x) + Q_j)\omega_{\alpha} \tag{51}$$

which must leave invariant an ε-perturbation of $W(\varphi)$, the classical gauge invariant action corresponding to $\mathscr{L}_{\text{inv}}(\Phi, \mathscr{A}_\mu)$ (Eq. (4)) at the point $\mathscr{A}_\mu^\alpha \equiv 0$. Indeed, as shown in Appendix B, the perturbed Slavnov transformations induce automatically the appropriate $\Phi.\Pi$. gauge fixing term, which is a perturbation of the original one; the remaining part, invariant under these transformations, contributes (except for the abelian photons mass term which we shall see later) only to the gauge invariant action where the dependence on the matter fields is completely identified by their global transformation properties.

In this way, the mentioned invariance condition puts further constraints on the admissible T_{ij}^α matrices and leads to the net result, proved in Appendix A, that the general T_{ij}^α is not equivalent to t_{ij}^α if and only if we can find at least one $\tau_\natural^{\alpha_A}$, linearly independent from the t^α's and commuting with them, such that $w(\varphi)$ is invariant under the transformation

$$\delta\varphi_i(x) = \tau_{\natural ij}^{\alpha_A}(\varphi_j(x) + Q_j)\omega_{\alpha_A} . \tag{52}$$

Remark that this result requires quite severe specifications the matter fields must meet for $\tau_\natural^{\alpha_A}$ to be necessarily zero; in particular it excludes the presence of baryonic or leptonic components of the matter field vector ϕ. In fact, in such a case, the generator of the baryon or lepton charge immediately furnishes a $\tau_\natural$ matrix which violates stability.

Before taking too seriously this source of instability let us recall that the possible existence in the theory of « hidden » parameters, may spoil its renormalizability if these parameters are explicitely needed to compensate some renormalization parts. Since we shall show in the next paper that the t^{α_A}'s are not renormalized thanks to a mechanism similar to the one leading to the well known Ward identity of Q. E. D., we can from now on forget about the $\tau_\natural^\alpha$ problem.

Concerning the stability of the external fields independent part of the action in Eq. (31), it follows from Appendix B, that the most general Slavnov invariant $\Phi.\Pi$. neutral functional of maximum degree four is given by

$$\Gamma(\psi) = \Gamma_{\text{inv}}(\phi, \mathscr{A}_\mu) + \int dx \left\{ -\frac{1}{2} g_\alpha \Lambda^{\alpha\beta} g_\beta + c_\alpha \Lambda^{\alpha\beta}(\mathscr{M}\bar{c})_\beta - \left(\frac{\mathscr{A}_{\nu\alpha}\mu^{\alpha\beta}\mathscr{A}_\beta^\nu}{2} + c_\alpha \mu^{\alpha\beta}\bar{c}_\beta \right) - \chi_i\varphi_i \right\}(x) \tag{53}$$

where $\Lambda^{\alpha\beta}$ and $\mu^{\alpha\beta}$ are real, symmetric matrices and $\Lambda^{\alpha\beta}$ is positive definite, $\mu^{\alpha\beta s} = 0$, $\mu^{\alpha_A\beta}\rho_{i\beta} = t_{ij}^{\alpha_A}\chi_j$, $\chi_j q_j^\alpha = 0$;

$$g_\alpha(x) = \partial_\mu \mathscr{A}_\alpha^\mu(x) + \rho_{\alpha i}\varphi_i(x) , \tag{54 a}$$

$$\mathscr{M}_\alpha^\beta(x, y) = \frac{\delta g_\alpha(x)}{\delta\omega_\beta(y)} . \tag{54 b}$$

and

$$\left.\frac{\partial\Gamma(\psi)}{\partial\varphi_i}\right|_{\substack{\psi=0\\ \eta=0}} = 0\,. \tag{55}$$

Observe that if

$$\mu^{\alpha A\beta}\rho_{\beta i} = t_{ij}^{\alpha A}\chi_j = 0 \tag{56}$$

the last term on the r. h. s. of Eq. (53) can be included in $\Gamma_{\text{inv}}(\phi, \mathscr{A}_\mu)$. It follows that all field renormalizations being by now fixed, $\Gamma^{\text{Cl}}(\psi)$ turns out to be stable under perturbations leaving the Slavnov transformations invariant only if it has the canonical form given in Eq. (53).

Comparing with Eq. (8) we notice that the matrix $\Lambda^{\alpha\beta}$ now replaces the parameter $\frac{1}{k}$ which thus appears to be a very particular choice of the gauge function; moreover the $\mu^{\alpha A\beta A}$ mass term in Eq. (53) is completely absent in the historical model. Some Authors (Ref. [12]) avoid these new parameters by prescribing the wave equation for the Φ.Π. fields as it is deduced from the classical model, thus ruling out massive Q. E. D. and in general all the models containing it. For the abelian H. K. model the introduction of such mass terms, although compatible with the Slavnov identity, spoils the gauge invariance of the model (Ref. [20]). It thus follows that a definition of the gauge models comprehensive of massive Q. E. D. must allow such mass terms with some special constraint. We shall see in the following Section that these constraints may be put in the form of mass normalization conditions which essentially amount to giving arbitrary masses to those Φ.Π. fields which are free.

A few remarks on the matrix $\Lambda^{\alpha\beta}$ appearing in Eq. (53) are now appropriate; $\Lambda^{\alpha\beta}$ can always be written as $\Lambda^{\alpha\beta} = \frac{1}{k}(h^{\text{T}}h)^{\alpha\beta}$ with k a positive number and h invertible. After the substitutions

$$\begin{gathered} c^\alpha \to (hc)^\alpha\,, \quad \bar{c}^\alpha \to (h\bar{c})^\alpha\,, \quad \mathscr{A}_\mu^\alpha \to h_\beta^\alpha \mathscr{A}_\mu^\beta\,, \quad t_{ij}^\alpha \to t_{ij}^\beta h_\beta^{-1\alpha}\,, \\ f_\alpha^{\beta\gamma} \to h_\alpha^{\alpha'} h_{\beta'}^{-1\beta} h_{\gamma'}^{-1\gamma} f_{\alpha'}^{\beta'\gamma'}\,, \quad \rho_i^\alpha \to h_\beta^\alpha \rho_i^\beta\,, \quad \mu^{\alpha\beta} \to h_\gamma^{-1\alpha} h_\gamma^{-1\beta} \mu^{\gamma\sigma} \end{gathered} \tag{57}$$

the integral on the r. h. s. of Eq. (53) becomes

$$\frac{1}{k}\int dx \left\{ -\frac{1}{2} g^\alpha g_\alpha + c_\alpha(\mathscr{M}\bar{c})^\alpha - \left(\frac{\mathscr{A}_{\nu\alpha}\mu^{\alpha\beta}\mathscr{A}_\beta^\nu}{2} + c_\alpha \mu^{\alpha\beta}\bar{c}_\beta \right) \right\}(x)\,. \tag{58}$$

Now the matrix $\Lambda^{\alpha\beta}$ does not appear explicitely in Eq. (58), but its arbitrariness has been transferred into the representations of $\mathscr{G}$ and the gauge parameters as it is clear from Eq. (57).

The alternatives Eq. (53) or Eq. (58) for the classical action are perfectly equivalent for what concerns the definition of the model and its stability, but while in Eq. (53) the wave operator of the transverse photons and matter fields maintains its naive form which allows an easy particle interpretation for the physical sector, the choice in Eq. (58) is more convenient when performing explicit calculations in the unphysical sector, thus allowing a simpler analysis of the unitarity of the scattering matrix.

4. MASS DEGENERACY IN THE GHOST SECTOR

This Section is devoted to the analysis of the unphysical one-particle states of the model with the aim of characterizing these states through the S. I. and the suitable mass normalization conditions.

The main reasons for this investigation are twofold: first to complete the stability check of the previous Section and second to state the rules which, together with the S. I., define the renormalized model.

The unphysical one-particle states are those associated with the Φ.Π. fields (Φ.Π. sector) and a subset of the scalar sector spanned by the coupled longitudinal photon-scalar matter fields. In the original model outlined in Section 2 the fields in the scalar sector comprehend the physical fields, whose masses are the positive eigenvalues of the matrix m_{ij} (Eq. 6) and the unphysical ones: the longitudinal photons and the Goldstone bosons. If N is the dimension of the algebra, n_S the number of the independent scalar matter fields, $n_{\mathscr{G}}$ the number of the Goldstone bosons; then the number of physical and unphysical fields in the scalar sector is respectively $n_S - n_{\mathscr{G}}$ and $N + n_{\mathscr{G}}$.

It is suggested by the B. R. S. analysis of the abelian H. K. model [*18*] that the above distinction of the scalar sector into physical and unphysical fields and the gauge invariance of the theory is not always extendible to a generic Slavnov invariant Lagrangian. To maintain gauge invariance the S. I. must be endowed with the normalization condition (mass degeneracy condition) that $N + n_{\mathscr{G}}$ states of the scalar sector be degenerate in mass with the N states of the Φ.Π. sector. It is precisely these $N + n_{\mathscr{G}}$ fields which we shall call unphysical.

Having in mind that our task is that of discussing the full renormalized model, we shall push the analysis of the masses of the particles in the scalar and Φ.Π. sectors beyond the tree approximation level, thus getting results valid to all orders.

Let us define the tools by which we shall work out our analysis to all orders.

We have seen in Section 3 that the general solution of Eq. (26) in the tree approximation leads, up to a field redefinition, to the following wave equation for the $\bar{c}$ fields:

$$\frac{\delta}{\delta c_\alpha(x)}\Gamma(\psi, \eta) = \left(\delta^{\alpha\beta_S}\partial_\mu \frac{\delta}{\delta\gamma_\mu^{\beta_S}(x)} + \rho_i^\alpha \frac{\delta}{\delta\gamma_i(x)}\right)\Gamma(\psi, \eta) + \delta^{\alpha\beta_A}\,\Box\,\bar{c}_{\beta_A}(x) + \mu^{\alpha\beta_A}\bar{c}_{\beta_A}(x)\,. \quad (59)$$

It is clear that Eq. (59) solves Eq. (26) also beyond the tree approximation and we shall prove in the next paper that it can be kept to all orders of the perturbation expansion.

Defining

$$\left.\frac{\delta^2\Gamma(\psi,\eta)}{\delta\bar{\bar{c}}_\beta(p)\delta\gamma^\mu_{\alpha s}(0)}\right|_{\psi,\eta=0} = ip_\mu\sigma^{\alpha s\beta}(p)\,,\quad \sigma^{\alpha_A\beta_A} = \delta^{\alpha_A\beta_A}\,, \tag{60 a}$$

$$\left.\frac{\delta^2\Gamma(\psi,\eta)}{\delta\bar{\bar{c}}_\beta(p)\delta\gamma_i(0)}\right|_{\psi,\eta=0} = \chi^\beta_i(p)\,, \tag{60 b}$$

we can write the Fourier transformed wave matrix of the Φ.Π. fields in the form

$$\Delta^{\alpha\beta}(p) = \left.\frac{\delta^2\Gamma(\psi,\eta)}{\delta\bar{\bar{c}}_\beta(p)\delta c_\alpha(0)}\right|_{\psi,\eta=0} = -\sigma^{\alpha\beta}p^2 + \rho^\alpha_i\chi^\beta_i + \mu^{\alpha_A\beta_A}\delta^{\alpha\alpha_A}\delta^{\beta\beta_A}\,. \tag{61}$$

In the scalar sector, we introduce the matrices

$$D^{\alpha\beta}_{\mu\nu}(p) = \left.\frac{\delta^2\Gamma(\psi,\eta)}{\delta\tilde{\mathscr{A}}^\mu_\alpha(p)\delta\mathscr{A}^\nu_\beta(0)}\right|_{\psi,\eta=0} \tag{62 a}$$

$$D^\alpha_{\mu j}(p) = \left.\frac{\delta^2\Gamma(\psi,\eta)}{\delta\tilde{\mathscr{A}}^\mu_\alpha(p)\delta\varphi_j(0)}\right|_{\psi,\eta=0} = -D^\alpha_{j\mu}(p) \tag{62 b}$$

$$d_{ij}(p) = \left.\frac{\delta^2\Gamma(\psi,\eta)}{\delta\tilde{\varphi}_i(p)\delta\varphi_j(0)}\right|_{\psi,\eta=0} \tag{62 c}$$

and define

$$\begin{pmatrix} D^{\alpha\beta} & D^\alpha_j \\ D^\beta_i & d_{ij}\end{pmatrix} = \begin{pmatrix} -i\dfrac{p^\mu}{p} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} D^{\alpha\beta}_{\mu\nu} & D^\alpha_{\mu j} \\ D^\beta_{i\nu} & d_{ij}\end{pmatrix}\begin{pmatrix} i\dfrac{p^\nu}{p} & 0 \\ 0 & 1\end{pmatrix} \tag{63}$$

The Fourier transformed wave equations for the Φ.Π. and the scalar sectors are respectively:

$$\Delta^{\alpha\beta}w_\beta = 0 \tag{64 a}$$

and

$$D^{\alpha\beta}v_\beta + D^\alpha_j u_j = 0 \tag{64 b}$$

$$D^\beta_i v_\beta + d_{ij}u_j = 0 \tag{64 c}$$

The masses of the particles in these sectors are given by the solutions of the equations:

$$\det|\Delta^{\alpha\beta}|(p) = 0 \tag{65 a}$$

$$\det\left|\begin{pmatrix} D^{\alpha\beta} & D^\alpha_j \\ D^\beta_i & d_{ij}\end{pmatrix}\right|(p) = 0 \tag{65 b}$$

with the parameters of the theory so chosen as to ensure their positivity.

The S. I. yields the following relations among the wave matrices, Eqs. (61), (63)

$$D^{\alpha}_{i}\chi^{\beta}_{i} - pD^{\alpha\gamma}\sigma^{\beta}_{\gamma} + p\Delta^{\alpha\beta} = 0 \qquad (66\ a)$$

$$d_{ij}\chi^{\alpha}_{j} - pD^{\gamma}_{i}\sigma^{\alpha}_{\gamma} + \rho_{i\gamma}\Delta^{\gamma\alpha} = 0\,. \qquad (66\ b)$$

Having now set the stage, we begin at the classical level where the mass degeneracy condition necessary to preserve the gauge invariance of the model can be immediately individuated and translated into a relation among the parameters of the Lagrangian.

In the tree approximation, where $\sigma^{\alpha\beta} = \delta^{\alpha\beta}$ and $\chi^{\alpha}_{i} = q^{\alpha}_{i}$, Eq. (61) becomes

$$\Delta^{\alpha\beta} = -\,p^{2}\delta^{\alpha\beta} + \rho^{\alpha}_{i}q^{\beta}_{i} - \delta^{\alpha}_{\alpha_{A}}\delta^{\beta}_{\beta_{A}}\mu^{\alpha_{A}\beta_{A}} \qquad (67)$$

and Eqs. (66 *a*), (66 *b*) are solved by

$$d_{ij} = p^{2}\delta_{ij} - m_{ij} - \rho^{\alpha}_{i}\rho_{\alpha j}\,, \qquad (68\ a)$$

$$D^{\alpha}_{i} = p(q^{\alpha}_{i} - \rho^{\alpha}_{i}) \qquad (68\ b)$$

$$D^{\alpha\beta} = -\,p^{2}\delta^{\alpha\beta} + q^{\alpha}_{i}q^{\beta}_{i} + \delta^{\alpha}_{\alpha_{A}}\delta^{\beta}_{\beta_{A}}\mu^{\alpha_{A}\beta_{A}} \qquad (68\ c)$$

with

$$m_{ij}q^{\alpha}_{j} = \delta^{\alpha}_{\alpha_{A}}\mu^{\alpha_{A}\beta_{A}}\rho_{\beta_{A}i}\,. \qquad (69)$$

Here Eq. (69) shows that for a generic choice of the $\mu^{\alpha_{A}\beta_{A}}$ matrix, the spontaneous symmetry breaking condition (Eq. (6)), upon which the original model was built, is violated, and that it can be restored by imposing

$$\mu^{\alpha_{A}\beta_{A}}\rho_{\beta_{A}i} = 0\,. \qquad (70)$$

From this equation it immediately follows that the $(\mu\bar{c})^{\alpha_{A}}$ fields are free since by Eqs. (59), (22), (70) we obtain

$$\square\,(\mu\bar{c})^{\alpha_{A}}(x) - \mu^{\alpha_{A}\beta_{A}}(\mu\bar{c})^{\beta_{A}}(x) = 0\,. \qquad (71)$$

The relevance of Eq. (71) for the gauge invariance of the theory can be understood by the following heuristic argument.

Let ψ_{phys} be the set of the physical fields; the Slavnov variations $\mathscr{S}^{(P)}\psi_{\text{phys}}$, from Eqs. (10 *a*), (10 *b*), (10 *c*) do not have linear contributions in the $\bar{c}$ fields. This insures the annihilation of the mass-shall Green functions with a $\mathscr{S}^{(P)}\psi_{\text{phys}}$ vertex.

The gauge invariance of the model, i. e. the independence of the mass-shall Green functions involving only physical fields from the k parameter in Eq. (8), requires that the insertion of the gauge fixing part of the Lagrangian $\left(\dfrac{g_{\alpha}g^{\alpha}}{2} - c_{\alpha}(\mathscr{M}\bar{c})^{\alpha}\right)$ into such a Green function, disconnects. Now the

Slavnov invariance says that for any given product of fields operators $\prod_{i=1}^{n} O_i(x_i)$, the following identity holds:

$$\sum_{j=1}^{n}(-1)^{f_j}\left\langle \prod_{i=1}^{j-1} O_i(x_i)\mathscr{I}^{(P)}O_j(x_j)\prod_{k=j+1}^{n} O_k(x_k)\right\rangle_+ = 0\,, \tag{72}$$

and hence

$$\left\langle \mathscr{I}^{(P)}\left[(c_\alpha g^\alpha)(x)\prod_{i=1}^{n}\psi_{i_{\text{phys}}}(x_i)\right]\right\rangle_+ = \left\langle\left[g^\alpha g_\alpha - c^\alpha(\mathscr{M}c)_\alpha\right](x)\prod_{i=1}^{n}\psi_{i_{\text{phys}}}(x_i)\right\rangle_+$$
$$- \left\langle (c_\alpha g^\alpha)(x)\mathscr{I}^{(P)}\left(\prod_{i=1}^{n}\psi_{i_{\text{phys}}}(x_i)\right)\right\rangle_+ = 0\,. \tag{73}$$

Furthermore, using the $\Phi.\Pi.$ equation of motion Eq. (59), we have

$$\left\langle c_\alpha(\mathscr{M}\bar{c})^\alpha(x)\prod_{i=1}^{n}\psi_{i_{\text{phys}}}(x_i)\right\rangle_+$$
$$= \left\langle c_{\alpha_A}(\mu\bar{c})^{\alpha_A}(x)\prod_{i=1}^{n}\psi_{i_{\text{phys}}}(x_i)\right\rangle_+ + \text{disc. terms} \tag{74}$$

where the first term on the r. h. s. of Eq. (74) is also disconnected, since by Eq. (71) $(\mu\bar{c})_{\alpha_A}$ is a free field. Comparing Eq. (74) with Eq. (73) and recalling that the Slavnov variations of the ψ_{phys} have no pole on the mass-shell, we get the desired result.

Note that from the Slavnov invariance alone we arrive at Eq. (73) which is not the statement of gauge invariance due to the 1/2 missing factor in the term $g^\alpha g_\alpha$. This fact has forced us to prove that the two terms $g^\alpha g_\alpha$ and $c^\alpha(\mathscr{M}\bar{c})_\alpha$ separately do not contribute to the mass-shell connected Greens functions and for this proof the validity of Eq. (70) and Eq. (71) is essential.

The condition given in Eq. (70) requires a short comment. In order to avoid massless particles in the model, the matrices

$$q_i^\alpha q_i^\beta + \delta^\alpha_{\alpha_A}\delta^\beta_{\beta_A}\mu^{\alpha_A\beta_A} \tag{75 a}$$

and

$$m_{ij} + \rho_i^\alpha \rho_{\alpha j} \tag{75 b}$$

must be positive definite. This is not compatible with Eq. (70) for an arbitrary choice of the ρ_i^α parameters (4). For the sake of brevity we shall only remark that 't Hooft's choice

$$\rho_i^\alpha = \lambda^{(\alpha)} q_i^\alpha \tag{76}$$

(4) In particular, defining through the ρ_i^α's a set of vectors in the scalar matter field space, the mass positivity condition together with Eq. (70) requires that the number of linearly independent vectors in this set be equal to $n_{\mathscr{G}}$, i. e. that of the q^α vectors. However this is not a sufficient condition.

is consistent with Eq. (70) and the absence of massless particles in the scalar sector.

We shall now investigate, to all orders of perturbation theory, the mentioned mass degeneracy condition between the particles of the scalar and the $\Phi.\Pi.$ sectors, showing that it is equivalent to Eq. (70). We shall here show that this equation implies the degeneracy of $N + n_{\mathscr{G}}$ masses of the scalar sector with the $\Phi.\Pi.$ masses, leaving to Appendix C the proof of its necessity.

Assume that for a given $p(p^2 > 0)$ the system Eqs. (64 *b*, *c*) has solution $\{ v_\alpha, u_i \}$; multiplying to the left Eqs. (66 *a*. *b*) by v_α, u_i respectively we get

$$v_\alpha D^\alpha_i \chi^\beta_i - p v_\alpha D^{\alpha\gamma}\sigma^\beta_\gamma + p v_\alpha \Delta^{\alpha\beta} = 0 \quad (77\,a)$$

$$u_i d_{ij} \chi^\alpha_j - p u_i D^\gamma_i \sigma^\alpha_\gamma + u_i \rho_{\gamma i} \Delta^{\gamma\alpha} = 0 \quad (77\,b)$$

and making use of Eqs. (64 *b*, *c*) we arrive at

$$v_\alpha D^\alpha_i \chi^\beta_i + p u_j D^\gamma_j \sigma^\beta_\gamma + p v_\alpha \Delta^{\alpha\beta} = 0 \quad (78\,a)$$

$$- v_\beta D^\beta_j \chi^\alpha_j - p u_j D^\gamma_j \sigma^\alpha_\gamma + u_j \rho_{\gamma j} \Delta^{\gamma\alpha} = 0 \quad (78\,b)$$

which, after summation, yield

$$(p v_\gamma + u_i \rho_{\gamma i}) \Delta^{\gamma\alpha} = 0\,. \quad (79)$$

Thus, provided

$$p v_\gamma + u_i \rho_{\gamma i} \neq 0 \quad (80)$$

we have a solution of Eq. (64 *a*), hence to every solution of Eqs. (64 *b*, *c*) which satisfies Eq. (80) there corresponds an unphysical state in the scalar sector whose mass is degenerate with a state in the $\Phi.\Pi.$ sector.

On the other hand, when

$$p v_\gamma + u_i \rho_{\gamma i} = 0\,, \quad (81)$$

it is easily seen, by Eqs. (77 *a*, *b*) that Eqs. (64 *b*, *c*) are no longer independent. Setting

$$d_{ij} = \hat{d}_{ij} - \rho^\alpha_i \rho_{\alpha j} \quad (82)$$

and substituting into Eq. (64 *c*) we get by Eq. (81):

$$\left[\frac{1}{p} D^\alpha_i \rho_{\alpha j} + \rho^\alpha_i \rho_{\alpha j} - \hat{d}_{ij} \right] u_j = 0\,. \quad (83\,a)$$

Similarly substituting Eq. (82) and the explicit form of $\Delta^{\alpha\beta}$ (Eq. (61)) into Eq. (66 *b*) yields:

$$\frac{1}{p} D^\alpha_i + \rho^\alpha_i = \frac{1}{p^2} [\hat{d}_{ij} \chi^\beta_j + \rho_{\gamma i} \mu^{\gamma\beta}] \sigma_\beta^{-1\alpha} \quad (83\,b)$$

since $\sigma^{\alpha\beta}$ is invertible in the tree approximation. The insertion of Eq. (83 *b*) into Eq. (83 *a*) finally gives

$$\hat{d}_{ij} \left[\frac{1}{p^2} \chi^\beta_j \sigma_\beta^{-1\gamma} \rho_{\gamma k} u_k - u_j \right] + \frac{1}{p^2} \rho_{\beta i} \mu^{\beta\gamma} \sigma_\gamma^{-1\alpha} \rho_{\alpha j} u_j = 0 \quad (84)$$

which, as we shall see in the following, contains all the necessary informations.

In fact, if Eq. (70) holds true, from Eq. (84) we have the alternatives: either

$$\chi_j^\beta \sigma_\beta^{-1\gamma} \rho_{\gamma k} u_k - p^2 u_j = 0 \tag{85}$$

or

$$\Omega_j = \chi_j^\beta \frac{\sigma_\beta^{-1\gamma}}{p^2} \rho_{\gamma k} u_k - u_j \neq 0 \tag{86}$$

and solves

$$\hat{d}_{ij} \Omega_j = 0 . \tag{87}$$

Let us discuss the first alternative. Multiplying to the left Eq. (85) by ρ_j^α and recalling Eq. (70), yields a solution of Eq. (64 *a*), individuating another state of the scalar sector mass degenerate with a Φ.Π. state. On the other hand, if Eqs. (86), (87) are verified we have a single particle physical state: indeed Eq. (87) in the tree approximation reads

$$(p^2 \delta_{ij} - m_{ij}) \Omega_j = 0 \tag{88}$$

so that our state is an eigenvector at $p^2 > 0$ of the m_{ij} matrix which, comparing with the naive model (Section 2), can be identified with a physical state. Conversely, from every solution Ω_j of Eq. (87) we can get a vector u_j satisfying Eq. (86) which together with v^α obtained from Eq. (81) solve the system Eqs. (64 *b*), (64 *c*), provided that the matrices m_{ij} and $\rho_i^\alpha q_i^\beta$ do not have accidental non zero equal eigenvalues.

Since the matrix m_{ij}, for a generic choice of the parameters compatible with Eq. (70) has $n_S - n_{\mathscr{G}}$ positive eigenvalues, we can deduce that, if $\mu^{\alpha\beta} \rho_{\beta i} = 0$, the situation in the scalar sector coincides with the one holding in the original model, namely that there are $N + n_{\mathscr{G}}$ states which are mass degenerate with the Φ.Π. sector, i. e. unphysical, and $n_S - n_{\mathscr{G}}$ physical states.

That all the Φ.Π. masses are degenerate with someone in the scalar sector follows directly from the S. I. (e. i. Eqs. (66 *a*), (66 *b*)). In fact a non zero solution w_β of Eq. (64 *a*), substituted into Eqs. (66 *a*, *b*) gives

$$D_i^\alpha \chi_i^\beta w_\beta - p D^{\alpha\gamma} \sigma_\gamma^\beta w_\beta = 0 \tag{89 a}$$

$$d_{ij} \chi_j^\alpha w_\alpha - p D_i^\gamma \sigma_\gamma^\alpha w_\alpha = 0 , \tag{89 b}$$

which compared with Eqs. (64 *b*, *c*) yield a solution of the scalar wave equations

$$v_\alpha = - p \sigma_\alpha^\beta w_\beta \tag{90 a}$$

$$u_i = \chi_i^\alpha w_\alpha . \tag{90 b}$$

It remains to be proved that the condition $\mu^{\alpha\beta} \rho_{\beta i} = 0$ follows from the mass degeneracy requirement. As mentioned before the proof of this assertion is carried out in Appendix C.

The complexity of the analysis developed in this section justifies a summary of the results on the masses of the particles in the unphysical sector. This analysis may also be clarifying in connection with the saturation of unitarity relation of the model.

If the semi-simple gauge symmetry is completely broken, $N - n_{\mathscr{G}}$ among the N $\Phi.\Pi.$ particles, are free and mass degenerate with an equal number of particles in the scalar sector. The remaining $n_{\mathscr{G}}$ $\Phi.\Pi.$ particles are mass degenerate with $n_{\mathscr{G}}$ pairs in the scalar sector. This exhausts the ghost sector, since the scalar unphysical particles are $N + n_{\mathscr{G}}$.

It is not difficult, by extending our method, to show that the situation remains essentially unchanged also in the case where some of the unphysical particles are massless, the only difference being that the massless $\Phi.\Pi.$ states belong to the first considered $N - n_{\mathscr{G}}$ ones and are not necessarily free.

5. CONCLUSIONS

Our task in writing this paper was to find a definition encompassing the largest possible class of gauge models and to specify their parameters. A by-product of this study has been an accurate analysis of the unphysical one-particle states of the model whose results are summarized at the end of the previous Section.

According to our point of view the gauge models are defined by the Slavnov identity implemented with the condition that the masses of any one-particle ghost state corresponding to the longitudinal photon and Goldstone boson degrees of freedom, be degenerate with at least one $\Phi.\Pi.$ mass.

We have also characterized the most general renormalizable classical Lagrangian possessing this property. Such a Lagrangian is built from a canonical form through the linear field transformations shown in Eqs. (27). Given the gauge group and a representation of it in the matter field space, the canonical form consists of the most general gauge invariant Lagrangian plus the $\Phi.\Pi.$ gauge fixing terms which correspond to the gauge parameters $\Lambda^{\alpha\beta}$, ρ_i^α and the mass term in Eq. (58) with $\mu^{\alpha\beta}\rho_{\beta i} = 0$ (see Eqs. (4), (53), (54)).

We further discussed the stability problem for such theories, i. e. whether two Lagrangians differing for small variations of the parameters correspond to the same gauge group and to equivalent representations in the matter field space. If this is true, then we can go from one Lagrangian to the other by varying the above mentioned parameters. The analysis shows that the only source of instability is in the representation of the abelian invariant subgroups of the gauge group. We also stated that this kind of instability is not dangerous for the quantum extension of the theory.

Let us finally remark that if one finds an appropriate regularization method this paper implicitely contains the proof of the renormalizability of the theory, for, in such a case, the stability properties guarantee that all the counterterms originate from variations of the parameters in the classical action.

In a more general approach (B. P. H. Z.) [*17*] to renormalization, stability asserts that all the quantum extensions of the classical theory can be reached from any given one through a finite change of the parameters. This property ensures the complete generality of those extensions which are obtained by the aid of supplementary renormalization conditions.

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APPENDIX A

In this appendix we analyze the stability to first order ε perturbations of the transformations of the φ_i fields alone.

The classical invariant action at the point $\mathscr{A}_\mu^\alpha \equiv 0$ has the following structure (see Eq. (4)).

$$W(\varphi) = \int dx(\varphi_i^\dagger K_{ij}\varphi_j + H(\varphi + q))(x) \tag{A.1}$$

where the first term on the r. h. s. is the kinetic term. The functional $W(\varphi)$ is invariant under the global version of the φ_i fields transformations (Eq. 1).

Along the lines of Section 3, our analysis is carried out in the hypothesis that a perturbation $W_\varepsilon(\varphi)$ of the $W(\varphi)$ functional must exist

$$W_\varepsilon(\varphi) = W(\varphi) + \varepsilon W'(\varphi) \tag{A.2}$$

and be invariant under the perturbed global transformation of the φ field (Eq. 51). These can be written after a suitable redefinition of the fields in the form

$$\delta\varphi_i(x) = (t_{ij}^\alpha + \varepsilon\delta_{\beta\mathrm{A}}^\alpha \tau_{\mathfrak{i}\mathfrak{j}}^{\beta\mathrm{A}})(\varphi_j(x) + q_j)\omega_\alpha \tag{A.3}$$

where the matrices $\tau_{\mathfrak{i}}^{\alpha\mathrm{A}}$'s commute with the matrices t^α's and are not linear combinations of the $t^{\alpha\mathrm{A}}$'s.

To study the transformation properties of the functional $W_\varepsilon(\varphi)$ let us introduce the functional differential operators

$$\mathscr{D}^\alpha = \int dx \left\{ (\varphi_i(x) + q_i)t_{ij}^\alpha \frac{\delta}{\delta\varphi_j(x)} \right\} \tag{A.4 a}$$

$$d^\alpha = \int dx \left\{ (\varphi_i(x) + q_i)\delta_{\beta\mathrm{A}}^\alpha \tau_{\mathfrak{i}\mathfrak{j}}^{\beta\mathrm{A}} \frac{\delta}{\delta\varphi_j(x)} \right\} \tag{A.4 b}$$

which obey the commutation rule

$$[\mathscr{D}^\alpha, d^\beta] = 0 \,. \tag{A.5}$$

The linear space of the functionals $W(\varphi)$ of natural dimensions less than or equal to four carries a completely reducible representation of the algebra $\mathscr{G}$ defined by the action of the $\mathscr{D}^\alpha$ operators.

The invariance, up to first order, of the functional $W_\varepsilon(\varphi)$ in Eq. (A.2) under the transformation Eq. (A.3) leads to

$$\mathscr{D}^\alpha W(\varphi) = 0 \tag{A.6 a}$$

$$\mathscr{D}^\alpha W'(\varphi) + d^\alpha W(\varphi) = 0 \,. \tag{A.6 b}$$

Let us now define through a non degenerate invariant symmetric form $k^{\alpha\beta}$ the quadratic Casimin operator

$$\mathscr{D}^2 = \mathscr{D}^\alpha K_{\alpha\beta}\mathscr{D}^\beta \tag{A.7}$$

whose null space (the functionals which satisfy $\mathscr{D}^2 W = 0$) exhausts the space of the invariant functionals i. e. those obeying Eq. (A.6 *a*). Multiplying Eq. (A.6 *b*) to the left with $\mathscr{D}^\beta k_{\beta\alpha}$ and making use of Eqs. (A.5), (A.6 *a*) yields

$$\mathscr{D}^2 W'(\varphi) = 0 \tag{A.8 a}$$

hence

$$\mathscr{D}^\alpha W'(\varphi) = -\, d^\alpha W(\varphi) = 0 \tag{A.8 b}$$

from which two main results can be deduced.

First the invariance, under the transformations generated by d^α, of the non-degenerate kinetic part of $W(\varphi)$ insures that the $\tau_{\text{H}}^{\alpha A}$ matrices are anti-hermitian, thus each of them defines on the φ space a completely reducible, non trivial representation of a one-parameter group commuting with the gauge group.

Seconds, Eq. (A.8 *b*) tells us that the functional $W(\varphi)$ is also invariant under the single action of such one parameter groups, thus completing the analysis of the kind of instability related to the presence of the $\tau_{\text{H}}^{\alpha A}$ matrices in Eq. (46 *a*).

APPENDIX B

In this Appendix we shall show that the most general Slavnov invariant $\Phi.\Pi.$ neutral external field independent functional of degree less than or equal to four, is given by Eq. (53).

Let $\Gamma(\psi)$ be such a functional; the $\Phi.\Pi.$ neutrality and dimensionality constraints allow us to write

$$\Gamma(\psi) = \Gamma(\Phi, \mathscr{A}^\mu) + \int dx\, c_\alpha(x)(\mathrm{K}\bar{c})^\alpha(x) + \int dx\, \mathrm{L}^{\alpha\beta,\gamma\delta}(c_\alpha c_\beta \bar{c}_\gamma \bar{c}_\delta)(x) \qquad \text{(B.1)}$$

where $\Gamma(\phi, \mathscr{A}_\mu)$ does not depend upon the $\Phi.\Pi.$ fields, $\mathrm{L}^{\alpha\beta,\gamma\delta}$ is antisymmetric in the indices $\alpha\beta$ and $\gamma\delta$, and $(\mathrm{K}\bar{c})^\alpha$ is explicitely given by

$$\begin{aligned}(\mathrm{K}\bar{c})^\alpha(x) = \{ \Lambda^{\alpha\beta} \Box\, \bar{c}_\beta + l^{\alpha\beta\gamma}\partial_\mu(\mathscr{A}^\mu_\beta \bar{c}_\gamma) \\ + \mathrm{N}^{\alpha\beta\gamma}\mathscr{A}^\mu_\gamma \partial_\mu \bar{c}_\beta + \varphi_i \mathrm{M}^{\alpha\beta}_{ij}\varphi_j \bar{c}_\beta + \mathrm{Q}^{\alpha\beta}_i \varphi_i \bar{c}_\beta \\ + \mathrm{R}^{\alpha\beta\gamma\delta}\mathscr{A}^\mu_\beta \mathscr{A}_{\mu\gamma}\bar{c}_\delta + \mu^{\alpha\beta}\bar{c}_\beta \}(x)\,.\end{aligned} \qquad \text{(B.2)}$$

The Slavnov invariance of Eq. (B.1) means

$$(\mathscr{S}^{(\mathrm{P})}\Gamma)(\psi) = 0 \qquad \text{(B.3)}$$

with $\mathscr{S}^{(\mathrm{P})}$ given by Eq. (12), and hence *a fortiori*

$$(\mathscr{S}^{(\mathrm{P})2}\Gamma)(\psi) = 0 \qquad \text{(B.4)}$$

By direct computation, recalling Eq. (9 *b*), Eq. (B.4) reads

$$\int dx(\mathscr{M}\bar{c})^\alpha(x)\frac{\delta\Gamma(\Psi)}{\delta c^\alpha(x)} = 0\,. \qquad \text{(B.5)}$$

The substitution of Eqs. (B.1), (B.2) into Eq. (B.5) yields, for the coefficients of the independent field monomials, the relations

$$\mathrm{L}^{\alpha\beta,\gamma\delta} = \mathrm{N}^{\alpha\beta\gamma} = \mathrm{M}^{\alpha\beta}_{ij} = \mathrm{R}^{\alpha\beta\gamma\delta} = 0 \qquad \text{(B.6 a)}$$

$$\Lambda^{\alpha\beta} = \Lambda^{\beta\alpha} \qquad \text{(B.6 b)}$$

$$l^{\alpha\beta\gamma} = \Lambda^{\alpha\delta}f^{\beta\gamma}_\delta \qquad \text{(B.6 c)}$$

$$\mathrm{Q}^{\alpha\beta}_i = \Lambda^{\alpha\gamma}\rho_{\gamma j}t^\beta_{ji} \qquad \text{(B.6 d)}$$

$$\mu^{\alpha\beta} = \mu^{\beta\alpha}, \qquad \mu^{\alpha\beta s} = 0 \qquad \text{(B.6 e)}$$

$$\mu^{\alpha\gamma}\rho_{\gamma i}t^\beta_{ij} = \mu^{\beta\gamma}\rho_{\gamma i}t^\alpha_{ij} \qquad \text{(B.6 f)}$$

Conditions (B.6 *e*) and Eq. (B.6 *f*) show that the term $\mu^{\alpha\mathrm{A}\gamma\mathrm{A}}\rho_{\gamma_{\mathrm{A}}i}$ is invariant under semisimple transformations. The further decomposition into abelian variant and invariant parts

$$(\mu\rho_i)^{\alpha\mathrm{A}} = (\mu\rho_i)^{\alpha\mathrm{A}}_\flat + (\mu\rho_i)^{\alpha\mathrm{A}}_\natural \qquad \text{(B.7)}$$

substituted into Eq. (B.6 *f*) yields the co-homological solution

$$(\mu\rho_i)^{\alpha\mathrm{A}}_\flat = t^{\alpha\mathrm{A}}_{ij}\chi_j \qquad \text{(B.8)}$$

while $(\mu\rho_i)^{\alpha\mathrm{A}}_\natural$ is left undetermined. Hence, by the necessary conditions $(\mathscr{S}^{(\mathrm{P})2}\Gamma)(\psi) = 0$, Eq. (B.1) assumes the form

$$\Gamma(\psi) = \Gamma(\phi, \mathscr{A}_\mu) + \int dx[c_\alpha(x)\Lambda^{\alpha\beta}(\mathscr{M}\bar{c})_\beta(x) - c_\alpha(x)\mu^{\alpha\beta}\bar{c}_\beta(x)]\,. \qquad \text{(B.9)}$$

Define now

$$\Gamma(\phi, \mathscr{A}_\mu) = \Gamma_{\text{inv}}(\phi, \mathscr{A}_\mu) - \frac{1}{2}\int dx[g_\alpha(x)\Lambda^{\alpha\beta}g_\beta(x)] + \Delta\Gamma(\phi, \mathscr{A}_\mu) \tag{B.10}$$

where $g_\alpha(x)$ is given by Eq. (9 *a*) and $\Gamma_{\text{inv}}(\phi, \mathscr{A}_\mu)$ is the most general gauge invariant functional of degree less than or equal to four. Substituting Eqs. (B.9), (B.10) into Eq. (B.3) we obtain

$$\int dx \frac{\delta(\Delta\Gamma)}{\delta\omega^\alpha(x)} \bar{c}^\alpha(x) + \int dx\, g_\alpha(x)(\mu\,\bar{c})^\alpha(x) = 0 \tag{B.11}$$

which by Eqs. (B.6 *e*) and (9 *a*) is equivalent to the system

$$\frac{\delta(\Delta\Gamma)}{\delta\omega^{\alpha s}(x)} = 0 \tag{B.12 a}$$

$$\frac{\delta(\Delta\Gamma)}{\delta\omega_{\alpha_A}(x)} = -\,\mu^{\alpha_A\beta_A}(\partial_\mu\mathscr{A}^\mu_{\beta_A}(x) - \rho_{\beta_A i}\varphi_i(x)) \tag{B.12 b}$$

The solution of Eq. (B.12 *a*) and the homogeneous solution of Eq. (B.12 *b*) give a gauge invariant functional which is included in the term $\Gamma_{\text{inv}}(\phi, \mathscr{A}_\mu)$, hence we have only to find a particular solution of Eq. (B.12 *b*). This is easily seen to be

$$\Delta\Gamma(\phi, \mathscr{A}_\mu) = -\int dx\left[\frac{\mathscr{A}_{\nu\alpha}\mu^{\alpha\beta}\mathscr{A}^\nu_\beta}{2} + \chi_i\varphi_i\right](x) \tag{B.13}$$

where χ_i satisfies Eq. (B.8), $\chi_i q^\alpha_i = 0$, and the term $(\mu\rho_i)^{\alpha_A}_i$ in Eq. (B.7) is zero.

The functional in Eq. (B.1) can thus be written as

$$\Gamma(\psi) = \Gamma_{\text{inv}}(\phi, \mathscr{A}_\mu) - \int dx\left[g_\alpha\frac{\Lambda^{\alpha\beta}}{2}g_\beta - c_\alpha\Lambda^{\alpha\beta}(\mathscr{M}\bar{c})_\beta\right](x)$$
$$- \int dx\left[\mathscr{A}_{\nu\alpha_A}\frac{\mu^{\alpha_A\beta_A}}{2}\mathscr{A}^\nu_{\beta_A} + c_{\alpha_A}\mu^{\alpha_A\beta_A}\bar{c}_{\beta_A}\right](x) - \int dx\,\chi_i\varphi_i(x)\,. \tag{B.14}$$

APPENDIX C

In this Appendix we prove that the condition Eq. (70)

$$\mu^{\alpha\beta}\rho_{\beta i} = 0 \tag{C.1}$$

follows from the mass degeneracy requirement discussed in Section 4.

For all purposes we have to show that Eq. (C.1) is stable under small perturbations preserving the Slavnov invariance of the Lagrangian if the masses of any one particle ghost state of the scalar sector are degenerate with at least one Φ.Π. mass.

We shall analyze the problem in the tree approximation, without any loss of generality. Indeed the tree approximation result implies the necessity of Eq. (C.1) to all orders since, under this condition, the $(\mu\bar{c})^\alpha$ fields turn out to be free.

To make the calculations more transparent we shall choose the gauge parameters $\rho^\alpha_{i,\mathrm{S}}$, considered as vectors in the scalar matter field space, to span the same subspace $\mathscr{S}_g$ of the vectors $q^\alpha_{i,\mathrm{S}}$, as e. g. in the 't Hooft gauge (Eq. (76)). According to this choice we shall write.

$$q^\alpha_i = \gamma^\alpha_\beta \rho^\beta_i \, . \tag{C.2}$$

As shown in Section 4 the masses of the scalar sector are automatically degenerate with some Φ.Π. mass, except, possibly, the non vanishing eigenvalues (p^2) of Eq. (84) which in the tree approximation writes:

$$\hat{d}_{ij}\left[u_j - \frac{1}{p^2} q^\alpha_j \rho_{\alpha k} u_k\right] = \rho_{\alpha i}\frac{\mu^{\alpha\beta}}{p^2}\rho_{\beta k}u_k \tag{C.3}$$

with

$$\hat{d}_{ij} = p^2\delta_{ij} - m_{ij} \tag{C.4}$$

and the symmetric matrix m_{ij} is constrained by (Eq. (69))

$$m_{ij}q^\alpha_j = \delta^\alpha_{\alpha_A}\mu^{\alpha_A\beta_A}\rho_{\beta_A i} \, . \tag{C.5}$$

Clearly, by this equation, the matrix m_{ij} leaves the subspace $\mathscr{S}_g$ invariant and we can decompose it as:

$$m_{ij} = \rho_{\alpha i}\mathrm{M}^{\alpha\beta}\rho_{\beta j} + (m_\perp)_{ij} \tag{C.6}$$

where $(m_\perp)_{ij}$ annihilates $\mathscr{S}_g$.

Following this decomposition we can separate from Eq. (C.3) two subsystems involving the components of the vector u along $\mathscr{S}_g\,(u^g_i)$ and its orthogonal complement $(u^g_{\perp i})$, i. e.

$$\left(\delta_{ik} - \rho_{\alpha i}\frac{\mathrm{M}^{\alpha\beta}}{p^2}\rho_{\beta k}\right)(\delta_{kj} - q^\alpha_k\rho_{\alpha j})u^g_j = \rho_{\alpha i}\frac{\mu^{\alpha\beta}}{p^2}\rho_{\beta k}u^g_k \tag{C.7 a}$$

$$(p^2\delta_{ij} - (m_\perp)_{ij})u^g_{\perp j} = 0 \, . \tag{C.7 b}$$

Recalling the analysis of Section 4 we notice that the eigenvalues of the matrix $m_\perp$, which are positive by hypothesis, correspond to physical states in the scalar sector. Therefore our degeneracy condition involves only the eigenvalues (p^2) of Eq. (C.7 *a*), which multiplied to the left by ρ^γ_i, after the substitutions

$$\mathrm{R}^{\alpha\beta} = \rho^\alpha_i\rho^\beta_i \tag{C.8 a}$$

$$v^\alpha = \rho^\alpha_i u_i \tag{C.8 b}$$

and making use of Eq. (C.5), becomes

$$[p^2\delta^{\alpha\beta} - (\mathrm{R}\gamma^{\mathrm{T}})^{\alpha\beta}]v_\beta = (\mathrm{RM})^{\alpha\beta}v_\beta \, . \tag{C.9}$$

Summarizing this rather involved introduction, the mass degeneracy condition requires that the eigenvalues (p^2) of this equation be degenerate with some of the $\Phi.\Pi$. equation of motion (Eq. (67)) which, in the same notation, writes:

$$[p^2\delta^{\alpha\beta} - (R\gamma^T)^{\alpha\beta} - \mu^{\alpha\beta}]w_\beta = 0 . \tag{C.10}$$

The comparison of Eq. (C.9) and Eq. (C.10) is performed in the limit of small $\mu^{\alpha\beta}\rho_{\beta i}$. In this limit the r. h. s. of Eq. (C.9) appears, by Eqs. (C.5), (C.6), as a perturbation; to evidentiate an analogous perturbation term in Eq. (C.10) we need some more work.

Let $P_{\alpha\beta}$ be the orthogonal projector acting on the algebra and such that $(1 - P)_{\alpha\beta}$ projects onto the null space of the matrix $R^{\alpha\beta}$ in Eq. (C.8 *a*). (In the following, whenever the meaning is clear, we shall omit the α, β, ..., indices and consider the corresponding operators.) Setting

$$Pw = w_1 \tag{C.11 a}$$

$$(1 - P)w = w_2 \tag{C.11 b}$$

and projecting Eq. (C.10) with P and $(1 - P)$, yields the systems

$$p^2w_1 - R\gamma^Tw_1 - R\gamma^Tw_2 - P\mu w_1 - P\mu w_2 = 0 \tag{C.12 a}$$

$$p^2w_2 - (1 - P)\mu w_1 - (1 - P)\mu w_2 = 0 \tag{C.12 b}$$

in which the terms proportional to $P\mu$ or μw_1 should be considered as small.

Now, when the perturbation vanishes, the eigenvectors of the systems (C.12 *a*), (C.12 *b*) can be separated into two classes, where the first comprehends those which are degenerate with a solution of Eq. (C.9) and satisfy $w_1 = v$, $w_2 = 0$. The remaining solutions, for which

$$[p^2 - (1 - P)\mu(1 - P)]w_2 \equiv (p^2 - \mu_\perp)w_2 = 0 \tag{C.13}$$

are in general non degenerate with those of Eq. (C.9). Consequently we shall consider only the eigenvectors of the system (C.12) whose unperturbed limit belongs to the first class.

For a choice of the parameters avoiding the accidental coincidence of two $\Phi.\Pi$. masses, the operator $p^2 - \mu_\perp$, with p^2 eigenvalue in the first class, is invertible; thus Eq. (C.12 *b*) can be solved as

$$w_2 = \frac{1}{p^2 - \mu_\perp}(1 - P)\mu w_1 . \tag{C.14}$$

The substitution of Eq. (C.14) into Eq. (C.12 *a*), keeping only the first order terms in the perturbation P_μ, gives

$$(p^2 - R\gamma^T)w_1 = P\mu w_1 + R\gamma^T\frac{1}{p^2 - \mu_\perp}(1 - P)\mu w_1 \tag{C.15}$$

where, in much the same way as in Eq. (C.9), the r. h. s. is proportional to $P\mu$, i. e. vanishes when $\mu^{\alpha\beta}\rho_{\beta i} = 0$.

Our task is now to show that Eq. (C.1) follows from the requirement that, to first order in the perturbation $P\mu$, the solutions of Eqs. (C.9), (C.15) corresponding to the same unperturbed eigenvalues $p^2 > 0$, remain degenerate.

To impose this constraint, we shall assume, without lack of generality, that the $R\gamma^TP$ operator in Eqs. (C.9), (C.15) is diagonalizable and has non degenerate eigenvalues. Then let $\{ Z_n \}$ be the basis which diagonalizes $R\gamma^TP$ in the subalgebra projected by P, i. e.

$$R\gamma^TP = \sum_n \lambda^{(n)}Z_n\tilde{Z}_n \tag{C.16}$$

where $\lambda^{(n)}$ are the eigenvalues and $\tilde{Z}_n$ is the dual basis satisfying $(\tilde{Z}_n, Z_m) = \delta_{nm}$. From Eqs. (C.5), (C.2), (C.6), (C.8 *a*) we get

$$M R \gamma^T = P\mu \tag{C.17}$$

which, multiplied to the right with P yields,

$$P\mu P = MR\gamma^T P = P\gamma RM \tag{C.18}$$

In terms of Eq. (C.16), the above equation writes

$$M \sum_n \lambda^{(n)} z_n \tilde{z}_n - \sum_n \lambda^{(n)} \tilde{z}_n z_n M = 0 . \tag{C.19}$$

Multiplying Eq. (C.19) to the right with z_m and to the left with z_n, gives

$$(\lambda^{(m)} - \lambda^{(n)})(z_n, M z_m) = 0 , \tag{C.20}$$

from which we see that the M operator assumes the form

$$M = \sum_n M^{(n)} \tilde{z}_n \tilde{z}_n \tag{C.21}$$

with $M^{(n)} = (z_n, M z_n)$.

Now, imposing that the perturbations in Eqs. (C.9), (C.15) produce the same shift on the degenerate unperturbated eigenvalues, we have

$$M^{(n)}(z_n, R\tilde{z}_n) = M^{(n)}\left(\tilde{z}_n, \left[\lambda^{(n)}\left(1 - \frac{1}{p^2 - \mu_\perp}(1 - P)\lambda^{(n)}\right)\right] z_n\right). \tag{C.22}$$

Again, for an arbitrary choice of the parameters, the matrix elements of the operator inside the square bracket on the r. h. s. of Eq. (C.22) are nonzero, hence we obtain the condition

$$M^{(n)} = 0 \tag{C.23}$$

which, from Eqs. (C.5), (C.6) implies

$$m_{ij} q^\alpha_j = \delta^\alpha_{\alpha A} \mu^{\alpha A \beta A} \rho_{\beta A i} = 0 \tag{C.24}$$

i. e. the validity of Eq. (C.1).

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