ANNALES DE L'I. H. P., SECTION A

M. FANNES

The entropy density of quasi-free states for a continuous boson system

Annales de l'I. H. P., section A, tome 28, nº 2 (1978), p. 187-196
<http://www.numdam.org/item?id=AIHPA_1978__28_2_187_0>

© Gauthier-Villars, 1978, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Henri Poincaré, Section A :
Vol. XXVIII, n° 2, 1978, p. 187-196. *Physique théorique.*

The entropy density of quasi-free states for a continuous boson system

by

M. FANNES (*)
Instituut voor Theoretische Fysica,
Katholieke Universiteit Leuven (**)

ABSTRACT. — We compute the entropy density for a gauge and euclidean invariant, quasi-free state of the continuous boson system.

1. INTRODUCTION

We compute in this paper the entropy density of a gauge and euclidean invariant quasi-free state for a continuous boson system. The result is expressed in terms of the Fourier transform of the operator associated to it's two-point function. In order to keep the notation as simple as possible we considered a 1-dimensional system but the generalization is straightforward. A lower bound for this density, which gives in fact the correct result, has been studied in [*4*] for a restricted class of states.

The method used in this paper is related to the one used in [*5*] and consists essentially in computing the entropy density for a « nice » class of states and extending afterwards the result to the general case. Instead of computing only the limit of the local densities we have formulated a more general lemma in the appendix. This lemma can then also be used to derive for instance, an analogous result for the continuous fermion system.

(*) Aangesteld Navorser NFWO, Belgium.
(**) Postal address: Celestijnenlaan 200, D, B-3030 Heverlee, Belgium.

2. PRELIMINARIES

We consider the C. C. R.-algebra $\mathscr{A}$ over the space $C_0^\infty(\mathbb{R})$ of the infinitely differentiable, complex-valued functions with compact support in $\mathbb{R}$. This algebra is generated by elements δ_ψ, $\psi \in C_0^\infty(\mathbb{R})$ which satisfy:

$$\delta_\psi \delta_\phi = \exp - \frac{i}{2} \sigma(\psi \mid \phi) \qquad \delta_{\phi+\psi}$$
$$\delta_\psi^* = \delta_{-\psi}$$

where $\sigma(\cdot \mid \cdot)$ is the symplectic form $\operatorname{Im} \langle \cdot \mid \cdot \rangle$ and $\langle \cdot \mid \cdot \rangle$ is the usual scalar product on $\mathscr{L}^2(\mathbb{R}, dx)$ [6].

$\mathscr{A}$ is a simple, non-separable, quasi-local C*-algebra:

$$\mathscr{A} = \overline{\bigcup_\Lambda \mathscr{A}_\Lambda}$$

where the union is taken over the compact subsets Λ of $\mathbb{R}$ and $\mathscr{A}_\Lambda$ is the C. C. R.-algebra over the space of infinitely differentiable, complex-valued functions with support contained in Λ.

The aim of this paper is to derive an explicit expression for the entropy density of a locally-normal, quasi-free state which is invariant under the action of both the gauge and the euclidean groups. The gauge automorphisms are induced by:

$$\delta_\psi \to \delta_{e^{i\theta}\psi} \qquad \psi \in C_0^\infty(\mathbb{R}),\ \theta \in [0, 2\pi[$$

and the euclidean automorphisms by:

$$\delta_\psi \to \delta_{U\psi} \qquad \psi \in C_0^\infty(\mathbb{R}),\ U \text{ euclidean}.$$

The gauge-invariant, quasi-free states on $\mathscr{A}$ are of the form:

$$\omega_A(\delta_\psi) = \omega_F(\delta_\psi) \exp - \frac{1}{2} A(\psi)$$

where

$$\omega_F(\delta_\psi) = \exp - \frac{1}{4} \langle \psi \mid \psi \rangle$$

is the Fock state and $\psi \to A(\psi)$ is a positive, quadratic form. We will assume in the following that there exists a positive self-adjoint operator A on $\mathscr{L}^2(\mathbb{R}, dx)$ such that

$$A(\psi) = \langle \psi \mid A\psi \rangle .$$

Imposing the condition that ω_A be euclidean invariant leads then to

$$A = a(-\Delta)$$

where Δ is the Laplace operator in one dimension and $a(\cdot)$ is a non-negative Borel function [7].

A state ω on $\mathscr{A}$ is said to be locally-normal if for all compacts $\Lambda \subset \mathbb{R}$, $\omega|_{\mathscr{A}_\Lambda}$ is normal with respect to $\omega_F|_{\mathscr{A}_\Lambda}$. For gauge-invariant, quasi-free states ω_B this condition amounts to asking that for all compacts $\Lambda \subset \mathbb{R}$, $E_\Lambda B E_\Lambda$ be a trace-class operator on $\mathscr{L}^2(\Lambda, dx)$, where E_Λ is the orthogonal projection from $\mathscr{L}^2(\mathbb{R}, dx)$ on $\mathscr{L}^2(\Lambda, dx)$. For the quasi-free states

$$\omega_A(\delta_\psi) = \omega_F(\delta_\psi) \exp -\frac{1}{2}\langle \psi \mid a(-\Delta)\psi \rangle$$

this condition is also equivalent to [3]

$$\int_{\mathbb{R}} dx\, a(x^2) < \infty \; [3]$$

Let Λ be a compact subset of $\mathbb{R}$ and ω a state on $\mathscr{A}$. One defines the entropy $S(\omega, \Lambda)$ of ω with respect to Λ as:

$S(\omega, \Lambda) = +\infty$ if $\omega|_{\mathscr{A}_\Lambda}$ is not normal with respect to $\omega_F|_{\mathscr{A}_\Lambda}$

$S(\omega, \Lambda) = -\mathrm{Tr}_{\mathscr{H}_F^\Lambda}\, \rho^\Lambda \ln \rho^\Lambda$ if $\omega|_{\mathscr{A}_\Lambda}$ is normal with respect to $\omega_F|_{\mathscr{A}_\Lambda}$.

In this last case ρ^Λ is the density matrix on $\mathscr{H}_F^\Lambda$ given by:

$$\omega(x) = \mathrm{Tr}_{\mathscr{H}_F^\Lambda}\, \Pi_F^\Lambda(x)\rho^\Lambda \qquad x \in \mathscr{A}$$

where $(\Pi_F^\Lambda, \mathscr{H}_F^\Lambda, \Omega_F^\Lambda)$ is the G. N. S. triplet associated to $\omega_F|_{\mathscr{A}_\Lambda}$.

Let ω be a translation-invariant state on $\mathscr{A}$. Its entropy density $s(\omega)$ is defined as:

$$s(\omega) = \lim_{l \to \infty} \frac{1}{2l} S(\omega, [-l, l])$$

The existence of this limit was shown in [1].

The aim of this paper is to prove that for states

$$\omega_A(\delta_\psi) = \omega_F(\delta_\psi) \exp -\frac{1}{2}\langle \psi \mid a(-\Delta)\psi \rangle \qquad (*)$$

$$s(\omega_A) = \frac{1}{2\pi}\int_{\mathbb{R}} dx\, e(a(x^2))$$

where

$$e(x) = (1 + x)\ln(1 + x) - x \ln x, \qquad x \geqslant 0$$

3. COMPUTATION OF $s(\omega_A)$ FOR A RESTRICTED CLASS OF STATES

Let A be a non-negative self-adjoint operator on an Hilbert space $\mathscr{H}$. We put

$$S(A) = \mathrm{Tr}_{\mathscr{H}}\, e(A).$$

THEOREM 3.1. — Let ω_B be a gauge-invariant, quasi-free state on $\mathscr{A}$ and $\Lambda \subset \mathbb{R}$ a compact set. Then:

$$S(\omega_B, \Lambda) = S(E_\Lambda B E_\Lambda)$$

where E_Λ is the orthogonal projection from $\mathscr{L}^2(\mathbb{R}, dx)$ on $\mathscr{L}^2(\Lambda, dx)$

Proof. — The proof of this theorem is well-known and consists in a straightforward computation of $S(\omega_B, \Lambda)$.

LEMMA 3.2. — The function $A \to S(A)$ is concave.

Proof. — Let A, B be self-adjoint non-negative operators on $\mathscr{K}$ and $0 < \alpha < 1$. Put $C = \alpha A + (1 - \alpha)B$. We have to show that:

$$\alpha S(A) + (1 - \alpha)S(B) \leqslant S(C)$$

Suppose that $S(A) = +\infty$, then also $S(\alpha A) = +\infty$ and as $C \geqslant \alpha A$ and $S(\cdot)$ is an increasing function (see Lemma 4.1) $S(C) = +\infty$.

Suppose next that S(A) and S(B) are finite. Applying Klein's inequality [9] to the function $x \to e(x)$ one gets

$$\mathrm{Tr}\,(1 + A)\ln(1 + A) - A\ln A \leqslant \mathrm{Tr}\,(1 + A)\ln(1 + C) - A\ln C$$

and

$$\mathrm{Tr}\,(1 + B)\ln(1 + B) - B\ln B \leqslant \mathrm{Tr}\,(1 + B)\ln(1 + C) - A\ln C$$

and so:

$$\alpha S(A) + (1 - \alpha)S(B) \leqslant S(C)$$

Let $x \in \mathbb{R} \to a(x^2) \in \mathbb{R}$ be a measurable, essentially non-negative function. We define $s'(a)$ as:

$$s'(a) = \frac{1}{2\pi}\int_{\mathbb{R}} dx\, e(a(x^2))$$

THEOREM 3.3. — Let $x \in \mathbb{R} \to a(x^2) \in \mathbb{R}$ be a non-negative, continuous function with compact support and ω_A the state on $\mathscr{A}$ corresponding to $a(\cdot)$ (see (*)) then:

$$s(\omega_A) = s'(a)$$

Proof. — 1) Let $l > 0$ and define a group $\{ U^l(s), s \in \mathbb{R} \}$ of unitaries on $\mathscr{L}^2([-l, l], dx)$ by:

$$(U^l(s)\phi)(x) = \phi((x + s) \bmod 2l).$$

Denote by E^l the orthogonal projection from $\mathscr{L}^2(\mathbb{R}, dx)$ on $\mathscr{L}^2([-l, l], dx)$ and let $A^l = E^l A E^l$. Define now the operator A^l_c on $\mathscr{L}^2([-l, l], dx)$ by:

$$A^l_c = \frac{1}{2l} s - \int_0^{2l} ds\, U^l(s) A^l U^{l*}(s) \qquad (a)$$

Then:

i) $0 \leqslant A^l_c$ as $A^l \geqslant 0$

ii) $\frac{1}{2l}\mathrm{Tr}\, A^l_c = \frac{1}{2l}\mathrm{Tr}\, A^l = \frac{1}{2\pi}\int_{\mathbb{R}} dx\, a(x^2) < \infty$

iii) $[A^l_c, U^l(s)] = 0, \ s \in \mathbb{R}$.

As the group $\{ U^l(s) \mid s \in \mathbb{R} \}$ has simple discrete spectrum:

$$U^l(x) \frac{1}{\sqrt{2l}} e^{\frac{i\pi n x}{l}} = e^{\frac{i\pi n s}{l}} \frac{1}{\sqrt{2l}} e^{\frac{i\pi n x}{l}} \qquad n \in \mathbb{Z},\ x \in [-l, l]$$

iii) implies that

$$A_c^l \frac{1}{\sqrt{2l}} e^{\frac{i\pi n x}{l}} = \alpha_c^l(n) \frac{1}{\sqrt{2l}} e^{\frac{i\pi n x}{l}}$$

and a simple calculation shows that

$$\alpha_c^l(n) = \frac{1}{\pi} \int_{\mathbb{R}} dx \frac{a(x^2) \sin^2 lx}{l\left(\frac{n\pi}{l} + x\right)^2} \qquad n \in \mathbb{Z}$$

Using (*a*), Theorem 3.1 and Lemma 3.2 one gets:

$$S(\omega_A, [-l, l]) \leqslant S(A_c^l) = \sum_{n \in \mathbb{Z}} e(\alpha_c^l(n))$$

Dividing both sides by $2l$, taking the limit $l \to \infty$ and using Lemma 5.1 this becomes:

$$s(\omega_A) \leqslant s'(a)$$

2) Consider next for $l > 0$, the gauge-invariant, locally-normal, quasi-free state ω_A^l with period $2l$ constructed on:

$$\delta_\psi \to \omega_F(\delta_\psi) \exp -\frac{1}{2} \langle \psi \mid A_c^l \psi \rangle \qquad \psi \in \mathscr{A}_{[-l,l]}$$

An easy computation shows that

$$w^* - \lim \omega_A^l = \omega_A,$$

and, as the entropy density is w^*-upper-semi-continuous [*8*] on the periodic, locally normal states, one gets

$$s(\omega_A) \geqslant \lim_{l \to \infty} s(\omega_A^l) = \lim_{l \to \infty} \frac{1}{2l} S(A_c^l) = s'(a)$$

using as above Lemma 5.1.

4. EXTENSION OF THE ENTROPY DENSITY FORMULA TO GENERAL LOCALLY-NORMAL GAUGE AND EUCLIDIAN INVARIANT, QUASI-FREE STATES

LEMMA 4.1. — Let $0 \leqslant A \leqslant B$ and $S(B) < \infty$, then:

$$0 \leqslant S(B) - S(A) \leqslant S(B - A)$$

Proof. — 1) One has

$$e(A) = \int_0^1 \{ \mathbb{1} + \ln (s\mathbb{1} + A) \} \, dx ,$$

and as $\ln(\cdot)$ is a monotone operator increasing function [2] $e(A) \leqslant e(B)$ and so $S(A) \leqslant S(B)$.

2) Define for $0 < \lambda \leqslant 1$

$$f(\lambda) = S(B - A + \lambda A) - S(\lambda A)$$

Then

$$f'(\lambda) = \operatorname{Tr} A \{ e'(B - A + \lambda A) - e'(\lambda A) \}$$

As

$$e'(A) = \int_0^1 \frac{ds}{s\mathbb{1} + A}$$

$e'(\cdot)$ is monotone operator decreasing and so $f'(\lambda) \leqslant 0$. Then:

$$\begin{aligned} S(B) - S(A) = f(1) &\leqslant f(\lambda) \qquad 0 < \lambda \leqslant 1 \\ &\leqslant \lim_{\lambda \to 0} f(\lambda) = S(B - A) \end{aligned}$$

LEMMA 4.2. — Suppose that

$x \in \mathbb{R} \to a(x^2)$ is non-negative and measurable

$x \in \mathbb{R} \to a_n(x^2)$, $n \in \mathbb{N}$ is a sequence of non-negative measurable functions, a. e. bounded by $a(\cdot)$ and such that

$$\lim_{n \to \infty} a_n(x^2) = a(x^2) \qquad \text{a. e.}$$

Let ω_A and ω_{A_n} be the states corresponding to $a(\cdot)$ and $a_n(\cdot)$ and suppose $s(\omega_A) < \infty$

Then:

$$\lim_{n \to \infty} s(\omega_{A_n}) = s(\omega_A)$$

Proof. — Using the same notation as in the proof of Theorem 3.3 one has for any interval $[-l, l]$, $l > 0$:

$$0 \leqslant A_n^l \leqslant A^l$$

and as $S(A^l) < \infty$ [1] one has by Lemma 1:

$$0 \leqslant S(A^l) - S(A_n^l) \leqslant S((A - A_n)^l)$$

and therefore

$$0 \leqslant s(\omega_A) - s(\omega_{A_n}) \leqslant s(\omega_{A - A_n})$$

Choose an $l > 0$. From the subadditivity of the entropy [1] one gets

$$s(\omega_{A - A_n}) \leqslant \frac{1}{2l} S(A - A_n)^l) .$$

So it is sufficient to show that

$$\lim_{n \to \infty} S(A - A_n)^l) = 0 .$$

Under the conditions of the lemma one has:

$$(A - A_n)^l \xrightarrow[n\to\infty]{s} 0$$

$$0 \leqslant (A - A_n)^l \leqslant A^l$$

$$S(A^l) < \infty$$

but this implies

$$\lim_{n\to\infty} S((A - A_n)^l) = 0$$

THEOREM 4.3. — Let $x \in \mathbb{R} \to a(x^2)$ be a non-negative bounded, measurable function and ω_A the state corresponding to $a(\cdot)$, then:

$$s(\omega_A) = s'(a)$$

Proof. — 1) Suppose that

$$\int_{\mathbb{R}} dx\, a(x^2) = +\infty$$

Then $s'(a) = +\infty$ and ω_A is not locally normal, but this implies $s(\omega_A) = +\infty$

2) Suppose that

$$\int_{\mathbb{R}} dx\, a(x^2) < +\infty$$

but $s'(a) = +\infty$. This case implies by an immediate application of the second part of Theorem 3.3 that $s(\omega_A) = +\infty$

3) Suppose that

$$\int_{\mathbb{R}} dx\, a(x^2) < +\infty \quad \text{and} \quad s'(a) < +\infty$$

Choose then a sequence $x \in \mathbb{R} \to a_n(x^2)$, $n \in \mathbb{N}$, satisfying:

i) $0 \leqslant a_n(x^2) \leqslant a(x^2)$ a. e.

ii) $x \to a_n(x^2)$ continuous with compact support

iii) $\lim_{n\to\infty} a_n(x^2) = a(x^2)$ a. e.

and let ω_{A_n} be the state corresponding to $a_n(\cdot)$.

Applying Theorem 3.3 and Lemma 4.2 one finds:

$$s(\omega_A) = \lim_{n\to\infty} s(\omega_{A_n}) = \lim_{n\to\infty} s'(a_n) = s'(a)$$

where the last equality follows from the Lebesgue dominated convergence theorem.

APPENDIX

LEMMA 5.1. — Suppose that

i) $x \in \mathbb{R} \to a(x) \in \mathbb{R}$ is continuous with compact support.

ii) $x \in \mathbb{R} \to g(l, x) \in \mathbb{R}$, $l > 0$ is a family of measurable functions satisfying:

— $\exists A > 0$ such that

$$\int_{\mathbb{R}} dx \mid g(l, x) \mid \leqslant A \qquad l > 0$$

— $\lim_{l \to \infty} \int_{\mathbb{R}} dx \; g(l, x) = 1$

— $\exists \alpha > 1$, $\beta > 0$ and $B > 0$ such that

$$\mid g(l, x) < B \quad \mid x \mid^{-\alpha} l^{-\beta} \; l > 0, \qquad x \in \mathbb{R}$$

iii) $x \in \mathbb{R} \to f(x) \in \mathbb{R}$ is continuous and $\exists \lambda > 0$ and $C > 0$ such that

$$\mid f(x) \mid \leqslant C \mid x \mid^{\gamma} \qquad x \in \mathbb{R}$$

Assume furthermore that $\gamma(\alpha - 1) < 1$ and that $\gamma\alpha > 1$, then:

$$\lim_{l \to \infty} \frac{1}{l} \sum_{n=-\infty}^{+\infty} f\left\{ \int_{\mathbb{R}} dx \, a\left(x + \frac{n}{l}\right) g(l, x) \right\} = \int_{\mathbb{R}} dx \, f \circ a(x).$$

Proof. — 1) Choose $x_0 > 0$ such that $a(x) = 0$ for $\mid x \mid \geqslant x_0$

We show that

$$\lim_{l \to \infty} \frac{1}{l} \sum_{n=lx_0+1}^{\infty} f\left\{ \int_{\mathbb{R}} dx \, a\left(x + \frac{n}{l}\right) g(l, x) \right\} = 0$$

$$\left| \frac{1}{l} \sum_{n=lx_0+1}^{\infty} f\left\{ \int_{\mathbb{R}} dx \, a\left(x + \frac{n}{l}\right) g(l, x) \right\} \right|$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\infty} \left| f\left\{ \int_{\mathbb{R}} dx \, a\left(x + \frac{n}{l}\right) g(l, x) \right\} \right|$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\infty} C \left| \int_{\mathbb{R}} dx \, a\left(x + \frac{n}{l}\right) g(l, x) \right|^{\gamma}$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\infty} C \mid\mid a \mid\mid_{\infty}^{\gamma} \left| \int_{-x_0 - \frac{n}{l}}^{x_0 - \frac{n}{l}} dx \mid g(l, x) \right|^{\gamma}$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\infty} C \mid\mid a \mid\mid_{\infty}^{\gamma} B^{\gamma} l^{-\beta\gamma} \left| \int_{-x_0 - \frac{n}{l}}^{x_0 - \frac{n}{l}} dx \mid x \mid^{\alpha} \right|^{\gamma}$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\gamma} \frac{C \mid\mid a \mid\mid_{\infty}^{\gamma} B^{\gamma} l^{-\beta\gamma}}{(\alpha - 1)^{\gamma}} \left\{ \frac{1}{\left(\frac{n}{l} - x_0\right)^{\alpha - 1}} - \frac{1}{\left(\frac{n}{l} + x_0\right)^{\alpha - 1}} \right\}^{\gamma}$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\infty} \frac{C \| a \|_\infty^\gamma B^\gamma l^{-\beta\gamma}}{(\alpha -)^\gamma} \left(\frac{n}{l}\right)^{-\gamma(\alpha-1)} \frac{x_0^\gamma \{ 2^{\alpha-1} + 2(\alpha - 1) \}^\gamma}{\left(\frac{n}{l} - x_0\right)^{\gamma(\alpha-1)}} \left(\frac{n}{l}\right)^{-\gamma} \left(\frac{n}{l}\right)^{\gamma(\alpha-1)}$$

$$\leqslant \frac{C \| a \|_\infty^\gamma B^\gamma x_0^\gamma \{ 2^{\alpha-1} + 2(\alpha - 1) \}^\gamma}{(\alpha - 1)^\gamma} l^{-1-\beta\gamma} \sum_{n=lx_0+1}^{\infty} \left(\frac{n}{l}\right)^{-\gamma} \left(\frac{n}{l} - x_0\right)^{-\gamma(\alpha-1)}$$

$$\leqslant \frac{C \| a \|_\infty^\gamma B^\gamma x_0^\gamma \{ 2^{\alpha-1} + 2(\alpha - 1) \}^\gamma}{(\alpha - 1)^\gamma} l^{-1-\beta\gamma} \int_{lx_0}^{\infty} dx \left(\frac{x}{l}\right)^{-\gamma} \left(\frac{x}{l} - x_0\right)^{-\gamma(\alpha-1)}$$

$$= \frac{C \| a \|_\infty^\gamma B^\gamma x_0^\gamma \{ 2^{\alpha-1} + 2(\alpha - 1) \}^\gamma}{(\alpha - 1)^\gamma} l^{-\beta\gamma} \int_{x_0}^{\infty} dx\, x^{-\gamma}(x - x_0)^{-\gamma(\alpha-1)}$$

and this last expression tends to 0 als $l \to \infty$

2) One shows in the same way that

$$\lim_{l\to\infty} \frac{1}{l} \sum_{n=-\infty}^{-lx_0-1} f\left\{\int_{\mathbb{R}} dx\, a\left(x + \frac{n}{l}\right) g(l, x)\right\} = 0$$

3) Computation of

$$\lim_{l\to\infty} \frac{1}{l} \sum_{n=-lx_0}^{lx_0} f\left\{\int_{\mathbb{R}} dx\, a\left(x + \frac{n}{l}\right) g(l, x)\right\}$$

Choose $\varepsilon > 0$ then there is a $\delta > 0$ such that:

$$\left.\begin{array}{l} | x - y | < \delta \\ x,\ y \in \text{Range } a(\cdot) \end{array}\right\} \to | f(x) - f(y) | < \varepsilon$$

Choose now a $\mu > 0$ such that

$$| x - y | < \mu \to | a(x) - a(y) | < \frac{1}{2A} \delta$$

and divide $[- x_0, x_0]$ in intervals $I_k = \left[\frac{k\mu}{2}, (k + 1)\frac{\mu}{2}\right]$

$$k = \frac{-2x_0}{\mu}, \ldots, \frac{2x_0}{\mu} - 1$$

For $\frac{n}{l}$, $y \in I_k$ one gets:

$$\left| \int_{\mathbb{R}} dx\, a\left(x + \frac{n}{l}\right) g(l, x) - a(y) \right| \leqslant$$

$$\left| \int_{-\mu/2}^{\mu/2} dx\, a\left(x + \frac{n}{l}\right) g(l, x) - a(y) \int_{-\mu/2}^{\mu/2} dx\, g(l, x) \right|$$

$$+ \left| \int_{\mathbb{R} - \left[\frac{-\mu}{2}, \frac{\mu}{2}\right]} dx\, a\left(x + \frac{n}{l}\right) g(l, x) \right| + \left| a(y) \int_{\mathbb{R} - \left[\frac{-\mu}{2}, \frac{\mu}{2}\right]} dx\, g(l, x) \right|$$

$$+ \left| a(y) \left\{ 1 - \int_{\mathbb{R}} dx\, g(l, x) \right\} \right|$$

$$\leqslant \frac{1}{2}\delta + \frac{2 \| a \|_\infty l^{-\beta} B 2^{2-\alpha}}{(1 - \alpha)\mu^{1-\alpha}} + \| a \|_\infty \left| 1 - \int_{\mathbb{R}} dx\, g(l, x) \right| \leqslant \delta$$

for $l \geqslant l_0$

Then also:

$$\left| \int_{\mathbb{R}} dx\, f \circ a(x) - \frac{1}{l} \sum_{n=-lx_0}^{lx_0} f\left\{ \int_{\mathbb{R}} dx\, a\left(x + \frac{n}{l}\right) g(l, x) \right\} \right| \leqslant$$

$$\sum_{k=-\frac{2x_0}{\mu}}^{\frac{2x_0}{\mu}-1} \left| \int_{I_k} dx\, f \circ a(x) - \frac{1}{l} \sum_{\frac{n}{l} \in I_k} f\left\{ \int_{\mathbb{R}} dx\, a\left(x + \frac{n}{l}\right) g(l, x) \right\} \right| \leqslant \varepsilon$$

ACKNOWLEDGMENTS

I wish to thank Prof. A. Verbeure for many helpfull discussions and stimulations during the preparation of this work.

REFERENCES

[1] H. ARAKI and E. H. LIEB, Entropy Inequalities, *Commun. Math. Phys.*, t. **18**, 1970, p. 160-170.

[2] J. BENDAT and S. SHERMAN, Monotone and Convex Operator Functions, *Trans. Ann. Math. Soc.*, t. **79**, 1955, p. 58-71.

[3] J. M. CHAIKEN, Number Operators for Representations of the Canonical Commutation Relations, *Commun. Math. Phys.*, t. **8**, 1968, p. 164-184.

[4] R. H. CRITCHLEY and J. T. LEWIS, The entropy Density of Quasi-Free States, *preprint*.

[5] M. FANNES, The Entropy Density of Quasi-Free States, *Commun. Math. Phys.*, t. **31**, 1973, p. 279-290.

[6] J. MANUCEAU, M. SIRUGUE, D. TESTARD and A. VERBEURE, The smallest C*-Algebra for Canonical Commutation Relations, *Commun. Math. Phys.*, t. **32**, 1973, p. 231-243.

[7] J. MANUCEAU and A. VERBEURE, Quasi-Free States of the C. C. R. Algebra and Bogoliubov Transformation, *Commun. Math. Phys.*, t. **9**, 1968, p. 293-302.

[8] D. W. ROBINSON, *The Thermodynamic Pressure in Quantum Statistical Mechanics*, Springer-Verlag, Heidelberg, 1971.

[9] D. RUELLE, *Statistical Mechanics*, W. A. Benjamin, Inc., New York, Amsterdam, 1969.

(Manuscrit reçu le 1er mars 1977)