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Section A :
Physique théorique.

Exact relativistic theory of wave propagation in prestressed nonlinear elastic solids

by

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ABSTRACT. — The propagation of weak discontinuities in general relativistic prestressed elastic solids is studied under the sole hypothesis that the material be isotropic in an ideally unstrained state. No limitations are placed upon the amplitude of stresses and deformation processes and the formulation has thermodynamical foundations. In this general framework where the state equation for the potential has a quite arbitrary form, it is shown that (*i*) principal wave fronts are either longitudinal or transverse (the propagation of longitudinal ones being impeded by the possible incompressibility of the material); (*ii*) in general, there may be two transverse waves with distinct speeds; (*iii*) the values of these speeds is expressible in terms of three (scalar) response functions (typical of the material) and of the initial stretches; (*iv*) it is possible to give a relative classification of these values and (*v*) in the case of propagation through an initial state of high hydrostatic pressure (case of dense stellar objects), there exists a universal relationship between the simple speed of longitudinal disturbances and the double speed of transverse ones and the speed of sound of a relativistic perfect fluid that would have a law of compression corresponding to the initial state. For a sensible special case of state equation and for an initial state of hydrostatic pressure, the speeds of propagation and the speed of sound referred to above are determined exactly in function of two fundamental scalars of the material and a density ratio. Taking account in supplement of perturbations in the geometry of space-time, the same formalism is applied to the construction of perturbation equations (generalized Hooke-Navier-Duhamel equations) valid either for isothermal or isentropic processes. The latter equations are those to be used either in the study of « elastic » gravitational wave detectors or in the study of small elastic oscillations of dense stellar objects.

RÉSUMÉ. — La propagation des discontinuités faibles dans les solides élastiques précontraints est étudiée dans le cadre de la relativité générale sous la seule hypothèse que le matériau soit isotrope dans un état idéal non déformé. Aucune restriction n'est imposée à l'amplitude des déformations et des contraintes et la formulation est basée sur la thermodynamique. Dans ce cadre général où l'équation d'état a une expression suffisamment arbitraire, il est montré que : (*i*) les ondes principales sont soit longitudinales soit transversales (l'incompressibilité possible du milieu empêchant la propagation des premières) ; (*ii*) il y a en général deux ondes transversales de vitesses distinctes ; (*iii*) la valeur de ces vitesses peut être exprimée en fonction de trois fonctions de réponse scalaires (typiques du matériau) et des élongations initiales ; (*iv*) il est possible de donner une classification relative de ces vitesses et, (*iv*) dans le cas où la propagation a lieu dans un état de forte pression hydrostatique (cas des objets stellaires denses), il existe une relation universelle entre la vitesse simple des perturbations longitudinales, la vitesse double des perturbations transversales et la vitesse du son d'un fluide parfait relativiste qui aurait une loi de compression correspondant à l'état initial. Pour une expression plausible de l'équation d'état et un état initial de pression hydrostatique, il est alors possible de déterminer exactement les vitesses de propagation ainsi que cette vitesse sonique en fonction de deux scalaires caractéristiques du matériau et d'un rapport de densités. De plus, prenant en compte les perturbations de la géométrie de l'espace-temps, le même formalisme est employé à la construction des équations de perturbation (équations généralisées de Hooke-Navier-Duhamel) valables pour des processus isothermes ou isentropiques. Les équations ainsi obtenues sont celles qui doivent être utilisées soit dans l'étude des vibrations élastiques des détecteurs d'ondes gravitationnelles, soit dans l'étude des petites oscillations élastiques des objets stellaires denses.

1. INTRODUCTION

In papers [*1*]-[*2*] recently published we have studied the propagation of infinitesimal discontinuities in certain simple classes of relativistic elastic (or hypoelastic) bodies. A study of the growth of the amplitude of such wave fronts along their ray has demonstrated the need for an exact theory requiring no hypotheses as far as the amplitude of strains is concerned and relying upon a sound thermodynamics. It is the purpose of this paper to present such an exact theory with the sole hypothesis that, in the case of arbitrary finite strains, the body be *isotropic* in an ideally unstrained state. The wave fronts, however, propagate through an initially strained state. For analytical convenience the study must be limited to that concerning a propagation along a principal direction of the initial state of stress. Since the state equations of the elastic bodies to which the present treatment applies are, for the least, badly known (e. g., in neutron stars), we concentrate upon the derivation of those results which may be said to be *universal* in the sense that they do not depend explicitly on the exact form of such a state equation.

Having recalled some basic notions of the theory of deformation processes for general relativistic matter in space-time in Section 2, we establish in Section 3 the *exact* form of constitutive equations for relativistic isotropic nonlinear thermoelastic solids. Following the definition of infinitesimal discontinuities in Section 4, we prove in Section 5 a series of lemmas, theorems and corollaries concerning universal (in the sense specified above) results pertaining to the longitudinal or transverse character of the wave fronts, the values of the speed of these waves, and the relative classification of these values. In general two transverse waves with distinct speeds and one longitudinal wave can propagate. In particular, a universal relationship between the propagation speeds of longitudinal and transverse wave fronts in an initial state of high hydrostatic pressure (as may occur in certain astrophysical objects) is proven in this general framework. The case of relativistic incompressible nonlinear elastic solids which requires special attention is briefly commented upon in Section 6 where the non-propagation of longitudinal wave fronts is proven for such bodies. The results are specialized in Section 7 for a special form of the free energy density. There the speeds of propagation of longitudinal and transverse wave fronts and the speed of sound in the case of an initial state of hydrostatic pressure are determined exactly in terms of two fundamental scalars (analogous to Lamé's moduli), which are characteristic of the material, and of the density ratio. By way of conclusion, we deduce in Section 8, from the exact expressions established before, the generalization of Hooke's law to be used in the treatment of small elastic oscillations either in « elastic » gravitational-wave detectors or in astrophysical objects acted upon by their own gravitational field. The Appendices provide lists of coefficients as also a brief comparison between constitutive equations proposed by different authors for the description of relativistic elastic matter (1). Basic results reported in this paper have been enunciated in a short *Note* [*4*]. Related previous works using a different formalism, which

(1) The brief review given in a preceding paper ([*3*], Appendix) written in 1971-1972 now is obsolete.

either are based on too specialized hypotheses or fall short of the conclusions reached in the present paper, are those of Bressan [5], Grot [6] (in special relativity), and Carter [7].

2. PREREQUISITES

2.1. Notation

Let $M = (V^4, g_{\alpha\beta})$ be a space-time of general relativity equipped with a normal hyperbolic metric $g_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3, 4$; index 4 time-like; Lorentzian signature $+, +, +, -$). u^α is the four-velocity such that $g_{\alpha\beta}u^\alpha u^\beta = -1$ ($c = 1$ for notational convenience). ∂_α and ∇_α denote the partial and covariant derivatives with respect to the local chart x^α of M. $D \equiv u^\alpha\nabla_\alpha$ is the invariant derivative in the direction of u^α. $P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ is the spatial projector which is used systematically in the following development to write down the local canonical space-time decomposition of any tensor field defined on M. The local spatial projection of any geometrical object **A** is noted $\mathbf{A}_\perp$ and admits $\boldsymbol{u}$ as zero vector for all its indices in a local chart. Objects such that $\mathbf{A} \equiv \mathbf{A}_\perp$ are said to be *spatial*. The *transverse* or *spatial* covariant derivative is defined by $\overset{\perp}{\nabla}_\alpha \equiv P_\alpha^{\cdot\beta}\nabla_\beta$.

2.2. Deformation of matter in space-time

Following Maugin [8] and Carter and Quintana [9], we admit that the motion of a relativistic continuum is described either by means of a canonical differentiable projection $\mathscr{P}$ such that $\mathscr{P} : \mathscr{T}[B] \to \mathscr{M}$ or with the aid of the space-time parametrized congruence of world lines $\mathscr{C} : \boldsymbol{x} = \mathscr{X}(\mathbf{X}, \tau)$, $\mathbf{X} \in B$, $\tau \in \mathbb{R}$. Here $\mathscr{T}[B]$ is the open tube of V^4 which is swept out by the material body B (whose constituents are the material « particles » **X**) and $\mathscr{M} = (V^3, G_{KL})$, $K, L = 1, 2, 3$, is the three-dimensional manifold which serves to describe the material continuum. B is an open region of $\mathscr{M}$. τ is the proper time of **X**. $\mathscr{M}$ is equipped with the local background metric G_{KL} and local charts X^K, $K = 1, 2, 3$. We have thus

$$\mathscr{P} : X^K = X^K(x^\alpha), \qquad \tau = \tau(x^\alpha), \qquad \mathscr{C} : x^\alpha = \mathscr{X}^\alpha(X^K, \tau). \tag{2.1}$$

These relations are assumed to possess a sufficient degree of continuity and differentiability in their arguments so as to allow for the forthcoming manipulations. For instance, one can define the inverse motion gradient X^K_α by

$$X^K_\alpha \equiv \partial_\alpha X^K \qquad \left(u^\alpha X^K_\alpha = DX^K = 0\right). \tag{2.2}$$

$g^{\alpha\beta}$ being the reciprocal of $g_{\alpha\beta}$, a space-time invariant noted $\overset{-1}{C}{}^{KL}$ — but tensor field on $\mathscr{M}$ — is constructed by applying the projection $\mathscr{P}$:

$$\overset{-1}{C}{}^{KL} \equiv g^{\alpha\beta}X^K_\alpha X^L_\beta = P^{\alpha\beta}X^K_\alpha X^L_\beta = \overset{-1}{C}{}^{LK}. \tag{2.3}$$

This defines the relativistic analogue of the Piola finite-strain tensor of classical continuum mechanics (Compare [*10*], Chap. I). Its geometrical significance is clear. It is the image of the space-time metric $g^{\alpha\beta}$ by the projection of the space-time on its quotient by the congruence $(2.1)_3$. Furthermore, assuming that the Jacobian determinant of $(2.1)_3$ keeps the same sign (e. g., plus) in the course of the relativistic motion of **X** and defining the direct motion gradient x^α_K by

$$x^\alpha_K = \left(\frac{\partial \mathscr{X}^\alpha}{\partial X^K}\right)_\perp \qquad (x^\alpha_K u_\alpha = 0), \tag{2.4}$$

the chain rule of differentiation yields

$$X^K_\alpha x^\alpha_L = \delta^K_L, \qquad X^K_\alpha x^\beta_K = P^{\cdot\beta}_\alpha, \tag{2.5}$$

and it is possible to define the strain tensor C_{KL} such that $\overset{-1}{C}{}^{PK} C_{KL} = \delta^P_L$ and

$$C_{KL} = g_{\alpha\beta} x^\alpha_K x^\beta_L = P_{\alpha\beta} x^\alpha_K x^\beta_L = C_{LK}. \tag{2.6}$$

Let $G_{\alpha\beta}$ be the image of G_{KL} in M by $\mathscr{P}$, i. e.,

$$G_{\alpha\beta} = G_{KL} X^K_\alpha X^L_\beta = G_{\beta\alpha}. \tag{2.7}$$

Then we can define two useful tensor fields which serve to measure finite strains either on $\mathscr{M}$ or on M, E_{KL} and $\mathscr{E}_{\alpha\beta}$, such that [*3*] [*11*]

$$E_{KL} \equiv \frac{1}{2}(C_{KL} - G_{KL}) = \mathscr{E}_{\alpha\beta} x^\alpha_K x^\beta_L = E_{LK}, \tag{2.8}$$

and

$$\mathscr{E}_{\alpha\beta} = \frac{1}{2}(P_{\alpha\beta} - G_{\alpha\beta}) = E_{KL} X^K_\alpha X^L_\beta = \mathscr{E}_{\beta\alpha}. \tag{2.9}$$

Elementary computations [*3*], [*12*] then allow one to establish the following results that relate propertime rates of change of different geometrical objects of interest:

$$(DX^K_\alpha)_\perp = - e^{\beta}_{\cdot\alpha} X^K_\beta, \qquad (Dx^\alpha_K)_\perp = e^{\alpha}_{\cdot\lambda} x^\lambda_K, \tag{2.10}$$

$$(DP_{\alpha\beta})_\perp = 0, \tag{2.11}$$

$$DG_{KL} = 0, \qquad \underset{u}{\pounds} G_{\alpha\beta} = 0, \tag{2.12}$$

$$DC_{KL} = 2 d_{\alpha\beta} x^\alpha_K x^\beta_L = 2 DE_{KL}, \tag{2.13}$$

and

$$2(\underset{u}{\pounds}\mathscr{E}_{\alpha\beta}) = (\underset{u}{\pounds} P_{\alpha\beta}) = 2 d_{\alpha\beta}, \tag{2.14}$$

where

$$e_{\alpha\beta} \equiv \overset{\perp}{\nabla}_\beta u_\alpha, \qquad d_{\alpha\beta} \equiv e_{(\alpha\beta)} \equiv \frac{1}{2}(\overset{\perp}{\nabla}_\beta u_\alpha + \overset{\perp}{\nabla}_\alpha u_\beta) \tag{2.15}$$

and

$$(\underset{u}{\pounds} A_{\alpha\beta})_\perp = (DA_{\alpha\beta})_\perp + A_{\gamma\beta} \overset{\perp}{\nabla}_\alpha u^\gamma + A_{\alpha\gamma} \overset{\perp}{\nabla}_\beta u^\gamma \tag{2.16}$$

for any spatial tensor $A_{\alpha\beta} \equiv (A_{\alpha\beta})_\perp$. $\underset{u}{\pounds}$ denotes the Lie derivative with respect to the field $\boldsymbol{u}$. In terms of the differentiable projection $\mathscr{P}$, we have

$$(\underset{u}{\pounds}\mathbf{A})_\perp(x) = \mathscr{P}^{-1}\left[\frac{\partial}{\partial\tau}\mathscr{P}(\mathbf{A})(\mathbf{X}, \tau)\right](x), \qquad \forall \mathbf{A}(x) = \mathbf{A}_\perp(x). \tag{2.17}$$

Equation $(2.12)_2$ is a consequence of Eqs. $(2.12)_1$, (2.7) and (2.10). This shows that $G_{\alpha\beta}$ is the background metric on M, which serves as a local standard to measure strains. According to Eqs. (2.12) and (2.14), the Herglotz-Born local condition of rigid-body motion is defined in differential form (Killing's theorem) by either one of the following conditions:

$$DC_{KL} = 0 \quad \text{on } \mathscr{M} \text{ for any } \tau, \qquad d_{\alpha\beta} = 0 \quad \text{on M for } x \in \mathscr{C}. \tag{2.18}$$

2.3. Field equations

In supplement to Einstein's field equations that relate linearly the Einstein tensor and the total energy-momentum tensor, we have

$$\nabla_\alpha(\rho u^\alpha) = 0 \qquad \text{(continuity)}, \tag{2.19}$$

$$\nabla_\beta T^{\alpha\beta} = 0, \qquad T^{[\alpha\beta]} \equiv \frac{1}{2}(T^{\alpha\beta} - T^{\beta\alpha}) = 0. \tag{2.20}$$

Here $\rho(x) = \rho(\mathbf{X}, \tau)$ is the mass per unit of proper volume. In absence of heat conduction, electromagnetic fields and spin the energy-momentum tensor $T^{\alpha\beta}$ admits the following simple canonical space-time decomposition:

$$T^{\alpha\beta} = \rho(1 + \varepsilon)u^\alpha u^\beta - t^{\alpha\beta}, \tag{2.21}$$

where $t^{\alpha\beta} = t^{\beta\alpha}$, as a consequence of Eq. $(2.20)_2$, is the spatial relativistic stress tensor, and ε is the internal energy per unit of proper mass. Taking account of Eq. (2.19) and of the fact that $t^{[\alpha\beta]} = 0$, we project Eq. $(2.20)_1$ along u_α and orthogonally to $\boldsymbol{u}$ to obtain

$$\rho D\varepsilon = t^{\alpha\beta}d_{\alpha\beta} \qquad \text{(energy equation)} \tag{2.22}$$

and

$$\rho f^{\alpha}_{.\beta} Du^\beta = P^{\alpha}_{.\gamma}\overset{\perp}{\nabla}_\beta t^{\gamma\beta} \qquad \text{(Euler-Cauchy equations)}, \tag{2.23}$$

where

$$f_{\alpha\beta} \equiv (1 + \varepsilon)P_{\alpha\beta} - \rho^{-1}t_{\alpha\beta} = f_{\beta\alpha} \equiv (f_{\alpha\beta})_\perp \tag{2.24}$$

is the *tensorial index* of the continuum, cf. Ref. [2]. In the present case the local statement of the second principle of thermodynamics reduces to the equation $D\eta = 0$, where η is the entropy per unit of proper mass. Introducing the specific free energy ψ by $\psi = \varepsilon - \eta\theta$, where θ is the proper thermodynamical temperature ($\theta > 0$, $\inf \theta = 0$), we can rewrite Eq. (2.22) in the form

$$\rho D\psi = -\rho\eta D\theta + t^{\alpha\beta}d_{\alpha\beta}. \tag{2.25}$$

3. CONSTITUTIVE EQUATIONS FOR ISOTROPIC THERMOELASTIC BODIES

A natural definition for general thermoelastic bodies (i. e., with *a priori* large deformations) is given by postulating a functional dependence of the form

$$(\psi, \eta, t^{\alpha\beta}) = \text{functions in the usual sense of } (X^K_\alpha, \theta). \tag{3.1}$$

That is, there are no hereditary effects since dependent and independent variables are considered at the same event point of M or, equivalently, for the same values of the four parameters (X^K, τ). Then we have the

THEOREM 3.1. — *The exact constitutive equations of an* anisotropic *thermoelastic body are given by*

$$t^{\alpha\beta} = -\rho\left(\frac{\partial\psi}{\partial X^K_{(\alpha}}\right)_\theta X^K_\mu P^{\beta)\mu}, \qquad \eta = -\left(\frac{\partial\psi}{\partial\theta}\right)_{X^K_\alpha}, \tag{3.2}$$

ψ satisfying the following set of first order linear partial differential equations:

$$\frac{\partial\psi}{\partial X^K_{[\alpha}} X^K_\mu P^{\beta]\mu} = 0. \tag{3.3}$$

It then follows the

COROLLARY 3.2. — *Equations* (3.2) *and* (3.3) *can be replaced by*

$$t_{\alpha\beta} = -2\rho\frac{\partial\psi}{\partial \overset{-1}{C}{}^{KL}} X^K_\alpha X^L_\beta, \qquad \eta = -\frac{\partial\psi}{\partial\theta}, \tag{3.4}$$

i. e., *$t_{\alpha\beta}$ is the image by $\mathcal{P}$ of* $T_{KL} \equiv (\partial\psi/\partial \overset{-1}{C}{}^{KL}) = T_{LK}$ *up to the factor* -2ρ.

Proof. — Consider $\psi = \psi(X^K_\alpha, \theta)$ to start with, then compute $D\psi$ with the help of Eq. $(2.10)_1$ to obtain

$$\rho D\psi = \rho(\partial\psi/\partial\theta)D\theta - \rho\left(\frac{\partial\psi}{\partial X^K_{(\alpha}} X^K_\mu P^{\beta)\mu}\right)d_{\alpha\beta} - \rho\left(\frac{\partial\psi}{\partial X^K_{[\alpha}} X^K_\mu P^{\beta]\mu}\right)\omega_{\alpha\beta}, \tag{3.5}$$

if $\omega_{\alpha\beta} \equiv e_{[\alpha\beta]}$. Furthermore, according to the principles of formulation set forth by the Author, ψ must be *objective*, i. e., its explicit form should not depend on the observer [*12*], [*13*]. We have shown that in the present case which does not involve hereditary effects, this invariance is equivalent to the rotational Lorentz invariance of $L^\uparrow_+$ in a local inertial frame, or else, to invariance under all generators of SO(3) in local nonholonomic spatial frames along $\mathscr{C}$. Studying such a form invariance for $\psi(X^K_\alpha, \theta)$ under infinitesimal transformations $x'_\alpha \overset{*}{=} (\delta_{\alpha\beta} + \varepsilon L_{\alpha\beta})x_\beta$ in special relativity or under infinitesimal rotations $d'_{(k)\alpha} = \varepsilon L_{\alpha\beta} d_{(k)\beta}$, $k = 1, 2, 3$, between rigid spatial triads $\boldsymbol{d}_{(k)}$ at an event point of M, where $*$ indicates validity in inertial frames only, ε is an infinitesimally small and $L_{\alpha\beta} = -L_{\beta\alpha}$ is arbitrary, and recasting the resulting equation in a complete covariant framework, we are led to the condition (3.3). On account of this, Eq. (3.5) simplifies and the expression of $\rho D\psi$ being carried in Eq. (2.25) which is posited to be valid for any $D\theta$ and all deformation fields that do not rigidify the continuum in the sense of Herglotz and Born (i. e., $d_{\alpha\beta} \neq 0$), completes the proof of Theorem 3.1. The system of differential equations (3.3) integrates immediately along its characteristics if ψ depends on X^K_α only through the space-time invariant combination $\overset{-1}{C}{}^{KL}$. Hence the proof of Corollary 3.2; *Q. E. D.* Equations (3.4) are the equations deduced previously from a variational principle by the Author [8].

Equations (3.4) and those equivalent equations which are discussed in Appendix I describe *anisotropic* thermoelastic bodies. The notion of material symmetry which relies upon cristallography is essentially a three-dimensional Euclidean notion, i. e., it concerns the study of the invariance of functions with respect to members of subgroups of the group O(3). Since, as a result of Corollary 3.2, ψ depends now only on arguments defined on $\mathscr{M}$, θ being a parameter, material symmetry must be discussed in the local tangent space to $\mathscr{M}$ at **X**. However, we shall avoid this complication in the sequel for we shall use only arguments defined on M to facilitate the analysis of wave-front propagation. In fact, using the result enunciated in Theorem 3.1, we can state the

THEOREM 3.3. — a) *The exact constitutive equations of an* isotropic *relativistic thermoelastic body are given by either*

$$t^{\alpha\beta} = \rho\left[\left(\frac{\partial\psi}{\partial\mathscr{E}_{\alpha\beta}}\right)_\theta - 2\mathscr{E}_\lambda^{\cdot(\alpha}\left(\frac{\partial\psi}{\partial\mathscr{E}_{\beta)\lambda}}\right)_\theta\right] = t^{\beta\alpha}, \qquad \eta = -\left(\frac{\partial\psi}{\partial\theta}\right)_{\mathscr{E}} \tag{3.6}$$

or

$$t^{\alpha\beta} = \rho\left[\left(\frac{\partial\varepsilon}{\partial\mathscr{E}_{\alpha\beta}}\right)_\eta - 2\mathscr{E}_\lambda^{\cdot(\alpha}\left(\frac{\partial\varepsilon}{\partial\mathscr{E}_{\beta)\lambda}}\right)_\eta\right] = t^{\beta\alpha}, \qquad \theta = \left(\frac{\partial\varepsilon}{\partial\eta}\right)_{\mathscr{E}} \tag{3.7}$$

depending on whether ψ or ε is used as thermodynamical potential.

b) *Furthermore, ψ and ε are* isotropic *functions* [in the sense of SO(3)] *of the relativistic finite-strain tensor $\boldsymbol{\mathscr{E}}$.*

Proof. — We can write Eq. (2.9) in the form

$$\mathscr{E}_{\mu\nu} = \frac{1}{2}(C_{MN} - G_{MN})X^M_\mu X^N_\nu . \tag{3.8}$$

Note that $\mathscr{E}_{\mu\nu}$ depends on X^K_μ via X^K_μ itself and via C_{MN} that depends on x^α_K, hence on its reciprocal X^K_α. It follows by varying Eq. $(2.5)_1$ that

$$(\partial x^\lambda_P/\partial X^K_\alpha)_\perp = -x^\lambda_K x^\alpha_P . \tag{3.9}$$

From Eq. (2.6) it follows that

$$(\partial C_{MN}/\partial x^{\lambda}_{P}) = P_{\varepsilon\lambda} x^{\varepsilon}_{N} \delta^{P}_{M} + P_{\lambda\varepsilon} x^{\varepsilon}_{M} \delta^{P}_{N} . \tag{3.10}$$

Then Eq. (3.8) yields

$$\left(\frac{\partial \mathscr{E}_{\mu\nu}}{\partial X^{K}_{\alpha}}\right)_{\perp} X^{K}_{\beta} = - P^{\alpha}_{\cdot(\mu}[P_{\nu)\beta} - 2\mathscr{E}_{\nu)\beta}] \tag{3.11}$$

on account of Eqs. (2.5). Then, if one makes the change of independent variables $\psi(X^{K}_{\alpha}, \theta) \rightarrow \psi(\mathscr{E}_{\alpha\beta}, \theta)$, Eq. (3.11) substituted in Eqs. $(3.2)_1$ and (3.3) yields Eq. $(3.6)_1$ and, on account of the symmetry of $\mathscr{E}_{\alpha\beta}$, the following system of first order linear partial differential equations

$$2\mathscr{E}_{\lambda}{}^{[\alpha}(\partial\psi/\partial\mathscr{E}_{\beta]\lambda})_{\perp} = 0 \, ; \tag{3.12}$$

similar equations involving ε instead of ψ are obtained by performing the partial Legendre transformation $\varepsilon = \psi + \eta\theta$. As is readily checked, Eq. (3.12) is but the covariant expression of the fact that ψ must be *objective*, i. e., form-invariant by SO(3) in a local inertial frame. In such a frame Eq. (3.12) is satisfied identically if ψ depends on $\boldsymbol{\mathscr{E}}$ only through its fundamental invariants $I_k \equiv \operatorname{tr} \boldsymbol{\mathscr{E}}^k$, tr = trace, $k = 1, 2, 3$. Since these are space-time invariants, the result holds good in fully covariant formalism. This means that ψ or ε is an *isotropic* function of its tensorial argument, θ or η acting as a simple parameter. The body thus described exhibits no preferred spatial direction as far as its response to deformations is concerned. It is *isotropic; Q. E. D.*

Applying the Cayley-Hamilton theorem it then is possible to restate the foregoing result as

COROLLARY 3.4. — *The exact constitutive equations of an isotropic relativistic thermoelastic body are given in intrinsic notation by*

$$\boldsymbol{t} = \rho \sum_{\Gamma=0}^{2} g_{\Gamma}(I_k, \eta)\boldsymbol{\mathscr{E}}^{\Gamma}, \qquad \theta = (\partial\varepsilon/\partial\eta)_{\boldsymbol{\mathscr{E}}} \tag{3.13}$$

where the g_Γ's *are space-time invariant scalars and, by convention,* $(\boldsymbol{\mathscr{E}}^0)^{\alpha\beta} = P^{\alpha\beta}$.

On account of the expression given in Appendix II for the scalars g_Γ, Eq. $(3.13)_1$ is the relativistic version of the constitutive equation derived by Murnaghan [*14*] in classical isotropic elasticity with finite deformations. Remark that this equation is *universal* in the sense that ε is a general function of the invariants I_k, whose expression can be constrained only by some regularity assumptions, some conditions of elastic stability and the conditions of relativistic causality and the required reality of wave speeds, the latter being determined in following sections.

Remark. — (*i*) The manner in which the equations above have been obtained guarantees that they are valid in special relativity, and at the nonrelativistic limit, in classical continuum mechanics, and that, in supplement to the objectivity requirement of the Author, they satisfy identically the *rheological invariance* proposed by Oldroyd [*15*] for general relativistic continuous matter.

Remark. — (*ii*) A direct proof of Eqs. (3.13) can be given by starting with the *a priori* functional dependence $\psi(\mathcal{E}_{\alpha\beta}, \theta)$. Then, in computing $D\psi$, one uses Eq. (2.16) to pass from $(D\mathcal{E}_{\alpha\beta})_\perp$ to $(\underset{u}{\pounds}\mathcal{E}_{\alpha\beta})_\perp$, hence to $d_{\alpha\beta}$ in virtue of Eq. (2.14). Taking account of the decomposition of $e_{\alpha\beta}$ in symmetric and skewsymmetric parts, one is thus led to

$$D\psi = \left\{ \frac{\partial\psi}{\partial\mathcal{E}_{\lambda(\beta}} [P^{\alpha)}_{\cdot\lambda} - 2\mathcal{E}^{\alpha)}_{\cdot\lambda}] \right\} d_{\alpha\beta} + \left(2\frac{\partial\psi}{\partial\mathcal{E}_{\gamma[\alpha}} \mathcal{E}^{\beta]}_{\cdot\gamma} \right) \omega_{\alpha\beta} + (\partial\psi/\partial\theta)D\theta . \quad (3.14)$$

Applying the same argument as that applied in the proof of Theorem 3.1, but for the objectivity of ψ as a function of $\mathcal{E}_{\alpha\beta}$, it results Eq. (3.12). Hence Eq. (3.14) simplifies, and it remains to substitute for the expression of $D\psi$, provided by this simplified equation, into Eq. (2.25) to arrive at the results (3.6).

4. DEFINITION OF INFINITESIMAL DISCONTINUITIES

We recall the definitions introduced in a previous work [*16*] (See also Lichnerowicz [*17*]). Let $W(x^\alpha) = 0$ be the time-like hypersurface that represents a discontinuity front which propagates in V^4 and thus separates $\mathcal{B} = \mathcal{C}[B]$ in two subregions $\mathcal{B}^+$ and $\mathcal{B}^-$ at each time. We set

$$l_\alpha \equiv \partial_\alpha W = L(\lambda_\alpha - \mathcal{U} u_\alpha), \quad (4.1)$$

$$\lambda_\alpha \equiv L^{-1} P_\alpha^{\cdot\beta} l_\beta, \qquad P^{\alpha\beta}\lambda_\alpha\lambda_\beta = 1, \quad (4.2)$$

and

$$L \equiv (P^{\alpha\beta} l_\alpha l_\beta)^{1/2} \neq 0, \qquad \mathcal{U} \equiv L^{-1}(u^\sigma l_\sigma). \quad (4.3)$$

$\mathcal{U}$ is the (nondimensional) speed of the discontinuity front measured relatively to the moving matter. l_α is oriented from the « minus » to the « plus » side of W. $\mathbf{A}^+$ and $\mathbf{A}^-$ being the uniform limits of $\mathbf{A}$ in approaching W on its two faces, we note $[\![\mathbf{A}]\!] = \mathbf{A}^+ - \mathbf{A}^-$. If $\mathbf{A}$, $g_{\alpha\beta}$ and u^α are continuous across W and if $\bar{\delta}$ denotes the Dirac distribution with compact support on W, then we can write

$$\bar{\delta}[\![\nabla_\alpha \mathbf{A}]\!] = l_\alpha \delta\mathbf{A}, \quad (4.4)$$

and

$$\bar{\delta}[\![\overset{\perp}{\nabla}_\alpha \mathbf{A}]\!] = L\lambda_\alpha \delta\mathbf{A}, \qquad \bar{\delta}[\![D\mathbf{A}]\!] = L\mathcal{U}\delta\mathbf{A}, \quad (4.5)$$

where the field $\delta\mathbf{A}$ is called the infinitesimal discontinuity of $\mathbf{A}$ through W. We call $H_{(\lambda)}$ the two-plane orthogonal to the unit spatial vector λ_α.

$S_{\alpha\beta} \equiv P_{\alpha\beta} - \lambda_\alpha\lambda_\beta$ is the covariant projector on to $H_{(\lambda)}$. The canonical decomposition of any *spatial* geometrical object along the direction of $\boldsymbol{\lambda}$ and on to $H_{(\lambda)}$ is obtained by applying the operator **S**, e. g., with an obvious notation and obvious properties for the elements of decomposition thus introduced,

$$\delta u^\alpha = \delta u^\alpha_\perp + \lambda^\alpha \delta u\,, \qquad \lambda_\alpha \delta u^\alpha_\perp = 0\,, \qquad \delta u = \lambda_\alpha \delta u^\alpha\,, \tag{4.6}$$

$$t_{\alpha\beta} = \bar{T}_{\alpha\beta} + 2\bar{T}_{(\alpha}\lambda_{\beta)} + \bar{T}\lambda_\alpha\lambda_\beta = t_{\beta\alpha}\,. \tag{4.7}$$

Similar decompositions hold good for $f_{\alpha\beta}$ and $\mathscr{E}_{\alpha\beta}$ with the elements of decomposition $(\bar{F}_{\alpha\beta}, \bar{F}_\alpha, \bar{F})$ and $(\bar{E}_{\alpha\beta}, \bar{E}_\alpha, \bar{E})$, respectively.

We call $\mathfrak{M}_0(x \in \mathscr{C}[B] \subset M) = \{ \rho, \varepsilon, \eta, u^\alpha, t^{\alpha\beta}, \mathscr{E}_{\alpha\beta}, g_{\alpha\beta} \}$ a solution of the system of equations formed by Einstein's field equations, Eqs. (2.19) and (2.23), the constitutive equations (3.13) and the condition $D\eta = 0$ (provided that such a solution exists; this difficult problem of existence is not approached in this paper). Then the wave fronts that we consider in the forthcoming sections satisfy the following set of hypotheses:

H_1 : any typical solution $\mathfrak{M}_0(x)$ is continuous across W;

H_2 : except for the metric $g_{\alpha\beta}$, all space-time derivatives of the first order of the fields of the solution $\mathfrak{M}_0(x)$ suffer discontinuities across W (the case where $[\![\partial_\gamma g_{\alpha\beta}]\!] \neq 0$ requires a special study);

H_3 : W is not a gravitational wave front, i. e., $\mathscr{U}^2 = 1$ is excluded;

H_4 : W is not a *material* wave front or, in other words, since $D\eta = 0$ yields $\mathscr{U}\delta\eta = 0$ in agreement with Eq. $(4.5)_2$, W is not an *entropy front*, i. e., $\mathscr{U} = 0$ is excluded so that $\delta\eta = 0$ necessarily.

In virtue of H_1, W is not a shock wave since $[\![u^\alpha]\!] \neq 0$. In virtue of H_3 and H_4 the admissible range for $\mathscr{U}$ is limited to the open interval $]0, 1[\subset \mathbb{R}$ if $\mathscr{U}$ is to be real and less than the light velocity in vacuum (relativistic causality).

We call *principal* wave fronts those wave fronts for which λ_α coincides with an eigenvector of the initial state of stress $t^{\alpha}_{\cdot\beta} \in \mathfrak{M}_0$. According to Eq. $(3.13)_1$, if W is such a wave front, then the corresponding λ_α coincides also with an eigenvector of the initial state of strain $\mathscr{E}^{\alpha}_{\cdot\beta} \in \mathfrak{M}_0$. Naturally, this holds true only for *isotropic* bodies. *Longitudinal* wave fronts are those wave fronts for which $(\delta u \neq 0,\ \delta u^\alpha_\perp = 0)$, and *transverse* wave fronts are those for which $(\delta u = 0,\ |\delta u^\alpha_\perp| \neq 0)$. We shall not consider general wave fronts which may be called *mixed* wave fronts (Cf. [*2*]).

5. PRINCIPAL WAVE FRONTS IN ISOTROPIC RELATIVISTIC THERMOELASTIC BODIES

We consider only *principal* wave fronts except in degenerate cases of initial state of stress where the character or principalness has no meaning. In general $t_{\alpha\beta} \in \mathfrak{M}_0(x)$ admits three distinct orthogonal (with respect to the metric $P_{\alpha\beta}$) eigenvectors (spatial unit four-vectors) $d^\beta_{(k)}$, $k = 1, 2, 3$, with corresponding eigenvalues $t_{(k)}$. For a principal wave front W, let $\boldsymbol{d}_{(1)}$ coincides with $\boldsymbol{\lambda}$. $\boldsymbol{d}_{(1)}$ is also an eigenvector of $\mathscr{E}_{\alpha\beta} \in \mathfrak{M}_0(\boldsymbol{x})$. Let $t_{(1)} = t_{//}$ be the corresponding stress eigenvalue and $\mathscr{E}_{//}$ the corresponding strain eigenvalue. Then $t_{//}$ and $\mathscr{E}_{//}$ are related by the equation

$$t_{//} = \rho \sum_{\Gamma=0}^{2} g_\Gamma \mathscr{E}_{//}^{\Gamma} . \tag{5.1}$$

The remaining two eigenvectors of both $t_{\alpha\beta}$ and $\mathscr{E}_{\alpha\beta}$, $\boldsymbol{d}_{(2)}$ and $\boldsymbol{d}_{(3)}$, form an orthonormal dyad on which can be projected any tensorial object **A** such that $\mathbf{S}(\mathbf{A}) \equiv \mathbf{A}$. Then we can set the following lemma.

LEMMA 5.1. — *Principal* (infinitesimal discontinuity) *wave fronts* W *that propagate in an isotropic relativistic thermoelastic body are either purely longitudinal or purely transversal.*

Proof. — A straightforward calculation yields the following expression (written in intrinsic formalism) for the right-hand side of Eq. (2.23) on account of Eq. $(3.13)_1$:

$$(\overset{\perp}{\nabla} . \boldsymbol{t})_\perp = \rho^{-1} \boldsymbol{t} . \overset{\perp}{\nabla} \rho + \rho \sum_{\Gamma=0}^{2} \left[\sum_{j=1}^{3} \mathrm{A}_{\Gamma j}(\mathrm{I}_k, \eta) \boldsymbol{\mathscr{E}}^\Gamma . \overset{\perp}{\nabla} \mathrm{I}_j + g_\Gamma (\overset{\perp}{\nabla} . \boldsymbol{\mathscr{E}}^\Gamma)_\perp \right]$$
$$+ \rho \left(\sum_{\Gamma=0}^{2} \bar{g}_\Gamma (\mathrm{I}_k, \eta) \boldsymbol{\mathscr{E}}^\Gamma \right) . \overset{\perp}{\nabla} \eta , \tag{5.2}$$

where $\bar{g}_\Gamma(\mathrm{I}_k, \eta) \equiv \partial g_\Gamma / \partial \eta$ and the nine scalars $\mathrm{A}_{\Gamma j}$, which are functions of I_k and η only, are listed in Appendix III.

Now consider the infinitesimal discontinuities of Eqs. (2.19), (2.23) and of $(\underset{u}{\pounds} \mathscr{E}_{\alpha\beta})_\perp$ on account of the definition (2.16) and of Eq. (5.2). Taking account of the fact that $f_{\alpha\beta}$ and $\mathscr{E}_{\alpha\beta}$ are continuous across W and using the definitions (4.1) through (4.5) and Eq. (5.1), we obtain, with $\mathscr{U} \neq 0$,

$$\delta\rho = - \rho^{-1} \mathscr{U} \delta u , \tag{5.3}$$

$$(\delta \mathscr{E}_{\alpha\beta})_\perp = \mathscr{U}^{-1} \left[\frac{1}{2} (\lambda_\alpha \delta u_\beta + \lambda_\beta \delta u_\alpha) - \mathscr{E}_{\gamma\beta} \lambda_\alpha \delta u^\gamma - \mathscr{E}_{\alpha\gamma} \lambda_\beta \delta u^\gamma \right], \tag{5.4}$$

$$\mathscr{E}^{\alpha\beta} (\delta \mathscr{E}_{\alpha\beta})_\perp = \mathscr{U}^{-1} \mathscr{E}_{//} (1 - 2\mathscr{E}_{//}) \delta u , \tag{5.5}$$

and

$$\mathscr{E}^{\alpha}_{\cdot\gamma} \mathscr{E}^{\gamma\beta} (\delta \mathscr{E}_{\alpha\beta})_\perp = \mathscr{U}^{-1} \mathscr{E}_{//}^2 (1 - 2\mathscr{E}_{//}) \delta u , \tag{5.6}$$

$$\begin{aligned} &\rho f^{\alpha}_{\cdot\beta} \mathscr{U}^2 \delta u^\beta - \rho \lambda^\alpha (1 - 2\mathscr{E}_{//}) \mathscr{A} \delta u + t_{//} \lambda^\alpha \delta u \\ &\quad - \rho \left\{ g_1 \left[\frac{1}{2} (\lambda^\alpha \delta u + \delta u^\alpha) - \mathscr{E}_{//} \lambda^\alpha \delta u - \mathscr{E}^{\alpha}_{\gamma} \delta u^\gamma \right] \right. \\ &\quad \left. + 2 g_2 \left[\frac{1}{2} (\mathscr{E}_{//} \lambda^\alpha \delta u + \mathscr{E}^{\alpha}_{\cdot\gamma} \delta u^\gamma) - \mathscr{E}_{//}^2 \lambda^\alpha \delta u - \mathscr{E}^{\alpha\mu} \mathscr{E}_{\mu\gamma} \delta u^\gamma \right] \right\} = 0 , \end{aligned} \tag{5.7}$$

where

$$\mathscr{A}(\mathrm{I}_k, \eta, \mathscr{E}_{//}) \equiv \sum_{\Gamma=0}^{2} \sum_{j=1}^{3} (j\mathscr{E}_{//}^{\Gamma+j-1}\mathrm{A}_{\Gamma j}). \tag{5.8}$$

In writing Eq. (5.7) we have taken account of the results (5.3), (5.5) and (5.6) and of the fact that $\delta\eta = 0$. Upon using the decompositions (4.6) and (4.7) and the analogous decompositions for $f_{\alpha\beta}$ and $\mathscr{E}_{\alpha\beta}$ and accounting for the fact that, λ_α being an eigenvector of $\mathscr{E}_{\alpha\beta}$ and $t_{\alpha\beta}$, it also is an eigenvector of $f_{\alpha\beta}$, the projection of Eq. (5.7) along the direction of $\boldsymbol{\lambda}$ yields

$$\rho[\mathscr{U}^2\bar{\mathrm{F}} - \{(1-2\mathscr{E}_{//})\mathscr{A} - \mathscr{B} + g_1(1-2\mathscr{E}_{//}) + 2g_2\mathscr{E}_{//}(1-2\mathscr{E}_{//})\}]\delta u = 0, \tag{5.9}$$

with

$$\mathscr{B}(\mathrm{I}_k, \eta, \mathscr{E}_{//}) \equiv \sum_{\Gamma=0}^{2} g_\Gamma \mathscr{E}_{//}^{\Gamma} = \rho^{-1}t_{//}, \tag{5.10}$$

whereas its projection onto $\mathrm{H}_{(\lambda)}$ reads

$$\rho\left[\mathscr{U}^2\bar{\mathrm{F}}^{\alpha}_{\cdot\beta} - g_1\left(\frac{1}{2}\mathrm{S}^{\alpha}_{\cdot\beta} - \mathrm{S}^{\alpha}_{\cdot\mu}\mathscr{E}^{\mu}_{\cdot\beta}\right) - 2g_2\left(\frac{1}{2}\mathrm{S}^{\alpha}_{\cdot\mu}\mathscr{E}^{\mu}_{\cdot\beta} - \mathrm{S}^{\alpha}_{\cdot\mu}\mathscr{E}^{\mu\lambda}\mathscr{E}_{\lambda\beta}\right)\right]\delta u^{\beta}_{\perp} = 0, \tag{5.11}$$

the mixed projection vanishing identically for $\bar{\mathrm{T}}_\alpha = \bar{\mathrm{F}}_\alpha = \bar{\mathrm{E}}_\alpha = 0$; hence the proof of Lemma 5.1. That is, we have uncoupling between longitudinal and transverse wave fronts because (*i*) of the *isotropy* of the body and (*ii*) of the principalness of the wave front.

Since $\rho \neq 0$, we can state at once the following theorem:

THEOREM 5.2. — *Longitudinal principal wave fronts that propagate in an isotropic relativistic thermoelastic body have a speed* $\mathscr{U}_{\mathrm{L}}$ *such that*

$$\mathscr{U}_{\mathrm{L}}^2 = (\bar{\mathrm{F}}v_{//}^2)^{-1}[(g_1 + 2\mathscr{E}_{//}g_2) + \mathscr{A} - v_{//}^2\mathscr{B}], \tag{5.12}$$

where

$$v_{//} \equiv (1 - 2\mathscr{E}_{//})^{-1/2} \tag{5.13}$$

is the principal stretch in the spatial direction λ_α, *whereas transverse principal wave fronts in general have two distinct speeds,* $\mathscr{U}_{\mathrm{T}2}$ *and* $\mathscr{U}_{\mathrm{T}3}$, *which are solutions of the equation*

$$\det_{2\times 2} |\mathrm{Q}_{\perp}{}^{\alpha}_{\cdot\beta}| = 0 \tag{5.14}$$

where

$$\mathrm{Q}_{\perp}{}^{\alpha}_{\cdot\beta}(\mathfrak{M}_0, \mathscr{U}_{\mathrm{T}}^2) = \mathscr{U}^2\bar{\mathrm{F}}^{\alpha}_{\cdot\beta} - \left[g_1\left(\frac{1}{2}\mathrm{S}^{\alpha}_{\cdot\beta} - \mathrm{S}^{\alpha}_{\cdot\mu}\mathscr{E}^{\mu}_{\cdot\gamma}\mathrm{S}^{\gamma}_{\cdot\beta}\right)\right.$$
$$\left. + 2g_2\left(\frac{1}{2}\mathrm{S}^{\alpha}_{\cdot\mu}\mathscr{E}^{\mu}_{\cdot\gamma} - \mathrm{S}^{\alpha}_{\cdot\mu}\mathscr{E}^{\mu\lambda}\mathscr{E}_{\lambda\gamma}\right)\mathrm{S}^{\gamma}_{\cdot\beta}\right]. \tag{5.15}$$

Equation (5.14) is solved immediately in the nonholonomic frame $(\boldsymbol{d}_{(2)}, \boldsymbol{d}_{(3)})$ where $\mathrm{Q}_{\perp}{}^{\alpha}_{\cdot\beta}$ diagonalizes. Setting $f_2 = \bar{\mathrm{F}}^{2}_{\cdot 2}$, $f_3 = \bar{\mathrm{F}}^{3}_{\cdot 3}$, $\mathscr{E}_2 = \bar{\mathrm{E}}^{2}_{\cdot 2}$, $\mathscr{E}_3 = \bar{\mathrm{E}}^{3}_{\cdot 3}$, and $v_j \equiv (1 - 2\mathscr{E}_j)^{-1/2}$, $j = 2, 3$, the solutions of (5.14) are given by

$$\mathscr{U}_{\mathrm{T}2,3}^2 = (2f_{2,3}v_{2,3}^2)^{-1}[g_1(\mathfrak{M}_0) + 2\mathscr{E}_{2,3}g_2(\mathfrak{M}_0)]. \tag{5.16}$$

For the wave speeds to be real and less than unity, the right-hand side of Eqs. (5.12) and (5.16) must be in the interval]0, 1[. This clearly imposes constraints on the initial state $\mathfrak{M}_0$ i. e., on rather complex combinations of the response functions g_Γ and the initial strains and stretches, $\mathscr{E}_k$ and v_k. Apart from those constraints, the results enunciated in the form of Eqs. (5.12) and (5.16) are *universal* since they do not depend on any assumption as regards the amplitude of strains (e. g., they are valid for *finite* strains) and on any particular functional dependence of the internal energy $\in(\mathrm{I}_k, \eta)$, which of course possesses a sufficient regularity. A general study of the constraints referred to above cannot be performed under the hypothesis of a general initial state. Neither can it be achieved a relative classification of the two transverse wave speeds in an exact manner in such a general framework. The approximate following results, however, can be established. Let us define A_k, $k = 2, 3$, by

$$\mathrm{A}_k^{-1/2} = \sqrt{f_k}\, v_k\,. \tag{5.17}$$

Then with the definition of v_k we can give the following form to the difference $\mathscr{U}_{\mathrm{T}2}^2 - \mathscr{U}_{\mathrm{T}3}^2$:

$$2(\mathscr{U}_{\mathrm{T}2}^2 - \mathscr{U}_{\mathrm{T}3}^2) = (\mathrm{A}_2 - \mathrm{A}_3)(g_1 + g_2) - g_2(f_2\mathrm{A}_2^2 - f_3\mathrm{A}_3^2)\,. \tag{5.18}$$

Of course, $f_2 \sim f_3 = 1 + 0(c^{-2}) > 0$, so that we can introduce a mean value f for f_2 and f_3 and rewrite (5.18) as

$$2(\mathscr{U}_{\mathrm{T}2}^2 - \mathscr{U}_{\mathrm{T}3}^2) \simeq (\mathrm{A}_2 - \mathrm{A}_3)\{\, g_1 + g_2[1 - f(\mathrm{A}_2 + \mathrm{A}_3)]\,\}\,. \tag{5.19}$$

The A_k are all positive from their very definition, and $v_i > v_j$ yields $\mathrm{A}_i < \mathrm{A}_j$. We have thus

COROLLARY 5.3. — a) *Transverse wave fronts with amplitude parallel to the axis of lesser transverse stretch travel at a greater absolute speed than others if, with* $g_1 > 0$,

i) *either* $g_2 < 0$ *and* $1 - f(\mathrm{A}_2 + \mathrm{A}_3) < g_1/|\,g_2\,|$,

ii) *or* $g_2 > 0$ *and* $f(\mathrm{A}_2 + \mathrm{A}_3) - 1 < g_1/g_2$.

b) *Transverse wave fronts with amplitude parallel to the axis of greater transverse stretch travel at a greater absolute speed than others if, with* $g_1 > 0$,

i) *either* $g_2 < 0$ *and* $f(\mathrm{A}_2 + \mathrm{A}_3) - 1 < g_1/g_2$,

ii) *or* $g_2 > 0$ *and* $1 - f(\mathrm{A}_2 + \mathrm{A}_3) < g_1/|\,g_2\,|$.

c) *The two types of wave front travel at the same absolute speed if and only if the corresponding transverse stretches are equal.*

The result *c*) is exact and does not require the approximation (5.19). Statements *a*) and *b*) follow from the discussion of the sign of the right-hand side of Eq. (5.19).

The reason why we have considered $g_1 > 0$ is made clear as follows. Another possibility for expressing the difference $\mathscr{U}_{\mathrm{T}2}^2 - \mathscr{U}_{\mathrm{T}3}^2$ is obtained by reintroducing the principal stresses $t_{(2)}$ and $t_{(3)}$ via an equation of the type of Eq. (5.1) for those quantities in terms of the eigenvalues $\mathscr{E}_2$ and $\mathscr{E}_3$. We have

$$f_2\mathscr{U}_{T2}^2 - f_3\mathscr{U}_{T3}^2 = (\mathscr{E}_3 - \mathscr{E}_2)(g_1 - g_2) + g_2(\mathscr{E}_3^2 - \mathscr{E}_2^2), \qquad (5.20)$$

and

$$g_2\mathscr{E}_j^2 = (t_{(j)}/\rho) - (g_0 + g_1\mathscr{E}_j), \qquad j = 2, 3. \qquad (5.21)$$

Hence, Eq. (5.20) takes the form

$$f_2\mathscr{U}_{T2}^2 - f_3\mathscr{U}_{T3}^2 = \rho^{-1}(t_{(3)} - t_{(2)}) - g_2(\mathscr{E}_3 - \mathscr{E}_2). \qquad (5.22)$$

If $f \sim f_2 \sim f_3$, Eq. (5.22) reduces to

$$\frac{\mathscr{U}_{T2}^2 - \mathscr{U}_{T3}^2}{c_T^2} = \frac{t_{(3)} - t_{(2)}}{\mu}, \qquad (5.23)$$

if

$$g_2 = 0 \quad \text{and} \quad c_T^2 = \mu/\rho f, \qquad (5.24)$$

where μ is Lamé's modulus and c_T is a typical transverse-wave speed. $g_2 = 0$ represents one part of the neo-Hookean assumption (stress-strain constitutive relation at most explicitly linear in $\boldsymbol{\mathscr{E}}$). Equation (5.23) is similar to an equation given in our previous work [*2*]. It says that transverse wave fronts with amplitude parallel to the axis of lesser transverse stress travel at a greater absolute speed than the others. The other part of the neo-Hookean assumption is obtained by looking at Eq. (5.18) which, with the approximation made above, takes the same form as Eq. (5.23) if and only if $\mu \simeq \rho g_1/2v^2$, where v is a typical transverse stretch. Since μ is experimentally shown to be positive (and must in fact be so according to the thermodynamics of neo-Hookean materials), and $v^2 > 0$, then g_1 must be greater than zero. By the same token the definition $(5.24)_2$ makes sense. In conclusion of this point a representation of neo-Hookean materials is obtained for

$$g_2 = 0, \qquad (\rho g_1/2v^2) = \mu. \qquad (5.25)$$

The statement *c*) of Corollary 5.3 holds good in certain degenerate cases of initial stresses and strains, for instance, *i*) if this state is a cylindrically symmetric one about the direction $\boldsymbol{\lambda}$ and *ii*) if this initial state is *spherical*, that is, fully degenerate, in which case the above-obtained results apply although the notion of principalness has lost its meaning. Such an initial state is, for instance, an initial state of high hydrostatic pressure, as can arise in the « geophysics » of neutron stars (See Ruderman [*18*]). Regarding this special case the following remarkable result can be arrived at.

THEOREM 5.4. — *The* (simple) *speed* $\mathscr{U}_{//}$ *and the* (double) *speed* $\mathscr{U}_{\perp}$ *of longitudinal and transverse wave fronts that propagate in an isotropic relativistic nonlinear elastic body, of which the initial state is one of high* *hydrostatic pressure* p_0 *(case of dense stellar objects), are related by the universal relationship*

$$\mathcal{U}_{//}^2 = \frac{4}{3}\mathcal{U}_\perp^2 + a^2(\mathfrak{M}_0), \tag{5.26}$$

where

$$a^2(\mathfrak{M}_0) = f^{-1}(\mathfrak{M}_0)\frac{\partial p}{\partial \rho}\bigg|_{\mathfrak{M}_0} \tag{5.27}$$

and

$$f \equiv 1 + \varepsilon(\mathfrak{M}_0) + p_0/\rho(\mathfrak{M}_0), \qquad \eta = \text{const.}, \tag{5.28}$$

a being the sound speed, and f the index (in the sense of Lichnerowicz), *of a relativistic perfect fluid that would have the same law of compression.*

Proof. — We are in a fully degenerate case for which

$$t^{\alpha\beta}(\mathfrak{M}_0) = -\, p_0(\mathfrak{M}_0)\mathrm{P}^{\alpha\beta}(\mathfrak{M}_0) \tag{5.29}$$

with

$$p_0 = -\,\rho(\mathfrak{M}_0)\sum_{\Gamma=0}^{2} g_\Gamma(\mathfrak{M}_0)\mathscr{E}^\Gamma, \tag{5.30}$$

where $\mathscr{E}_{//} = \mathscr{E}_2 = \mathscr{E}_3 \equiv \mathscr{E}$. Set $v = (1 - 2\mathscr{E})^{-1/2}$ the isotropic stretch in the state $\mathfrak{M}_0$. Then the matter proper density $\rho(\mathfrak{M}_0)$ and the same density in an ideally unstrained state, $\rho_{(i)}$, are related by the equation $\rho(\mathfrak{M}_0) = \rho_{(i)}v^{-3}$. We deduce thus

$$\sum_{j=1}^{3} j\mathscr{E}^{j-1}\frac{\partial}{\partial \mathrm{I}_j} = \frac{1}{3}\frac{\partial}{\partial \mathscr{E}} \tag{5.31}$$

and

$$\frac{\partial}{\partial \mathscr{E}}\bigg|_{\mathfrak{M}_0} = -\,3\rho(\mathfrak{M}_0)v^2(\mathfrak{M}_0)\frac{\partial}{\partial \rho}\bigg|_{\mathfrak{M}_0} \tag{5.32}$$

by applying the chain rule of differentiation. It follows from (5.30) and (5.32) that

$$\frac{\partial p_0}{\partial \rho}\bigg|_{\mathfrak{M}_0} = \frac{p_0}{\rho(\mathfrak{M}_0)} + \frac{1}{3v^2}\left[\frac{\partial g_0}{\partial \mathscr{E}} + \mathscr{E}\frac{\partial g_1}{\partial \mathscr{E}} + \mathscr{E}^2\frac{\partial g_2}{\partial \mathscr{E}} + g_1 + 2\mathscr{E}g_2\right]_{\mathfrak{M}_0}, \tag{5.33}$$

whereas Eqs. (5.12) and (5.16) reduce to

$$\mathcal{U}_{//}^2 = \mathcal{U}_{\mathrm{L}}^2 = \frac{1}{fv^2}\left[g_1 + 2\mathscr{E}g_2 + \frac{1}{3}\left(\frac{\partial g_0}{\partial \mathscr{E}} + \mathscr{E}\frac{\partial g_1}{\partial \mathscr{E}} + \mathscr{E}^2\frac{\partial g_2}{\partial \mathscr{E}} + v^2\frac{p_0}{\rho}\right)\right]_{\mathfrak{M}_0}, \tag{5.34}$$

and

$$\mathcal{U}_\perp^2 = \mathcal{U}_{\mathrm{T}2}^2 = \mathcal{U}_{\mathrm{T}3}^2 = \frac{1}{2fv^2}(g_1 + 2\mathscr{E}g_2)_{\mathfrak{M}_0}, \tag{5.35}$$

respectively, on account of Eq. (5.31) and of the definition $(5.28)_1$. Substituting from (5.33) into (5.34) and combining (5.34) and (5.35) completes the proof. *Q. E. D.*

The *exact* result (5.26) valid within the relativistic framework of finite-strain theory is *universal* for no hypotheses need be made concerning the explicit functional form of the internal energy function. It consists in the general relativistic generalization of a classical result due to Truesdell [*19*]. In the neo-Hookean case described by Eq. (5.25) it reduces to the equation proposed by Carter [*7*]. It is reasonable to assume that $(\partial p_0/\partial \rho)_\eta > 0$. Therefore, in general, $\mathcal{U}_{//}^2 > \frac{4}{3}\mathcal{U}_\perp^2$. Relativistic causality thus imposes that $(4/3)\mathcal{U}_\perp^2 + a^2(\mathfrak{M}_0) < 1$. That is,

$$\frac{2}{3v^2(\mathfrak{M}_0)f(\mathfrak{M}_0)}\left[g_1(\mathfrak{M}_0) + \left(1 - \frac{1}{v^2}\right)g_2(\mathfrak{M}_0)\right] + a^2(\mathfrak{M}_0) < 1\,. \quad (5.36)$$

It is difficult to establish the reality of $\mathcal{U}_\perp$, but the following can be pointed out:

COROLLARY 5.5. — *If transverse wave fronts can propagate at all in an isotropic relativistic nonlinear elastic body in an initial state of hydrostatic pressure, then longitudinal wave fronts can propagate as well.*

Indeed, if $\mathcal{U}_\perp^2 > 0$, then $\mathcal{U}_{//}^2 > a^2 > 0$. However, if $\mathcal{U}_\perp^2 < 0$ (no propagation of transverse fronts), then $\mathcal{U}_{//}^2 < a^2$ and $\mathcal{U}_{//}$ can be zero or imaginary, so that the case $\mathcal{U}_{//}^2 < 0$ (no propagation of longitudinal wave fronts) cannot *a priori* be excluded.

In the neo-Hookean case the causality condition (5.36) takes on the simple form (with $\tilde{\mu} \equiv \mu/\rho$)

$$(4\tilde{\mu}/3) + (\partial p_0/\partial \rho)_{\mathfrak{M}_0} < f(\mathfrak{M}_0)\,. \quad (5.37)$$

For a body unable to support shearing effects, hence for $\tilde{\mu} \equiv 0$, this last inequality reduces to that given in relativistic hydrodynamics (Cf. Israel [*20*]).

6. REMARK ON THE INCOMPRESSIBLE CASE

Typical materials for which the foregoing development applies are those which make up the thick crust of neutron stars, of which the outer portion probably resembles terrestrial matter except that it is about 10^{18} times more rigid than steel and much more incompressible, so that it is easier to jiggle it than to compress it (Cf. [*8*], [*21*]). Conclusions regarding this limiting case can be drawn directly from the results of previous sections. If the relativistic elastic body is incompressible, then the deformations it suffers are *isochoric*. This is expressed in terms of the strain tensor $\mathcal{E}_{\alpha\beta}$ by the condition

$$\mathrm{A} \equiv \det_{3\times 3} |\, \mathrm{P}^{\alpha}_{.\beta} - 2\mathcal{E}^{\alpha}_{.\beta} \,| = \text{const.} \quad (6.1)$$

This states that there exists a scalar relationship between the fundamental invariants of $\mathscr{E}_{\alpha\beta}$. Only two of these are independent. Then, in computing the propertime derivative of the internal energy density $\varepsilon(\mathrm{I}_k, \eta)$, one must account for the constraint (6.1) by introducing a scalar Lagrange multiplier $\bar{p}$. Taking the propertime derivative of Eq. (6.1) we have

$$\left(\frac{\partial \mathrm{A}}{\partial \mathrm{I}_1} \mathrm{P}^{\alpha\beta} + 2\frac{\partial \mathrm{A}}{\partial \mathrm{I}_2} \mathscr{E}^{\alpha\beta} + 3\frac{\partial \mathrm{A}}{\partial \mathrm{I}_3} \mathscr{E}^{\alpha}_{.\mu}\mathscr{E}^{\mu\beta}\right)(\mathrm{D}\mathscr{E}_{\alpha\beta})_{\perp} = 0 . \tag{6.2}$$

Reintroducing the Lie derivative of $\mathscr{E}_{\alpha\beta}$, hence $d_{\alpha\beta}$ and $\omega_{\alpha\beta}$, we can rewrite Eq. (6.2) as [in intrinsic notation, the « : » meaning double contraction]

$$\left[\left(\sum_{j=1}^{3} j\frac{\partial \mathrm{A}}{\partial \mathrm{I}_j} \boldsymbol{\mathscr{E}}^{j-1}\right) . (\mathbf{P} - 2\boldsymbol{\mathscr{E}})\right] : \boldsymbol{d} = 0 \tag{6.3}$$

with $(\boldsymbol{\mathscr{E}}^0)^{\alpha\beta} = \mathrm{P}^{\alpha\beta}$ [$\omega_{\alpha\beta}$ does not contribute for its factor is symmetrical in α and β in virtue of the isotropy]. Then on account of this constraint and for *isotropic* incompressible bodies, the constitutive equations (5.1) are replaced by

$$\boldsymbol{t} = \sum_{j=1}^{3} \left[\rho g_{j-1}(\mathrm{I}_k, \eta)\mathbf{P} - j\bar{p}\frac{\partial \mathrm{A}(\mathrm{I}_k)}{\partial \mathrm{I}_j}(\mathbf{P} - 2\boldsymbol{\mathscr{E}})\right] . \boldsymbol{\mathscr{E}}^{j-1} . \tag{6.4}$$

$\bar{p}$ is an indeterminate multiplier to be determined upon solving a well-posed boundary-value problem. Of crouse, the Cayley-Hamilton theorem (II.1) can be used to eliminate the third power of $\boldsymbol{\mathscr{E}}$.

By the same token and for a principal wave front, Eq. (6.2) yields the infinitesimal discontinuity

$$\mathscr{U}^{-1}\left[\sum_{j=1}^{3} j\frac{\partial \mathrm{A}}{\partial \mathrm{I}_j} \mathscr{E}_{//}^{j-1}\right](1 - 2\mathscr{E}_{//})\delta u = 0 \tag{6.5}$$

on account of equations of the type of Eqs. (5.5) and (5.6). The component of Eq. (6.4) along $\boldsymbol{\lambda}$ yields

$$t_{//} = \sum_{j=1}^{3} \rho g_{j-1}\mathscr{E}_{//}^{j-1} - \bar{p}\left[\sum_{j=1}^{3} j\frac{\partial \mathrm{A}}{\partial \mathrm{I}_j} \mathscr{E}_{//}^{j-1}(1 - 2\mathscr{E}_{//})\right] . \tag{6.6}$$

The last quantity within brackets cannot be zero for it is the quantity that distinguishes the present case from the general one [compare Eq. (5.1)]. Of course, $(1 - 2\mathscr{E}_{//})$ is not zero for $(1 - 2\mathscr{E}_{//}) = v_{//}^2$, and $v_{//}^{-2}$ and the other stretches v_2 and v_3 at the same power cannot be zero since the constraint (6.1) written down in an adapted system of coordinates, where $(P^{\beta}_{.\alpha} - 2\mathscr{E}^{\beta}_{.\alpha})$ diagonalizes, reads

$$\prod_{j=1}^{3}(1 - 2\mathscr{E}_j) = \prod_{j=1}^{3} v^{-2} = 1 . \tag{6.7}$$

Since $\mathscr{U}^{-1} \neq 0$, it follows from (6.5) that δu vanishes necessarily. Hence we have the

LEMMA 6.1. — *The incompressibility of an isotropic relativistic nonlinear elastic solid impedes the propagation of any longitudinal disturbance.*

We do not pursue further the incompressible case.

7. NEO-HOOKEAN THERMOELASTIC MATERIALS

So far, no particular expression for ε or ψ has been given. In order to get some insight in the approximation that will follow in Section 8, we consider the special case where ψ has the following expression in terms of the invariants I_k of $\mathscr{E}_{\alpha\beta}$ and of the temperature variable $\bar{\theta} \equiv \theta - \theta_0$, where θ_0 is some reference thermodynamical temperature such that thermal equilibrium in a stationary gravitational field corresponds to constant « red-shifted » temperature, i. e., $T \equiv \theta_0\sqrt{-g_{44}} = \text{const.}$ (Cf. [*22*]):

$$\psi(I_k, \bar{\theta}) = \psi_0(\bar{\theta}) - \alpha\bar{\theta}\tilde{\Lambda}I_1 + \frac{\tilde{\Lambda}}{2}I_1^2 + \frac{\tilde{\mu}}{3}(3I_2 - I_1^2), \tag{7.1}$$

where α is the thermal expansion coefficient, $\tilde{\Lambda}$ is the bulk modulus (per unit of proper mass) and $\tilde{\mu}$ is a material constant. Then the exact constitutive equation (3.6), which we can rewrite as

$$t^{\alpha\beta} = \rho\left[\left(\frac{\partial\psi}{\partial\mathscr{E}_{\alpha\beta}}\right)_\theta - 2P^{\alpha\gamma}\mathscr{E}_{\lambda\gamma}\left(\frac{\partial\psi}{\partial\mathscr{E}_{\beta\lambda}}\right)_\theta\right], \qquad \eta = -\left(\frac{\partial\psi}{\partial\bar{\theta}}\right)_{\mathscr{E}}, \tag{7.2}$$

yield

$$t^{\alpha\beta} = \rho[(\tilde{\lambda}I_1 - \alpha\tilde{\Lambda}\bar{\theta})P^{\alpha\beta} + 2(\tilde{\mu} - \tilde{\lambda}I_1 + \alpha\tilde{\Lambda}\bar{\theta})\mathscr{E}^{\alpha\beta} - 4\tilde{\mu}\mathscr{E}^{\alpha}_{.\lambda}\mathscr{E}^{\lambda\beta}], \tag{7.3}$$

and

$$\eta = \eta_0 - \alpha\tilde{\Lambda}I_1, \tag{7.4}$$

where

$$\eta_0 \equiv - d\psi_0/d\bar{\theta}, \qquad \tilde{\lambda} \equiv \frac{1}{3}(\tilde{\Lambda} - 2\tilde{\mu}). \tag{7.5}$$

Therefore, the g_Γ 's of Eq. $(3.13)_1$ are given by

$$\begin{aligned} g_0(I_k, \bar{\theta}) &= \tilde{\lambda}I_1 - \alpha\tilde{\Lambda}\bar{\theta}, \\ g_1(I_k, \bar{\theta}) &= 2(\tilde{\mu} - \tilde{\lambda}I_1 + \alpha\tilde{\Lambda}\bar{\theta}), \\ g_2(I_k, \theta) &= -4\tilde{\mu} \Rightarrow (\partial g_2/\partial I_k) = 0, \quad \forall I_k . \end{aligned} \tag{7.6}$$

A dual formulation holds good with ψ and θ replaced by ε and $-\eta$. Equations (7.3) and (7.4) constitute a relativistic *nonlinear* version of Duhamel's equations of thermoelasticity (Compare [*10*], Chap. 8). These equations are highly nonlinear, not only because of the presence of product terms in $\boldsymbol{\mathscr{E}}$, but also as a consequence of the presence of the factor ρ, which can be shown to depend on all invariants I_k. If $\rho_{(i)}$ is the proper density of matter in an ideally unstrained state, it is shown, following along the lines of the classical case ([*23*], p. 226) that

$$\rho = \rho_{(i)}\left[1 + 2I_1 + 2(I_1^2 - I_2 + 2I_1I_2) - \frac{4}{3}I_1^3 - \frac{8}{3}I_3\right]^{1/2}. \quad (7.7)$$

[for an isotropic deformation, hence isotropic strains and stretches, $\mathscr{E}$ and v, this yields $\rho_{(i)} = \rho(1 - 2\mathscr{E})^{-3/2} = \rho v^3$, hence the formula used in the proof of Theorem 5.4]. If, however, one discards temperature effects, $\alpha = 0$, and assumes, $\mathscr{E}_{\alpha\beta}$ being a spatial tensor, that $|\boldsymbol{\mathscr{E}}| = (\mathrm{tr}\,\boldsymbol{\mathscr{E}}^2)^{1/2} \ll 1$, then Eq. (7.3) can be approximated by (compare Eq. (4.11) in Ref. [*2*])

$$t^{\alpha\beta} = \rho(\tilde{\lambda}I_1P^{\alpha\beta} + 2\tilde{\mu}\mathscr{E}^{\alpha\beta}), \quad (7.8)$$

which we have referred to as the stress constitutive equation for a relativistic neo-Hookean material. The corresponding relativistic study of the propagation of infinitesimal discontinuities has been performed in Ref. [*2*]. If, now, we consider the more involved case of Eq. (7.3), but with $\alpha = 0$, then the results of Section 5 specialize to the following ones on account of Eq. (7.6).

a) *For an initially unstrained state* $\mathfrak{M}_0$: Then, for this $\mathfrak{M}_0$, $g_1 = 2\tilde{\mu}$, $v_2 = v_3 = v_{//} = 1$, $\mathscr{E}_2 = \mathscr{E}_3 = \mathscr{E}_{//} = 0$, $\Psi_1 = \tilde{\lambda}I_1$, $\Psi_2 = \tilde{\mu}$, $\Psi_{11} = \tilde{\lambda}$, $\mathscr{B} = 0$, $\mathscr{A} = A_{01} = \Psi_{11} = \tilde{\lambda}$ and $f(\mathfrak{M}_0) = 1$, so that

$$\mathscr{U}_L^2 = \tilde{\lambda} + 2\tilde{\mu}, \qquad \mathscr{U}_{T2}^2 = \mathscr{U}_{T3}^2 = \tilde{\mu}\,; \quad (7.9)$$

b) *For an initial state* $\mathfrak{M}_0$ *of hydrostatic pressure* p_0 *corresponding to a density* ρ_0 : Then, for this $\mathfrak{M}_0$, $g_1 = 2(\tilde{\mu} - \tilde{\lambda}I_1)$, $g_2 = -4\tilde{\mu}$, $A_{01} = \tilde{\lambda}$, $A_{11} = -2\tilde{\lambda}$, $\mathscr{B} = -p_0/\rho_0$, $\mathscr{A} = A_{01} + \mathscr{E}A_{11} = \tilde{\lambda}v^{-2}$, $I_1(\mathfrak{M}_0) = 3\mathscr{E}$, $I_2(\mathfrak{M}_0) = 3\mathscr{E}^2$, $\mathscr{E} = \frac{1}{2}(1 - v^{-2})$ and $v^{-2} = [\rho_0/\rho_{(i)}]^{2/3}$, so that

$$\begin{aligned}\mathscr{U}_{//}^2 &= 2\mathscr{U}_\perp^2 + f_0^{-1}(v^{-2}\mathscr{A} - \mathscr{B}), \\ &= f_0^{-1}\left(\frac{\rho_0}{\rho_{(i)}}\right)^{2/3}\left\{\tilde{\lambda} + 2\tilde{\mu} + 4\left[\left(\frac{\rho_0}{\rho_{(i)}}\right)^{2/3} - 1\right](\tilde{\lambda} + \tilde{\mu})\right\} + (p_0/f_0\rho_0) \quad (7.10)\end{aligned}$$

and

$$\mathscr{U}_\perp^2 = \mathscr{U}_{T3}^2 = \mathscr{U}_{T2}^2 = f_0^{-1}\left(\frac{\rho_0}{\rho_{(i)}}\right)^{2/3}\left\{\tilde{\mu} + \left[\left(\frac{\rho_0}{\rho_{(i)}}\right)^{2/3} - 1\right]\left(\frac{3\tilde{\lambda} + 2\tilde{\mu}}{2}\right)\right\}, \quad (7.11)$$

where

$$f_0 = 1 + \frac{p_0}{\rho_0} + \frac{3}{4}\left[1 - \left(\frac{\rho_0}{\rho_{(i)}}\right)^{2/3}\right]\left\{\tilde{\lambda} + \tilde{\mu}\left[1 - \left(\frac{\rho_0}{\rho_{(i)}}\right)^{2/3}\right]\right\}. \quad (7.12)$$

Comments. — *i*) The results (7.9) are identical with those of the classical Hooke-Lamé theory of nonrelativistic linear elasticity. Since they correspond to the case $p_0 = 0$, then, if $(\partial p_0/\partial\rho)(\mathfrak{M}_0)$ also vanishes, the universal relationship (5.26) holds good in this limiting case if and only if $\mathcal{U}_{//}^2 = \frac{4}{3}\mathcal{U}_{\perp}^2$. That is, according to Eq. $(7.5)_2$, if and only if the reduced bulk modulus $\tilde{\Lambda} = 3\tilde{\lambda} + 2\tilde{\mu}$ vanishes identically. This condition is analogous to Stokes' hypothesis for viscous fluids.

ii) As p_0 goes to zero and ρ_0 goes to $\rho_{(i)} \neq 0$, the results (7.10) and (7.11) approach the results (7.9). The universal relationship (5.26) holds good *a priori* for the results (7.10)-(7.11). This allows one to deduce the speed of sound in the matter under study as

$$a^2(\mathfrak{M}_0) = \frac{2}{3}\mathcal{U}_{\perp}^2 + f_0^{-1}\left(\frac{\tilde{\lambda}}{v^4} + \frac{p_0}{\rho_0}\right) \quad (7.13)$$

from the first part of Eq. (7.10). That is, on account of Eqs. (7.11) and $(7.5)_2$,

$$a^2(\mathfrak{M}_0) = f_0^{-1}\left(\frac{\rho_0}{\rho_{(i)}}\right)^{2/3}\left[2\left(\frac{\rho_0}{\rho_{(i)}}\right)^{2/3} - 1\right]\frac{\tilde{\Lambda}}{3} + \frac{p_0}{\rho_0}. \quad (7.14)$$

This formula agrees with the comments (*i*). Equation (7.14) transforms further on account of Eq. (5.30), which reads $(p_0/\rho_0) = v^{-2}(v^{-2} - 1)\tilde{\Lambda}$. So that, finally,

$$a^2(\mathfrak{M}_0) = f_0^{-1}\left(\frac{\rho_0}{\rho_{(i)}}\right)^{2/3}\left[\frac{5}{4}\left(\frac{\rho_0}{\rho_{(i)}}\right)^{2/3} - 1\right]\frac{4\tilde{\Lambda}}{3}. \quad (7.15)$$

For $\rho_0 \# \rho_{(i)}$ (infinitesimally deformed initial state $\mathfrak{M}_0$), $a^2 \# \tilde{\Lambda}/3f_0$ (for $c \to \infty$: classical result). In this approximation $a^2 < \mathcal{U}_{//}^2$ for $\tilde{\mu} > 0$, and $\mathcal{U}_{\perp}^2 > 0$ as a consequence of Eq. (5.26). Thus, with $\tilde{\Lambda} > 0$ and $\tilde{\mu} > 0$, as can be proven on the basis of classical thermodynamical arguments (requirement that the thermodynamical potential be positive for any $\mathcal{E}_{\alpha\beta}$ ([2])), both transverse and longitudinal wave fronts can propagate in this initial state.

([2]) For isothermal processes Eq. (7.1) yields $\frac{\tilde{\Lambda}}{2}\mathrm{I}_1^2 + \tilde{\mu}\left(\mathrm{I}_2 - \frac{1}{3}\mathrm{I}_1^2\right) > 0$; But $\mathrm{I}_1^2 > 0$ and $\mathrm{J}_2^2 \equiv \mathrm{I}_2 - \frac{1}{3}\mathrm{I}_1^2 > 0$ are independent for they are, respectively, the squared trace and the squared deviatoric part of the same tensor, hence $\tilde{\mu} > 0$ and $\tilde{\Lambda} > 0$ (if strict positiveness is required).

8. GENERALIZED HOOKE'S LAW IN GENERAL RELATIVITY

By Hooke's law must be understood in the present context a relationship between infinitesimally small strains and stresses of weak intensity. Such a relationship can be deduced from the general constitutive equations (3.6) or (3.7) for an ideally isotropic body. Since the *operator* of infinitesimal discontinuity, δ, is a *derivation* (See Lichnerowicz [*17*]), it is sufficient to apply this operator to both sides of, e. g., Eq. (3.6) or (7.2). Now we account for gravitational effects explicitly and must therefore vary the space-time metric $g_{\alpha\beta}$ (hypothesis H_3 relaxed). The variation performed thus is analogous to that made in a previous paper [*8*] since we vary simultaneously *i*) the space-time metric, *ii*) the world line of the material particle **X** and *iii*) the thermodynamical temperature θ. We set thus

$$(\delta P_{\alpha\beta})_\perp = (\delta g_{\alpha\beta})_\perp \equiv h_{\alpha\beta} = h_{\beta\alpha}, \qquad \xi^\lambda \equiv \delta x^\lambda. \tag{8.1}$$

Then Eqs. (2.5) yield

$$(\delta X^K_\alpha)_\perp = -P^{\mu\lambda} X^K_\mu (\nabla_\alpha \xi_\lambda)_\perp, \tag{8.2}$$

and it follows from Eq. (2.7) and the fact that G_{KL} remains invariant in the variation procedure that

$$(\delta G_{\alpha\beta})_\perp = -P^{\lambda\mu}[G_{\mu\beta}(\nabla_\alpha\xi_\lambda)_\perp + G_{\alpha\mu}(\nabla_\beta\xi_\lambda)_\perp]. \tag{8.3}$$

Upon using again Eq. (2.9) and carrying the resulting expression and Eq. (8.1) in the variation $(\delta\mathscr{E}_{\alpha\beta})_\perp$, we are led to

$$(\delta\mathscr{E}_{\alpha\beta})_\perp = \frac{1}{2}\{ h_{\alpha\beta} + P^{\lambda\mu}[(P_{\mu\beta} - 2\mathscr{E}_{\mu\beta})(\nabla_\alpha\xi_\lambda)_\perp + (P_{\alpha\mu} - 2\mathscr{E}_{\alpha\mu})(\nabla_\beta\xi_\lambda)_\perp]\}. \tag{8.4}$$

The factors of $(\nabla_\alpha\xi_\lambda)_\perp$ and $(\nabla_\beta\xi_\lambda)_\perp$ are to be evaluated at $\mathfrak{M}_0$ and the spatial projection to be performed with the help of $P^{\alpha}_{\cdot\beta}(\mathfrak{M}_0)$. In particular, for an initially unstrained state, Eq. (8.4) reduces to

$$\varepsilon_{\alpha\beta} \equiv (\delta\mathscr{E}_{\alpha\beta})_\perp = \frac{1}{2}[h_{\alpha\beta} + (\nabla_\alpha\xi_\beta)_\perp + (\nabla_\beta\xi_\alpha)_\perp]. \tag{8.5}$$

This last tensor is spatial and symmetric, so that in a rectangular frame $\{ e_i ; i = 1, 2, 3 \}$, we have the strain tensor

$$\varepsilon_{ij} = \frac{1}{2} h_{ij} + \xi_{(i,j)}. \tag{8.6}$$

This has the same form as the classical linearized Eulerian strain tensor of linear elasticity theory except for the contribution $h_{ij}/2$, which represents a purely general relativistic effect. The tensor ε_{ij} is that introduced previously by several authors (e. g., Dyson [*24*], Papapetrou [*25*], Maugin [*3*], Glass and Winicour [26]-[27]; See also Soper [28], p. 202-205) using different arguments, all in fact different from that used herein above. In particular, the more general expression (8.4) is that to be used for an initially strained body. Within the same general framework the variation of Eq. (7.2) yields

$$(\delta t^{\alpha\beta})_{\perp} = C_{\theta}^{\alpha\beta\mu\nu}(\mathfrak{M}_0)(\delta\mathscr{E}_{\mu\nu})_{\perp} + A_{\theta}^{\alpha\beta\mu\nu}(\mathfrak{M}_0)h_{\mu\nu} - \Theta^{\alpha\beta}(\mathfrak{M}_0)\delta\theta\,, \qquad (8.7)$$

and

$$\theta_0\delta\eta = C_{\mathscr{E}}(\mathfrak{M}_0)\delta\theta - \theta_0\rho_0^{-1}A^{\alpha\beta}(\mathfrak{M}_0)(\delta\mathscr{E}_{\alpha\beta})_{\perp}\,, \qquad (8.8)$$

where θ_0 and ρ_0 are the proper temperature and matter density of the initial state $\mathfrak{M}_0$, and we have set

$$C_{\theta}^{\alpha\beta\mu\nu}(\mathfrak{M}_0) \equiv c_{\theta}^{\alpha\beta\mu\nu}(\mathfrak{M}_0) - 2c_{\theta}^{\lambda(\alpha|\mu\nu|}(\mathfrak{M}_0)\mathscr{E}^{\beta)}_{\cdot\lambda}(\mathfrak{M}_0) - 2\rho_0\Psi^{\mu(\alpha}(\mathfrak{M}_0)P_0^{\beta)\nu} - t_0^{\alpha\beta}P_0^{\mu\nu}\,, \qquad (8.9)$$

$$A_{\theta}^{\alpha\beta\mu\nu}(\mathfrak{M}_0) \equiv 2\rho_0\mathscr{E}^{\mu}_{\cdot\lambda}(\mathfrak{M}_0)\Psi^{\lambda(\beta}(\mathfrak{M}_0)P_0^{\alpha)\nu} = A_{\theta}^{\alpha\beta\nu\mu}\,, \qquad (8.10)$$

$$\Theta^{\alpha\beta}(\mathfrak{M}_0) \equiv -\rho_0[\kappa^{\alpha\beta}(\mathfrak{M}_0) - 2P_0^{\gamma(\alpha}\kappa^{\beta)\lambda}(\mathfrak{M}_0)\mathscr{E}_{\lambda\gamma}(\mathfrak{M}_0)] = \Theta^{\beta\alpha}\,, \qquad (8.11)$$

and

$$C_{\mathscr{E}}(\mathfrak{M}_0) \equiv -\theta_0(\partial^2\psi/\partial\theta^2)_{\mathscr{E}}(\mathfrak{M}_0)\,, \qquad (8.12)$$

where

$$c_{\theta}^{\alpha\beta\mu\nu}(\mathfrak{M}_0) \equiv \rho_0(\partial\Psi^{\alpha\beta}/\partial\mathscr{E}_{\mu\nu})_{\perp}(\mathfrak{M}_0) = \rho_0(\partial\Psi^{\mu\nu}/\partial\mathscr{E}_{\alpha\beta})_{\perp}(\mathfrak{M}_0) = c_{\theta}^{(\alpha\beta)(\mu\nu)}\,, \qquad (8.13)$$

$$\kappa^{\alpha\beta}(\mathfrak{M}_0) \equiv (\partial\Psi^{\alpha\beta}/\partial\theta)_{\perp}(\mathfrak{M}_0) = -(\partial\eta/\partial\mathscr{E}_{\alpha\beta})_{\perp}(\mathfrak{M}_0) = \kappa^{\beta\alpha}\,, \qquad (8.14)$$

$$A^{\alpha\beta}(\mathfrak{M}_0) \equiv \rho_0\kappa^{\alpha\beta}(\mathfrak{M}_0)\,, \qquad (8.15)$$

$$\Psi^{\alpha\beta} \equiv (\partial\psi/\partial\mathscr{E}_{\alpha\beta})_{\perp}\,, \qquad P_0^{\alpha\beta} \equiv P^{\alpha\beta}(\mathfrak{M}_0)\,, \qquad t_0^{\alpha\beta} \equiv t^{\alpha\beta}(\mathfrak{M}_0)\,. \qquad (8.16)$$

In deriving the equations (8.7) and (8.8) we have used the fact that

$$\delta\rho = -\rho_0P_0^{\alpha\beta}(\delta\mathscr{E}_{\alpha\beta})_{\perp}\,, \qquad (8.17)$$

and

$$(\delta g^{\mu\nu})_{\perp} = -P_0^{\mu\alpha}P_0^{\nu\beta}h_{\alpha\beta}\,. \qquad (8.18)$$

The second contribution in the right-hand side of Eq. (8.7) vanishes either in special relativity ($h_{\mu\nu} \equiv 0$) or in general relativity for an initially unstrained state. The spatial symmetric tensor $\Theta^{\alpha\beta}$ is the tensor of thermoelastic moduli. $C_{\theta}^{\alpha\beta\mu\nu}(\mathfrak{M}_0)$ is the (spatial) tensor of *apparent elasticities* at constant temperature. $C_{\mathscr{E}}(\mathfrak{M}_0)$ is the heat capacity at constant strains at the initial state $\mathfrak{M}_0$. On account of Eq. (8.4), Eq. (8.7) is the *exact* perturbation equation for the relativistic stress about the state $\mathfrak{M}_0$. Its expression can be reduced on account of the exact representation $\psi = \psi(I_k, \theta)$. We shall not give this expression here. We however note that, for an *isothermal process*, the purely gravitational effect on $t^{\alpha\beta}$ and η is given by

$$(\delta t^{\alpha\beta})_{\perp}^{gr} = \left[A_{\theta}^{\alpha\beta\mu\nu}(\mathfrak{M}_0) + \frac{1}{2}C_{\theta}^{\alpha\beta\mu\nu}(\mathfrak{M}_0)\right]h_{\mu\nu} \qquad (8.19)$$

and

$$\delta\eta = -\frac{1}{2}\kappa^{\alpha\beta}(\mathfrak{M}_0)h_{\alpha\beta} \qquad (8.20)$$

on account of Eq. (8.4). Whereas for *isentropic processes* Eq. (8.8) in general yields

$$\delta\theta = [\theta_0/C_{\mathscr{E}}(\mathfrak{M}_0)]\kappa^{\alpha\beta}(\delta\mathscr{E}_{\alpha\beta})_\perp , \tag{8.21}$$

so that Eq. (8.7) can be rewritten in the form

$$(\delta t^{\alpha\beta})_\perp = C_\eta^{\alpha\beta\mu\nu}(\mathfrak{M}_0)(\delta\mathscr{E}_{\mu\nu})_\perp + A_\eta^{\alpha\beta\mu\nu}(\mathfrak{M}_0)h_{\mu\nu} , \tag{8.22}$$

where

$$\left.\begin{aligned} C_\eta^{\alpha\beta\mu\nu}(\mathfrak{M}_0) &\equiv C_\theta^{\alpha\beta\mu\nu}(\mathfrak{M}_0) - [\theta_0/C_{\mathscr{E}}(\mathfrak{M}_0)]\Theta^{\alpha\beta}(\mathfrak{M}_0)\kappa^{\mu\nu}(\mathfrak{M}_0) , \\ A_\eta^{\alpha\beta\mu\nu}(\mathfrak{M}_0) &\equiv A_\theta^{\alpha\beta\mu\nu}(\mathfrak{M}_0) . \end{aligned}\right\} \tag{8.23}$$

For an *initially unstrained* (hence, unstressed) state $\mathfrak{M}_0$, we deduce immediately from the general equations (8.7) through (8.16) that

$$t^{\alpha\beta} = t_0^{\alpha\beta} + (\delta t^{\alpha\beta})_\perp \equiv (\delta t^{\alpha\beta})_\perp , \qquad (t_0^{\alpha\beta} \equiv 0) ,$$

i. e.,

$$t^{\alpha\beta} = C_\theta^{\alpha\beta\mu\nu}(0)\varepsilon_{\mu\nu} - \alpha(0)\Lambda(\delta\theta)P_0^{\alpha\beta} , \tag{8.24}$$

and

$$\theta_0\delta\eta = C_{\mathscr{E}}(0)\delta\theta + \theta_0\rho_0^{-1}\Lambda\alpha(0)\varepsilon^{\alpha}_{\cdot\alpha} , \tag{8.25}$$

where

$$\begin{aligned} C_\theta^{\alpha\beta\mu\nu}(0) &= c_\theta^{\alpha\beta\mu\nu}(0) \\ &= \rho_0[(\mathscr{U}_{//}^2 - 2\mathscr{U}_\perp^2)P_0^{\alpha\beta}P_0^{\mu\nu} + \mathscr{U}_\perp^2(P_0^{\alpha\mu}P_0^{\beta\nu} + P_0^{\alpha\nu}P_0^{\beta\mu})] , \end{aligned} \tag{8.26}$$

and

$$\alpha(0) = -\rho_0\kappa(0)\Lambda^{-1} \tag{8.27}$$

with [we are the case where Eqs. (7.9) hold good independently of the exact form of ψ]

$$\begin{gathered} \mathscr{U}_\perp^2 = \tilde{\mu} \equiv \frac{\partial\psi}{\partial I_2}(\mathfrak{M}_0) , \qquad \mathscr{U}_{//}^2 = \tilde{\lambda} + 2\tilde{\mu} , \\ \tilde{\lambda} \equiv (\partial^2\psi/\partial I_1^2)(\mathfrak{M}_0) , \qquad \Lambda \equiv \rho_0(3\tilde{\lambda} + 2\tilde{\mu}) , \\ \kappa(0) = (\partial^2\psi/\partial\theta\partial I_1)(\mathfrak{M}_0) , \qquad C_{\mathscr{E}}(0) \equiv -\theta_0(\partial^2\psi/\partial\theta^2)(\mathfrak{M}_0) . \end{gathered} \tag{8.28}$$

Thermodynamical arguments require that $\tilde{\mu} > 0$, $\tilde{\Lambda} > 0$ and $C_{\mathscr{E}} > 0$. Nothing can be decided as to the sign of $\kappa(0)$, hence the sign of the thermal expansion coefficient $\alpha(0)$, which depends on whether the body expands or contracts on heating. For *isentropic processes*, Eq. (8.24) becomes

$$t^{\alpha\beta} = C_\eta^{\alpha\beta\mu\nu}(0)\varepsilon_{\mu\nu} , \tag{8.29}$$

where

$$C_\eta^{\alpha\beta\mu\nu}(0) = C_\theta^{\alpha\beta\mu\nu}(0) + \left[\frac{\Lambda^2}{\rho_0 C_{\mathscr{E}}}\alpha^2(0)\right]P_0^{\alpha\beta}P_0^{\mu\nu} , \tag{8.30}$$

hence an alteration in Lamé's moduli.

The generalized Hooke's laws (8.24) and (8.29) thus obtained for thermoelastic bodies within the general relativistic framework offer a justification, on the one hand, for the equations used in the treatment of elastic gravitational-wave detectors (Cf. [*3*]) and, on the other hand, for the equations used by Cattaneo and Gerardi [*29*] ([3]) in their treatment, at the post-Newtonian approximation, of the equilibrium of an elastic sphere in its own gravitational field, in which, as shown above if $\alpha = 0$, only two material constants $\tilde{\lambda}$ and $\tilde{\mu}$ intervene. They could be used to study, for astrophysical purposes and by generalizing the work of Taub [*31*] in relativistic hydrodynamics, the small isentropic oscillations of a self-gravitating elastic sphere. The equations that generalize the above-deduced equations for dielectric elastic (e. g., piezoelectric) bodies — with the purpose of examining in a better fashion the oscillations of elastic gravitational-wave detectors — and for magnetoelastic bodies — with the purpose of examining the small isentropic magnetoelastic oscillations of a solid-like stellar object in its own gravitational and magnetic fields (the latter are known to be very intense) — are given elsewhere [*32*].

([3]) See also previous works of Cattaneo [*30*].

APPENDIX I

Different formulations of general relativistic elasticity

The formulation given in Theorem 3.2 for relativistic elasticity appears to be the primitive one for it makes use of the primitive independent variable X^K_α as far as deformation processes are concerned and there is no restriction placed on the symmetry of the matter. The formulation (3.4) is the same as that given in Maugin [8]. Up to the formalism, it appears to be the same as Souriau's [33], which appeared in a Journal of small diffusion. It is possible to show that dual equations using x^α_K and C_{KL} or E_{KL} as independent variables can be constructed. That is, equivalently to Eqs. (3.2), (3.3) and (3.4) we have:

a) if $\psi = \psi(x^\alpha_K, \theta)$,

$$t^{\alpha\beta} = \rho \left.\frac{\partial\psi}{\partial x^\gamma_K}\right|_\theta P^{\gamma(\alpha} x^{\beta)}_K , \qquad \eta = -\left(\frac{\partial\psi}{\partial\theta}\right)_{x^\alpha_K} , \qquad \left.\frac{\partial\psi}{\partial x^\gamma_K}\right|_\theta P^{\gamma[\alpha} x^{\beta]}_K = 0 \tag{I.1}$$

on account of Eq. $(2.10)_2$;

b) if $\psi = \psi(C_{KL}, \theta)$: Then $(I.1)_3$ is satisfied identically and (I.1) yields

$$t^{\alpha\beta} = 2\rho \left.\frac{\partial\psi}{\partial C_{KL}}\right|_\theta x^\alpha_K x^\beta_L , \qquad \eta = -\left(\frac{\partial\psi}{\partial\theta}\right)_{C_{KL}=\text{const.}} ; \tag{I.2}$$

c) if $\psi = \psi(E_{KL}, \theta)$:

$$t^{\alpha\beta} = \rho \left.\frac{\partial\psi}{\partial E_{KL}}\right|_\theta x^\alpha_K x^\beta_K , \qquad \eta = -\left(\frac{\partial\psi}{\partial\theta}\right)_{E_{KL}=\text{const.}} . \tag{I.3}$$

Here we have used Eq. (2.13). Equations $(I.2)_1$, $(I.3)_1$ and (3.4) are equivalent on account of the fact that

$$\delta \overset{-1}{C}{}^{KP} = - \overset{-1}{C}{}^{KL} \overset{-1}{C}{}^{PQ} \delta C_{LQ} , \qquad \delta C_{MN} = - C_{MA} C_{NB} \delta \overset{-1}{C}{}^{AB} .$$

Equation $(I.2)_1$ in special relativity is essentially the result obtained at the beginning of the century by several authors (e. g., Herglotz [*34*], Born [*35*], Ignatowsky [*36*] who defined the relativistic rigid-body motion by the local condition $DC_{KL} = 0$ (for any proper time of the « particle » **X**). It also is the equation arrived at by several authors in recent years for general relativistic systems (e. g., Bressan [*37*], Schöpf [*38*] and Lianis [*39*]). The formulation (I.3) is that used by Maugin [*3*] and Barrabes [*40*]. All above-mentioned formulations allow the consideration of a restricted symmetry (and not only isotropy) for the relativistic matter. An approximation of the Hookean type (8.29) for infinitesimal strains and weak perturbations in the space-time metric was deduced in Ref. [*3*] from the formulation (I.3) This can be obtained by considering a quadratic expansion of ψ in terms of E_{KL}, assuming isotropy so as to reduce the form of the tensorial coefficients present in this expansion and noting that, by using a variational process similar to that used in Section 8 but applied to the definition (2.8) of the relativistic Lagrangian strain E_{KL} on account of the fact that

$$(\delta x^\alpha_K)_\perp = - x^\alpha_L x^\beta_K (\delta X^L_\beta)_\perp ,$$

we obtain

$$\delta E_{KL} = \frac{1}{2} [h_{\alpha\beta} + P^{\lambda}_{\cdot\beta}(0)(\nabla_\alpha \xi_\lambda)_\perp + P^{\lambda}_{\cdot\alpha}(0)(\nabla_\beta \xi_\lambda)_\perp] x^\alpha_K(0) x^\beta_L(0) \tag{I.4}$$

for an *initially unstrained* state $\mathfrak{M}_0$. Thus $x^\alpha_K(0)$ is evaluated in such conditions. In a local spatial orthonormal frame Eq. (I.4) thus yields

$$\delta E_{KL} = \varepsilon_{ij} \delta^i_K \delta^j_L , \tag{I.5}$$

where ε_{ij} is the strain tensor defined by Eq. (8.6), since then $x^i_K(0) = \delta^i_K$ in the first approximation. The corresponding Hooke's law generalized so as to account for viscosity in gravitational — wave detectors has been established by Maugin [*41*] and used subsequently by different authors (e. g., Gambini [*42*]).

Somewhat similar to the formulation of Theorem 3.3 several other formulations have been proposed which make use only of independent variables (strains) defined on the space-time manifold (and not on $\mathscr{M}$). The first of these is Rayner's [*43*], but this author specializes from the start to an expression for $t^{\alpha\beta}$ linear in $\mathscr{E}_{\alpha\beta}$. Bennoun's formulation [*44*], although based on thermodynamical arguments, appears to be unsound for it uses as independent variable the projector $P_{\alpha\beta}$ itself. This tensor, however, does not contain the whole information needed to describe deformation processes. In fact, it contains only the effect of the space-time metric, so that Bennoun's theory reduces to nought in special relativity [Indeed, with $P_{\alpha\beta}$ as sole variable, the approximation (8.5) cannot be obtained for, then, $(\delta P_{\alpha\beta})$ provides only $h_{\alpha\beta}$ and nothing concerning the displacement (4)]. The same comment applies to the formulation of Section 3 in Carter and Quintana [*9*]. Mrs Lamoureux and Choquet-Bruhat [*45*] do not recognize the fact that their equation $t^{\alpha\beta} = \rho(\partial\psi/\partial\mathscr{E}_{\alpha\beta})_\perp$ is valid only for *isotropic* bodies and, furthermore, forget the second part present in Eqs. $(3.6)_1$ and $(3.7)_1$ (5), so that their equation does not reduce to that of classical nonlinear elasticity at the nonrelativistic limit. The same comments hold true as regards the formulation of Section 3 in Ref. [*9*]. The equation of Cattaneo [*30*], however, is correct only in *convected* coordinates (similar to the X^K's), in which case his equation is none other than $\tilde{t}^{KL} \equiv t^{\alpha\beta}X^K_\alpha X^L_\beta = \rho(\partial\psi/\partial E_{KL})$ [compare Eq. $(I.3)_1$]. In the local chart of M it should take the form $(3.6)_1$ for an isotropic body.

An original formulation due to Hernandez [*46*] makes use of the (3 + 1)-dimensional formalism of Arnowitt, Deser and Misner instead of the operation of projection used herein. We refer to Ref. [*2*] for a brief discussion of constitutive equations of the rate-type form (so-called hypoelastic bodies).

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(4) This criticism does not apply to the tensor $\overset{-1}{C}{}^{KL}$ which, although constructed from $P^{\alpha\beta}$ by Eq. (2.3), contains the classical deformation effect via the X^K_α.

(5) In deriving constitutive equations from a *scalar* potential, care must be taken to take first the *invariant* derivative, D, of the potential and then pass from the invariant derivative (whose form does not depend on the tensorial order of the object to which it applies) of the strain tensor to its Lie derivative (whose explicit form in local charts obviously depends on the tensorial order of the object).

APPENDIX II

Scalar coefficients of eq. (3.13)

With $I_k = \mathrm{tr}\,\boldsymbol{\mathcal{E}}^k$, $k = 1, 2, 3$, the Cayley-Hamilton theorem written in index-free notation for the second-order symmetric spatial tensor $\boldsymbol{\mathcal{E}}$ reads [Compare Gol'denblat [*23*], Eq. (1.102)]

$$\boldsymbol{\mathcal{E}}^3 - I_1\boldsymbol{\mathcal{E}}^2 + \frac{1}{2}(I_1^2 - I_2)\boldsymbol{\mathcal{E}} - \left(\frac{1}{3}I_3 - \frac{1}{2}I_1I_2 + \frac{1}{6}I_1^3\right)\mathbf{P} = \mathbf{0}. \qquad \text{(II.1)}$$

This provides $\boldsymbol{\mathcal{E}}^3$ as a function of $\mathbf{P}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{E}}^2$ and the invariants I_k. Then, considering either $\varepsilon = \varepsilon(I_k, \eta)$ or $\psi = \psi(I_k, \theta)$ and taking account of (II.1), Eq. $(3.13)_1$ obtains with the *response functions* g_Γ given by

$$g_0 = \Psi_1 - \Psi_3(2I_3 - 3I_1I_2 + I_1^3), \qquad \text{(II.2)}$$

$$g_1 = 2(\Psi_2 - \Psi_1) + 3\Psi_3(I_1^2 - I_2), \qquad \text{(II.3)}$$

$$g_2 = 3\left[\Psi_3(1 - 2I_1) - \frac{4}{3}\Psi_2\right], \qquad \text{(II.4)}$$

where

$$\Psi_k \equiv (\partial\varepsilon/\partial I_k)_\eta \qquad [\text{or } (\partial\psi/\partial I_k)_\theta]. \qquad \text{(II.5)}$$

APPENDIX III

Scalar coefficients A_{Γ_j} ($\Gamma = 0, 1, 2$; $j = 1, 2, 3$)

$$A_{01} = \Psi_{11} - (2I_3 - 3I_1I_2 + I_1^3)\Psi_{13} + 3\Psi_3(I_2 - I_1^2),$$
$$A_{02} = \Psi_{12} - (2I_3 - 3I_1I_2 + I_1^3)\Psi_{23} + 3\Psi_3I_1,$$
$$A_{03} = \Psi_{13} - (2I_3 - 3I_1I_2 + I_1^3)\Psi_{33} - 2\Psi_3,$$
$$A_{11} = 2(\Psi_{12} - \Psi_{11}) + 3\Psi_{13}(I_1^2 - I_2) + 6\Psi_3I_1,$$
$$A_{12} = 2(\Psi_{22} - \Psi_{12}) + 3\Psi_{23}(I_1^2 - I_2) - 3\Psi_3,$$
$$A_{13} = 2(\Psi_{23} - \Psi_{13}) + 3\Psi_{33}(I_1^2 - I_2),$$
$$A_{21} = 3\left[\Psi_{13}(1 - 2I_1) - 2\Psi_3 - \frac{4}{3}\Psi_{12}\right],$$
$$A_{22} = 3[\Psi_{23}(1 - 2I_1) - (4/3)\Psi_{22}],$$
$$A_{23} = 3[\Psi_{33}(1 - 2I_1) - (4/3)\Psi_{23}],$$

where

$$\Psi_{ij} = \left(\frac{\partial^2\varepsilon}{\partial I_i\partial I_j}\right)_\eta = \Psi_{ji}.$$

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