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Section A :
Physique théorique.

Quantization and global properties of manifolds

by

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ABSTRACT. — The quantization conditions are formulated for the systems with curved phase space $\mathscr{M}$, and relations of these conditions to different bundle spaces over $\mathscr{M}$ and to the Feynman integrals over paths are analysed. A topological model of the scalar particle production is proposed as an attaching of the 3-dimensional cell.

By quantization of a classical system is usually understood the construction of such a quantum system which describes the phenomena of microworld in more detail. The quantization is connected with the introduction of a fundamental quantity $\hbar$ such that for $\hbar \to 0$ the limit of the quantum system is just the corresponding classical system. Even intuitively, it is clear that this limiting transition is not unique because the quantum system describing microworld in more detail can be of different structure though having the same classical limit. Many different quantization schemes exist, however it seems that all these can be splitted into two classes.

We call the first of them the local method. It employs the canonical variables (q, p) and introduces differential operators describing the quantum-mechanical system. A mathematically rigorous formulation of the condition for quantization of this kind has been given by G. Weil in the form [*1*]

$$\exp \{ i\alpha p \} . \exp \{ i\beta q \} = \exp \{ i\hbar\alpha\beta \} . \exp \{ i\beta q \} . \exp \{ i\alpha p \}, \quad (1)$$

different from the Heisenberg commutation relations

$$[p, q] = \frac{1}{i} \hbar \quad (2)$$

which are mathematically noncorrect due to unboundedness of operators p and q. The Weil scheme of quantization, however, is applicable only to the classical systems with flat phase space since it is based on the use of canonical variables (q, p).

We call the second class the global method because it is based on the global properties of manifolds. Widely known the original Bohr quantization condition

$$\oint pdq = (2n + 1)\pi h \tag{3}$$

belongs to the second class, where the integral is taken over some plane in the phase space. However, the most beautiful and complete realization of the global method is the « Feynman method of summation over trajectories » [2]. The idea of the method is that the probability amplitude $\psi(x, t)$ is expressed via the probability amplitude $\psi(x', t')$ by using the propagation function which can be obtained from the classical function of action by calculating it for all possible paths between points (x, t) and (x', t'), more exactly it means that

$$\psi(x, t) = \int \langle x, t \mid x', t' \rangle \psi(x', t')dx' ,$$

where

$$\langle x, t \mid x', t' \rangle = \int \exp\left\{ \frac{i}{\hbar} \int_{x(t)}^{x'(t')} \mathrm{L}[x(\tau), \dot{x}(\tau)]d\tau \right\} \mathscr{D}[x(\tau)] . \tag{4}$$

An alternative way to calculate (4) is to define the skeleton history by indicating the sequence of intermediate moments of time $t < t_1 < t_2 < \ldots t'$ and giving, at these moments, the configurations $x, x_1, x_2, \ldots, x'$. The actual calculations are performed using finite differences of the type

$$\frac{x_{k+1} - x_k}{t_{k+1} - t_k} \tag{5}$$

in the function of action, instead of derivatives. There is, however, some arbitrariness in expressing $\dot{x}(\tau)$ through these differences. Such an arbitrariness connected with the ordering of cofactors in Lagrangian results in that the different choices of the skeleton histories give different expressions for the propagation function. In other words, such a quantization method is not unique. What is the nature of this nonuniqueness? Is it connected with an incorrect definition of quantization or from the very beginning underlies in its mathematical structure? To answer this question, consider in more detail the global properties of quantum conditions using the methods of differential geometry and topology. The main idea of these methods is as follows: the quantization consists in introducing the linear connection in a certain bundle space, the Hilbert space of states being constructed of sections of the given bundle space. These methods have been proposed by A. A. Kirillov [3] and B. Kostant [4] who applied the idea of quantization in the theory of representations of Lie groups. Quantization of the systems with a curved phase space has been considered by P. Dirac [5]. Some models of quantization of the system with phase space of a rather large class are analysed by F. A. Berezin [6]. The methods of application of the continual integration to the quantized fields are due to E. Nelson [23].

MATHEMATICAL DESCRIPTION OF CLASSICAL SYSTEMS

In the description of classical systems the concept of phase space is of primary importance. Points of that space correspond to the possible states of a system and functions on it specify different physical quantities related to this system. Mathematically, the phase space is a symplectic manifold, that is a smooth manifold of even dimension on which is given a nondegenerated closed 2-form. Cover the basic space $\mathcal{M}$ by a set of neighbourhoods $\{ U_i \}$ and for every U_i define the coordinate system φ_{ix}. For any point x on the overlapping of two neighbourhoods, i. e. $x \in U_i \cap U_j$, the map

$$\varphi_{ix}^{-1} \cdot \varphi_{jx} : \mathcal{Y} \to \mathcal{Y} \tag{1.1}$$

transforms the fiber into itself, i. e. it defines an element of a group G acting on the fiber

$$\varphi_{ix}^{-1} \cdot \varphi_{jx} : g_{ij}(x) : U_i \cap U_j \to G . \tag{1.2}$$

Functions $g_{ij}(x)$ possess the following properties [7] :

$$\begin{aligned} &a)\ g_{kj}(x) g_{ji}(x) = g_{ki}(x), \text{ for } x \in U_i \cap U_j \cap U_k \\ &b)\ g_{ii}(x) \equiv 1 \text{ (identity element of G)}, \ x \in U_i \\ &c)\ g_{ij}(x) g_{ji}(x) = 1, \ x \in U_i \cap U_j \\ &d)\ g_{ij}(x) g_{jk}(x) g_{ki}(x) = 1, \ x \in U_i \cap U_j \cap U_k . \end{aligned} \tag{1.3}$$

Section f over U is given by the set of functions f_i :

$$f_i : U \cap U_i \to \mathcal{Y} ,$$

where

$$U \subset \mathcal{M} , \tag{1.4}$$

called components of f and obeying the condition

$$f_i(x) = g_{ij}(x) f_j(x) , \qquad x \in U \cap U_j \cap U_i . \tag{1.5}$$

Let $\mathfrak{S}$ be an arbitrary smooth manifold of dimension n which we call configurational space. If $V_q(\mathfrak{S})$ denotes a tangent space to $\mathfrak{S}$ at point $q \in \mathfrak{S}$ and $\left\{ e_i = \dfrac{\partial}{\partial q_i} \right\}$ is its basis in a certain coordinate neighbourhood $\tilde{U}_i$

with coordinates $\{ q_i \}$, then the basis in the space $V_q^*(\mathfrak{S})$ dual to the space $V_q(\mathfrak{S})$ is defined by the following equation

$$dq^j \left\{ \frac{\partial}{\partial q^i} \right\} = \delta_i^j , \tag{1.6}$$

and any 1-form on $\tilde{U}_j$ can be represented in a unique way [8] as follows:

$$\sigma_{U_j} = \sum_i \mathscr{A}_i dq^i , \tag{1.7}$$

where $\mathscr{A}_i$ are called the vector coordinates in $V_q^*(\mathfrak{S})$. The manifold $V^*(\mathfrak{S}) = \bigcup_q V_q^*(\mathfrak{S})$ can be represented in every neighbourhood as a set of pairs $\{ q^i, p_i \}$, where $p_i \in V_q^*(\mathfrak{S})$. The map

$$d\pi_{(q,p)}^* : V_q^*(\mathfrak{S}) \to V_{(q,p)}^* \tag{1.8}$$

allows a unique definition of a covariant vector field on $V^*(\mathfrak{S})$

$$\sigma_0 = \sum_k p_k dq^k \tag{1.9}$$

called fundamental vector field of the manifold [9].

Further, geometry gives the following theorem [10]:

Let ω be closed 2-form on a $2n$-dimensional manifold $\mathscr{M}$, throughout having rank $2n$. Then near each point from $\mathscr{M}$ one may introduce such coordinates $\{ q^i, p_i \}$ that

$$\omega = \sum_i dp_i \wedge dq^i , \tag{1.10}$$

i. e. ω will equal to the outer derivative of (1.9).

However, such a global separation of variables into p and q may not exist. Geometrically, this separation means to define the Lagrangian manifold, i. e. to define at each point $x \in U_i$ the manifold the dimension of which is equal to that of the configurational space and on which

$$\sum_\alpha \left\{ \frac{\partial p_\alpha}{\partial \beta_i} \frac{\partial q^\alpha}{\partial \beta_j} - \frac{\partial p_\alpha}{\partial \beta_j} \frac{\partial q^\alpha}{\partial \beta_i} \right\} = 0 , \tag{1.11}$$

where $\{ \beta_i \} = \beta$ are parameters of the Lagrangian manifold, i. e. the Lagrange brackets of $p(\beta)$, $q(\beta)$ are zero. The equation for this manifold is as follows

$$p = p(q) . \tag{1.12}$$

Also it may be shown [11] that if the Lagrangian manifold is one-to-one projected onto q-space, it is given by a certain generating function S, so that

$$p_i = \partial S/\partial q^i . \tag{1.13}$$

Let the system state at an instant t_0 be specified by a point x of phase space $\mathcal{M}$, then at instant $t > t_0$ the state of the given system will be defined by point $x = U_t(x_0)$, the following equality

$$U_{t_1+t_2} = U_{t_1} . U_{t_2} \tag{1.14}$$

being fulfilled; in other words, the totality of all U_t composes a transformation semigroup, so-called dynamical semigroup. The manifold of all points $U_t(x_0)$ at fixed x_0 and varying t reproduces a trajectory in phase space. The infinitesimal generator of the dynamical semigroup is the vector field defining the tanget vector at every point of the trajectory. This vector field ξ is called Hamiltonian if the induced by it a set of transformations conserves the form

$$L_\xi \omega = 0 , \tag{1.15}$$

where L_ξ is the Lie derivative [*12*] along the vector field ξ. In virtue of (1.15) the dynamical semigroup with such a field transforms a Lagrangian manifold again into a Lagrangian one. However, it may happen that at some t' into one point q of the configurational space $\mathfrak{S}$ several points can be projected from the new Lagrangian manifold, these points are called « critical » [*13*]. By introducing an operator i_ξ lowering the degree of the differential form, the Lie derivative can be written in the form [*14*]

$$L_\xi = d . i_\xi + i_\xi . d \tag{1.16}$$

and equation (1.15) is represented as

$$d\omega_\xi = 0 , \tag{1.17}$$

where

$$\omega_\xi = i_\xi \omega . \tag{1.18}$$

Relation (1.17) means that the form ω_ξ is closed. We call ξ strictly Hamiltonian if ω_ξ is the exact form, i. e.

$$\omega_\xi = dC \tag{1.19}$$

takes place for an arbitrary function C on $\mathcal{M}$. Generally, the form ω_ξ only locally has the form dC, being continued onto the whole manifold it may be many-valued. And the set of all closed forms composes the vector space where the exact forms constitute a subspace; the dimension of a coset with respect to the given subspace depends on the topology of a manifold only [*15*].

If the vector ξ in local coordinates $\{ q^i, p_i \} \subset U_i$ has the form

$$\xi = \sum_i \left\{ a^i \frac{\partial}{\partial q^i} - b_i \frac{\partial}{\partial p_i} \right\} \tag{1.20}$$

then from (1.18) and (1.10) it follows

$$\omega_\xi = \sum_k \{ b_k dq^k - a^k dp_k \} . \tag{1.21}$$

From (1.19) and (1.21) it is possible to express the Hamiltonian field components in terms of the generating function C :

$$b_k = \frac{\partial C}{\partial q^k}, \qquad a^k = - \frac{\partial C}{\partial p_k} . \tag{1.22}$$

If in phase space $\mathcal{M}$ for a certain coordinate neighbourhood U_i with coordinates $\{ q^i, p_i \}$ one defines the trajectory with infinitesimal generator ξ_c, then the equations for it are represented in the form

$$\frac{dq^i}{d\tau} = \frac{\partial C}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = - \frac{\partial C}{\partial q^i} . \tag{1.23}$$

Equations (1.23) determine the one-parameter group of homogogeneous tangent transformations in which C is an arbitrary analytic function of variables (q^i, p_i), homogeneous of first degree in p_i [*16*]. If now in the configurational space $\mathfrak{S}$ one defines the Riemannian metrics

$$ds^2 = \sum_{ij} g_{ij} dq^i dq^j \tag{1.24}$$

then as the generating function one may take

$$C = \sqrt{\sum_{ij} g^{ij} p_i p_j} . \tag{1.25}$$

In this case eqs. (1.23) are

$$\frac{dq^i}{ds} = \sum_j g^{ij} p_j , \qquad \frac{dp_i}{ds} = - \frac{1}{2} \sum_{jk} \frac{\partial g^{jk}}{\partial q^i} p_j p_k \tag{1.26}$$

at $\tau = s$, g_{ij} being the metrical tensor.

In general, every real function f given on a manifold $\mathcal{M}$ may be treated as a generating function of the strict Hamiltonian field ξ_f defined in the local coordinate system $\{ q^i, p_i \}$ by the formula

$$\xi_f = \sum_i \left\{ \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} \right\} . \tag{1.27}$$

Any two smooth functions f and g given on the manifold obey the equality

$$\{ f, g \} = \omega(\xi_f, \xi_g) = \xi_f(g) = - \xi_g(f) . \tag{1.28}$$

The value of expression (1.28) is called the Poisson brackets. They have the following properties

$$\{ f, g \} = - \{ g, f \}$$
$$\{ f\{ g, h \} \} + \{ g \{ h, f \} \} + \{ h \{ f, g \} \} = 0 , \qquad (1.29)$$

therefore the space $C^{\infty}(\mathcal{M}, \mathcal{R})$ of smooth real functions on $\mathcal{M}$ composes a Lie algebra infinite-dimensional relative to these brackets [3].

QUANTIZATION

The procedure of quantization consists in the following: among a great number of physical quantities specifying the behaviour of a classical system there is separated such a subset $\{ f_i \}$ which produces a Lie algebra with respect to the Poisson brackets (1.28). Then to each f_i one makes corresponds its quantum analog

$$\begin{aligned} \hat{\mathcal{A}}(f_i) &= \hat{f}_i \\ \hat{\mathcal{A}}(\mathrm{I}) &= \hat{e} \qquad (\hat{e}\text{-identity operator}) \end{aligned} \qquad (2.1)$$

so that the relation

$$\exp i\hat{\mathcal{A}}(f) \exp i\hat{\mathcal{A}}(g) = \exp i\hbar d\sigma(\xi_f, \xi_g) \exp i\hat{\mathcal{A}}(g) \exp i\hat{\mathcal{A}}(f) \qquad (2.2)$$

is fulfilled, or, in the infinitesimal form,

$$[\hat{\mathcal{A}}(f), \hat{\mathcal{A}}(g)] = i\hbar\, \mathcal{A}[\{ f, g \}] = i\hbar\, \omega(\xi_f, \xi_g) = i\hbar\, d\sigma(\xi_f, \xi_g) . \qquad (2.3)$$

Let $\mathcal{M}$ be the symplectic $2n$-dimensional manifold with 1-form $\sigma = \sum_i \sigma_i dx^i$, then for any two vectors X and Y on $\mathcal{M}$ the relation [8]

$$2d\sigma(\mathrm{X}, \mathrm{Y}) = \mathrm{X}\sigma(\mathrm{Y}) - \mathrm{Y}\sigma(\mathrm{X}) - \sigma([\mathrm{X}, \mathrm{Y}]) \qquad (2.4)$$

takes place.

Now let us construct the bundle space $\mathcal{L}$ over $\mathcal{M}$ with the fiber c (the complex plane)

$$\begin{array}{ccc} c & \rightarrow & \mathcal{L} \\ & & \downarrow \pi \\ & & \mathcal{M} \end{array}$$

through introducing the linear connexion

$$\nabla_{\mathrm{X}} f = \frac{\sigma(\mathrm{X})}{i\hbar} f, \qquad (2.5)$$

where X is a vector on $\mathcal{M}$ and f is a section. In this case for every function $\varphi \in c$ one has [4]

$$\nabla_{\varphi \mathrm{X}} = \varphi \, . \, \nabla_{\mathrm{X}} ,$$
$$\nabla_{\mathrm{X}}(\varphi f) = (\mathrm{X}\varphi) f + \varphi \nabla_{\mathrm{X}} f \, . \qquad (2.6)$$

The Hilbert space of states of a physical system is constructed from sections f of this bundle space.

By using (2.5) and (2.6), relation (2.4) can be rewritten in the form

$$2d\sigma(X, Y) = i\hbar \{ [\nabla_X \nabla_Y] - \nabla_{[XY]} \} = i\hbar\, \mathscr{R}(X, Y) \tag{2.7}$$

where $\mathscr{R}(X, Y)$ is the curvature tensor. Substituting (2.7) into (2.3) we get

$$[\hat{\mathscr{A}}(f), \hat{\mathscr{A}}(g)] = -\frac{\hbar^2}{2} \mathscr{R}(\xi_f, \xi_g). \tag{2.8}$$

Consider now in $\mathscr{M}$ the curve $\mathscr{C}(\tau)$ given by eqs. (1.23). Using (2.5) one may define the covariant derivative

$$\frac{\mathscr{D} f}{\partial \tau} = \xi_{\mathscr{C}}(f) + \frac{1}{i\hbar} \sigma(\xi_{\mathscr{C}}) f \tag{2.9}$$

of section $f \in \Gamma(\mathscr{L}, \mathscr{M})$ along this curve. Or, it reads also

$$\frac{\mathscr{D} f}{\partial \tau} = \frac{df}{d\tau} + \frac{1}{i\hbar} \sum_i \sigma_i \frac{dx^i}{d\tau} f. \tag{2.10}$$

The equality of the covariant derivative to zero gives the condition for translation of a differentiable path from the basic space $\mathscr{M}$ into the bundle space $\mathscr{L}$ and thus defines the connexion in the latter:

$$df + \frac{1}{i\hbar} \sum_i \sigma_i dx^i f = 0. \tag{2.11}$$

Solving this equation, we obtain

$$f = \exp\left\{ \frac{i}{\hbar} \int_{\mathscr{C}_i} \sum_k \sigma_k dx^k \right\}, \tag{2.12}$$

where the integral is taken over the path $\mathscr{C}_i(\tau)$ in the neighbourhood $U_i \subset \mathscr{M}$. Provided that as a 1-form the fundamental vector field (1.9) is taken, one has

$$f = \exp\left\{ \frac{i}{\hbar} \int_{\mathscr{C}_i} \sum_k p_k dq^k \right\}. \tag{2.13}$$

If the values for dq^k are taken from the first eqs. of system (1.26) and substituted into (2.13), then

$$f = \exp\left\{ \frac{i}{\hbar} \int_{\mathscr{C}_i} \mathscr{H} d\tau \right\} \tag{2.14}$$

where $\mathscr{H} = \sum_{ij} g^{ij} p_i p_j$.

Taking different paths $\mathscr{C}_i \in \Omega_i$, one obtains different f. Thus, expression (2.12) defines the representation of a manifold of paths Ω_i of neighbourhood U_i into a manifold of the unit modulus complex numbers Z which composes the group T_1 with respect to multiplication of these numbers, i. e.

$$f : \Omega_i \rightarrow T_1 . \tag{2.15}$$

In other words, we have found the main bundle space $\mathscr{L}$ with basis $\mathscr{M}$ and structure group T_1 (see (1.2)). Hence, the problem of uniqueness of quantization reduces to the algebraic topology problem on classification of bundle spaces with a given specified basis and fiber. This classification is performed by connecting a system of topological invariants with each equivalence class of bundle spaces.

Let us connect with each differentiable representation the main bundle space $(\mathscr{L}, \pi, \mathscr{M}, T_1)$ with basis $\mathscr{M}$ and structure group T_1, then $\mathscr{L}$ is a space of representation without fixed points, with orbits being fibers. Cover manifold $\mathscr{M}$ by coordinate neighbourhoods $\{ U_i \}$, then to each bundle space there corresponds the system of transition functions

$$g_{ij} : U_i \cap U_j \rightarrow T_1 \tag{2.16}$$

with properties (1.3). Using these properties, it may be shown that the system of maps g_{ij} composes a cocycle of dimension 1 and there is fulfilled the theorem A [*17*]:

The equivalence classes of the main bundle spaces with basis $\mathscr{M}$ and group T_1 make up a set $\mathscr{E}(\mathscr{M}, T_1)$ which is in one-to-one correspondence with the set $H^2(\mathscr{M}, Z)$ where $H^2(\mathscr{M}, Z)$ is the two-dimensional cohomology group of de Rham on the manifold $\mathscr{M}$ with iteger coefficients.

Let two cocycles $\{ g_{ij} \}$ and $\{ g'_{ij} \}$ be given, then the functions

$$g''_{ij} = g_{ij} \cdot g'_{ij} \tag{2.17}$$

compose new cocycles. If the given cocycles correspond to the equivalence classes $\xi, \xi' \in \mathscr{E}(T_1, \mathscr{M})$, the one containing the bundle space defined by cocycle $\{ g''_{ij} \}$ will be $\xi \otimes \xi'$ and is called the tensor product of classes ξ and ξ'. The set $\mathscr{E}(\mathscr{M}, T_1)$ with such an operation is the Abelian group isomorphic to group $H^2(\mathscr{M}, Z)$, i. e.

$$ch : \mathscr{E}(\mathscr{M}, T_1) \rightarrow H^2(\mathscr{M}, Z) . \tag{2.18}$$

This map makes correspond to each class ξ of the bundle space $\mathscr{L}$ with basis $\mathscr{M}$ the class of integer cohomologies $ch(\xi)$ of dimension 2. This class is called the characteristic class of bundle spaces. From exp. (1.3) and theorem A it follows that for the bundle space $(\mathscr{L}, \pi, \mathscr{M}, T_1) \in \xi$ and covering $\{ U_i \}$ the class $ch(\xi)$ is represented by the cocycle $\{ g_{ijk} \}$ defined by the formula [*18*]

$$\ln g_{ijk} = \ln g_{ik} - \ln g_{jk} + \ln g_{ij} \tag{2.19}$$

where $x \in U_i \cap U_j \cap U_k$, the function $\ln g_{ij}(x)$ being assumed to be determined for each pair (U_i, U_j). This is not a constraint if one supposes that $(U_i \cap U_j)$ are small enough so that

$$T_1 \neq g_{ij}(U_i \cap U_j). \tag{2.19'}$$

Let the covering $\{ U_i \}$ be chosen so that the intersections $(U_i \cap U_j)$ are simply connected and $(U_i \cap U_j \cap U_k)$ connected. Since $\{ g_{ij} \}$ is a cocycle, from (1.3) it follows that ln g_{ijk} are integer numbers independent of $x \in U_i \cap U_j \cap U_k$ and composing a two-dimensional cocycle. By the de Rham theorem [*15*], to this cocycle the cohomological class of external forms of the second order corresponds. It can be shown that form ω being the form of curvature of infinitesimal connexion (2.11) belongs to that class, i. e. the image of the characteristic class of the bundle space $(\mathscr{L}, \pi, \mathscr{M}, T_1)$ under the de Rham isomorphism contains the form of curvature of infinitesimal connection. These forms may be introduced into the bundle space by means of 1-forms σ defined on $\mathscr{M}$ and invariant under the group T_1. In a neighbourhood of the bundle space $\pi^{-1}(U_i)$ with coordinates $\{ x^i, z \}$ these 1-forms are

$$\sigma = \sum_i a_i dx^i + \omega_0\left(d\varphi - \sum_i \sigma_i dx_i \right), \qquad (z = \exp \{ 2\pi i\varphi \}) \tag{2.20}$$

where a_i, ω_0, σ_i do not depend on φ.

Then ω is given by the formula

$$\omega = \frac{1}{2} \sum_{ij} \left\{ \frac{\partial \sigma_i}{\partial x^j} - \frac{\partial \sigma_j}{\partial x^i} \right\} dx^i \wedge dx^j . \tag{2.21}$$

By assuming that there exists an invariant under the dynamical group Lagrangian manifold $\mathscr{L}$ lying at the energy $\mathscr{H} = \text{const}$, the condition for the form ω to be integer means that

$$\int_{\mathscr{L}} \omega = n \, . \, \hbar \tag{2.22}$$

whence, by the Stokes theorem, we obtain that for every closed contour γ on $\mathscr{L}$ the relation

$$\oint_\gamma \sum_i p_i dq^i = n \, . \, \hbar \tag{2.23}$$

holds, it being equivalent to the Bohr quantization condition (3). If the characteristic class of bundle spaces $(\mathscr{L}, \pi, \mathscr{M}, T_1)$ for covering $\{ U_i \}$ equals zero, then the cocycle $\{ g_{ijk} \}$ is a coboundary. In this case the bundle space is trivial, i. e. equivalent to the bundle space $(\mathscr{L}', \pi', \mathscr{M})$ in which

$\mathscr{L}' = \mathscr{M} \otimes T_1$ and π' is the projection of $\mathscr{L}'$ on the first factor. All trivial bundle spaces belong to one equivalence class.

Consequently, the bundle space $(\mathscr{L}, \pi, \mathscr{M}, T_1)$ with the integrable connexion, i. e. with that the curvature form of which is zero, in general, is trivial. It is nontrivial only if the group $H^2(\mathscr{M}, Z)$ contains nonzeroth periodic elements. The bundle spaces, for which these elements are characteristic classes, are spaces associated with the universal covering of manifold $\mathscr{M}$ and representations of the homotopic group $\pi_1(\mathscr{M})$ in group T_1 [*19*]. Let us assume that the manifold $\mathscr{M}$ is covered by a system of the neighbourhoods $\{ U_i \}$ and let in a neighbourhood U_i at point x_0 and instant t_0 a section $f_i \in \Gamma(\mathscr{L}, \mathscr{M})$ be given. Since the dynamical system is defined on the Lagrangian manifold (1.11), then in virtue of (1.12), the sections depend only on the projection $\pi : \mathscr{M} \to (\mathfrak{S})$ of points of phase space onto the configurational one, i. e. they are functions of coordinates only. Hence, according to (1.5), (2.13) and (2.14) the value of section at point $x' \in \pi(U_i)$ and instant t' is expressed through the value of section at point $x \in \pi(U_i)$ and instant t_0 by the formula

$$f_i(x', t') = \left[\exp\left\{ \frac{i}{\hbar} \int_{x_0}^{x'} \sum_k p_k dq^k \right\} . \exp\left\{ \frac{i}{\hbar} \int_{t_0}^{t'} \mathscr{H} d\tau \right\} \right] . f_i(x_0, t_0)$$
$$= \left[\exp\left\{ \frac{i}{\hbar} \int_{x_0(t_0)}^{x'(t')} L_{U_i} d\tau \right\} \right] . f_i(x_0, t_0), \quad (2.24)$$

where integrals are taken over some path $\mathscr{C}_i \subset \Omega_i$ on U_i and U_i should obey condition (2.19). Continuiting the 1-form σ_{U_i} defined on U_i onto the whole space covered by neighbourhoods, we find the transition function for sections from one point x_0 to another x of manifold $\pi(\mathscr{M})$ as the following limit [*20*]

$$G(x, t; x_0, t_0) = \lim_{n \to \infty} \int_{\mathscr{B}} \cdots \int_{\mathscr{B}} \left\{ \prod_k^n \exp\left[\frac{i}{\hbar} \int_{x_k}^{x_{k+1}} L_{U_k} d\tau \right] \right\} d\sigma(x_1) \ldots d\sigma(x_n)$$
$$= \int_{x_0(t_0)}^{x(t)} \exp\left[\frac{i}{\hbar} \int_{t_0}^{t} L d\tau \right] \mathscr{D}[\mathscr{C}(\tau)] \quad (2.25)$$

where $\sigma(x_i)$ is measure on $\mathscr{B} \subset \mathfrak{S}$ and the latter expression is the continual integral taken over all possible paths $\mathscr{C} \in \Omega_{x_0 x}$ on $\mathscr{M}$ joining two given points. Then the difference between two nearby paths going through different neighbourhoods (see Fig. 1) is

$$\exp \frac{i}{\hbar} \int_{x_0}^{x} \sigma + \exp \frac{i}{\hbar_t} \int_{Q} d\sigma$$
$$= \exp \left\{ - \frac{i}{\hbar} \int_{t_0}^{t} \mathscr{H} d\tau \right\} . \exp \left\{ \frac{i}{\hbar} \int_{x_0}^{x} \sigma \right\} . \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t} \mathscr{H} d\tau \right\} . \quad (2.26)$$

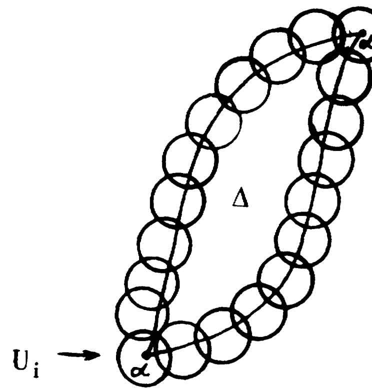

FIG. 1.

As follows from the presented above, for the bundle space $\mathscr{L}$ to be not trivial, it is necessary that its characteristic class be not zero, i. e. the form of connexion should not be integrable. Consequently, only those paths contribute into the continual integral for which

$$\int_Q d\sigma \neq 0 . \tag{2.27}$$

If $\sigma = \sum_k p_k dq^k$, then taking account of (2.7), from the integrity condition we get

$$\int_Q dp \wedge dq \sim \frac{n\hbar}{2}, \tag{2.28}$$

whence

$$\Delta p \,.\, \Delta q \sim \frac{\hbar}{2} . \tag{2.29}$$

Let all the paths joining two given points in phase space be projected on a configurationa space $\pi : \mathscr{M} \to \mathfrak{S}$ which is assumed to be smooth, and let α, α' be projections of these points. Then points α and α' are joined by a totality of piecewise smooth paths lying on $\mathfrak{S}$. This totality will be denoted by $\Omega(\alpha, \alpha', \mathfrak{S})$.

Let us assume that in $\mathfrak{S}$ the metric (1.24) is defined. For the path $\mathscr{C} \in \Omega$ we define the action on $\mathscr{C}$ from α to α' :

$$S = \int_\alpha^{\alpha'} \left| \frac{d\mathscr{C}}{d\tau} \right|^2 d\tau . \tag{2.30}$$

If $\bar{\beta} : U \to \Omega$ is the two-parameter variation of a geodesic γ to which there correspond the variation vector fields

$$w_i = \frac{\partial \beta}{du_i}(0, 0) \in V\Omega_\gamma , \qquad i = 1, 2$$

then the second derivative of the function of action, $\frac{1}{2}\frac{\partial^2 S}{\partial u_1 \partial u_2}$, equals [21]

$$- \sum_{\tau} \left\langle w_2, \Delta_\tau \frac{\mathscr{D} w_1}{d\tau} \right\rangle - \int_{\alpha}^{\alpha'} \left\langle w_2, \frac{\mathscr{D}^2 w_1}{d\tau^2} + \mathscr{R}(\mathrm{V}, w_1)\mathrm{V} \right\rangle d\tau, \quad (2.31)$$

where

$$\mathrm{V} = \frac{d\gamma}{d\tau} \quad \text{and} \quad \Delta_\tau \frac{\mathscr{D} w_1}{d\tau} = \frac{\mathscr{D} w_1}{d\tau}(\tau^+) - \frac{\mathscr{D} w_1}{d\tau}(\tau^-)$$

refers to the derivative's discontinuity at one of the points of discontinuity, $\mathscr{R}$ is the tensor of curvature.

Assuming that a point α' is conjugate to α along à geodesic γ, the variation vector field w obeys the Jacobi equation

$$\frac{\mathscr{D}^2 w}{d\tau^2} - \mathscr{R}(\mathrm{V}, w)\mathrm{V} = 0. \quad (2.32)$$

If along γ one chooses the orthonormal parallel vector field $\{ \hat{e}_i \}$, then at $w = \sum_i \varphi_i \hat{e}_i$ eq. (2.32) takes the form

$$\frac{d^2 \varphi^i}{d\tau^2} + \sum_j a^i_j \varphi^j = 0, \qquad i = 1, 2, \ldots, \quad (2.33)$$

where $a^i_j = \langle \mathscr{R}(\mathrm{V}, \hat{e}_j)\mathrm{V}, \hat{e}_i \rangle$. With notations

$$\mathrm{K_V}(w) = \mathscr{R}(\mathrm{V}, w)\mathrm{V} \quad (2.34)$$

properties of the tensor of curvature give that the operator $\mathrm{K_V}(w)$ is self-conjugated, i. e.

$$\langle \mathrm{K_V}(w), w' \rangle = \langle w, \mathrm{K_V}(w') \rangle \quad (2.35)$$

Hence, the orthonormalized basis $\{ \hat{e}_i \}$ van be chosen so that

$$\mathrm{K_V} \hat{e}_i = \omega_i^2 \hat{e}_i. \quad (2.36)$$

Now eq. (2.33) reads

$$\frac{d^2 \varphi^i}{d\tau^2} + \omega_i^2 \varphi^i = 0, \qquad i = 1, 2, \ldots \quad (2.37)$$

Let us show that to a conjugate point α' there corresponds a stationary point (q, p) of phase space of the dynamical system i. e. a point such that

$$\mathrm{U}_t(q, p) = (q, p). \quad (2.38)$$

Relation (2.38) holds only when

$$p = 0, \qquad (d\mathfrak{U})_q = 0 \quad (2.39)$$

(where $\mathfrak{U}$ is a potential and U_t is a semigroup satisfying (1.14).

The Lagrangian in a neighbourhood of such a point has the form

$$L = \sum_{ij} [a_{ij}\dot{q}^i\dot{q}^j - b_{ij}q^iq^j] \tag{2.40}$$

where a_{ij}, b_{ij} are constants. By the known theorem of the linear algebra, there exists such a linear transformation which diagonalized both the quadratic forms, and consequently, the function L in new coordinates reads

$$L = \sum_{i} [(\dot{\varphi}^i)^2 - \omega_i^2(\varphi^i)^2] \tag{2.41}$$

and the Lagrange equation is of the form

$$\ddot{\varphi}^i + \omega_i^2\varphi^i = 0\,, \qquad i = 1, 2, \ldots \tag{2.42}$$

and thus coincides with eq. (2.37).

For the dynamical system with infinite number of degrees of freedom eq. (2.42) takes the form

$$\ddot{\varphi}(\vec{k}) + \omega^2(\vec{k})\varphi(\vec{k}) = 0\,. \tag{2.43}$$

The basis in this case is provided by the functions

$$\{\, \hat{e}_k = e^{i\vec{k}\vec{x}} \,\}\,. \tag{2.44}$$

Multiplying both sides of eq. (2.43) by these functions, and integrating over k, we get

$$\int \ddot{\varphi}(\vec{k})e^{i\vec{k}\vec{x}}d\vec{k} + \int \omega^2(\vec{k})\varphi(\vec{k})e^{i\vec{k}\vec{x}}d\vec{k} = 0\,. \tag{2.45}$$

Keeping the idea that $\omega(\vec{k})$ is a frequency and that it should be invariant under rotations in the three-dimensional space, one can assume that $\omega(\vec{k}) = \vec{k}^2$. Then eq. (2.45) may be written as follows

$$\frac{1}{c^2}\frac{\partial^2\varphi(x)}{\partial t^2} - \frac{\partial^2\varphi(x)}{\partial x_1^2} - \frac{\partial^2\varphi(x)}{\partial x_2^2} - \frac{\partial^2\varphi(x)}{\partial x_3^2} = 0, \tag{2.46}$$

i. e. one arrives at the wave equation. This equation is adequate to the Jacobi equations for the variation vector field, also infinite-dimensional.

Since the dynamical process is realized in space and time, to each point of configurational space $\mathfrak{S}$ of the dynamical system there corresponds a point of space-time continuum $v : \mathfrak{S} \to \mathcal{N}$. If now the action (2.30) is treated as a function of the second point α, then it, in its turn, becomes a function of space-time coordinates $(\vec{x}, x_0) \in \mathcal{N}$; in other words

$$S : \mathcal{N} \to R\,.\ (\text{R are real numbers})\,. \tag{2.47}$$

Denote $\mathcal{N}^a = S^{-1}(-\infty, a)$, $\{ p \in \mathcal{N}, S(p) \leqslant a \}$. If different hypersurfaces $S(x) = \text{const}$ are considered, then equations of the orthogonal trajectories having in the local coordinate system $\{ x^i \}$ the form

$$\frac{dx^i}{d\tau} = \sum_j g^{ij} \frac{\partial S}{\partial x^j} \tag{2.48}$$

define the shift $\mathcal{N}^a \to \mathcal{N}^b$ without changing the homotopy type of manifold $\mathcal{N}^b$, if the set $S^{-1}[a, b]$ does not contain the critical points (these points correspond to the conjugate ones). If the level $S(x) = \delta$ is critical, i. e. contains the critical point p, the orthogonal trajectories in neighbourhoods of noncritical points of this level behave like in other points of manifold $\mathcal{N}$, since

$$\frac{\partial S}{\partial \tau} = |\operatorname{grad} S|^2 > \varepsilon > 0 \tag{2.49}$$

for all points lying outside small enough cylindrical neighbourhood of the critical point (see Fig. 2) [*21*], which is left fixed by those trajectories.

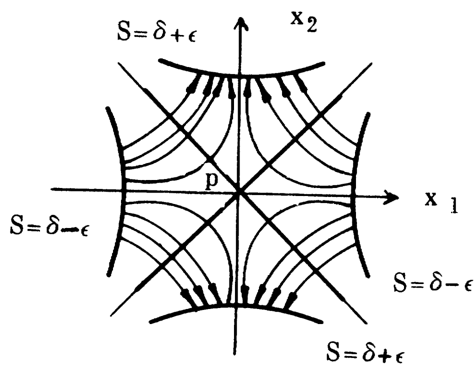

FIG. 2. — *The cylindrical neighbourhood of point* : $S^2 = -x_1^2 + x_2^2$.

In a neighbourhood of the critical point there holds the Theorem B [*22*] :

Let $S : \mathcal{N} \to R$ *be a smooth function and* p *its nondegenerated critical point with index* λ. *Assume that the set* $S^{-1}[\delta - \varepsilon, \delta + \varepsilon]$ *where* $\delta = S(p)$ *is compact and does not contain the critical points of* S *other than* p *for some* $\varepsilon > 0$. *Then for all sufficiently small* $\varepsilon > 0$ *the set* $\mathcal{N}^{\delta+\varepsilon}$ *has the homotopy type* $\mathcal{N}^{\delta-\varepsilon}$ *with an attached cell of dimension* λ.

Now let us make the assumption (*) that « *production* » *of an elementary particle with mass m topologically is equivalent to attaching the three-dimensional cell* e^3

$$x_1^2 + x_2^2 + x_3^2 \leqslant \varepsilon < \left(\frac{\hbar}{mc} \right)^2 \tag{2.50}$$

in the neighbourhood of the critical point of the function of action.

Then, by Theorem B, the particle production is adequate to transition through the conjugate point defined by the Jacobi equation (2.32). Indeed, physically, the elementary particle is realized at the point where wave phases are in interference. Since the action (2.30) plays the role of phase, then it is just the point at which geodesics intersect with different $\omega(\vec{k})$ but equal S, i. e. the conjugate point.

By a Morse lemma [22], in a neighbourhood U of the critical point p there exists such a local system of coordinates $\{ x^i \}$ in U that the identity

$$S^2(x) = S^2(p) + a^2 \{ x_0^2 - x_1^2 - x_2^2 - \dots - x_\lambda^2 \} \tag{2.51}$$

holds, where λ is the index of S at point p, S(p) called the critical value of S. By an appropriate choice of a constant one can obtain $S(p) = 0$. Keeping in mind that S plays the role of phase and comparing (2.51) with the action for free particle we find that $a^2 = \left(\frac{mc}{\hbar}\right)^2$. Since, in our case also $\lambda = 3$ the final expression for $S^2(x)$ is as follows

$$S^2(x) = \left(\frac{mc}{\hbar}\right)^2 \{ x_0^2 - x_1^2 - x_2^2 - x_3^2 \}. \tag{2.52}$$

Let us take $\varepsilon > 0$ small enough so that

1. the neighbourhood of the critical level does not contain critical points other than p,
2. the image of U under the imbedding

$$\{ x^i \} : U \rightarrow R^4 \tag{2.53}$$

does contain the closed sphere

$$\{ (x_0, x_1, x_2, x_3) : \sum_i (x_i)^2 \leqslant 2\varepsilon \}. \tag{2.54}$$

Define now e^3 as a set of points from U, where

$$x_1^2 + x_2^2 + x_3^2 \leqslant \varepsilon, \qquad x_0 = 0. \tag{2.55}$$

The configuration obtained is schematically drawn in Fig. 3. Note that $e^3 \cap \mathcal{N}^{\delta-\varepsilon}$ is exactly the boundary $\dot{e}^3$ and thus the cell e^3 is attached to $\mathcal{N}^{\delta-\varepsilon}$ in the topological sense [22]. From the assumption (*) it follows that the equation for a « produced » elementary particle can be obtained from the condition that the eigenvalue μ of the quadratic functional of the second variation of the function of action (2.31) with the normalization

$$\int_\alpha^{\alpha'} | w |^2 d\tau = 1 \tag{2.56}$$

be nonzero. In other words, instead the Jacobi equation (2.32) there takes

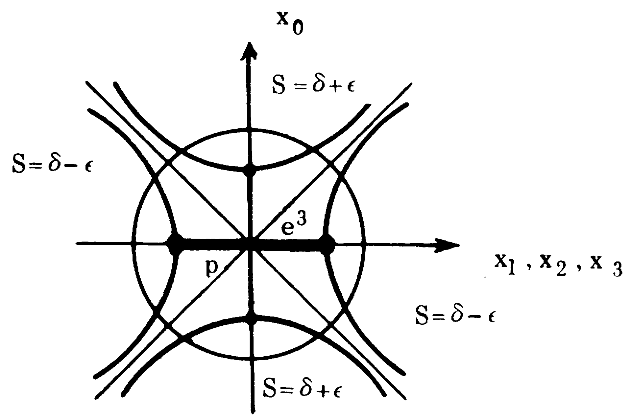

FIG. 3.

place the Sturm-Liuville equation for a system with infinite number of degrees of freedom

$$\ddot{\varphi}(\vec{k}) + \omega(\vec{k})\varphi(\vec{k}) = - |\mu| \varphi(\vec{k}) . \tag{2.57}$$

Here μ should be negative as the interval (α, α') includes a conjugate point. On the other hand, since μ is an eigenvalue of the functional of the second variation of S, then

$$|\mu| = \langle \text{grad } S \rangle^2 \tag{2.58}$$

whence, by using (2.52) we find

$$|\mu| = \left(\frac{mc}{\hbar}\right)^2 . \tag{2.59}$$

Multiplying both sides of eq. (2.57) by $e^{i\vec{k}\vec{x}}$ and integrating over $\vec{k}$ we arrive at the Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x_1^2} - \frac{\partial^2 \varphi}{\partial x_2^2} - \frac{\partial^2 \varphi}{\partial x_3^2} + \left(\frac{mc}{\hbar}\right)^2 \varphi = 0 . \tag{2.60}$$

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