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Section A :
Physique théorique.

Reductions in a class of dissipative dynamical systems of macroscopic physics

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ABSTRACT. — A common formal structure is extracted from a large class of the families of dynamical systems introduced phenomenologically in kinetic theory, fluid mechanics and non-equilibrium thermodynamics. The family of abstract dynamical systems that possesses this structure is called the family of dissipative dynamical systems of macroscopic physics. Some qualitative properties of its phase portrait are derived and their physical interpretation is discussed. Relationships between two independent families of dissipative dynamical systems is established—one family is reduced to the other—by relating some qualitative properties of their phase portraits. An example of the reduction of m-component Enskog-Vlasov kinetic theory to m-component fluid mechanics illustrates the general theory.

RÉSUMÉ. — Nous extrayons une structure commune à une vaste classe de systèmes dynamiques introduits phénoménologiquement en théorie cinétique, en mécanique des fluides ainsi qu'en thermodynamique hors èquilibre.

La classe des systèmes dynamiques abstraits possédant cette structure est appelée la famille des systèmes dynamiques dissipatifs de la physique macroscopique. Nous décrivons quelques propriétés de son portait de phase et en discutons l'interprétation physique.

Nous établissons ensuite une relation entre deux familles indépendantes de systèmes dynamiques dissipatifs et montrons comment ramener l'une

à l'autre en mettant en correspondance certaines propriétés qualitatives de leurs portaits de phase. En illustration de la théorie générale nous ramenons la théorie cinétique d'Enskog-Vlasov à m composantes à la dynamique des fluides à m composantes.

I. INTRODUCTION

A common structure has been extracted form the families of dynamical systems introduced phenomenologically in kinetic theory and fluid mechanics [*11*]. The family of abstract dynamical systems that possesses this structure is called the family of dissipative dynamical systems of macroscopic physics (hereafter called DDS). An abstract, geometrical definition of DDS, qualitative properties of its phase portrait, and reduction of one DDS to another DDS are the problems that were outlined and illustrated on examples in [*11*] [*10*]. In this paper, we want to proceed from the general outline and illustrations to a systematic development of the theory.

We start with an abstract definition of DDS. Let $\mathcal{Q}$ denotes the phenomenological quantities (also called the fundamental quantities) through which the individuality of the physical systems considered is expressed. To each $q \in \mathcal{Q}$ a dynamical system $(\mathcal{H}, \mathcal{P})$ is attached; $\mathcal{H}$ denotes the set of all admissibles states (the elements f of $\mathcal{H}$ characterize completely the admissible states of the physical system considered) and $\mathcal{R}$ is a vector field defined on $\mathcal{H}$. Complete and admissible is meant with respect to the set of observations and the measurements that form the empirical basis for the dynamical theory considered. A family of dynamical systems $(\mathcal{H}, \mathcal{P})$ parametrized by $\mathcal{Q}$, denoted by $(\{\mathcal{H}\}, \{\mathcal{P}\}, \mathcal{Q})$, is called the family of dissipative dynamical systems of macroscopic physics (abbreviated by DDS) if the following properties (I) (A), (I) (B), (I) (C), and (II) (A), (II) (B), (II) (C) are satisfied:

(I)

(A) $\mathcal{H}$ is a smooth, possibly infinite dimensional manifold. The tangent space to $\mathcal{H}$ at f for all $f \in \mathcal{H}$ is a Hilbert space.

(B) An involution $J : \mathcal{H} \to \mathcal{H}$; $J^2 = \mathrm{id}$; is defined on $\mathcal{H}$ (by id we denote the identity operator). We define $\mathcal{H}^{(+)} = \{ f \in \mathcal{H};\ Jf = f \}$; $\mathcal{H}^{(-)} = \{ f \in \mathcal{H};\ Jf = -f \}$.

(C) A sufficiently smooth function $S : \mathcal{H} \to \mathbb{R}$ is defined on $\mathcal{H}$. We assume that $JS(Jf) = S(f)$ for all $f \in \mathcal{H}$ and that $D_f^{(2)}S$ (i. e. the second derivative of S evaluated at f) is a positive definite bounded linear operator defined everywhere on $T_f\mathcal{H}$ (the tangent space to $\mathcal{H}$ at f) for all $f \in \mathcal{H}$.

(II)

We shall define $\mathscr{R}^{(\pm)} = \frac{1}{2}(\mathscr{R} \pm J\mathscr{R}J)$

(A)
$$\left(\frac{\partial S}{\partial t}\right)_+ \leq 0, \tag{I.1}$$

where $\left(\frac{\partial S}{\partial t}\right)_+$ denotes the change of S in time if only $\mathscr{R}^{(+)}$ generates the time evolution. The equality in (I.1) holds if and only if

$$f \in \mathscr{M} = \{ f \in \mathscr{H}, \mathscr{R}^{(+)}f = 0 \}.$$

(B)
$$\left.\begin{array}{l} f \in \mathscr{H}^{(+)} \\ f \in \mathscr{M} \\ [\mathscr{R}^{(-)}f]_{\mathscr{H}^{(+)}\cap\mathscr{M}} = 0 \\ \text{boundary conditions} \end{array}\right\} \tag{I.2}$$

is equivalent to the variational problem

$$\frac{\partial \mathbb{V}}{\partial f} = 0, \tag{I.3}$$

where $\mathbb{V} = \mathscr{H} \times \Xi \to \mathbb{R}_{(-)}$; $\mathbb{V} = S + \sum_{i=1}^{m} \sigma_i w_i$; $[\mathscr{R}^{(-)}f]_{\mathscr{H}^{(+)}\cap\mathscr{M}}$ means $\mathscr{R}f$ restricted to $\mathscr{H}^{(+)} \cap \mathscr{M} \subset \mathscr{H}$, $(\sigma_1, \ldots, \sigma_m) \equiv \sigma \in \Xi$, σ_i, $i = 1, \ldots, m$ are real numbers entering the boundary conditions, $w_i : \mathscr{H} \to \mathbb{R}, i = 1, \ldots, m$ are sufficiently smooth functions, m is a positive integer. The solutions to (I.2), or equivalently (I.3), are called equilibrium states.

The set of all equilibrium states, for all $\sigma \in \Xi$ is denoted by $\mathscr{F}$.

(C) Let $F \in \mathscr{F}$ is regular. The linear operator P defined as the Hessian of the vector field $\mathscr{R}$ evaluated at F is an Onsager symmetric operator. A state F is said to be regular if (*i*) for given $\sigma \in \Xi$ there is only one solution to (I.3); all such σ compose $\Xi_{\text{reg}} \subset \Xi$, (*ii*) $A \stackrel{\text{def}}{=} D_F^{(2)}\mathbb{V}$ (the second derivative of $\mathbb{V}$ with respect to f evaluated at F) is bounded and positive definite, (*iii*) F is uniform. All the regular states form $\mathscr{F}_{\text{reg}} \subset \mathscr{F}$. In the context of the kinetic theory and fluid dynamics, F is uniform if F is independent of the position vector r.

The family of linear dynamical systems satisfying (II) (C) is called the family of the local dissipative dynamical systems (abbreviated by LDDS). The concept of the Onsager symmetry introduced in (II) (C) above is defined in Section II. The main purpose of this paper is to deduce some consequences of (II) (C) and to use these consequences in the study of reduction of one DDS to another DDS.

The first three propositions in Section II sum up the results of the theory of indefinite inner product spaces that are directly relevant to LDDS and to the problem of reduction. In Proposition 4 we state and prove (in

Appendix) the theorem of Maltman and Laidlaw [*15*]. Our proof is simpler and is developed in the spirit of the proofs of other propositions in [*1*] [*14*]. Proposition 5 sums up our results about the canonical form of the Fourier transformed m-component local kinetic theory and fluid dynamics introduced in Examples 5 and 6.

The results obtained in Section II are used in Section IV for the study of reduction of one DDS to another DDS. The process of reduction becomes clearly defined and well understood from both the physical and the mathematical points of view. Two dynamical systems are related by relating certain well defined qualitative properties of their trajectories. In our discussion of reduction, we want in particular to avoid and *ad hoc* relation between the quantities characterizing states in the two theories that are to be related [*4*]. Such a relation should appear as a result of the discussion and not as its input (see also [*6*]). The reduction of m-component Enskog-Vlasov kinetic theory to m-component fluid dynamics illustrates the general theory and the problems that remain to be resolved.

Before proceeding with a detailed study of (II) (C) (i. e. detailed study of LDDS), we shall make a remark about physical interpretation of the concepts introduced in the definition of DDS. The above abstract structure defining DDS has been extracted from the well established (relation of the theory to certain well defined observations and measurements is established) Boltzmann-Enskog-Vlasov kinetic theory and the Navier-Stokes-Fourier fluid dynamics. In these theories the quantities introduced in the above definition of DDS have the following physical interpretation. Solutions to (I.2) define equilibrium states. Among all stationary states (i. e. the states for which $\mathscr{P}f = 0$) those satisfying moreover (I.2) are the equilibrium states. The quantities $\sigma_1, \ldots, \sigma_m$ have physical meaning of thermodynamic fields determining a state of a $m - 1$ component system in thermodynamics and $-\mathbb{V}|_{\mathscr{F}}$ (i. e. $-\mathbb{V}$ restricted to the solutions of (I.2)) is the $(m + 1)th$ thermodynamic field expressed as a function of $\sigma_1, \ldots, \sigma_m$. If we restrict ourselves to regular equilibrium states then indeed $-\mathbb{V}|_{\mathscr{F}_{\text{reg}}}$ is a single-valued function. For a general equilibrium state a kind of « Maxwell's rule » has to be used to construct σ_{m+1}. Another dynamical information has to be used to obtain such a rule. We restrict ourselves in this paper to regular equilibrium states only, we thus avoid any difficulty of this type. The condition (*i*) in the definition of a regular equilibrium state is physically interpreted as restriction to one phase state the condition (*ii*) as the condition of thermodynamic stability. The function $\mathbb{V}$ is then naturally called the nonequilibrium extension of thermodynamic potential. The property (II)(A) coincides in the case of the Boltzmann equation with Boltzmann's H-theorem. The property (II)(A) can be thus considered as an abstract H-theorem. The Onsager symmetry introduced above coincides in the particular context of nonequilibrium thermodynamics with the well known Onsager's symmetry relations [see Section III].

II. SOME CONSEQUENCES OF ONSAGER'S SYMMETRY

Let F be a fixed regular equilibrium state and $H = T_F\mathscr{H}$ (the tangent space to $\mathscr{H}$ at F). The linear space H is a Krein space (A denotes $D_F^{(2)}\mathbb{V}$ introduced in (II) (C) and J the involution introduced in (I) (B)).

The linear space H (real or complex) is called a Krein space [*1*] if H is a Hilbert space equipped with the inner product $(.,.)_A$ and with the fundamental decomposition $H = H^{(+)} \oplus H^{(-)}$. The inner product $(.,.)$ denotes a fixed inner product (e. g. Euclidean inner product in the case of finite dimensional H or L_2 inner product in the case H is a space of functions) $(.,.)_A = (.,A\,.)$, A is a positive definite completely invertible bounded linear operator defined everywhere on H. The linear operator A commutes with the fundamental symmetry J (i. e. $AJ = JA$), where $J = \Pi^{(+)} - \Pi^{(-)}$, $\Pi^{(+)}H = H^{(+)}$, $\Pi^{(-)}H = H^{(-)}$, $H^{(+)}$ and $H^{(-)}$ are two complete subspaces of H (notice $J^2 = \mathrm{id}$, where id denotes the identity operator in H). Thus, two operators A and J are always associated with a Krein space H. Instead of H, we shall also write H_{AJ}.

Let B is a bounded linear operator defined everywhere on H with only trivial nullspace. We say that an operator P is B-selfadjoint in the Hilbert space H if P is selfadjoint with respect to the non-degenerate, but in general indefinite inner product $(.,.)_B$ (i. e. $(Pf, Bg) = (f, BPg)$ for every pair f, $g \in \mathscr{D}(P)$; $\mathscr{D}(P)$ denotes domain of P) where $(.,.)$ denotes the inner product in H.

A linear operator P defined on a dense subset $\mathscr{D}(P)$ of H_{AJ} is said to be dissipative in H_{AJ} if $(f, Pf)_A + (Pf, f)_A \leq 0$ for all $f \in \mathscr{D}(P)$. We shall introduce the operators $P^{(s)} = \frac{1}{2}(P + P^*)$; $P^{(a)} = \frac{1}{2}(P - P^*)$, where P^* denotes the adjoint (with respect to the inner product $(.,.)_A$) of P. Domains of all the operators appearing in this paper will be assumed to be dense in H.

DEFINITION. — A linear operator P defined on a dense subspace of a Krein space H_{AJ} is called an Onsager symmetric operator if *i*) P is selfadjoint with respect to the indefinite inner product $(.,AJ.)$ and *ii*) P is dissipative in H_{AJ}.

The following notation is used: $\Sigma_p(P)$, $\Sigma_c(P)$, $\Sigma_r(P)$, $\rho(P)$ denote the point spectrum, continuous spectrum, residual spectrum and resolvent set respectively; $\lambda \in \mathbb{C}$, $\bar{\lambda}$ is complex conjugate of λ. In the case when H is finite dimensional, we shall denote by $H_\lambda(P)$ the principal subspace of λ $\left(\text{i. e. } H_\lambda(P) = \bigcup_{j=0}^{\infty} N((P - \lambda\,\mathrm{id})^j)\right)$, where N(Q) denotes the nullspace of the operator Q. $H_{\lambda}^{\perp_J}(P)$ denotes the orthocomplement of $H_{\lambda}(P)$ with respect to the inner product $(.,.)_J$. Matrix $Q_{\{e_j\}}$ is the representation of Q in the basis $\{e_j\}$.

PROPOSITION 1. — *i)* If P is J-selfadjoint with respect to the inner product $(.,.)_A$, then AP is J-selfadjoint or P is AJ-selfadjoint with respect to the inner product $(.,.)$.

ii) If P is J-selfadjoint then $P^{(s)} = P^{(+)}$ and $P^{(a)} = P^{(-)}$, where

$$P^{(\pm)} = \frac{1}{2}(P \pm JPJ).$$

iii) Let H is finite dimensional. An AJ-selfadjoint operator with respect to the inner product $(.,.)$ is similar to a J-selfadjoint operator P' with respect to the same inner product $(.,.)$. In fact $P' = A^{-\frac{1}{2}}PA^{\frac{1}{2}}$, where $A^{\frac{1}{2}}$ is defined by $A^{\frac{1}{2}}A^{\frac{1}{2}} = A$.

Proof. — The proof is elementary.

PROPOSITION 2. — *i)* If P is an AJ-selfadjoint operator with respect to the inner product $(.,.)$ then P is closed.

ii) If P is dissipative and bounded, then $\lambda \in \rho(P)$ implies $\mathrm{Re}(\lambda) \leq 0$.

iii) If P is J-selfadjoint operator in the Krein space $H_{\mathrm{id}\,J}$, then $\lambda \in \rho(P)$ implies $\bar{\lambda} \in \rho(P)$; $\lambda \in \Sigma_c(P)$ implies $\bar{\lambda} \in \Sigma_c(P)$; $\lambda \in \Sigma_r(P)$ implies $\bar{\lambda} \in \Sigma_p(P)$; $\lambda \in \Sigma_p(P)$ implies $\bar{\lambda} \in \Sigma_p(P) \cup \Sigma_r(P)$.

iv) Existence of the flow generated by an Onsager symmetric operator. If P is an Onsager symmetric operator, then there exists unique bounded solution to the initial value problem $\frac{\partial f}{\partial t} = Pf$; $f(0) \in \mathscr{D}(P)$, $\| f(0) \| < \infty$ for all $t \geq 0$. The operators $U(t)$ defined by $f(t) = U(t)f(0)$ form a strongly continuous semigroup of contraction operators. The strong limit of $U(t)f(0)$ as $t \to 0$ equals to $f(0)$, $f \in H$.

Proof. — Proofs of all statements in this Proposition are straightforward modifications of the proofs existing in literature. *i)* is proved in [*1*], Chap. VI, § 2; *ii)* is the Bendixon theorem, see [*8*], Chap. IV, § 4 or [*23*]; *iii)* is proved in [*1*], Chap. VI, § 6; *iv)* is the Hille-Yoshida-Phillips theorem, see [*23*].

Some other results about J-selfadjoint operators can be found in Chap. VI, § 6 of [*1*]. If at least one of the space $H^{(+)}$ or $H^{(-)}$ is finite-dimensional, H is then called a Pontrjagin space, stronger results are available [*7*], [*1*]. In the next proposition, we shall assume that both $H^{(+)}$ and $H^{(-)}$ are finite dimensional.

PROPOSITION 3. — Let P be a J-symmetric operator in a finite-dimensional

Krein space H_{id_J}. Beside the results in Proposition 1 and Proposition 2 the following is true:

i) $\lambda \neq \bar{\mu}$ implies $H_\lambda(P)$ is J-orthogonal to $H_\mu(P)$.

ii) $H_\lambda(P) \cap H_{\bar{\lambda}}^{\perp_J}(P) = 0$, let $\{ e_j \}$ be a basis in $H_\lambda(P)$ and $\{ f_j \}$ basis in $H_{\bar{\lambda}}(P)$, $(e_j, f_k)_J = \delta_{jk}$, then the matrices $P_{\lambda\{e_j\}}$ and $P_{\bar{\lambda}\{f_j\}}$ are dual conjugates $(P_\lambda = P |_{H_\lambda(P)})$.

iii) If E is a subspace of $H_{id\,J}$ invariant with respect to P, then its J-orthocomplement is also invariant with respect to P.

iv) $N(P - \lambda\,\mathrm{id})$ is definite (i. e. $(f, g)_J > 0$ for every non-zero pair f, $g \in N(P - \lambda\,\mathrm{id})$ or $(f, g)_J < 0$ for every non-zero pair f, $g \in N(P - \lambda\,\mathrm{id})$) implies λ is real and semi-simple.

v) Number of complex eigenvalues $\leq 2 \min (\dim (H^{(+)}), \dim (H^{(-)}))$.

vi) The number of non-semi-simple eigenvalues $< \min (\dim (H^{(+)}), \dim (H^{(-)}))$.

vii) The length of a Jordan chain (size of Jordan block in Jordan canonical form) corresponding to a real eigenvalue $< 2 \min (\dim (H^{(+)}), \dim (H^{(-)})) + 1$.

Proof. — Proofs of the statements *i)*-*iv)* can be found in [*1*], Chap. II, § 3; statements *v)*-*vii)* can be found in [*1*], Chap. IX, § 5; see also [*14*], Chap. VII, § 108. The result *v)* has also been obtained by Lekkerkerker, Laidlaw, McLennan [*13*], [*17*] without the help of [*14*], [*1*].

PROPOSITION 4. — Let P be a J-symmetric operator on a finite dimensional non-degenerate indefinite inner product space $H = H^{(+)} \oplus H^{(-)}$ with $\dim (H^{(+)}) = n$, $\dim (H^{(-)}) = 1$. Then P is diagonalizable iff $Q(\lambda)$ has distinct roots, where $Q(\lambda)$ is the quotient of the characteristic polynominal of P by the product of linear factors corresponding to the common eigenvalues of P and $P^{(+)} |_{H^{(+)}}$.

Proof. — This Proposition is due to Maltman and Laidlaw [*15*]. Our proof (in Appendix) is shorter and is developed in the spirit of the theory of indefinite inner product spaces. Statements *v)*-*vii)* of Proposition 3 will also be derived as a corollary of the proof.

COROLLARY. — Let P be a J-symmetric operator in a finite dimensional non-degenerate indefinite inner product space

$$H = H^{(+)} \oplus H^{(-)} \ (\dim (H^{(+)}) = n, \dim (H^{(-)}) = 1).$$

If P and $P^{(+)} |_{H^{(+)}}$ have no common eigenvalues then P is diagonalizable iff P is simple. If in addition $P^{(+)} |_{H^{(+)}}$ is simple then P has $n - 1$ distinct real eigenvalues.

Proof. — The first statement is a direct consequence of Proposition 4. The second statement follows from the proof of the corollary in the Appendix.

III. EXAMPLES OF DDS AND LDDS

Example 1. — Linear non-equilibrium thermodynamics.
H is a $(n + m)$-dimensional Krein space, $H \in f \equiv (\alpha, \beta)$, where

$$\alpha \equiv (\alpha_1, \ldots, \alpha_n) \in H^{(+)} \quad \text{and} \quad \beta \equiv (\beta_1, \ldots, \beta_m) \in H^{(-)}.$$

The matrix A is the second derivative of entropy. The vector Af is called generalized thermodynamic force, the vector $\frac{df}{dt}$ is called generalized thermodynamic flux. With this specification and notation our definition of LDDS coincides with the non-equilibrium thermodynamics introduced in Chap. IV, § 3 of [*3*], see also [*18*]. This example also explains why we have chosen the name Onsager symmetry.

Example 2. — Classical mechanics in the Liouville-Koopman representation.

The space H is the L_2 Hilbert space, $(\,,\,)$ denotes the standard L_2 inner product. The elements of H are functions $f : \mathbb{R}^{6N} \mapsto \mathbb{R}$,

$$(\underline{r}_1, \ldots, \underline{r}_N, \underline{p}_1, \ldots, \underline{p}_N) \mapsto f(\underline{r}_1, \ldots, \underline{r}_N, \underline{p}_1, \ldots, \underline{p}_N),$$

where N is the number of particles in the physical system considered, (r_i, p_i) is the position and impluse vector of the i-th particle. The functions f are symmetric with respect to a change in the labeling of the particles. The operator A is the identity operator in H, the fundamental symmetry J is defined as $f(\underline{r}_1, \ldots, \underline{r}_N, \underline{p}_1, \ldots, \underline{p}_N) \overset{J}{\mapsto} f(\underline{r}_1, \ldots, \underline{r}_N, -\underline{p}_1, \ldots, -\underline{p}_N)$. The time evolution in H is defined by

$$U_t(f(\underline{r}_1, \ldots, \underline{r}_N, \underline{p}_1, \ldots, \underline{p}_N)) = f(u_{-t}(\underline{r}_1, \ldots, \underline{r}_N, \underline{p}_1, \ldots, \underline{p}_N)),$$

where u_t is the operator of the time evolution of N classical particles generated by the Hamilton equations, $t \in \mathbb{R}$ denotes the time. The set of phenomenological quantities $\mathcal{Q}$ is now a set of functions $h : \mathbb{R}^{6N} \to \mathbb{R}$, called Hamiltonians, that satisfy $Jh = h$ (say

$$h(\underline{r}_1, \ldots, \underline{r}_N, \underline{p}_1, \ldots, \underline{p}_N) = \sum_{i=1}^{N} p_i^2 + V_{pot}(\underline{r}_1, \ldots, \underline{r}_N)$$

where V_{ot} denotes the potential energy). The infinitesimal generator of U_t, the Liouville operator, is indeed the Onsager symmetric operator. The Liouville operators satisfy moreover the property $P^{(+)} = 0$. Thus classical mechanics in the Liouville-Koopman representation is an example of LDDS, there is however no DDS that would bring the Liouville-Koopman representation of classical mechanics as its linearization at fixed points.

If we would add to the Liouville operator a dissipation $P^{(+)}$ satisfying the following two properties *i*) the nullspace of $P^{(+)}$ is one-parameter (denoted $\alpha_1^{(L)}$) family of functions of the type

$$n(\underline{r}_1, \ldots, \underline{r}_N) \exp\left(-\sigma_1^{(L)}(\underline{p}_1^2 + \ldots + \underline{p}_N^2)\right),$$

the function $n : \mathbb{R}^{3N} \to \mathbb{R}_+$ is arbitrary ($\mathbb{R}_+$ denotes positive real line), *ii*) the linearized vectorfield $P^{(+)}$ at F is a nonpositive selfadjoint operator, then the resulting family of dynamical system would be an example of DDS. The potential $\mathbb{V}^{(L)}$ would be ($\underline{r} \equiv (\underline{r}_1, \ldots, \underline{r}_N)$, $\underline{p} \equiv (\underline{p}_1, \ldots, \underline{p}_N)$)

$$\mathbb{V}^{(L)} = \int d^3\underline{r}_1, \ldots, d^3\underline{r}_N \int d^3\underline{p}_1, \ldots, d^3\underline{p}_N[-f(\underline{r}, \underline{p}) \ln f(\underline{r}, \underline{p}) + \sigma_1^{(L)} f(\underline{r}, \underline{p}) h(\underline{r}, \underline{p}) - \sigma_2^{(L)} f(\underline{r}, \underline{p})].$$

Example 3. — Classical dynamics of N particles with friction.

The state of the system is characterizes by $(\underline{r}_1, \ldots, \underline{r}_N, \underline{p}_1, \ldots, \underline{p}_N) \equiv (\underline{r}, \underline{p})$ representing the position vectors and momenta respectively. The fundamental symmetry J is defined by

$$(\underline{r}_1, \ldots, \underline{r}_N, \underline{p}_1, \ldots, \underline{p}_N) \xrightarrow{J} (\underline{r}_1, \ldots, \underline{r}_N, -\underline{p}_1, \ldots, -\underline{p}_N)$$

The time evolution of N particles is generated by the equations

$$\frac{d\underline{r}}{dt} = \frac{\partial h}{\partial \underline{p}}$$

$$\frac{d\underline{p}}{dt} = -\frac{\partial h}{\partial \underline{r}} - \lambda \underline{p}$$

where

$$h(\underline{r}_1, \ldots, \underline{r}_N, \underline{p}_1, \ldots, \underline{p}_N) = \sum_{i=i}^{N} p_i^2 + V_{pot}(\underline{r}_1, \ldots, \underline{r}_N),$$

$\lambda > 0$ is a constant and $V_{pot}(\underline{r}_1, \ldots, \underline{r}_N)$, is a real valued function of $\underline{r}$. The fixed points F are clearly identical with the extremal points of the function h (i. e. $\mathbb{V} \equiv h$). Linearizing the above equations at F at Which $\mathbb{V}$ reaches its non-degenerate minimum, we obtain indeed an Onsager symmetric operator.

Example 4. — The Cahn-Hilliard equation.

The states of the system are described by a function $C : \Omega \to \mathbb{R}_+$, $\underline{r} \mapsto C(\underline{r})$, where Ω is a subset of $\mathbb{R}^3$ in which the system is confined, $\mathbb{R}_+$ denotes positive real line. In physical interpretation $C(\underline{r})$, is mass concentration at $\underline{r}$ of one of the components of a binary solution. $H \equiv H^{(+)}$ (i. e. J is identity operator) $\mathbb{V}^{(CH)} = \int_\Omega d^3\underline{r}\left(\frac{1}{2}K(\nabla_\alpha C)^2 + f(C)\right)$ is the so called Ginzburg-Landau potential. The time evolution of $C(\underline{r})$ is governed by the Cahn-Hilliard equation [2]

$$\frac{\partial C(\underline{r})}{\partial t} = - \nabla_\alpha M \nabla_\alpha \frac{\delta \mathbb{V}^{(CH)}}{\partial C(r)} .$$

The phenomenological quantities entering the theory are M, K and f. Clearly all the properties of DDS and LDDS are satisfied, moreover $\mathcal{H} \equiv \mathcal{H}^{(+)}$ and thus also $P^{(-)} \equiv 0$.

Example 5. — Local m-component Navier-Stokes-Fourier fluid mechanics.

The state of the system is characterized by $m + 1$ functions $C_1, \ldots, C_{m-1}$, E, N of the type $\Omega \to \mathbb{R}_+$ ($\Omega \subset \mathbb{R}^3$ is the region in which the physical system considered is confined; we shall assume that volume of Ω is equal to one, $\mathbb{R}_+$ denotes the positive real line) and one function $U : \Omega \to \mathbb{R}^3$. With respect to the fluid mechanics measurements, $C_1, \ldots, C_{m-1}$ have the meanings of densities of relative concentrations of components $1, \ldots, m-1$ of the fluid respectively, E denotes inner energy density, N denotes mass density, U has the meaning of density of the mean velocity of the fluid. The fundamental symmetry is defined by

$$f \equiv (C_1, \ldots, C_{m-1}, E, N, \underline{U}) \overset{J}{\mapsto} Jf \equiv (C_1, \ldots, C_{m-1}, E, N, - \underline{U}) .$$

The potential $\mathbb{V}$ is [11]

$$\mathbb{V}^{(FM)}(C_1, \ldots, C_{m-1}, E, N, U ; \sigma^{(FM)}) = \int_\Omega d^3r \left[- S(C_1, \ldots, C_{m-1}, E, N) \right.$$
$$\left. - \sum_{i=1}^{m-1} \sigma_i^{(FM)} C_i - \sigma_m^{(FM)} E - \sigma_{m+1}^{(FM)} \frac{1}{N} - \frac{1}{2} \sigma_m^{(FM)} U^2 \right] \quad (1)$$

The upper index (FM) denotes Fluid Mechanics. The vectorfields $\mathcal{P}$ and P are written in full detail in [11]. For simplicity, we shall consider in this paper only Fourier transformed (with respect to $\underline{r} \in \mathbb{R}^3$; $\underline{k}$ denotes the variable introduced by the Fourier transform) local fluid dynamics with $\underline{k}$ fixed and with only one component of $\underline{U}$ (in the direction of $\underline{k}$). The fixed point F at which LDDS is constructed is assumed to be independent of $\underline{r}$. The space H is then $(m + 2)$-dimensional, the fundamental symmetry belonging to the space H is

$$J = \begin{array}{c} \\ m+1 \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \\ \\ \end{array} \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 & \ldots & 0 \end{matrix}}^{m+1} & 0 \\ \begin{matrix} 0 & 1 & \ldots & 0 \end{matrix} & 0 \\ \begin{matrix} \vdots & & & \vdots \end{matrix} & \vdots \\ \begin{matrix} 0 & 0 & \ldots & 1 \end{matrix} & 0 \\ \begin{matrix} 0 & & \ldots & 0 \end{matrix} & -1 \end{bmatrix}$$

The operators P and A have the following form

$$\mathrm{A} = \begin{array}{c} \\ m+1 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \\ \end{array} \begin{array}{c} \overbrace{\hphantom{0 \ldots 0}}^{m+1} \\ \begin{bmatrix} & & & 0 \\ & & & \vdots \\ & \mathrm{A}_0 & & 0 \\ 0 & \ldots & 0 & \mathrm{B}_3 \end{bmatrix} \end{array}$$

$$\mathrm{A}_0 = \begin{bmatrix} a_{11} & & a_{1\,m+1} \\ \vdots & & \vdots \\ a_{1\,m+1} & \ldots & a_{m+1\,m+1} \end{bmatrix} > 0$$

$\mathrm{P} = -ik\mathrm{P}^{(-)} - k^2\mathrm{P}^{(+)}$
$\mathrm{P}^{(-)} = \mathrm{L}^{(-)}\tilde{\mathrm{A}}\,;\ \mathrm{P}^{(+)} = \mathrm{L}^{(+)}\mathrm{A}$

$$\mathrm{L}^{(-)} = \begin{array}{c} m+1 \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \\ \\ \\ \end{array} \begin{array}{c} \overbrace{\hphantom{0 \ldots 0}}^{m+1} \\ \begin{bmatrix} 0 & \ldots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & 0 & \mathrm{B}_1 \\ 0 & \ldots & 0 & \mathrm{B}_2 \\ 0 & \ldots & \dfrac{1}{\mathrm{B}_2} & 0 \end{bmatrix} \end{array}; \quad \mathrm{L}^{(+)} = \begin{array}{c} \overbrace{\hphantom{0 \ldots 0 \ 0}}^{m+1} \\ \begin{bmatrix} & & & 0 & 0 \\ & l & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 \\ 0 & \ldots\ldots & & 0 & \eta \end{bmatrix} \end{array} \tag{2}$$

$$l = \begin{bmatrix} l_{11} & \ldots & l_{1m} \\ \vdots & & \vdots \\ l_{1m} & \ldots & l_{mm} \end{bmatrix} > 0$$

$$\tilde{\mathrm{A}} = \begin{array}{c} m+1 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \\ \end{array} \begin{array}{c} \overbrace{\hphantom{0 \ldots 0}}^{m+1} \\ \begin{bmatrix} & & & 0 \\ & \tilde{\mathrm{A}}_0 & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & \mathrm{B}_3 \end{bmatrix} \end{array}; \quad \tilde{\mathrm{A}}_0 = \begin{bmatrix} a_{11} & \ldots & a_{1\,m+1} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{m\,m+1} \\ \tilde{a}_1 & \ldots & \tilde{a}_{m+1} \end{bmatrix};$$

$a_i = \mathrm{B}_2(\mathrm{B}_2 a_{i\,m+1} + \mathrm{B}_1 a_{im})$.

Direct calculation shows that the operators P introduced in (2) are indeed Onsager symmetric operators (show $(\mathrm{AP})^* = \mathrm{JAPJ}$). $\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3, \mathrm{A}_0, l, \eta$ are the phenomenological quantities $\mathscr{Q}^{(\mathrm{FM})}$ entering the local fluid dynamics, $\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3$ are real positive numbers, A_0 is the second derivative of the potential $\mathbb{V}^{(\mathrm{FM})}$ evaluated at the given fixed point. The condition $\mathrm{A}_0 > 0$ is the condition expressing that the fixed point at which A_0 is evaluated corresponds to a thermodynamically stable equilibrium state. l, η are values of the kinetic coefficients at the fixed point. We assume $l > 0, \eta > 0$. All phenomenological quantities B_1, B_2, B_3, A_0, l, η depend on the thermodynamical variables $\sigma^{(FM)} \equiv (\sigma_1^{(FM)}, \ldots, \sigma_{m+1}^{(FM)})$ characterizing the fixed points.

The four propositions now apply. In order to calculate the eigenvalues explicitely, we shall assume that k is small and the perturbation theory [*12*], [*16*] will be used. If we limit ourselves only to the order k^2, we have

$$(-ikP^{(-)} - k^2P^{(+)})(f + ikg) = (-ik\Lambda - k^2\kappa)(f + ikg). \tag{3}$$

Comparing the terms proportional to k, we obtain

$$P^{(-)}f = \Lambda f \tag{4}$$

Straightforward calculation shows that $\Lambda = 0$ is m-times degenerate eigenvalue. The other two eigenvalues of (4) are

$$\Lambda_{1,2} = \pm \Lambda_0 = \pm \left(B_1B_3 \frac{a_m}{B_2} + B_2B_3 \frac{a_{m+1}}{B_2} \right)^{\frac{1}{2}}.$$

The eigenvectors (not normalized) $f_{1,2}$ corresponding to $\Lambda_{1,2}$ are

$$f_{1,2} = (0, \ldots, 0, B_1B_3, B_2B_3, \pm \Lambda_0).$$

The eigenspace $H_0 \in H^{(+)}$ corresponding to $\Lambda = 0$ is spanned by m linearly independent (not normalized) vectors

$$f_{i+2} = (\underbrace{0, \ldots, 0, a}_{i}{}_{m+1}, 0, \ldots, -a_i, 0), \qquad i = 1, \ldots, m$$

By using the Proposition 2 and 3 or by direct calculations one shows that f_1 is A-orthogonal to f_2 and the two dimensional space spanned by f_1, f_2 is A-orthogonal to H_0. We can thus choose in H an A-orthonormal basis $f_1^{(n)}, \ldots, f_{m+2}^{(n)}$ in which A will be represented as a unit matrix and (as one can easily show, see also [*14*])

$$J = \begin{array}{cc} & m+2 \\ m+2 & \begin{bmatrix} 0 & 1 & 0 & & \cdots\cdots & 0 \\ 1 & 0 & 0 & & \cdots\cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ & & & & & \\ 0 & \cdots & & & 0 & 1 \end{bmatrix} \end{array} \tag{5}$$

Comparing the terms in (3) of order k^2, we obtain

$$P^{(-)}g - P^{(+)}f = \Lambda_0 g - \kappa f \tag{6}$$

If we multiply (6) subsequently by f_1 and f_2, and use the fact that P is AJ-symmetric we obtain

$$\kappa_0 = \kappa_1 = \kappa_2 = \frac{(f_1, P^{(+)}f_1)_A}{(f_1, f_1)_A} = \frac{(f_2, P^{(+)}f_2)_A}{(f_2, f_2)_A}.$$

If we multiply (6) subsequently by $f_3, \ldots, f_{m+2}$, we see that the m eigenvalues $\kappa_3, \ldots, \kappa_{m+2}$ can be obtained as the eigenvalues of the following generalized eigenvalue problem

$$\mathrm{AL}^{(+)}\mathrm{A}f = \kappa \mathrm{A}f, \tag{7}$$

where $f \in \mathrm{H}_0$ and H_0 is considered now as m-dimensional Euclidean space with the Euclidean inner product $(.,.)$. From the properties of $\mathrm{L}^{(+)}$ and A described above and by using the theory of generalized eigenvalue problem [5] one can see easily that all $\kappa_3, \ldots, \kappa_{m+2}$ are real and positive.

Example 6. — m-component local Enskog-Vlasov kinetic theory.

The state of the system is characterized by $f \equiv (f_1, \ldots, f_m)$ where $f_i : \Omega \times \mathbb{R}^3 \to \mathbb{R}_+ ;\ (\underline{r}, \underline{v}) \mapsto f_i(\underline{r}, \underline{v})$. The fundamental symmetry J is defined by $f(\underline{r}, \underline{v}) \overset{\mathrm{J}}{\mapsto} f(\underline{r}, -\underline{v})$. The functions $f_i(r, v)$ have, in terms of the measurements on which the kinetic theory is based, the meaning of a one-particle distribution function of particles of the i-th component of the fluid. The family of vector fields $\mathscr{P}$ is written in detail in [11] and [10]. Generalization to an arbitrary m is straightforward. The potential $\mathbb{V}^{(\mathrm{KT})}$ is

$$\begin{aligned}\mathbb{V}^{(\mathrm{KT})}(f, \sigma^{(\mathrm{KT})}) = \sum_{i,j=1}^{m} \int_\Omega d^3\underline{r} \int d^3\underline{v}\Big(&- f_i \ln f_i - f_i \frac{\delta\theta}{\delta n_i} \\ &- \sigma_i^{(\mathrm{KT})} m_i f_i + \frac{1}{2}\sigma_{m+1}^{(\mathrm{KT})}\Big[m_i v^2 f_i + \int_\Omega d^3\underline{r}_1 \int d^3\underline{v}_1 \mathrm{V}_{\mathrm{pot}\, ij}(|r - r_1|) \\ &f_i(r, v) f_j(r_1, v_1)\Big]\Big),\end{aligned} \tag{8}$$

where the upper index (KT) denotes kinetic theory, $n_i = \int d^3\underline{v} f_i(\underline{r}, \underline{v})$, θ is a sufficiently many times differentiable function of $n_1, \ldots, n_m$, $\mathrm{V}_{\mathrm{pot}\, ij}(r)$ ($m \times m$ matrix) are sufficiently smooth functions, sufficiently fast decreasing to zero as $r \to \infty$; all quantities in which they appear are well defined and m_i, $i = 1, \ldots, m$ are positive real numbers (m_i denotes the mass of one particle of the i-th component of the fluid). The quantities θ, $\mathrm{V}_{\mathrm{pot}\, ij}$, m_i, $i, j = 1, \ldots, m$ and other quantities entering $\mathscr{P}^{(+)}$, i. e. the Boltzmann collision operators are the phenomenological quantities $\mathscr{Q}^{(\mathrm{KT})}$ entering the kinetic theory. The fixed point F at which (LDDS) is constructed is assumed to be independent of $\underline{r}$. The following general properties, in addition to the properties defining the Onsager operator, can be extracted from the concrete form of $\mathscr{P}$ defined by the Enskog-Vlasov kinetic theory [11]: *i)* $\mathrm{P}^{(+)}$ are the m-component linearized Boltzmann collision operators. *ii)* $\lambda = 0$ is an $(m + 4)$-degenerate eigenvalue of $\mathrm{P}^{(+)}$. The corresponding eigenspace H_a is spanned by functions $h_i = \underbrace{(0, \ldots, 1}_{i}, \ldots, 0)$ for $i = 1, \ldots, m$, $h_{m+1} = (m_1 v^2, \ldots, m_m v^2)$, $h_{m+1+j} = (m_1 v_j, \ldots, m_m v_j)$, $j = 1, 2, 3$. *iii*) $P^{(+)} = L^{(+)}A$, where A is the second derivative of $\mathbb{V}^{(KT)}$ evaluated at the fixed point considered, $L^{(+)}$ is a selfadjoint, non-positive linear operator. *iv*) Between Re $(\lambda) = 0$ and Re $(\lambda) = -\nu_0$ (ν_0 is a positive real number that enters $P^{(+)}$ as its phenomenological parameter) can appear only points of the point spectrum of P.

In order to calculate the eigenvalues of P that are closest to $\lambda = 0$, we shall limit ourselves again to the Fourier transform of P with $\underline{k}$ fixed and (for the sake of simplicity) with only one component of $\underline{v}$ (in the direction of $\underline{k}$). The operators P are then

$$P = -ikP^{(-)} + P^{(+)} \tag{9}$$

The Boltzmann collision operator for hard spheres $P^{(+)}$ satisfies the conditions in Kato's perturbation theory [*16*] and thus the perturbation theory can be used to calculate the $m + 2$ eigenvalues that are closest to $\lambda = 0$ and that unfold from the eigenvalue $\lambda = 0$ of the unperturbed operator $P^{(+)}$. If we limit ourselves only to the order k^2, we have

$$(P^{(+)} - ikP^{(-)})(f + ikg) = (-ik\Lambda - k^2\kappa)(f + ikg) \tag{10}$$

Comparing the terms proportional to 1, we have

$$P^{(+)}f = 0,$$

thus $f \in H_a$.

The terms proportional to k give

$$P^{(-)}f - P^{(+)}g = \Lambda f \tag{11}$$

Let $h_i^{(n)}$, $i = 1, \ldots, m + 2$ be the A-orthonormal basis in H_a obtained from the vectors h_i, $i = 1, \ldots, m + 2$, introduced in the property *ii*) above, by a standard orthonormalization procedure. Multiplying (10) by $h_i^{(n)}$, $i = 1, \ldots, m + 2$, and using the AJ-selfadjointness of P, we obtain that $\Lambda = 0$ is the m-times degenerate eigenvalue. The remaining two eigenvalues are $\Lambda_{1,2} = \pm\Lambda_0 = \pm\left(\sum_{i=1}^{m+1} w_i^2\right)^{\frac{1}{2}}$, where $w_i = (h_i^{(n)}, P^{(-)}h_{m+2}^{(n)})_A$. The eigenfunctions corresponding to $\Lambda_{1,2}$ are $f_{1,2} = (w_1, \ldots, w_{m+1}, \pm\Lambda_0)$; the m-dimensional eigenspace $H_0 \subset H^{(+)}$ corresponding to $\Lambda = 0$ is spanned by the vectors $f_{i+2} = \underbrace{(0, \ldots, w_{m+1}}_{i}, 0, \ldots, -w_i, 0)$. By application of the Propositions introduced in Section II or by direct calculations one can show that f_1 is A-orthogonal to f_2 and both f_1 and f_2 are A-orthogonal to H_0. It is thus again possible to choose a basis in H_0 such that A in that basis is a unit operator and J is the same as in (5). By comparing in (10) the terms proportional to k^2, we obtain

$$P^{(-)}g = \kappa f + \Lambda g \tag{12}$$

From (11) we have $g = (P^{(+)})^{-1}(P^{(-)} - \Lambda)f$, thus

$$\kappa_0 = \kappa_1 = \kappa_2 = \frac{(f_1, (P^{(-)} - \Lambda_1)(P^{(+)})^{-1}(P^{(-)} - \Lambda_1)f_1)_A}{(f_1, f_1)_A} = \frac{(f_2, (P^{(-)} - \Lambda_2)(P^{(+)})^{-1}(P^{(-)} - \Lambda_2)f_2)_A}{(f_2, f_2)_A} \tag{13}$$

The eigenvalues $\kappa_3, \ldots, \kappa_{m+2}$ can be obtained similarly as in the previous Example 3, as the eigenvalues of the following generalized eigenvalue problem

$$P^{(-)}(L^{(+)})^{-1}P^{(-)}f = \kappa A f, \tag{14}$$

where $f \in H_0$ and H_0 is considered now as m-dimensional subspace of L_2 space with the standard L_2 inner product $(.,.)$. From the properties of $L^{(+)}$, $P^{(-)}$ and A describe above and from the theory of the generalized eigenvalue problem [5] one obtains that $\kappa_3, \ldots, \kappa_{m+2}$ are real and positive. Detailed calculations and results for $m = 1$ and $m = 2$ are in [*10*].

In the six examples discussed above, we have introduced six different LDDS arising from six different types of experiences. We have seen that indeed the infinitesimal generators of the families of dynamical systems are always Onsager symmetric operators. We have seen however that beside the fundamental decomposition $H = H^{(+)} \oplus H^{(-)}$ of the space H there are other decompositions. An element f of H has been written in Examples 1, 3, 4, 5 as n-tuple of numbers (or functions), each component in the n-tuple represents a result of some particular measurement from the set of all measurements that form the experimental base of the theory (e. g. density distribution of inner energy, etc.). Also the phenomenological quantities (e. g. l, η, A in Example 5) are defined only if f is represented as the particular n-tuple. From the mathematical point of view, the representation of f as a fixed n-tuple reflects an additional structure in H, namely that H is equipped with the structure of other decompositions of H on the components of the n-tuple. The question of whether and how this rich structure of H (arising if a direct relation of elements of H and measurements is required) can be relaxed remains open. If only thermodynamics is considered (only the operator A is considered) Tisza [*21*] has suggested that, since both σ and the Legendere transformation of σ are directly measurable, the rich structure can be relaxed to the extent that A becomes always diagonal (so called Tisza's diagonalization). Whether and how a similar idea applies to DDS or LDDS is an interesting problem that we intend to study in future.

The results obtained in the discussion of the Examples 5 and 6 will be summed up in the Proposition 5.

PROPOSITION 5. — *i*) The operators A, P introduced in the Example 5 (fluid dynamics) can be brought by a similarity transformation, that leaves invariant the subspaces $H^{(+)}$ and $H^{(-)}$, into the following canonical form

$$P^c = -ikP^{c(-)} - k^2P^{c(+)}, \tag{15}$$

where

$$P^{c(-)} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ \vdots & \cdots & 0 & 0 & 0 \\ \vdots & \cdots & 0 & 0 & \Lambda_0 \\ 0 & \cdots & 0 & \Lambda_0 & 0 \end{bmatrix},$$

$$P^{c(+)} = \begin{bmatrix} \kappa_1 & 0 & \cdots & & & 0 \\ 0 & \kappa_2 & & & & \\ \vdots & & & \kappa_m & 0 & 0 \\ \vdots & & & 0 & \kappa_0 & 0 \\ 0 & & & 0 & 0 & \kappa_0 \end{bmatrix} > 0$$

and A^c is the identity operator. Only terms up to and including the order k^2 are considered in (15). The numbers Λ_0, $\kappa_1, \kappa_2, \ldots, \kappa_m$ are obtained from (4), (6) and (7).

ii) The restrictions of the operators A, P introduced in the Example 6 (kinetic theory) to the $(m + 2)$ dimensional asymptotic invariant subspace of P can be brought by a similarity transformation, that leaves invariant the subspaces $H^{(+)}$ and $H^{(-)}$, to the same canonical form (15) provided only the terms up to and including the order k^2 are considered. The numbers Λ_0, κ_0, $\kappa_1, \ldots, \kappa_m$ are obtained from (11), (13) and (14).

IV. REDUCTION

Let us consider two families of dissipative dynamical systems $(\{\mathscr{H}\}^{(1)}, \{\mathscr{P}\}^{(1)}, \mathscr{Q}^{(1)})$ and $(\{\mathscr{H}\}^{(2)}, \{\mathscr{P}\}^{(2)}, \mathscr{Q}^{(2)})$ that arose from two independent experiences. The family of dynamical systems indexed by (1) arose from the experience obtained by approaching a class of physical systems $\mathscr{C}^{(1)}$ with a set of observations and measurements $\mathscr{M}^{(1)}$, the family indexed by (2) arose from the experience obtained by approaching $\mathscr{C}^{(2)}$ with $\mathscr{M}^{(2)}$. We shall assume that the intersection of $\mathscr{C}^{(1)}$ and $\mathscr{C}^{(2)}$ is non-empty and that if $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(2)}$ are considered only as sets, then $\mathscr{H}^{(2)} \subset \mathscr{H}^{(1)}$. In other words we assume that there are physical systems that are observed and measured by both $\mathscr{M}^{(1)}$ and $\mathscr{M}^{(2)}$. We shall assume moreover that

$\mathscr{M}^{(1)}$ consists of measurements and observations more detailed, more microscopic, than those in $\mathscr{M}^{(2)}$. To every $(\{\mathscr{H}\}, \{\mathscr{P}\}, \mathscr{Q})$ we associate its phase portrait defined as the collection of all phase portraits of $(\mathscr{H}, \mathscr{P})$ for all $q \in \mathscr{Q}$.

The family of dissipative dynamical systems $(\{\mathscr{H}\}^{(1)}, \{\mathscr{P}\}^{(1)}, \mathscr{Q}^{(1)})$ is reduced to the family $(\{\mathscr{H}\}^{(2)}, \{\mathscr{P}\}^{(2)}, \mathscr{Q}^{(2)})$ if a subset $\mathscr{Q}^{(12)}$ of $\mathscr{Q}^{(1)}$, the mapping $\rho_{\mathscr{Q}} : \mathscr{Q}^{(12)} \to \mathscr{Q}^{(2)}$ and the mappings $\rho_{\mathscr{H}} : \mathscr{H}^{(1)} \to \mathscr{H}^{(2)}$; $\rho_{\mathscr{P}} : \mathscr{P}^{(1)} \to \mathscr{P}^{(2)}$: for all $q \in \mathscr{Q}^{(12)}$ are found. The set $\mathscr{Q}^{(12)}$ expresses the range of validity of the second family of dynamical systems in terms of the first family of dynamical systems.

In other words $\mathscr{Q}^{(12)}$ determines $\mathscr{C}^{(1)} \cap \mathscr{C}^{(2)}$. The reduction mappings $\rho_{\mathscr{H}}, \rho_{\mathscr{P}}, \rho_{\mathscr{Q}}$ will be obtained be relating some qualitative properties (patterns) of the phase portraits corresponding to the first and the second families of dissipative dynamical systems.

The first pattern of the phase portrait that plays the fundamental role even in the definition of DDS and also plays the fundamental role in the study of reduction is the set $\mathscr{F}$ of the orbits consisting of single points that are elements of $\mathscr{H}^{(+)}$ physically interpreted as the thermodynamically equilibrium states. We shall limit ourselves again to $\mathscr{F}_{\text{reg}}$ consisting of F_{reg} (see Introduction) and the subscript reg will be dropped. To every $\mathscr{F}$ the function $\sigma_{m+2} : \Xi \to \mathbb{R}$ is associated. Identification of $\sigma_{m+2}^{(1)}$ and $\sigma_{m+2}^{(2)}$ (physically the identification of thermodynamical potentials derived from $(\mathrm{DDS})^{(1)}$ and $(\mathrm{DDS})^{(2)}$) brings the first information about the reduction mappings $\rho_{\mathscr{Q}}, \rho_{\mathscr{H}}, \rho_{\mathscr{P}}$.

The second pattern of the phase portrait that we shall consider is the pattern of the long time tails of the orbits approaching the fixed points $F \in \mathscr{F}$. At every F, we construct the corresponding local dynamical systems and find the asymptotic invariant subspaces $H_{as}^{(1)}$ and $H_{as}^{(2)}$ that are isomorphic as Krein spaces. The vectorfields $P^{(1)}$ and $P^{(2)}$ restricted to $H_{as}^{(1)}$ and $H_{as}^{(2)}$ respectively are identified. This identification provides another independent information about the reduction mappings. In Example 7 which illustrates the reduction, we shall have $H_{as}^{(2)} \equiv H^{(2)}$. The process of identification of $P^{(1)}$ restricted to $H_{as}^{(1)}$ and $P^{(2)}$ is well explained schematically on the following diagram.

$$\begin{tikzcd} (\mathrm{LDDS})^{(2)} \arrow[r, "\alpha^{(2)}"] & (\mathrm{LDDS})^{c(2)} \\ (\mathrm{LDDS})^{(1)} \arrow[u, "\rho"] \arrow[r, "\alpha^{(1)}"'] & (\mathrm{LDDS})^{c(1)} \arrow[u, "r^{(12)}"'] \end{tikzcd} \tag{16}$$

where $(\mathrm{LDDS})^c$ denotes the (LDDS) in its canonical form (a spectral decomposition of $P_{(1)}$ see e. g. (15)) $r^{(12)}$ is the projection onto the invariant subspace H_{as} of $H^{(1)}$. The Krein space $H_{as}^{(1)}$ is isomorphic to the Krein space $H^{(2)}$. The reduction mappings are obtained by requiring that the diagram (16) is commutative, i. e. formally, $\rho = (\alpha^{(2)})^{-1} \circ r^{(12)} \circ \alpha^{(1)}$. The five propositions introduced in Section II can be used to find the mappings $\alpha^{(1)}$, $\alpha^{(2)}$, $r^{(12)}$. In Example 7, we shall use Proposition 5. The ideas of McLennan [16], Scharf [20], Resibois [19] and Grmela [10] are generalized and further developed in the above formulation of reduction.

Example 7. — Reduction of m-component Enskog-Vlasov kinetic theory $(\{\mathscr{H}\}^{(1)}, \{\mathscr{P}\}^{(1)}, \mathscr{Q}^{(1)})$ to m-component fluid mechanics $(\{\mathscr{H}\}^{(2)}, \{\mathscr{P}\}^{(2)}, \mathscr{Q}^{(2)})$. The potentials $\mathbb{V}^{(\mathrm{KT})}$ and $\mathbb{V}^{(\mathrm{FM})}$ were introduced in (1) and (8) respectively. We shall assume that the relation of $\sigma_i^{(\mathrm{KT})}$ and $\sigma_i^{(\mathrm{FM})}$ $(i = 1, \ldots, m + 2)$ to the thermodynamic fields that are directly related to the thermodynamic measurements $\mathscr{M}_{\mathrm{th}}$ is known:

$$\begin{aligned} \sigma_i^{(\mathrm{FM})} &= \frac{1}{\mathrm{T}}(\mu_i - \mu_m) \qquad \text{for } i = 1, \ldots, m-1 \\ \sigma_m^{(\mathrm{FM})} &= -\frac{1}{\mathrm{T}} \\ \sigma_{m+1}^{(\mathrm{FM})} &= -\frac{p}{\mathrm{T}} \\ \sigma_{m+2}^{(\mathrm{FM})} &= \frac{\mu_m}{\mathrm{T}} \end{aligned} \tag{17}$$

$$\begin{aligned} \sigma_i^{(\mathrm{KT})} &= \frac{\mu_i}{\mathrm{T}} - \frac{3}{2}\ln\left(\frac{m_i}{2\pi\mathrm{T}}\right) \qquad \text{for } i = 1, \ldots, m \\ \sigma_{m+1}^{(\mathrm{KT})} &= \frac{1}{\mathrm{T}} \\ \sigma_{m+2}^{(\mathrm{KT})} &= +\frac{p}{\mathrm{T}}, \end{aligned} \tag{18}$$

where T is the temperature, p is the pressure and μ_i is the chemical potential of the i-th component, $i = 1, \ldots, m$. The equations (17), (18), (1), (8) and the equality $\frac{\partial\sigma_{m+2}}{\partial\sigma_i} = \left.\frac{\partial\mathbb{V}}{\partial\sigma_i}\right|_{\mathrm{F}(\sigma_1,\ldots,\sigma_{m+1})}$, that follows immediately from the definition of σ_{m+2} in terms of $\mathbb{V}$, imply

$$\mathrm{E}_{\mathrm{th}} = \int_\Omega d^3\underline{r}\,\mathrm{E}(\underline{r}) = \int_\Omega d^3\underline{r}\,\frac{1}{2}\sum\sum_{i,j=1}^{m}\left[\int d^3\underline{v}\left(\frac{1}{3}\delta_{ij}m_i v^2 f_i \right.\right.$$
$$\left.\left.\int_\Omega d^3\underline{r}_1\int d^3\underline{v}_1\,\mathrm{V}_{\mathrm{pot},ij}(|\underline{r}-\underline{r}_1|)f_i(\underline{r},\underline{v})f_j(\underline{r}_1,\underline{v}_1)\right)\right] \tag{19}$$

$$\mathrm{C}_{i-\mathrm{th}} = \int_\Omega d^3\underline{r}\,\mathrm{C}_i(\underline{r}) = \left(\sum_{i=1}^{m} m_i\int d^3\underline{v}f_i(\underline{r},\underline{v})\right)^{-1} m_i\int d^3\underline{v}f_i(\underline{r},\underline{v})$$

$$\mathrm{V}_{\mathrm{th}} = \int_\Omega d^3\underline{r}\,\frac{1}{\mathrm{N}(\underline{r})} = \left(\sum_{i=1}^{m} m_i\int d^3\underline{v}f_i(\underline{r},\underline{v})\right)^{-1},$$

where E_{th}, C_{i-th}, V_{th} are the thermodynamic energy, concentrations and volume, respectively.

From (19), it follows that

$$C_i(r) = \left(\sum_{i=1}^{m} m_i \int d^3\underline{v} f_i(\underline{r}, \underline{v})\right)^{-1} m_i(d^3\underline{v} f_i(\underline{r}, \underline{v}))$$

$$E(\underline{r}) = \frac{1}{2}\sum\sum_{i,j=1}^{m} \left[\int d^3\underline{v}\left(\frac{1}{3} m_i v^2 f_i \delta_{ij}\right.\right. \tag{20}$$

$$\left.\left. + \int_{\Omega} d^3 r_1 \int d^3\underline{v}_1 V_{\text{pot},ij}(|\underline{r} - \underline{r}_1|) f_i(\underline{r}, \underline{v}) f_j(\underline{r}_1, \underline{v}_1)\right)\right]$$

$$N(\underline{r}) = \sum_{i=1}^{m} m_i \int d^3\underline{v} f_i(\underline{r}, \underline{v}) .$$

The equation (20) determines the reduction map $\rho_{\mathscr{H}}$. If (20) is assumed to be known (e. g. on the basis of the comparison of $\mathscr{M}^{(KT)}$ and $\mathscr{M}^{(FM)}$) the relationship between $\sigma^{(KT)}$ and $\sigma^{(FM)}$ would appear as the result.

The mappings $\alpha^{(1)}$, $\alpha^{(2)} r^{(12)}$ needed in the diagram (16) of the reduction of kinetic theory to fluid dynamics have been obtained in Example 5 and Example 6 (see Proposition 5). To obtain the quantities l and η as functions of phenomenological quantities $Q^{(1)}$ entering the kinetic theory, we can proceed as follows.

From Proposition 5, we know that there is an $(m + 2)$ dimensional subspace $H^{(1)}_{as}$ (hereafter called just H) of $H^{(1)}$ such that H is invariant (up to and including the terms proportional to k^2) with respect to $P^{(1)}$ and moreover (up to and including the terms proportional to k^1) the positive definite quadratic form $(\varphi, A^{(1)}\varphi) = (x, Ax) + (\psi, \tilde{A}\psi)$, where $x \in H$, $A = A^{(1)}|_H$, ψ is an element of the complement $\tilde{H}$ of H in $H^{(1)}$ and $\tilde{A} = A^{(1)}|_{\tilde{H}}$. We know also that there is a basis $e^c_1, \dots, e^c_{m+2}$ in H such that in this basis $A = \text{id} = A^c$ and $P = P^{(1)}|_H = P^c$ given in (15). The transformation from an orthonormal (with respect to (.,.)) basis $e_1, \dots, e_{m+2}$ to $e^c_1, \dots, e^c_{m+2}$ will be denoted T_c. It follows from Proposition 5 that T_c does not mix $H^{(+)}$ and $H^{(-)}$. The problem is to find a new basis $\xi_1, \dots, \xi_{m+2}$ in H such that the linear functionals $\zeta_1 = (\xi_1, \varphi), \dots, \zeta_{m+2} = (\xi_{m+2}, \varphi)$, $\varphi \in H^{(1)}$ will be identified with the local fluid dynamics variables and the operator P acting on $\zeta_1, \dots, \zeta_{m+2}$ will assume the form (2). Comparison of P acting on $\zeta_1, \dots, \zeta_{m+2}$ with (2) then gives l, η expressed as functions of $\sigma^{(1)}_1, \dots, \sigma^{(1)}_{m+1}$ and $\mathscr{Q}^{(1)}$.

Let T_ξ denotes the transformation in H that carries the basis $(e_1, \dots, e_{m+2})$ into $(\xi_1, \dots, \xi_{m+2})$. The transformation $\alpha = T_\xi T_c^{-1}$ carries the basis $e^c_1, \dots, e^c_{m+2}$ into $\xi_1, \dots, \xi_{m+2}$, the matrix $A^c = \text{id}$ into

$$A_\alpha = \alpha^{*-1}\alpha^{-1} , \tag{21}$$

and the operator P^c into

$$P^c_\alpha = \alpha P^c \alpha^{-1}. \tag{22}$$

Let

$$\alpha = \alpha_0(1 + ik\alpha_1), \tag{23}$$

where

$$\alpha_0 = \begin{pmatrix} \alpha_{11} & \cdots\cdots & \alpha_{1\ m+1} & 0 \\ \vdots & & \vdots & \vdots \\ \alpha_{m+1\ 1} & \cdots & \alpha_{m+1\ m+1} & 0 \\ 0 & & 0 & \alpha_{m+2} \end{pmatrix} \tag{24}$$

and

$$\alpha_1 = \begin{pmatrix} 0 & 0 & v_1 \\ \vdots & & \\ 0 & 0 & v_{m+1} \\ w_1 & w_{m+1} & 0 \end{pmatrix}; \tag{25}$$

α_{ij}, $i, j = 1, \ldots, m+1$, v_i, w_i, $i = 1, \ldots, m+1$ are real numbers.

Thus (up to and including the terms proportional to k)

$$A_\alpha = (\alpha_0^T)^{-1}\alpha_0^{-1} + ik(\alpha_0^T)^{-1}(\alpha_1^T - \alpha_1)\alpha_0^{-1} \tag{26}$$

and (up to and including the terms proportional to k^2)

$$P_\alpha = -ik\alpha_0 P^{c-}\alpha_0^{-1} - k^2\alpha_0(P^{c+} + P^{c-}\alpha_1 - \alpha_1 P^{c-})\alpha_0^{-1}; \tag{27}$$

(α^T denotes the transpose of α).

The identifications are thus the following:

$$A^{(2)} = (\alpha_0^T)^{-1}\alpha_0^{-1} \tag{28}$$

$$P^{(2)-} = (\alpha_0^T)^{-1}P^{c-}\alpha_0^T \tag{29}$$

$$P^{(2)+} = (\alpha_0^T)^{-1}(P^{c+} + \alpha_1^T P^{c-} - P^{c-}\alpha_1^T)\alpha_0^T. \tag{30}$$

It is easy to see that

$$A^{(2)}P^{(2)-} = (\alpha_0^T)^{-1}B^{-1}P^{c-}B\alpha_0^{-1} \tag{31}$$

has zero entries everywhere except the entries $(i, m+2)$, $(m+2, i)$; $i = 1, \ldots, m+1$ and

$$A^{(2)}P^{(2)+} = (\alpha_0^T)^{-1}B^{-1}(P^{c+} + \alpha_1^T P^{c-} - P^{c-}\alpha_1^T)B\alpha_0^{-1} \tag{32}$$

has zero entries for $(i, m+2)$, $(m+2, i)$, $i = 1, \ldots, m+1$. In (31) and (32), we used the notation

$$B = \alpha_0^T\alpha_0 \tag{33}$$

From (2), we know that $A^{(2)}P^{(2)+} = A^{(2)}LA^{(2)}$, therefore

$$L^{(+)} = \alpha_0 B^{-1}(P^{c+} + \alpha_1^T P^{c-} - P^{c-}\alpha_1^T)B\alpha_0^T. \tag{34}$$

We shall assume that α_0 is known from (33) or from $\alpha_0 = T_\xi T_c^{-1}$. Since $L^{(+)}$ has the particular form (see (2)), the $2m + 1$ unknown quantities $v_1, \ldots, v_m, w_1, \ldots, w_m, w_{m+1} - v_{m+1} = v$ are determined by equating the entries $(m + 1, i)$, $(i, m + 1)$, $i = 1, \ldots, m + 1$ of the r. h. s. of (34) to zero. The equation (34) then determines l and η in terms of $\sigma_1^{(1)}, \ldots, \sigma_{m+1}^{(1)}$ and $\mathscr{D}^{(1)}$.

Calculation of l and η in the case of $m = 1$, resp. $m = 2$ in [*10*] serves as an illustration of the method sketched above. The explicit calculations for $m = 3, 4, \ldots$ become complicated because of the difficulty to solve explicitly (14) that is needed for obtaining $\varkappa_1, \ldots, \varkappa_m$ and the basis vectors $e_1^c, \ldots, e_m^c$ and consequently the matrix $\alpha_0 = T_\xi T_c^{-1}$.

Finally, we mention some questions that remain to be without an answer:

(*i*) The matrix α_0 can be obtained from the identification of the thermodynamics implied by $(DDS)^{(1)}$ and $(DDS)^{(2)}$. The transformation T_ξ is obtained from (19) and since T_c is obtained from (8)-(11) we know $\alpha_0 = T_\xi T_c^{-1}$. The question is as to whether this matrix α_0 is consistent with (29). Explicit calculations for the case $m = 1$, $m = 2$ provide an affirmative answer. We are not able however to prove this property in general.

(*ii*) The matrix $L^{(+)}$ defined by the r. h. s. of (34) is not symmetric unless α_0 possesses some special properties. We do not have again any proof that α_0 obtained from T_ξ, T_c and from (29) will guarantee symmetry of (34). For example the r. h. s. of (34) is symmetric if B (see (33)) is a diagonal matrix with entries $(b_1, \ldots, b_m, b_{m+1}, b_{m+1})$ where $b_1, \ldots, b_{m+1}$ are real numbers. In the case of $m = 1$ the matrix B has indeed this form (see [*10*]). If this property would follow from some independent considerations then α_0 would be completely determined by (29).

(*iii*) It would be of interest to know the behaviour of l and η in the vicinity of a critical point (i. e. the point for which the matrix A becomes singular). In the case $m = 1$ explicit calculations (see [*10*]) show that l and η remain finite and positive when approaching the critical point. It might be possible to extract this type of information from (34) without going through all the explicit calculations leading to the formulae for l and η.

V. CONCLUSION

The class of DDS and (LDDS) introduced in the six examples in Section III is large enough to justify the abstraction and creation of the concept of the family of dissipative dynamical of macroscopic physics. Beside the six examples in Section III, the following observations indicate a possible generality of the structure of DDS. The original (Onsager's) considerations [*18*] extended by van Kampen [*22*] lead to the dynamical systems possessing the structure of DDS. The existence of a common structure for dynamical systems of macroscopic physics can be supported by the following physical argument. Every macroscopic physical system can be approached by the observations and measurements that would result in a dynamical system having the structure of Hamiltonian dynamics (the detailed observation of the « elementary » particles composing the system). Every macroscopic physical system can be approached by the observation and measurements that would result in thermodynamics (observations and measurements of specially prepared system). The mathematical setting provided by DDS is general enough to include large class of kinetic equations and the equations arising in fluid dynamics and at the same time specific enough to imply interesting consequences about some qualitative properties of the phase portrait (Section II). Solutions to all realizations of DDS, thus in particular solutions to a large class of kinetic equations and equations of fluid dynamics, will possess these properties.

The abstract mathematical setting gives us a possibility of seeing the fundamental problem of reductions of dynamical systems in a new light. Both clarity (physical and mathematical) and simplicity is gained. Our objective is to relate two independent phenomenological theories by relating some common patterns of their phase portraits.

Finally, we mention another two interesting open problems related to DDS. All discussions in this paper have been restricted to the study of regular situation (only Ξ_{reg} have been considered). Study of singular fixed points (in physical interpretation critical equilibrium states) and the behaviour of the orbits in their neighborhood is an interesting open problem. In the special case, when $\mathscr{H} \equiv \mathscr{H}^{(+)}$ and the Cahn-Hilliard equation is used to define the vectorfield $\mathscr{P}$ (Example 4), some results in this direction have been obtained by Langer [2]. The second open problem is the problem of reduction of a family of Hamiltonian dynamical systems to a family of DDS. In this reduction the patterns in the phase portraits which must be compared will be different from those used in the reduction of one DDS to another in Section IV.

APPENDIX

$Y_{(\alpha,\beta)}$ denotes an $\alpha \times \beta$ matrix Y. In case $\alpha = \beta$ we shall write just Y_α.

Theorem. — Let P be a J-symmetric operator on a finite dimensional non-degenerate indefinite inner product space $H = H^{(+)} \oplus H^{(-)}$ with $\dim (H^{(+)}) = n$, $\dim (H^{(-)}) = 1$. Then P is diagonalizable iff $Q(\lambda)$ has distinct roots, where $Q(\lambda)$ is the quotient of the characteristic polynomial of P by the product of linear factors corresponding to the common eigenvalues of P and $P^{(+)}|_{H^{(+)}}$.

Proof. — The relation $(Px, y)_J = (x, Py)_J$ for all $x, y \in H$ implies that $P^* = JPJ$. With respect to some orthogonal, (with respect to standard inner product on $\mathbb{C}^{n+1}$), basis $\{ e_j \}$ we then have

$$[J]_{\{e_j\}} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{bmatrix},$$

which, using $P^* = JPJ$, implies that

$$[P]_{\{e_j\}} = \begin{bmatrix} & & & c_1 \\ & A_n & & \vdots \\ & & & c_n \\ -\bar{c}_1 & \dots & -\bar{c}_n & c_{n+1} \end{bmatrix},$$

where A_n is an $(n \times n)$ Hermitian matrix; c_{n+1} is real. A_n can be diagonalized via a unitary transformation U_n. Changing basis in H via

$$U = \left(\begin{array}{c|c} U_n & 0 \\ 0 & 1 \end{array}\right);$$

we note that matrix of J is left invariant and the matrix of P becomes

$$\begin{bmatrix} a_1 & \dots & 0 & b_1 \\ \vdots & \ddots & & \vdots \\ 0 & & a_n & b_n \\ -\bar{b}_1 & \dots & -\bar{b}_n & a_{n+1} \end{bmatrix};$$

where the $\{ a_j \}_{j=1}^{n+1}$ are real. Without loss of generality we can assume that the $\{ a_j \}_{j=1}^{n}$ are increasing. Suppose $a_1 = a_2 = \dots = a_{n_1} < a_{n_1+1} = \dots = a_{n^2} < \dots < a_{n^{p-1}+1} = \dots = a_{n^p}$, where n_j is the multiplicity of the eigenvalue a_j of $P^{(+)}|_{H^{(+)}}$, and $n^j = \sum_{k=1}^{j} n_k$. For simplicity of notation we shall write a^j for a_{n^j} $(1 \leqslant j \leqslant p)$ and also a^{p+1} for a_{n+1}. Then it may be established by induction, or otherwise, that the characteristic polynomial of P is:

$$p(\lambda) = \prod_{i=1}^{n+1}(a_i - \lambda) + \sum_{j=1}^{n} |b_j|^2 \prod_{i \neq j}^{n}(a_i - \lambda)$$

$$= \left[\prod_{i=1}^{p}(a^i - \lambda)^{n_i - 1}\right]\left[\prod_{i=1}^{p+1}(a^i - \lambda) + \sum_{j=1}^{p} c_j^2 \prod_{i \neq j}^{p}(a^i - \lambda)\right],$$

where $c_j^2 = \sum_{k=n^{j-1}+1}^{n^j} |b_k|^2$, $(n^\circ \equiv 0)$. We call the latter factor of $p(\lambda)$, $\tilde{Q}(\lambda)$. Note that if $P^{(+)}|_{H^{(+)}}$ is simple i. e. if the a_i are distinct $(1 \leq i \leq n)$, then $\tilde{Q}(\lambda) = p(\lambda)$. Finally, we can assume that no $b_j = 0$, otherwise we could restrict attention to the factor space H/Ue_j since Ue_j would then be an eigenvector of P with eigenvalue a_j. Thus no a^i is a root of $\tilde{Q}(\lambda)$. $Q(\lambda)$ is the quotient of $\tilde{Q}(\lambda)$ by all factors $(a^i - \lambda)$ for which $b_i = 0$. Thus we are assuming without loss of generality that $\tilde{Q}(\lambda) = Q(\lambda)$.

Let λ be an eigenvalue of P. If λ is a root of $Q(\lambda)$ then $\lambda \neq a^i$ $(1 \leq i \leq p)$ and we row reduce $P - \lambda I$ as follows:

$$\begin{bmatrix} a^1-\lambda & & & & b_1 \\ & \ddots & & & \vdots \\ & & a^1-\lambda & & \vdots \\ & & & \ddots & \vdots \\ & & & a^p-\lambda & \vdots \\ & & & \ddots & \vdots \\ & & & a^p-\lambda & b_n \\ -\bar{b}_1 & \cdots & \cdots & -\bar{b}_n & a^{p+1}-\lambda \end{bmatrix} \quad R_n \to R_n + \sum_{i=1}^{n} \frac{\bar{b}_i}{a_i - \lambda} R_i \quad \begin{bmatrix} a^1-\lambda & & & b_1 \\ & \ddots & & \vdots \\ & & a^p-\lambda & b_n \\ 0 & \cdots & 0 & x \end{bmatrix}$$

where

$$x = (a^{p+1} - \lambda) + \sum_{k=1}^{p} \sum_{j=n^{k-1}+1}^{n^k} \frac{|b_j|^2}{a^k - \lambda}$$

$$= (a^{p+1} - \lambda) + \sum_{k=1}^{p} \frac{c_k^2}{a^k - \lambda}$$

$$= \frac{1}{\prod_{i=1}^{p}(a^i - \lambda)} Q(\lambda) = 0$$

So $A - \lambda I$ has a nullity of one. Note that if the a_j's are distinct, we are done. If $\lambda = a^i$, $1 \leq i \leq p$ then without loss of generality $i = 1$ and $P - \lambda I$ row reduces as follows:

Case (i) $n_1 = n$:

$$\begin{bmatrix} & & b_1 \\ & 0_n & \vdots \\ & & b_n \\ -\bar{b}_1 & \cdots \ -\bar{b}_n & (a_{n+1} - a_1) \end{bmatrix} \quad \begin{array}{l} R_j \to R_j - \dfrac{b_n}{b_1} R_1, \quad (2 \leq j \leq n) \\ R_{n+1} \to R_{n+1} - \dfrac{(a_{n-1} - a_1)R_1}{b_1} \\ R_1 \leftrightarrow R_2 \\ R_{n+1} \leftrightarrow R_1 \end{array} \quad \begin{bmatrix} -\bar{b}_1 & \cdots & -\bar{b}_n & 0 \\ & & 0 & b_1 \\ & 0_n & \vdots & 0 \\ & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

Thus the nullity is $n + 1 - 2 = n - 1 = n_1 - 1$.

Case (*ii*) $1 \leq n_1 < n$:

$$\begin{bmatrix} 0_{n_1} & 0_{(n_1,n-n_1)} & b_1 \\ 0_{(n-n_1,n_1)} & \begin{bmatrix} a^2-a^1 & & \\ & \ddots & \\ & & a^p-a^1 \end{bmatrix}_{n-n_1} & \vdots \\ & & b_n \\ -\bar{b}_1 & \cdots\cdots\cdots \quad -\bar{b}_n & a_{n+1}-a_1 \end{bmatrix} \quad (\mathrm{R}_1 \leftrightarrow \mathrm{R}_{n+1})$$

$$\begin{bmatrix} -\bar{b}_1 & \cdots\cdots\cdots & -\bar{b}_n(a_{n+1}-a_1) \\ & & b_2 \\ 0_{(n_1-1,n_1)} & 0_{(n_1-1,n-n_1)} & \vdots \\ & \begin{bmatrix} a^2-a^1 & & \\ & \ddots & \\ & & a^p-a^1 \end{bmatrix}_{n-n_1} & \vdots \\ 0_{(n-n_1,n_1)} & & b_n \\ 0 \cdots\cdots 0 & \cdots\cdots 0 & b_1 \end{bmatrix}$$

$$\text{(multiple block-row switch)} \begin{bmatrix} -\bar{b}_1 & \cdots\cdots\cdots & -\bar{b}_n(a_{n+1}-a_1) \\ 0_{(n-n_1,n_1)} & \begin{bmatrix} a^2-a^1 & & \\ & \ddots & \\ & & a^p-a^1 \end{bmatrix}_{n-n_1} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ 0_{n_1} & 0_{(n_1,n-n_1)} & \vdots \\ & & * \end{bmatrix} \text{(multiple block-column switch)}$$

$$\begin{bmatrix} -\bar{b}_1 & * \cdots\cdots\cdots & * & (a_{n+1}-a_1) \\ 0 & \begin{bmatrix} a^2-a^1 & & \\ & \ddots & \\ & & a^p-a^1 \end{bmatrix}_{n-n_1} & 0_{(n-n_1,n_1-1)} & \begin{matrix} * \\ \vdots \end{matrix} \\ \vdots & 0_{(n_1-1,n_1)} & 0_{(n_1-1,n-n_1)} & \vdots \\ 0 & \cdots\cdots\cdots & 0 & * \end{bmatrix} \quad \begin{matrix} \mathrm{R}_{n+1} \leftrightarrow \mathrm{R}_{n+1-n_1} \\ \mathrm{C}_{n+1} \leftrightarrow \mathrm{C}_{n+1-n_1} \end{matrix}$$

$$\begin{bmatrix} -\bar{b}_1 & * \cdots\cdots\cdots & & & * \\ 0 & \begin{bmatrix} a^2-a^1 & & \\ & \ddots & \\ & & a^p-a^1 \end{bmatrix}_{n-n_1} & & 0_{(n-n_1,n_1-1)} & \vdots \\ 0 \cdots\cdots & 0 & x & 0 \cdots\cdots & * \\ \vdots & 0_{(n_1-2,n_1)} & & 0_{(n_1,n-n_1)} & \vdots \\ 0 \cdots\cdots & & & \cdots\cdots & 0 \end{bmatrix}$$

, where x is some b_j and stars denote entries which do not effect our calculations.

Therefore $\mathrm{P} - \lambda\mathrm{I}$ has nullity $n_1 - 1$.

Thus each a^i $(1 \leq i \leq p)$ has $n_i - 1$ linearly independent eigenvectors, so collectively they generate a subspace of dimension $\sum_{i=1}^{p} (n_i - 1) = n - p$. Each eigenvalue of $\mathrm{Q}(\lambda)$ is semi-simple and since Q is of degree $p + 1$ (under assumption $\tilde{\mathrm{Q}} = \mathrm{Q}$) it is both necessary and sufficient for the diagonalizability of P (i. e. existence of $(n + 1)$ linearly independent eigenvectors of P) that Q have $p + 1$ distinct roots.

COROLLARY. — Let P be a J-symmetric operator on a finite dimensional non-degenerate indefinite inner product space H, with dim $\mathrm{H}^{(-)} = 1$. Then the Jordan canonical form of P has one of the following forms:

i) One 3×3 block; the rest diagonal.

ii) One 2×2 block; the rest diagonal.

iii) Diagonal form.

Proof. — In the notation of the above proof with $\{ a_n \}$ increasing and $\tilde{Q} = Q$ we have $Q(a^1) > 0$, $Q(a^2) < 0$; $Q(a^2) < 0$, $Q(a^3) > 0$; ... ; $Q(a^{p-1})$ has opposite sign of $Q(a^p)$. There are $(p - 1)$ of these paired relations giving us $(p - 1)$ distinct real roots of $Q(\lambda)$. The remaining 2 roots can cause one of the three degeneracies listed above. Note that, using *vi*) of Proposition 3, we can exclude the case of two 2×2 blocks. Note that if $\tilde{Q} = Q$, we obtain just as many relationships lost as eigenvectors gained, so no net loss.

COROLLARY. — Let P be a J-symmetric operator in a finite dimensional non-degenerate indefinite inner product space $H = H^{(+)} \oplus H^{(-)}$, with $\dim H^{(+)} = n$, $\dim H^{(-)} = 1$. If P and $P^{(+)}|_{H^{(+)}}$ have no common eigenvalues then P is diagonalizable iff P is simple. If in addition $P^{(+)}|_{H^{(+)}}$ is simple then P has $n - 1$ distinct real eigenvalues.

Proof. — The first statement is a direct consequence of Proposition 4. The second statement follows from the proof of the above corollary.

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