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Section A :
Physique théorique.

A model of atomic radiation

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ABSTRACT. — We consider a non-relativistic quantum mechanical particle in an external potential well, coupled to an infinite free quantum field. We prove rigorously that with certain cut-offs and in the weak coupling limit, the particle decays exponentially between its bound states as predicted by perturbation theory. We also prove the existence of a « dynamical phase transition » for a particle attracted to two widely separated potential wells and also weakly coupled to an infinite reservoir.

RÉSUMÉ. — Nous considérons une particule quantique non-relativiste dans un potentiel, couplée à un champ libre infini. Avec certaines régularisations et dans une limite faible, nous montrons que la particule passe exponentiellement entre ses états liés selon les prédictions de la théorie des perturbations. Nous montrons l'existence d'une « transition de phase dynamique » pour une particule attractée par deux potentiels très éloignés et aussi couplée faiblement à un réservoir infini.

§ 1. DESCRIPTION OF THE MODEL

We consider a single quantum mechanical particle with Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3)$ and Hamiltonian

$$H_s = -\Delta/2M + V$$

where

$$V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$$

so that H_s may be easily defined as a self-adjoint operator on $\mathscr{H}$ as described in [*11*] [*18*].

We also consider an infinite free spinless boson quantum field, in a quasi-free state as described in [1]. The single-particle space is $\mathcal{V} = L^2(\mathbb{R}^3)$ with single-particle Hamiltonian

$$S = -\Delta/2m.$$

One has a representation of the CCRs on a Hilbert space $\mathcal{F}$ (not necessarily Fock space) with a cyclic vector Ω satisfying

$$\langle W(f)\Omega, \Omega \rangle = \exp\left[-\frac{1}{4}\| f \|^2 - \frac{1}{2}\langle Df, f \rangle\right]$$

where

$$(Df)^\wedge(k) = d(k) f^\wedge(k)$$

in the momentum space picture, and we assume for definiteness that d is a non-negative polynomially bounded C^∞ function on $\mathbb{R}^3$. For the Fock representation $d \equiv 0$ and more generally d determines the particle density for different momenta k. If the reservoir is in a Gibbs state (which is by no means the only interesting case) then according to [1]

$$d_{\beta,\mu}(k) = (2\pi)^{-3}\left[\exp\left(\frac{\beta k^2}{2m} - \beta\mu\right) - 1\right]^{-1}.$$

The parameters β, μ determine the temperature and density, and we need $\mu < 0$ to avoid having to consider the phenomenon of Bose-Einstein condensation [14].

The Weyl operator $W(f)$ is related to the field $\Phi(x)$ defined as an operator-valued distribution by

$$W(f) = \exp\left\{ i \int_{\mathbb{R}^3} f(x)\Phi(x)d^3x \right\} = \exp\{ i\Phi(f) \}.$$

The free Hamiltonian H_b on $\mathcal{F}$ satisfies

$$H_b\Omega = 0$$

and

$$e^{iH_bt}W(f)e^{-iH_bt} = W(e^{iSt}f)$$

for all $f \in \mathcal{V}$ and $t \in \mathbb{R}$.

The Hamiltonian for the interaction between the particle and quantum field is formally

$$H_s + H_b + \lambda \int_{\mathbb{R}^3} \delta(x)\Phi(x)d^3x$$

where $\delta(x)$ is the (singular) operator on $\mathcal{H}$ given as a quadratic form by

$$\langle \delta(x)f, g \rangle = f(x)\overline{g(x)}.$$

For technical reasons we introduce space and ultraviolet cut-offs in the interaction term and consider the self-adjoint operator H on $\mathscr{H} \otimes \mathscr{F}$ defined by

$$\mathrm{H} = \mathrm{H}_s \otimes 1 + 1 \otimes \mathrm{H}_b + \lambda \int_{|x| \leqslant a} \mathrm{A}_x \otimes \Phi(f_x) d^3x \tag{1.1}$$

where f is a function in Schwartz space and

$$f_x(y) = f(y - x)$$

and

$$\mathrm{A}_x \psi = g_x \langle \psi, g_x \rangle$$

for another function g in Schwartz space. The original Hamiltonian is then obtained by letting f and g tend to the delta function at the origin and letting $a \to \infty$, but we shall deal only with the regularised Hamiltonian H from now on.

Given that at time $t = 0$ the particle and field are uncorrelated and in a (mixed) state ρ and the (pure) vacuum state v respectively, the state of the particle at time $t \geqslant 0$ is

$$\rho_\lambda(t) = \mathrm{tr}_{\mathscr{F}} \left[e^{-i\mathrm{H}t} (\rho \otimes v) e^{i\mathrm{H}t} \right]. \tag{1.2}$$

One expects that for small λ the particle evolution will contain dissipative terms of order λ^2 and that as λ gets smaller the dissipation will become more nearly exponential. The precise result is

THEOREM 1.1. — For all states ρ on $\mathscr{H}$ and all $\tau_0 \geqslant 0$

$$\lim_{\lambda \to 0} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} \| \rho_\lambda(t) - \exp\{ (\mathrm{Z} + \lambda^2 \mathrm{K}) t \} \rho \|_{\mathrm{tr}} = 0 \tag{1.3}$$

where the operator Z on the Banach space V of trace-class operators on $\mathscr{H}$ is defined by

$$\mathrm{Z}(\rho) = -i[\mathrm{H}_s, \rho] \tag{1.4}$$

with the natural domain [*6*], and the bounded operator K on V is given by Eqs. (1.6-1.11).

Proof. — The problem is of exactly the type solved in [*2*] [*3*] [*17*], the fact that $\mathscr{H}$ is infinite-dimensional being of no importance. The crucial estimate needed is that on the field two-point function

$$\begin{aligned} h(x, t) &= \langle \Phi(e^{i\mathrm{S}t} f_x) \Phi(f) \Omega, \Omega \rangle \\ &= \frac{1}{2} \langle \mathrm{D} e^{i\mathrm{S}t} f_x, f \rangle + \frac{1}{2} \langle (1 + \mathrm{D}) f, e^{i\mathrm{S}t} f_x \rangle \\ &= \frac{1}{2} \int_{\mathbb{R}^3} e^{ik^2 t/2m + ix.k} | f^\wedge(k) |^2 d(k) d^3k \\ &+ \frac{1}{2} \int_{\mathbb{R}^3} e^{-ik^2 t/2m - ix.k} | f^\wedge(k) |^2 \{ 1 + d(k) \} d^3k . \end{aligned}$$

One may show by Fourier analysis that h satisfies

$$|h(x,t)| \leqslant c\{1+|t|\}^{-3/2} \tag{1.5}$$

for all $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. The uniformity of this estimate with respect to x deals with the space integrations arising from the interaction term because of the space cut-off. According to [2] [3] Eq. (1.3) holds with

$$K = K_1 + K_2 + K_3 + K_4 \tag{1.6}$$

where

$$K_1(\rho) = -\int_{t=0}^{\infty}\int_{|x|\leqslant a}\int_{|y|\leqslant a} A_x^t A_y \rho h(x-y,t) d^3x d^3y dt \tag{1.7}$$

$$K_2(\rho) = \int_{t=0}^{\infty}\int_{|x|\leqslant a}\int_{|y|\leqslant a} A_y \rho A_x^t h(x-y,t) d^3x d^3y dt \tag{1.8}$$

$$K_3(\rho) = \int_{t=0}^{\infty}\int_{|x|\leqslant a}\int_{|y|\leqslant a} A_x^t \rho A_y h(y-x,-t) d^3x d^3y dt \tag{1.9}$$

$$K_4(\rho) = -\int_{t=0}^{\infty}\int_{|x|\leqslant a}\int_{|y|\leqslant a} \rho A_y A_x^t h(y-x,-t) d^3x d^3y dt \tag{1.10}$$

and

$$A_x^t = e^{iH_s t} A_x e^{-iH_s t}. \tag{1.11}$$

The influence of the field on the time evolution of the system is therefore entirely determined by the two-point function h.

§ 2. ANALYSIS OF THE PARTICLE EVOLUTION

Although unambiguous the interpretation of Theorem 1.1 is somewhat obscure because Z and $\lambda^2 K$ do not have the same order of magnitude as $\lambda \to 0$. There are several methods of dealing with this problem [3] [4] [5] [12] but none of them is directly applicable. If H_s had pure point spectrum then by [4] the strong operator limit in $\mathscr{L}(V)$

$$K^\natural = \lim_{a\to\infty} \frac{1}{a}\int_0^a e^{-Zt} K e^{Zt} dt$$

would exist and we could replace Eq. (1.3) by

$$\lim_{\lambda\to 0} \sup_{0\leqslant\lambda^2 t\leqslant\tau_0} \| \rho_\lambda(t) - e^{(Z+\lambda^2 K^\natural)t}\rho \|_{tr} = 0$$

which has the advantage that Z and $K^\natural$ commute. We have, however, to take account of the fact that our H_s will generally have both continuous and discrete spectrum. The following example shows that we cannot expect $K^\natural$ to exist in this situation.

EXAMPLE 2.1. — Let $\mathscr{H} = \mathbb{C} \oplus L^2(\mathbb{R})$ and define the operators H and A by

$$H\{\alpha \oplus f(x)\} = 0 \oplus xf(x)$$
$$A\{\alpha \oplus f(x)\} = 0 \oplus \alpha\phi(x)$$

where

$$\phi(x) = e^{-x^2}$$

for all $x \in \mathbb{R}$. Then

$$A = |\psi\rangle\langle\phi|$$

where $\psi = 1 \oplus 0$. We define the operators Z and K on the space V of trace-class operators on $\mathscr{H}$ by

$$\begin{aligned} Z(\rho) &= -i[H, \rho] \\ K(\rho) &= -\frac{1}{2}A^*A\rho + A\rho A^* - \frac{1}{2}\rho A^*A \\ &= -\frac{1}{2}|\psi\rangle\langle\psi|\rho - \frac{1}{2}\rho|\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|\langle\rho\psi, \psi\rangle. \end{aligned}$$

A direct calculation shows that

$$\frac{1}{a}\int_0^a e^{-Zt}Ke^{Zt}\rho dt = -\frac{1}{2}|\psi\rangle\langle\psi|\rho - \frac{1}{2}\rho|\psi\rangle\langle\psi| + \langle\rho\psi, \psi\rangle\frac{1}{a}\int_0^a e^{iHt}|\phi\rangle\langle\phi|e^{-iHt}dt.$$

The last term converges weakly to zero, but its trace remains constant. Therefore it does not converge in trace norm and $K^\natural$ does not exist as a strong operator limit.

Notice that K happens to be a bounded operator on the Hilbert space $\mathscr{K}$ of Hilbert Schmidt operators on $\mathscr{H}$ and that e^{Zt} is a unitary group on $\mathscr{K}$.

PROPOSITION 2.2.

$$K^\natural(\rho) = \lim_{a\to\infty}\frac{1}{a}\int_0^a e^{-Zt}Ke^{Zt}\rho dt$$

exists as a limit in Hilbert-Schmidt norm for all $\rho \in \mathscr{K}$.

Proof. — We first note from Eq. (1.7) that

$$K_1(\rho) = X\rho$$

where X is a compact operator on $\mathscr{H}$. Hence

$$\frac{1}{a}\int_0^a e^{-Zt}K_1e^{Zt}\rho dt = \left\{\frac{1}{a}\int_0^a e^{iH_st}Xe^{-iH_st}dt\right\}\rho.$$

The term inside { } converges in operator norm as $a \to 0$ by [*12*], so $K_1^\natural$ exists. Similarly $K_4^\natural$ exists. Finally K_2 and K_3 are compact operators on $\mathscr{K}$ so $K_2^\natural$ and $K_3^\natural$ exist again by [*12*].

The disparity between this Hilbert-Schmidt result and that for the more physically relevant trace norm means that we have to proceed with some caution.

We let P be the orthogonal projection on $\mathscr{H}$ whose range is the discrete spectral subspace of H_s and let P_0 be the projection

$$P_0(\rho) = P\rho P$$

on V. We put $P_1 = 1 - P_0$ and let $V_i = P_iV$ for $i = 0, 1$. Since P commutes with H_s, P_0 commutes with Z and we put $Z_i = P_iZ$ for $i = 0, 1$. For any bounded operator L on V we put

$$L_{ij} = P_iLP_j .$$

LEMMA 2.3. — Both $K_1^\natural$ and $K_4^\natural$ exist as operators on V and

$$\lim_{\lambda\to 0} \sup_{0\leqslant\lambda^2 t\leqslant\tau_0} \| \exp\{(Z + \lambda^2 K)t\}\rho - \exp\{(Z + \lambda^2 K_1^\natural + \lambda^2 K_2 + \lambda^2 K_3 + \lambda^2 K_4^\natural)t\}\rho \|_{tr} = 0 \quad (2.1)$$

for all $\rho \in V$ and $\tau_0 \geqslant 0$.

Proof. — We can write

$$K_1(\rho) = X\rho, \qquad K_4(\rho) = \rho X^*$$

where X is a compact operator on $\mathscr{H}$. The operator norm limit

$$X^\natural = \lim_{a\to\infty} \frac{1}{a}\int_0^a e^{iH_s t}Xe^{-iH_s t}dt$$

exists by [*12*] so $K_1^\natural$ and $K_4^\natural$ exist as strong operator limits on V with

$$K_1^\natural(\rho) = X^\natural\rho, \qquad K_4^\natural(\rho) = \rho X^{\natural *}. \quad (2.2)$$

By [*12*] we also have

$$X^\natural = PX^\natural = X^\natural P \quad (2.3)$$

which implies that

$$P_0K_i^\natural = K_i^\natural P_0$$

for $i = 1, 4$. We define

$$K_1 + K_4 = A, \qquad K_2 + K_3 = B \quad (2.4)$$

and observe from Eqs. (1.8) and (1.9) that B is a compact operator on V.

To prove Eq. (2.1) we use the infinite Dyson expansion

$$e^{(Z+\lambda^2 A+\lambda^2 B)t} = e^{(Z+\lambda^2 A)t} + \lambda^2\int_{s=0}^t e^{(Z+\lambda^2 A)(t-s)}Be^{(Z+\lambda^2 A)s}ds + \lambda^4\int_{s=0}^t\int_{u=0}^s e^{(Z+\lambda^2 A)(t-s)}Be^{(Z+\lambda^2 A)(s-u)}Be^{(Z+\lambda^2 A)u}duds + \dots$$

Subtracting from this the similar expansion with A replaced by $A^\natural$ and putting

$$D(\lambda, t) = e^{(Z+\lambda^2 A)t} - e^{(Z+\lambda^2 A^\natural)t}$$

we obtain

$$e^{(Z+\lambda^2 A+\lambda^2 B)t} - e^{(Z+\lambda^2 A^\natural+\lambda^2 B)t} = D(\lambda, t)$$
$$+ \lambda^2 \int_{s=0}^{t} D(\lambda, t-s) B e^{(Z+\lambda^2 A)s} ds + \lambda^2 \int_{s=0}^{t} e^{(Z+\lambda^2 A^\natural)(t-s)} B D(\lambda, s) ds$$
$$+ \lambda^4 \int_{s=0}^{t} \int_{u=0}^{s} D(\lambda, t-s) B e^{(Z+\lambda^2 A)(s-u)} B e^{(Z+\lambda^2 A)u} du ds + \ldots \quad (2.5)$$

Now if $\rho \in V$ and

$$\beta(\lambda) = \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} \| D(\lambda, t)\rho \|$$

then $\beta(\lambda) \to 0$ as $\lambda \to 0$ by [*4*] [*12*]. If

$$\alpha(\lambda) = \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} \| D(\lambda, t) B \|$$

then it follows from the compactness of B that $\alpha(\lambda) \to 0$ as $\lambda \to 0$.

By Eq. (2.5) the left-hand side of Eq. (2.1) is bounded by

$$\beta(\lambda) + c\lambda^2 \{ \alpha(\lambda) + \beta(\lambda) \}/1\,! + c^2\lambda^4 \{ 2\alpha(\lambda) + \beta(\lambda) \}/2\,! + \ldots$$

which converges to zero as $\lambda \to 0$.

The value of the above reduction resides in the fact that by Eqs. (1.4) and (2.2)

$$e^{(Z+\lambda^2 A^\natural)t}\rho = e^{(-iH_s+\lambda^2 X^\natural)t} \rho e^{(iH_s+\lambda^2 X^{\natural *})t} \quad (2.6)$$

where $X^\natural$ satisfies Eq. (2.3). We now restrict attention to the time evolution within the subspace V_0 of bound states. The assumption of absolute continuity in the following lemma is known to be satisfied very generally [*13*].

LEMMA 2.4. — If $0 \leqslant \tau_0 < \infty$ and $\rho \in V_0$ then

$$\lim_{\lambda \to 0} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} \| P_0 e^{(Z+\lambda^2 A^\natural+\lambda^2 B)t}\rho - e^{(Z+\lambda^2 A^\natural+\lambda^2 B_{00})t}\rho \|_{tr} = 0$$

provided H_s has no singular continuous spectrum.

Proof. — We use the finite expansion

$$e^{(Z+\lambda^2 A^\natural+\lambda^2 B)t} = e^{(Z+\lambda^2 A^\natural+\lambda^2 B_{00})t}$$
$$+ \lambda^2 \int_{s=0}^{t} e^{(Z+\lambda^2 A^\natural+\lambda^2 B_{00})(t-s)} (B_{01} + B_{11} + B_{10}) e^{(Z+\lambda^2 A^\natural+\lambda^2 B_{00})s} ds$$
$$+ \lambda^4 \int_{s=0}^{t} \int_{u=0}^{s} e^{(Z+\lambda^2 A^\natural+\lambda^2 B)(t-s)} (B_{01} + B_{11} + B_{10})$$
$$. \, e^{(Z+\lambda^2 A^\natural+\lambda^2 B_{00})(s-u)} (B_{01} + B_{11} + B_{10}) e^{(Z+\lambda^2 A^\natural+\lambda^2 B_{00})u} du ds .$$

Multiplying on the left and right by P_0 and using the fact that Z, $A^\natural$, B_{00} all commute with P_0 we obtain the crude estimate

$$\| P_0 e^{(Z+\lambda^2 A^\natural + \lambda^2 B)t} P_0 - P_0 e^{(Z+\lambda^2 A^\natural + \lambda^2 B_{00})t} P_0 \|$$
$$\leqslant c\lambda^4 \int_{s=0}^{t} \int_{u=0}^{s} \| (B_{01} + B_{11}) e^{(Z+\lambda^2 A^\natural + \lambda^2 B_{00})(s-u)} B_{10} \| \, duds$$
$$\leqslant c\lambda^4 \int_{s=0}^{t} \int_{u=0}^{s} \| B e^{(Z+\lambda^2 A^\natural)(s-u)} B_{10} \| \, duds .$$
$$\leqslant c \int_{s=0}^{t} \int_{u=0}^{s} \| B e^{(\lambda^{-2} Z + A^\natural)(s-u)} B_{10} \| \, duds .$$

By the dominated convergence theorem it is therefore sufficient to show that for all $\tau > 0$

$$\lim_{\lambda \to 0} \| B e^{(\lambda^{-2} Z + A^\natural)\tau} B_{10} \| = 0$$

and by the compactness of B even enough to show that for all ρ in some dense subset of V_1 and all $\tau > 0$

$$\lim_{\lambda \to 0} \| B e^{(\lambda^{-2} Z + A^\natural)\tau} \rho \| = 0 .$$

The space V_1 is generated by vectors $\rho = | \phi \rangle \langle \psi |$ where at least one of ϕ, ψ (we henceforth assume it is ϕ) lies in $(1 - P)\mathcal{H}$. By Eqs. (1.8), (1.9) and (2.4) B has the abstract form

$$B(\rho) = \int_{\Omega} C'_\omega \rho C^*_\omega d\omega$$

where C_ω and C'_ω are compact operators on $\mathcal{H}$ and

$$\int_{\Omega} \| C'_\omega \| \, \| C_\omega \| \, d\omega < \infty .$$

Hence for all ρ of the above form

$$\| B e^{(\lambda^{-2} Z + A^\natural)\tau} \rho \|$$
$$\leqslant \int_{\Omega} \| C'_\omega e^{(-i\lambda^{-2} H_s + X^\natural)\tau} \phi \| \, \| C_\omega e^{(-i\lambda^{-2} H_s + X^\natural)\tau} \psi \| \, d\omega$$
$$\leqslant \int_{\Omega} \| C'_\omega e^{(-i\lambda^{-2} H_s + X^\natural)\tau} \phi \| \, \| C_\omega \| \, e^{\| X^\natural \| \tau} \| \psi \| \, d\omega$$
$$= c \int_{\Omega} \| C'_\omega e^{-i\lambda^{-2} H_s \tau} \phi \| \, \| C_\omega \| \, d\omega$$

by Eq. (2.3) and the fact that $\phi \in (1 - P)\mathcal{H}$.

By the Lebesgue dominated convergence theorem we are finally left with proving that

$$\lim_{\sigma \to \infty} \| C'_\omega e^{-iH_s \sigma} \phi \| = 0 .$$

This is a consequence of the compactness of C'_ω and the fact that ϕ lies in $(1 - P)\mathscr{H}$, which is the absolutely continuous spectral subspace of H_s since that operator is assumed to have no singular continuous spectrum.

THEOREM 2.5. — If $\rho \in V_0$, $0 \leqslant \tau_0 < \infty$ and H_s has no singular continuous spectrum then

$$\lim_{\lambda \to 0} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} \| P_0 e^{(Z + \lambda^2 K)t} \rho - e^{(Z + \lambda^2 C)t} \rho \| = 0 \tag{2.7}$$

where the bounded operator $C = (K_{00})^\natural$ commutes with Z and P_0.

Proof. — Putting Lemmas 2.3 and 2.4 together yields

$$\lim_{\lambda \to 0} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} \| P_0 e^{(Z + \lambda^2 K)t} \rho - e^{(Z + \lambda^2 C_1)t} \rho \| = 0 \tag{2.8}$$

where

$$C_1 = A^\natural + B_{00} .$$

Since $\rho \in V_0$ and Z, $A^\natural$ and B_{00} all commute with P_0 we can replace C_1 in Eq. (2.8) by

$$C_2 = (A^\natural)_{00} + B_{00} .$$

The restriction of Z to V_0 has pure point spectrum so $C \equiv C_2^\natural$ exists by [*4*] and we can replace C_1 in Eq. (2.8) by C. Moreover

$$C = C_2^\natural = \{ (A^\natural)_{00} \}^\natural + (B_{00})^\natural = (A_{00})^\natural + (B_{00})^\natural$$
$$= \{ (A + B)_{00} \}^\natural = \{ K_{00} \}^\natural . \tag{2.9}$$

Eq. (2.7) may be rewritten in the interaction picture using the fact that C commutes with Z. The result is that if $\rho \in V_0$ and $0 \leqslant \tau < \infty$ then

$$\lim_{\lambda \to 0} \| P_0 e^{-\lambda^{-2} Z\tau} e^{(\lambda^{-2} Z + K)\tau} \rho - e^{C\tau} \rho \| = 0 . \tag{2.10}$$

The operator C, which describes both the second order energy level shifts and the decay of the bound states of the system, may be explicitly computed from Eqs. (2.9) and (1.6) in the same manner as in [*2*].

§ 3. LIMITATIONS OF THE THEORY

We have given a rigorous treatment of the decay of a quantum particle which corresponds to the usual calculations of second order perturbation theory. It is very difficult to obtain rigorous higher order results, although some progress is made in [*5*] [*15*].

The model we have described can be modified by using a relativistic quantum field of arbitrarily small positive mass *m*. The single particle Hamiltonian S of the field is given in the momentum space picture by

$$(Sf)^\wedge(k) = (m^2 + k^2)^{1/2} f^\wedge(k)$$

and estimates similar to Eq. (1.5) can be proved by the method of stationary phase [*8*] [*9*].

The removal of the space cut-off in the interaction term of the Hamiltonian H is, however, more difficult.

PROPOSITION 3.1. — If the space cut-off is omitted by putting $a = \infty$, then the integrals defining K are generally not norm absolutely convergent.

Proof. — We take the Fock representation by putting $d \equiv 0$ so that

$$h(x - y, t) = \frac{1}{2}\langle f_y, e^{iSt}f_x \rangle .$$

We put $m = M = 1$, $f = g$ and $\rho = |\psi\rangle\langle\psi|$. Then according to Eq. (1.7)

$$K_1(\rho) = -\frac{1}{2}\int_{t=0}^{\infty}\int_{x\in\mathbb{R}^3}\int_{y\in\mathbb{R}^3}\langle \psi, g_y \rangle\langle g_y, e^{iSt}g_x \rangle\langle g_y, e^{iSt}g_x \rangle \\ |e^{iSt}g_x\rangle\langle\psi|\, d^3x d^3y dt .$$

The integral whose finiteness is at stake is therefore

$$I = \int_{t=0}^{\infty}\int_{x\in\mathbb{R}^3}\int_{y\in\mathbb{R}^3} |\langle \psi, g_y \rangle|\,|\langle g_y, e^{iSt}g_x \rangle|^2\, d^3x d^3y dt .$$

Since

$$\int_{\mathbb{R}^3} |g_x\rangle\langle g_x|\, d^3x = G$$

is a bounded operator which commutes with space translations and therefore with S

$$\begin{aligned} I &= \int_{t=0}^{\infty}\int_{y\in\mathbb{R}^3} |\langle \psi, g_y \rangle|\,\langle Ge^{-iSt}g_y, e^{-iSt}g_y \rangle\, d^3y dt \\ &= \int_{t=0}^{\infty}\int_{y\in\mathbb{R}^3} |\langle \psi, g_y \rangle|\,\langle Gg_y, g_y \rangle\, d^3y dt = \infty \end{aligned}$$

because the integrand is independent of t.

The above Proposition does not prove the non-existence of a suitable operator K but does suggest that considerably different techniques are needed to deal with the problem without the space cut-off.

We finally comment that one may solve certain problems similar to those of this paper with a reservoir of self-interacting particles provided again that the self-interaction has a space cut-off [7].

§ 4. COUPLING BETWEEN DISTANT POTENTIAL WELLS

One can study the evolution of a particle attracted by two widely separated potential wells and simultaneously coupled to an infinite reservoir by choosing the system Hamiltonian H_s to be

$$H_s = -\Delta + V_\mu$$

on $\mathscr{H} = L^2(\mathbb{R}^3)$, where V is a suitably regular potential, $\| a \| = 1$ and

$$V_\mu(x) = V(\mu^{-1}a - x) + V(\mu^{-1}a + x).$$

As $\mu \to 0$ the point spectrum of H_s becomes doubly degenerate and the eigenvectors of H_s, which are either symmetric or antisymmetric with respect to the operator

$$(S\psi)(x) = \psi(-x)$$

become less and less localised in space.

The time evolution with respect to the Hamiltonian H of Eq. (1.1) when λ and μ are both very small, should be studied by letting them converge to zero simultaneously but one could not hope to obtain a physically relevant answer without first removing the space cut-off. For this reason we consider a modified model which retains many of the features of H and H_s, at least as far as the discrete spectrum of the latter is concerned.

We take the system space to be

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_0$$

where the spaces $\mathscr{H}_0$ represent the separate potential wells. $\mathscr{H}$ is provided with a Hamiltonian H_0 with discrete spectrum. Its eigenvalues $\{ \lambda_n \}_{n=1}^{\infty}$ are supposed to have multiplicity two and to be strictly increasing with n, and to be associated with eigenvectors $e_n \oplus 0$ and $0 \oplus e_n$ forming an orthonormal basis of $\mathscr{H}$. The two wells are coupled by a Hamiltonian H_1 defined by

$$H_1(e_n \oplus 0) = -\beta_n(0 \oplus e_n)$$
$$H_1(0 \oplus e_n) = -\beta_n(e_n \oplus 0)$$

where $0 < \beta_n \leqslant \beta$ for all n so that H_1 is a bounded operator. The system Hamiltonian

$$H_s = H_0 + \mu H_1 \tag{4.1}$$

is then symmetric with respect to S where

$$S(\phi \oplus \psi) = \psi \oplus \phi$$

and the eigenvectors of H_s fall into symmetric and antisymmetric pairs

$$H_s(e_n \oplus e_n) = (\lambda_n - \mu\beta_n)(e_n \oplus e_n)$$
$$H_s(e_n \oplus - e_n) = (\lambda_n + \mu\beta_n)(e_n \oplus - e_n).$$

We shall also use the symmetry T on $\mathscr{H}$ defined by

$$T(\phi \oplus \psi) = \phi \oplus (-\psi)$$

and the induced symmetry $\bar{T}$ on V defined by

$$\bar{T}(\rho) = T\rho T^*$$

although T does not commute with H_s.

The system is coupled to an infinite free reservoir of the type already described, the Hamiltonian on $\mathscr{H} \otimes \mathscr{F}$ being

$$H = H_s \otimes 1 + 1 \otimes H_B + \lambda\{ A_{-1} \otimes \Phi(f_\mu^{-1}) + A_1 \otimes \Phi(f_\mu^1)\} \quad (4.2)$$

where A is a bounded self-adjoint operator on $\mathscr{H}_0$, f lies in Schwartz space, $\| a \| = 1$

$$A_{-1} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$$

and

$$f_\mu^{-1}(x) = f(x - \mu^{-1}a), \qquad f_\mu^1(x) = f(x + \mu^{-1}a)$$

for all $x \in \mathbb{R}^3$. We study the system in the limit as $\lambda \to 0$ and $\mu \to 0$ simultaneously. As $\mu \to 0$ the two system subspaces $\mathscr{H}_0$ become more weakly coupled to each other via H_1, and also are coupled to more remote parts of the quantum field. We show that the asymptotic evolution of the system depends on the relative speed at which λ and μ converge to zero. If we put $\mu = \lambda^\beta$ where $0 < \beta < \infty$ then the time evolution of the system changes discontinuously at $\beta = 2$. The significance of this « dynamical phase transition » is explained at the end of the section.

We start by showing that the results of [2] are uniform with respect to μ.

LEMMA 4.1. — If $\rho \in V$ and $0 \leqslant \tau_0 < \infty$ and

$$\rho_\lambda(t) = \mathrm{tr}_{\mathscr{F}}\, [e^{-iHt}(\rho \otimes v)e^{iHt}]$$

then

$$\lim_{\lambda \to 0} \sup_{|\mu| \leqslant 1} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} \| \rho_\lambda(t) - \exp\{ (Z_\mu + \lambda^2 K_\mu)t \} \rho \|_{\mathrm{tr}} = 0 \quad (4.3)$$

where the operators Z_μ and K_μ on V are defined by

$$Z_\mu(\rho) = -\, i[H_s, \rho]$$

and Eqs. (4.5-4.10) respectively.

Proof. — We show that the estimates of [2] are uniform with respect to μ by examining the two ways in which μ enters the calculations. The first is via H_s for which we use only the estimate

$$\| e^{iH_s t} \| = 1$$

which is valid for all μ and t. The second is via the field two-point functions

$$h_\mu^{ij}(t) = \langle \Phi(e^{iSt} f_\mu^i)\Phi(f_\mu^j)\Omega, \Omega \rangle .$$

By an analysis identical to that used for Eq. (1.5) one may obtain the estimate

$$| h_\mu^{ij}(t) | \leqslant (1 + | t |)^{-3/2} \quad (4.4)$$

for all $\mu, t \in \mathbb{R}$ and $i, j = \pm 1$. The analysis of [2] [3] now establishes Eq. (4.3) with

$$K_\mu = K_{1\mu} + K_{2\mu} + K_{3\mu} + K_{4\mu} \quad (4.5)$$

and

$$K_{1\mu}(\rho) = - \sum_{i,j=\pm 1} \int_{t=0}^{\infty} A_{i\mu}^t A_j \rho h_\mu^{ij}(t) dt \tag{4.6}$$

$$K_{2\mu}(\rho) = \sum_{i,j=\pm 1} \int_{t=0}^{\infty} A_j \rho A_{i\mu}^t h_\mu^{ij}(t) dt \tag{4.7}$$

$$K_{3\mu}(\rho) = \sum_{i,j=\pm 1} \int_{t=0}^{\infty} A_{i\mu}^t \rho A_j h_\mu^{ji}(-t) dt$$

$$K_{4\mu}(\rho) = - \sum_{i,j=\pm 1} \int_{t=0}^{\infty} \rho A_j A_{i\mu}^t h_\mu^{ji}(-t) dt \tag{4.9}$$

and

$$A_{i\mu}^t = e^{iH_s t} A_i e^{-iH_s t}. \tag{4.10}$$

Note that K_μ depends on μ both through h and through H_s. For the rest of the section we suppose that $\mu = \lambda^\beta$ where $0 < \beta < \infty$.

LEMMA 4.2. — If $\rho \in V$ and $0 \leqslant \tau_0 < \infty$ then

$$\lim_{\lambda \to 0} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} \| \rho_\lambda(t) - \exp\{(Z_0 + \lambda^\beta Z_1 + \lambda^2 K)t\} \rho \|_{tr} = 0 \tag{4.11}$$

where

$$Z_j(\rho) = - i[H_j, \rho]$$

for $j = 0, 1$, the operator K, which is independent of μ, is given by Eqs. (4.13-4.17) and commutes with $\bar{T}$.

Proof. — By using the infinite Dyson expansion with the estimate

$$\| e^{(Z_0 + \lambda^\beta Z_1)t} \| = 1$$

valid for all λ, t we find that in order to replace K_μ by K in Eq. (4.3) it is sufficient to prove that

$$\lim_{\mu \to 0} \| K_\mu - K \| = 0. \tag{4.12}$$

We examine each of the terms K_μ separately. By Eq. (4.4) and pointwise convergence

$$\lim_{\mu \to 0} \int_{-\infty}^{\infty} | h_\mu^{ij}(t) | dt = 0$$

if $i \neq j$, while if $i = j$

$$h_\mu^{ij}(t) = h(t) \equiv \langle \Phi(e^{iSt} f) \Phi(f) \Omega, \Omega \rangle.$$

Since H_1 is bounded

$$\lim_{\mu \to 0} \| e^{iH_s t} - e^{iH_0 t} \| = 0$$

for all $t \in \mathbb{R}$ so

$$\lim_{\mu \to 0} \| A_{i\mu}^t - A_i^t \| = 0$$

where

$$A_j^t = e^{iH_0 t} A_j e^{-iH_0 t}.$$

It follows by the Lebesgue dominated convergence theorem that

$$\lim_{\mu \to 0} \| K_{i\mu} - K_i \| = 0$$

where

$$K_1(\rho) = -\int_{t=0}^{\infty} (A_{-1}^t A_{-1} \rho + A_1^t A_1 \rho) h(t) dt \tag{4.13}$$

$$K_2(\rho) = \int_{t=0}^{\infty} (A_{-1} \rho A_{-1}^t + A_1 \rho A_1^t) h(t) dt \tag{4.14}$$

$$K_3(\rho) = \int_{t=0}^{\infty} (A_{-1}^t \rho A_{-1} + A_1^t \rho A_1) h(-t) dt \tag{4.15}$$

$$K_4(\rho) = -\int_{t=0}^{\infty} (\rho A_{-1} A_{-1}^t + \rho A_1 A_1^t) h(-t) dt. \tag{4.16}$$

Eq (4.12) is therefore valid with

$$K = K_1 + K_2 + K_3 + K_4. \tag{4.17}$$

Finally since A_i commute with T and H_0 also commutes with T, it may be seen that K commutes with $\bar{T}$.

THEOREM 4.3. — If $2 < \beta < \infty$ and $0 \leqslant \tau_0 < \infty$ and $\rho \in V$ then

$$\lim_{\lambda \to 0} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} \| \rho_\lambda(t) - \exp \{ (Z_0 + \lambda^2 K^\natural) t \} \rho \|_{tr} = 0$$

where the bounded operator

$$K^\natural = \lim_{a \to \infty} \frac{1}{a} \int_0^a e^{-Z_0 t} K e^{Z_0 t} dt$$

on V commutes with Z_0. Moreover both Z_0 and $K^\natural$ commute with $\bar{T}$.

Proof. — Since

$$e^{(Z_0 + \lambda^\beta Z_1 + \lambda^2 K)t} - e^{(Z_0 + \lambda^2 K)t} = \lambda^\beta \int_{s=0} e^{(Z_0 + \lambda^\beta Z_1 + \lambda^2 K)(t-s)} Z_1 e^{(Z_0 + \lambda^2 K)s} ds$$

and

$$\| e^{(Z_0 + \lambda^\beta Z_1 + \lambda^2 K)s} \| \leqslant e^{\lambda^2 \|K\| |s|}$$
$$\| e^{(Z_0 + \lambda^2 K)s} \| \leqslant e^{\lambda^2 \|K\| |s|}$$

for all $\lambda, s \in \mathbb{R}$

$$\| e^{(Z_0 + \lambda^\beta Z_1 + \lambda^2 K)t} - e^{(Z_0 + \lambda^2 K)t} \| \leqslant c\lambda^\beta t \leqslant c\lambda^{\beta - 2} \tau_0$$

which converges to zero as $\lambda \to 0$. We may therefore drop the term $\lambda^\beta Z_1$ in Eq. (4.11). Since Z_0 has pure point spectrum, K exists by [*4*] and we may replace K by $K^\natural$ in Eq. (4.11).

The case $0 < \beta < 2$ is more difficult because the term $\lambda^\beta Z_1$ is no longer negligible. We shall rely mainly on the following properties of Z_0 and Z_1. They commute and $\exp\{(Z_0 + \lambda^\beta Z_1)t\}$ is an isometry for all λ, $t \in \mathbb{R}$. Also Z_0 and Z_1 have joint pure point spectrum in the sense that the linear span of the set of simultaneous eigenvectors is dense in V.

LEMMA 4.4. — If $0 < \beta < 2$ then for all $\alpha \in \mathbb{R}^2$ the strong operator limit

$$P_\alpha = \lim_{\lambda \to 0} \lambda^2 \int_0^{\lambda^{-2}} \exp\{(i\alpha_0 + i\lambda^\beta\alpha_1 - Z_0 - \lambda^\beta Z_1)v\} dv \qquad (4.18)$$

exists and defines a projection P_α with range

$$V_\alpha = \{\rho \in V : Z_0\rho = i\alpha_0\rho \text{ and } Z_1\rho = i\alpha_1\rho\}.$$

If $\rho \in V$ and $0 \leqslant \tau_0 < \infty$ then

$$\lim_{\lambda \to 0} \sup_{0 \leqslant \sigma \leqslant \tau_0} \| \sigma P_\alpha\rho - \lambda^2 \int_0^{\lambda^{-2}\sigma} \exp\{(i\alpha_0 + i\lambda^\beta\alpha_1 - Z_0 - \lambda^\beta Z_1)v\}\rho \| dv = 0. \qquad (4.19)$$

Proof. — Since its right-hand side is uniformly bounded in norm as a function of λ, we need only prove Eq. (4.18) when applied to a state ρ lying in one of the subspaces V_γ. For such ρ

$$\lambda^2 \int_0^{\lambda^{-2}} \exp\{(i\alpha_0 + i\lambda^\beta\alpha_1 - Z_0 - \lambda^\beta Z_1)v\}\rho\, dv = \begin{cases} \rho & \text{if } \alpha = \gamma \\ \lambda^2\rho \dfrac{\exp i\{(\alpha_0 - \gamma_0)\lambda^{-2} + (\alpha_1 - \gamma_1)\lambda^{-2+\beta}\} - 1}{i(\alpha_0 - \gamma_0) + (\alpha_1 - \gamma_1)\lambda^\beta} \end{cases}$$

which converges to $\delta_{\alpha,\gamma}\rho$ as $\lambda \to 0$. The formula

$$P_\alpha\rho = \delta_{\alpha,\gamma}\rho$$

for all $\rho \in V_\gamma$ is sufficient to show both that P_α is a projection and that its range is V_α.

By uniform boundedness it is again sufficient to prove Eq. (4.19) when $\rho \in V_\gamma$. If $\alpha = \gamma$ then

$$\lambda^2 \int_0^{\lambda^{-2}\sigma} \exp\{(i\alpha_0 + i\lambda^\beta\alpha_1 - Z_0 - \lambda^\beta Z_1)v\}\rho\, dv = \sigma\rho$$

while if $\alpha \neq \gamma$

$$\begin{aligned} \| \lambda^2 \int_0^{\lambda^{-2}\sigma} & \exp\{(i\alpha_0 + \lambda^\beta\alpha_1 - Z_0 - \lambda^\beta Z_1)v\}\rho\, dv \| \\ &= \lambda^2 \|\rho\| \left| \frac{\exp i\sigma\{(\alpha_0 - \gamma_0)\lambda^{-2} + (\alpha_1 - \gamma_1)\lambda^{-2+\beta}\} - 1}{i(\alpha_0 - \alpha_1) + (\gamma_0 - \gamma_1)\lambda^\beta} \right| \\ &\leqslant 2\lambda^2 \|\rho\| \, |(\alpha_0 - \alpha_1) + (\gamma_0 - \gamma_1)\lambda^\beta|^{-1} \end{aligned}$$

which converges to zero as $\lambda \to 0$ uniformly with respect to σ.

LEMMA 4.5. — If K is a bounded operator on V then the strong operator limit

$$K^{\#} = \lim_{\lambda \to 0} \lambda^2 \int_0^{\lambda^{-2}} e^{-(Z_0 + \lambda^\beta Z_1)v} K e^{(Z_0 + \lambda^\beta Z_1)v} dv \qquad (4.20)$$

exists and is given formally by

$$K^{\#} = \sum_\alpha P_\alpha K P_\alpha . \qquad (4.21)$$

Moreover $K^{\#}$ commutes with Z_0 and Z_1. If (4.22)

$$J(\lambda, \sigma) = \lambda^2 \int_0^{\lambda^{-2}\sigma} e^{-(Z_0 + \lambda^\beta Z_1)v} K e^{(Z_0 + \lambda^\beta Z_1)v} dv$$

then

$$\lim_{\lambda \to 0} \sup_{0 \leqslant \sigma \leqslant \tau_0} \| \sigma K^{\#} \rho - J(\lambda, \sigma)\rho \| = 0$$

for all $\rho \in V$ and $0 \leqslant \tau_0 < \infty$.

Proof. — In order to prove the existence of $K^{\#}$ it is sufficient by the uniform boundedness in norm of the right-hand side of Eq. (4.20) to consider the case $\rho \in V_\alpha$. If $\rho \in V_\alpha$ then

$$\lambda^2 \int_0^{\lambda^{-2}} e^{-(Z_0 + \lambda^\beta Z_1)v} K e^{(Z_0 + \lambda^\beta Z_1)v} \rho dv$$
$$= \lambda^2 \int_0^{\lambda^{-2}} \exp \{ (i\alpha_0 + i\lambda^\beta \alpha_1 - Z_0 - \lambda^\beta Z_1)v \} (K\rho) dv$$

which converges to $P_\alpha K \rho$ as $\lambda \to 0$ by Lemma 4.3.

The formula

$$K^{\#}\rho = P_\alpha K \rho$$

for all $\rho \in V_\alpha$ establishes the validity of Eq. (4.21) when applied to any ρ in the dense subspace

$$\mathcal{D} = \text{lin} \{ V_\alpha : \alpha \in \mathbb{R}^2 \}$$

of V. By Eq. (4.21) we obtain

$$e^{Z_0 t} K^{\#} \rho = K^{\#} e^{Z_0 t} \rho, \qquad Z_1 K^{\#} \rho = K^{\#} Z_1 \rho$$

for all $\rho \in \mathcal{D}$ and hence all $\rho \in V$ by the boundedness of the operators involved. Hence $K^{\#}$ commutes with Z_0 and Z_1. Eq. (4.22) is deduced from Eq. (4.19).

THEOREM 4.6. — If $0 < \beta < 2$, $0 \leqslant \tau_0 < \infty$ and $\rho \in V$ then

$$\lim_{\lambda \to 0} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} \| \rho_\lambda(t) - \exp \{ (Z_0 + \lambda^\beta Z_1 + \lambda^2 K^{\#}) t \} \rho \|_{tr} = 0$$

where the operators Z_0, Z_1 and $K^{\#}$ all commute.

Proof. — We let $\mathscr{C}$ denote the Banach space of all norm-continuous V-valued functions on $[0, \tau_0]$, with the sup-norm $|||\ |||$.

Given $\rho \in V$ we define f and f_λ in $\mathscr{C}$ by

$$f(\tau) = e^{K^{\#}\tau}\rho$$
$$f_\lambda(\tau) = e^{-(\lambda^{-2}Z_0+\lambda^{-2+\beta}Z_1)\tau}e^{(\lambda^{-2}Z_0+\lambda^{-2+\beta}Z_1)\tau}\rho\,.$$

Since Z_0, Z_1 and $K^{\#}$ commute our problem is to prove that

$$\lim_{\lambda\to 0} |||\, f_\lambda - f \,||| = 0\,.$$

The equation

$$e^{(Z_0+\lambda^\beta Z_1+\lambda^2 K)t} = e^{(Z_0+\lambda^\beta Z_1)t} + \lambda^2 \int_{s=0}^{t} e^{(Z_0+\lambda^\beta Z_1)(t-s)}Ke^{(Z_0+\lambda^\beta Z_1+\lambda^2 K)s}ds$$

implies, upon substituting $\lambda^2 t = \tau$ and $\lambda^2 s = \sigma$, that

$$f_\lambda(\tau) = \rho + \int_{\sigma=0}^{\tau} e^{-(Z_0+\lambda^\beta Z_1)\lambda^{-2}\sigma}Ke^{(Z_0+\lambda^\beta Z_1)\lambda^{-2}\sigma}f_\lambda(\sigma)d\sigma$$

which may be rewritten as an operator equation

$$f_\lambda = \rho + \mathscr{H}_\lambda f_\lambda \tag{4.23}$$

on $\mathscr{C}$. Similarly the equation

$$f(\tau) = \rho + \int_{\sigma=0}^{\tau} K^{\#}f(\sigma)d\sigma$$

may be rewritten as an operator equation

$$f = \rho + \mathscr{H} f$$

on $\mathscr{C}$.

We show that $\mathscr{H}_\lambda$ converges in the strong operator topology on $\mathscr{C}$ to $\mathscr{H}$. Since $\mathscr{H}_\lambda$ are uniformly bounded in norm it is sufficient to show that

$$\lim_{\lambda\to 0} |||\, \mathscr{H}_\lambda g - \mathscr{H} g \,||| = 0$$

when g lies in the dense set of continuously differentiable functions from $[0, \tau_0]$ into V.

Integration by parts yields for such g

$$(\mathscr{H}_\lambda g)(\tau) = [J(\lambda, \sigma)g(\sigma)]_0^\tau - \int_{\sigma=0}^{\tau} J(\lambda, \sigma)g'(\sigma)d\sigma$$

which by Eq. (4.22) converges uniformly to

$$\tau K^{\#}g(\tau) - \int_{\sigma=0}^{\tau} \sigma K^{\#}g'(\sigma)d\sigma = \int_0^{\tau} K^{\#}g(\sigma)d\sigma = (\mathscr{H} g)(\tau)\,.$$

The strong convergence of $\mathscr{H}_\lambda$ to $\mathscr{H}$ as $\lambda \to 0$ implies that

$$\lim_{\lambda\to 0} ||| \mathscr{H}_\lambda^n g - \mathscr{H}^n g ||| = 0 \tag{4.24}$$

for all $g \in \mathscr{C}$ and integers $n \geqslant 0$. Since $\mathscr{H}_\lambda$ are integral operators of Volterra type the solution of Eq. (4.23) is

$$f_\lambda = \sum_{n=0}^{\infty} \mathscr{H}_\lambda^n \rho$$

where

$$|| \mathscr{H}_\lambda^n || \leqslant c^n/n\,! \tag{4.25}$$

for some constant c independent of n. Eqs. (4.24) and (4.25) imply that

$$\lim_{\lambda\to 0} f_\lambda = \sum_{n=0}^{\infty} \mathscr{H}^n \rho = f$$

as required.

We say that a state $\rho \in V$ is an asymptotic equilibrium state if for all $0 \leqslant \tau_0 < \infty$

$$\lim_{\lambda\to 0} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} || \rho_\lambda(t) - \rho || = 0 .$$

THEOREM 4.7. — If $0 < \beta < 2$ the asymptotic equilibrium states ρ are precisely those for which

$$Z_0\rho = Z_1\rho = K^\#\rho = 0 . \tag{4.26}$$

If $2 < \beta < \infty$ they are those for which

$$Z_0\rho = K^\natural\rho = 0 . \tag{4.27}$$

Proof. — Suppose $0 < \beta < 2$. By Theorem 4.6 every solution of Eq. (4.26) is an asymptotic equilibrium state. Conversely if ρ is an asymptotic equilibrium state then

$$\lim_{\lambda\to 0} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} || e^{(Z_0 + \lambda^\beta Z_1 + \lambda^2 K^\#)t}\rho - \rho || = 0 . \tag{4.28}$$

This implies the weaker result

$$\lim_{\lambda\to 0} \sup_{0 \leqslant t \leqslant t_0} || e^{(Z_0 + \lambda^\beta Z_1 + \lambda^2 K^\#)t}\rho - \rho || = 0$$

for all $0 \leqslant t_0 < \infty$, which can be reduced to

$$\sup_{0 \leqslant t \leqslant t_0} || e^{Z_0 t}\rho - \rho || = 0 .$$

Therefore $Z_0\rho = 0$, and Eq. (4.28) can be rewritten in the form

$$\lim_{\lambda\to 0} \sup_{0 \leqslant \lambda^2 t \leqslant \tau_0} || e^{(\lambda^\beta Z_1 + \lambda^2 K^\#)t}\rho - \rho || = 0$$

which implies

$$\lim_{\lambda \to 0} \sup_{0 \leq \lambda^{2-\beta} t \leq \tau_0} \| e^{(Z_1 + \lambda^{2-\beta} K^{\#})t} \rho - \rho \| = 0 .$$

Repeating the argument we obtain $Z_1 \rho = 0$ and then $K^{\#} \rho = 0$.

The case $2 < \beta < \infty$ is derived similarly from Theorem 4.3.

Theorems 4.3, 4.6 and 4.7 show that the asymptotic evolution of the system has different forms in the cases $0 < \beta < 2$ and $2 < \beta < \infty$; the special cases $\beta = 0$ and $\beta = 2$ can be solved by similar methods. Physically the result may be summarised by the statement that if $2 < \beta < \infty$ the interaction between the two potential wells has no effect for times of order λ^{-2}, while if $0 < \beta < 2$ it has a direct effect and also causes a modification to the decay resulting from the system-reservoir interaction. The result is that the equilibrium states in the two cases are quite distinct.

The symmetry $\bar{T}$, which appears only when $2 < \beta < \infty$, may be regarded as a superselection rule forbidding the appearance of superpositions between the states of the two potential wells. Thus the calculations we have presented are relevant to the Einstein-Podolski-Rosen and Schrodinger's cat « paradoxes » [*10*] and to the problems associated with the existence of stable isomers in quantum chemistry [*16*] [*19*] in that they show that « unphysical » superpositions of states of a complex closed system may be excluded if the system is considered as open and the various couplings have appropriate relative magnitudes.

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