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Two-body relativistic systems

by

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ABSTRACT. — The manifestly covariant hamiltonian formalism is displayed in view of application to a large class of interactions.

A relativistic generalization of central forces is more specially considered.

The relationship between the positions and the canonical variables is elucidated at least in the case of a solvable example. This example is a covariant generalization of the harmonic oscillator.

I. INTRODUCTION. NOTATIONS

Relativistic dynamics based upon second-order differential equations of motion has slowly emerged during these last years. It is sometimes called *Finitely Predictive Mechanics* because phase space has a finite number of dimensions and therefore, the evolution of a system can be, in principle, predicted when initial positions and velocities are known. In the old non-covariant approach [1] Currie-Hill conditions are to be satisfied in order to insure the relativistic invariance. In the manifestly covariant formulation [2] [3] which involves N proper times for N particles, the equations of motion are integrable and permit world lines to exist, provided the Predictivity Conditions [2] are satisfied. Since neither these conditions nor the Currie-Hill conditions are linear, it has been a long time almost impossible to construct explicit models with satisfactory physical features.

We have introduced [4] [5] [6] a covariant hamiltonian formalism and applied it for the construction of systems.

We call hamiltonians the (scalar!) generating functions giving rise to the equations of motion. Accordingly we have as many hamiltonians as we have particles, since these equations, in a somehow redundant way,

involve all the proper times (or N more general parameters). The hamiltonians cannot be confused with the *energy* which is the time component of the conserved total linear momentum. So far we have always identified them with $1/2$ of the squared masses. Due to the Currie-Jordan-Sudarshan zero interaction Theorem [1] which can be given a covariant form [7], we know that the positions cannot be canonical. They cannot be arbitrarily related to the canonical variables, but must satisfy the *Position Equations* (to simplify we assume $N = 2$)

$$(1.1) \quad \{ H, x^{\alpha'} \} = \{ H', x^{\alpha} \} = 0.$$

We have previously indicated how a predictive relativistic system can be constructed:

i) Let us *a priori* start from an *Abstract hamiltonian formulation i. e.* consider H and H' as Poincaré invariant functions of the canonical variables, submitted to the condition

$$(1.2) \quad \{ H, H' \} = 0$$

without specifying the relationship between the positions and the *canonical* variables.

ii) Then specify this relationship by solving (1.1) with respect to x, x' in a suitable way. We have shown that this procedure, in which (1.2) plays the role of the predictivity condition, automatically provides us with a 2nd order system of equations of motions:

$$(1.3) \quad \frac{d^2 x^{\alpha}}{d\tau^2} = \xi^{\alpha} \quad \frac{d^2 x^{\alpha'}}{d\tau'^2} = \xi^{\alpha'}$$

If by chance $\left(\frac{dx}{d\tau}\right)^2$ and $\left(\frac{dx'}{d\tau'}\right)^2$ appear to be constant in the motion, they are identified with the squared masses.

If, not, which is generally the case, an appropriate change of parameters is always possible. Eq. (1.3) gets transformed into another system, and the constancy of masses is restored.

More precisely we define the affine parameters σ and σ' by

$$(1.4) \quad \frac{d\tau}{d\sigma} = \sqrt{\frac{2H}{v^2}} \quad \frac{d\tau'}{d\sigma'} = \sqrt{\frac{2H}{v'^2}}$$

$$(1.5) \quad \text{where} \quad v = dx/d\tau \quad , \quad v' = dx'/d\tau'$$

$$(1.6) \quad \text{Then} \quad \left(\frac{dx}{d\sigma}\right)^2 = 2H \quad \text{and} \quad \left(\frac{dx'}{d\sigma'}\right)^2 = 2H'$$

and they are identified with the squared masses.

The change from τ, τ' to σ, σ' will be referred to as the *mass-parameter correction*. It is always possible, on the invariant subsystem defined by the

conditions $H > 0$, $H' > 0$ in phase space. So, the question of mass constancy is settled [8].

In view of constructing a system, we made a first attempt by adding to the free-particle hamiltonians an interacting term similar to that of a harmonic oscillator [5] [6]. At his time we had some difficulties with the interpretation, so we had first abandoned this model and worked with Hamilton Jacobi (H-J) co-ordinates (allowed to vary on the real line). We have exhibited all the systems one can reach by this method [8], but without undertaking the study of a specified example (as it could be done easily, since, expressed in terms of H-J coordinates, the hamiltonians are *formally* similar to the free particle ones, and all the interaction is carried by the deviation of the positions from the H-J co-ordinates [9]). But this does not exhaust all the hamiltonian systems. In fact, although any hamiltonian predictive system admit such H-J-coordinates [10] in general this is true only *locally*. Thus the systems constructed as in ref. [8] and the systems constructed otherwise will not coincide globally in general, and their qualitative picture can be quite different.

Besides, it is not usual, in classical mechanics, to construct systems by using H-J-coordinates, and we think that in the beginning, analogy with a classical model is a precious guide. That is why we go back to our initial attempt and consider again hamiltonians with an interacting term. The canonical variables q, q' cannot be the positions because of the zero interaction theorem. But, in this paper we shall make them to differ from the positions « as little as possible » in some sense. This point of view permits to find the symmetries, first integrals, etc., by analogy with classical mechanics, and sometimes with one-body relativistic mechanics. Fortunately things are improved, and we are now able to find a satisfactory relation between q, q' and x, x' at least in the oscillator-like case.

In Section II we give results and formulae valid for a general type of interaction. Section IV is specially devoted to the case where the interacting term suggest a harmonic oscillator. A suitable solving of (1.1) actually leads to the qualitative features of the harmonic oscillator, namely we have a bound state, the relative motion being elliptic in the appropriate frame.

Notations

Space-time M_4 , signature + - - -.

One-particle phase space is $T(M_4)$, identified with the product of M_4 by the space of four-vectors. Two-particle phase space $T(M_4) \times T(M_4)$.

Ordinary co-ordinates in phase-space $x^\alpha, v^\alpha, x^{\alpha'}, v^{\alpha'}$.

Canonical co-ordinates $q^\alpha, p^\alpha, q^{\alpha'}, p^{\alpha'}$.

When possible, the Greek indices are omitted, for instance x stands for x^α , etc. Scalar product written in compact form : $v \cdot v'$ stands for $v^\alpha v'_\alpha$, etc. We write v^2 instead of $v \cdot v$, etc.

The equations of motion involve the parameters τ , τ' that is to say we have

$$v = dx/d\tau \quad , \quad v' = dx'/d\tau'$$

Note that the four vectors v , v' are not constrained. When v^2 and v'^2 are constants of the motion we identify $v = mu$, $v' = m'u'$, where m , m' are the masses, u , u' are the unit four-velocities.

Thus (*up to a power of c*) v , v' have the dimension of a mass. Necessarily τ and τ' have the dimension of a surface. In case of constant v^2 and v'^2 we have $\tau = s/m$, $\tau' = s'/m'$ (otherwise we manage to have $\sigma = s/m$, $\sigma' = s'/m'$ by the mass-parameter correction as seen above).

Hamiltonians H , H' . They are simply $v^2/2$ and $v'^2/2$ in the free particle case, and in general they have the dimension of a squared mass. Quantities of unusual dimension result from the use of unconstrained v , v' , and the fact that the masses are not taken *a priori* constant, but rather considered as constants of the motion. This is necessary in order to have a *symplectic* (thus even dimensional) single-particle phase-space without introducing an universal constant. We assume that q , q' have the dimension of a length, p , p' that of a mass.

We take $c = \hbar = 1$. We set $\partial_\alpha = \partial/\partial x^\alpha$, $\partial_{\alpha'} = \partial/\partial x^{\alpha'}$.

Contravariant vector fields of phase-space are identified with linear differential operators.

Duality of skew-symmetric tensors of M_4 is noted $*$. For instance

$$*M^{\mu\nu} = \frac{1}{2} \eta^{\mu\nu\rho\sigma} M_{\rho\sigma}$$

If a , b , c are four vectors

$$(a \wedge b \wedge c)^\alpha = \eta^\alpha_{\mu\nu\rho} a^\mu b^\nu c^\rho$$

Product of a tensor by a vector: for instance

$$(L \cdot M)_\alpha = L^\mu M_{\mu\alpha}$$

Indices are moved by the Minkowski metric $\eta_{\mu\nu}$.

We separate *external* and *internal variables* by

$$\begin{aligned} r &= x - x' \\ Q &= \frac{1}{2}(q + q') & z &= q - q' \\ P &= p + p' & y &= \frac{1}{2}(p - p') \end{aligned}$$

Angular momentum

$$(1.7) \quad M = Q \wedge P + z \wedge y = q \wedge p + q' \wedge p'$$

Applying the space projector

$$(1.8) \quad \Pi^\alpha_\beta = \delta^\alpha_\beta - P^\alpha P_\beta / P^2$$

to anything will be noted \sim .

Example

$$\tilde{z}^\alpha = \Pi^\alpha_\beta z^\beta$$

Standard Poisson brackets

$$(1.9) \quad \begin{aligned} \{q^\alpha, p_\beta\} &= \{q^{\alpha'}, p_{\beta'}\} = \delta^\alpha_\beta \\ \{q, q'\} &= \{p, p'\} = 0 \end{aligned}$$

Thus

$$(1.10) \quad \{Q^\alpha, P_\beta\} = \delta^\alpha_\beta \quad \{z^\alpha, y_\beta\} = \delta^\alpha_\beta$$

The other brackets of Q, P, z, y are zero.

\mathcal{P} = Poincaré Group.

II. THE « A PRIORI » HAMILTONIAN APPROACH

We start from the hamiltonians H and H', assuming that they are the free particle hamiltonians plus an interacting term. For the sake of simplicity let us add the same term to both.

So we have

$$(2.1) \quad H = \frac{1}{2} p^2 + V \quad H' = \frac{1}{2} p'^2 + V$$

For convenience we shall call V the *potential*, although it has not the dimension of energy (For a given system, anyway, it will be possible to divide V by a mass and obtain a quantity with the correct dimension). Poisson brackets are defined by the symplectic form $dq \wedge dp + dq' \wedge dp'$. In other words we postulate the standards P. B. (1.9).

Let X and X' be the vector fields generated by H and H' on phase-space. This means that for any phase-space function $f_{(q,p,q',p')}$ we define

$$(2.2) \quad Xf = \{f, H\} \quad X'f = \{f, H'\}.$$

In order to insure the predictivity condition $[X, X'] = 0$ we require Eq. (1.2).

The Hamilton equations of motion are analogous in form to Heisenberg equations. But we keep in mind that, when we shall go back to the ordinary co-ordinates of phase-space, q (resp. q') will be, in general, a function of all these co-ordinates, and not only of x, v (resp. x', v').

Therefore, the equations of motion involve all the parameters, and we have [6]

$$(2.3) \quad \begin{aligned} \partial q / \partial \tau &= \{q, H\} & \partial q' / \partial \tau &= \{q', H\} \\ \partial p / \partial \tau &= \{p, H\} & \partial p' / \partial \tau &= \{p', H\} \\ & & (\text{Resp. } \tau', H'). & \end{aligned}$$

From (2.2) and (2.3) we see that

$$(2.4) \quad X = \partial / \partial \tau \quad X' = \partial / \partial \tau'.$$

Moreover we require that V is invariant under exchange of particles and

Poincaré invariant. Since we shall manage later that the change from canonical to ordinary co-ordinates does not break Poincaré invariance, this invariance is expressed by saying that V is a function of six scalars

$$(2.5) \quad z^2, p^2, p'^2, z.p, z.p', p.p'$$

First we see that (1.2) becomes simply

$$(2.6) \quad \{y.P, V\} = 0.$$

Using instead of (2.5) the equivalent scalars

$$(2.7) \quad \tilde{z}^2, P^2, \tilde{y}^2, z.P, \tilde{z}.\tilde{y}, P.y$$

we find easily that

(2.8) THEOREM. — The most general possible V is a function of the five scalars

$$\tilde{z}^2, P^2, \tilde{y}^2, \tilde{z}.\tilde{y}, P.y$$

The quantities \tilde{z}, \tilde{y} are the *spatial relative variables*.

Note that

$$(2.9) \quad \{y.P, \tilde{z}\} = \{y.P, \tilde{y}\} = 0$$

But

$$(2.10) \quad y.P = H - H'$$

thus eq. (2.9) means the vanishing of $\{\tilde{z}, H - H'\}$ and $\{\tilde{y}, H - H'\}$. By (2.4) this means $(\partial/\partial\tau - \partial/\partial\tau')\tilde{z} = (\partial/\partial\tau - \partial/\partial\tau')\tilde{y} = 0$. Thus.

(2.11) THEOREM. — In the motion, the relative spatial variables depend on $\tau + \tau'$ only.

Constants of the motion

The constants of the motion (or first integrals) are characterized by having a vanishing Poisson bracket with both H and H' .

Example: always H and H' themselves.

From \mathcal{P} -invariance we have immediatly P and M . Also $y.P$ by (2.10). From $y.P$ and P^2 we find $P.p$ and $P.p'$. Since $P^2, P.p$ and $P.p'$ are constant in the motion we shall without trouble restrict the system by the condition

$$(2.12) \quad P^2 > 0 \quad P.p > 0 \quad P.p' > 0.$$

Thus Π will always exist and will be the projector onto the space like hyperplane orthogonal to P . This point legitimates the name « spatial variable » given to \tilde{z} and \tilde{y} .

The *angular momentum vector* is as usually

$$(2.13) \quad L = P.*M/|P|$$

with $|P| = \sqrt{P^2}$.

The (spacelike) direction of the 2-plane orthogonal to \mathbf{P} and \mathbf{L} remains constant in the motion. We shall call it the *Canonical orbital plane* because the spatial relative canonical co-ordinates \tilde{z} and \tilde{y} remain in this plane during the motion (Recall $\mathbf{P} \cdot \mathbf{M} = \mathbf{P} \cdot (\mathbf{z} \wedge \mathbf{y})$ thus $\mathbf{L} \perp$ to \tilde{z} and \tilde{y}).

The external co-ordinates

Define the center of mass by the formula which is well-known in the absence of interaction [11]

$$(2.14) \quad \Gamma = \frac{\mathbf{P} \cdot p\mathbf{q} + \mathbf{P} \cdot p'\mathbf{q}'}{P^2}$$

or equivalently

$$(2.15) \quad \Gamma = Q + \frac{y \cdot \mathbf{P}}{P^2} z$$

Although q, q', Γ, Q are not vectors, we shall use the same notations in the calculations.

The relation with angular momentum is obtained from

$$(2.16) \quad \mathbf{P} \cdot \mathbf{M}/P^2 = (\mathbf{P} \cdot \mathbf{Q}/P^2)\mathbf{P} + (\mathbf{P} \cdot \mathbf{z}/P^2)\mathbf{y} - \Gamma.$$

In the free particle case $(\mathbf{P} \cdot \mathbf{z}/P^2)\mathbf{y}$ and $(\mathbf{P} \cdot \mathbf{Q}/P^2)\mathbf{P}$ vary linearly in the proper times.

For a general choice of V according to (2.8) we can say nothing about $\mathbf{P} \cdot \mathbf{Q}$. We shall see later that for a certain type of V , $\mathbf{P} \cdot \mathbf{Q}$ will again be linear in proper times.

Projecting (2.16) we obtain

$$(2.17) \quad \widetilde{\mathbf{P} \cdot \mathbf{M}}/P^2 = (\mathbf{P} \cdot \mathbf{z}/P^2)\tilde{y} - \tilde{Q} - (y \cdot \mathbf{P}/P^2)\tilde{z}$$

Since the left-hand side of (2.17) is constant we see that the evolution of \tilde{Q} is explicitly determined by the motion of the relative variables. Since \mathbf{P} is constant we can say finally:

(2.18) THEOREM. — Except (perhaps) for $\mathbf{P} \cdot \mathbf{Q}$ the evolution of the external co-ordinates is explicitly given when the motion of the internal (or relative) co-ordinates z, y is known.

Principles for abstract interaction

It is clear that, in so far as the position equations (1.1) have not yet been solved, the system is not completely specified.

But many results can be obtained « abstractly », that is to say with no more information than the form of the hamiltonians. In some sense eq. (2.1) define an « abstract system ». By *abstract integration* we mean the solving

of eq. (2.3) regardless of its future interpretation in terms of x, x' . The classical method using first integrals can be applied provided one is aware of the fact that X and X' define a *two*-parameter flow in space-time. In other words X and X' define (at least *locally*) the 2-parameter group of « translations in proper-times » and solving (2.3) means to determine 2-dimensional surfaces, the orbits of this local group.

Since phase-space has sixteen dimensions *we can find at most 14 independent first integrals*. When we know them, the orbits solutions of (2.3) are known in geometric form, *i. e.* the parameters τ, τ' being eliminated. In order to determine also the curves orbit of X (resp. X') alone, it is necessary to introduce one more quantity constant on these curves. This fifteenth function cannot be also invariant by X' (resp. X).

So we have introduced [12]:

DEFINITION. — *A partial integral* relative to X (resp. X') is a phase-space function which is invariant by X (resp. X') but not by X' (resp. X). Its Poisson bracket with H (resp. H') vanishes but its bracket with H' (resp. H) does not. We see in (1.1), the positions x^α (resp. x'^α) must be partial integrals of X' (resp. X).

III. CENTRAL-LIKE POTENTIAL

We shall now assume that the « potential » V has the form

$$(3.1) \quad V = G(P^2)F(\tilde{z}^2)$$

which is acceptable by (2.8).

In so far as our canonical co-ordinates will differ from the ordinary ones only by some corrections due to the interaction, $-\tilde{z}^2$ is a covariant generalization of the spatial squared distance involved in non relativistic force laws [13].

Potentials of the above type will be called *central-like* because of their similarity with the central potential, of non relativistic dynamics.

Actually they have several properties of these classical potentials. From (3.1) we are going to derive some useful formulae. Half of the calculations are avoided by use of the symmetry of (3.1) under *particle exchange*, which leaves Q and P invariant, changes z and y into $-z$ and $-y$, etc.

First, remarking that $\{z, P\}$ and $\{z, \Pi\}$ vanish we calculate $\{z, H\}$ and find

$$(3.2) \quad Xz = p \quad X'z = -p'$$

like in the free-particle case.

It is practically useful to introduce

$$(3.3) \quad \theta = \frac{z \cdot P}{P \cdot p} \quad \theta' = -\frac{z \cdot P}{P \cdot p'}$$

Thus we have from (3.2)

$$(3.4) \quad X\theta = 1 \quad X'\theta' = 1.$$

This formula shows that θ (resp. θ') is canonically conjugated to H (resp. H'). By use of (3.2) one finds also

$$(3.5) \quad X'\theta = -P.p'/P.p \quad X\theta' = -P.p/P.p'.$$

It is remarkable that they are constant in the motion and have the same expression as for free particles.

Note that θ and θ' are equal, up to a factor which is a constant of the motion. Their evolution in time is determined by the evolution of $P.z$. Multiplying (3.2) scalarly by P and integrating with respect to the parameters (Recall (2.4)) one finds

$$(3.6) \quad P.z = P.p\tau - P.p'\tau' + \text{Const.}$$

This quantity is analogous to a sort of relative proptime (with weights). We are able to complete the result (2.18) about the evolution of the external co-ordinates.

Consider

$$X(Q.P) = \left\{ Q, \frac{1}{2}P^2 \right\} .P + \{ Q, V \} .P$$

Thus

$$X(Q.P) = \frac{1}{2}P.p + \{ Q, V \} .P$$

But from (3.1) and (1.10) we have

$$\{ Q, V \} = \{ Q, G \} F + G(dF/d\tilde{z}^2) \{ Q, \tilde{z}^2 \}$$

where

$$(3.7) \quad \{ Q, \tilde{z}^2 \} = -2(P.z/P^2)\tilde{z}^\alpha$$

Thus, in the *particular case* where the function G is taken constant, we have simply

$$X(Q.P) = \frac{1}{2}P.p \quad X'(Q.P) = \frac{1}{2}P.p'$$

The integration is straightforward and yields a linear evolution

$$(3.8) \quad Q.P = \frac{1}{2}P.p\tau + \frac{1}{2}P.p'\tau' + \text{Const}$$

Let us go back to the more general case (3.1) where G can depend on P^2 .

(3.9) THEOREM. — It is always possible to require that $\widetilde{x - q}$ and $\widetilde{x' - q'}$ belong to the canonical orbital plane.

Proof. — The most general $\widetilde{x' - q'}$ possible can be developed onto the vectors \tilde{y}, \tilde{z}, L .

But the coefficient relative to L is necessarily constant since:

$$X(q' - x') = Xq' = XQ - \frac{1}{2}p = \{Q, V\}$$

and

$$\{Q^z, V\} = 2F \frac{dG}{dP^2} P^z - 2G \left(\frac{dF}{d\tilde{z}^2} \right) \frac{P \cdot z}{P^2} \tilde{z}^z$$

Thus $\{\tilde{Q}, V\}$ is colinear with \tilde{z} and $X(q' - x')$ has the same property. Finally $X(q' - x') \cdot L = 0$. The constant quantity $(q' - x')$. L can always be chosen to be zero.

Considering now $\tilde{r} - \tilde{z}$ we find.

(3.10) CONSEQUENCE. — If we make this requirement, the physical spatial separation

$$\tilde{r} = \widetilde{x - x'}$$

stays in the canonical orbital plane. In other words, the relative spatial motion takes place in this plane that we have now the right to call simply the *Orbital Plane*.

Returning to the canonical co-ordinates, we have useful formulae for the evolution of the relative variables which span the orbital plane.

Projection of (3.2) gives

$$(3.11) \quad X\tilde{z} = \tilde{y}$$

Note that $\tilde{p} = -\tilde{p}' = \tilde{y}$.

Since $Xy = \{y, V\}$ we compute from (3.1)

$$(3.12) \quad Xy^z = -2G(P^2) \frac{dF}{d\tilde{z}^2} \tilde{z}^z$$

thus

$$(3.13) \quad X\tilde{y} = -2G \frac{dF}{d\tilde{z}^2} \tilde{z}$$

Recall that \tilde{z} and \tilde{y} depend on the sum $\tau + \tau'$ only.

Practically eq. (3.11) (3.13) give a second-order equation in terms of \tilde{z} . Since $\{P^2, \tilde{z}\} = 0$, finding the evolution of \tilde{z} is equivalent to the solving of a one-body plane classical problem.

IV. OSCILLATOR-LIKE POTENTIAL

In a previous Note [5] we have suggested a potential of the form

$$(4.1) \quad V = KP^2\tilde{z}^2 \quad K = \text{const.}$$

The advantage of (4.1) is that this expression is a polynomial in the canonical variables, thus its quantization is more easy. One could also take $V = \text{const.} |P| \tilde{z}^2$ because in this case the coupling constant has the same dimension as for the classical oscillator.

But we prefer, as we did in more recent articles [12] [14], to consider the potential

$$(4.2) \quad V = k\tilde{z}^2 \quad k = \text{const.} > 0$$

which is the most simple for the calculations we have in mind.

For the comparison with a classical model we may consider the motions corresponding to a certain value of P^2 , and identify $k/|P|$ with the non relativistic constant.

The abstract integration can be carried out, in principle, by use of first integrals.

Beside obvious integrals

$$P^\alpha, \quad M_{\mu\nu}, \quad y \cdot P$$

we have the relative inertia-momentum tensor

$$(4.3) \quad N_{\alpha\beta} = \tilde{y}_\alpha \tilde{y}_\beta + 2k\tilde{z}_\alpha \tilde{z}_\beta$$

Proof: either by computation or by the final remark of Section III: The evolution equation for \tilde{z} is the same as for a classical one-body problem, for which we know the constants of motion [15]. Note that N defined in (4.3) is not traceless, we have

$$(4.4) \quad N_\alpha^\alpha = -I/4$$

where [6]

$$(4.5) \quad I = P^2 + 4(H - H')^2/P^2 - 4(H + H')$$

it is proportional to the relative energy. The constancy of N_α^α shows that \tilde{z} describes an ellipse. The axes can be determined by looking for the eigenvectors of N in the orbital plane. If these eigenvectors have the form $\alpha\tilde{y} + \beta\tilde{z}$ one finds

$$(4.6) \quad (\beta/\alpha)^2 \tilde{y} \cdot \tilde{z} + (\beta/\alpha)(\tilde{y}^2 - 2k\tilde{z}^2) - 2k\tilde{z} \cdot \tilde{y} = 0$$

If one tries to deduce from $P, M, y \cdot P, N$ the initial values of q, p, q', p' one finds that they involve 13 independent quantities only.

Thus, one should now look for partial integrals.

In practice it is better to perform the separation into external and internal variables.

Then P is constant, \tilde{Q} is given by (2.17) as a function of $P \cdot z, \tilde{y}, \tilde{z}$ and first integrals. But $P \cdot z$ is given by (3.6), $Q \cdot P$ is given by (3.8) and $y \cdot P$ is constant.

Finally the only remaining problem is to determine the evolution of \tilde{z} and \tilde{y} .

In the present case (3.11) (3.13) become

$$(4.7) \quad X\tilde{z} = \tilde{y} \quad X\tilde{y} = -2k\tilde{z}$$

Define

$$(4.8) \quad \omega = \sqrt{2k}$$

Remembering (2.11) the integration is straightforward and yields

$$(4.9) \quad \tilde{z} = A \sin [\omega(\tau + \tau') + C] + B \cos [\omega(\tau + \tau') + C]$$

where C is a constant, A and B constant orthogonal vectors of the orbital plane.

By derivation of (4.9) with respect to τ we find

$$(4.10) \quad \tilde{y} = A\omega \cos [\omega(\tau + \tau') + C] - B\omega \sin [\omega(\tau + \tau') + C]$$

For the configuration where the sinus vanishes we have $\tilde{z} = \pm B$ and $\tilde{y} = \pm A\omega$. Thus by (4.3) (4.8)

$$(4.11) \quad N_{\alpha\beta} = 2k(A_\alpha A_\beta + B_\alpha B_\beta).$$

But both handsides of (4.11) are constants of the motion, thus (4.11) holds true for any configuration. \tilde{z} moves on an ellipse the axes of which have lengths $2|A|$ and $2|B|$.

Solution of the position equation

Now we have to solve eq. (1.1). It can be written on the form

$$(4.12) \quad X(x' - q') = -Xq' \quad (\text{resp. } X', x, q).$$

We have seen (Section III) that $Xq' = \{Q, V\}$. From (4.1) we have

$$\{Q, V\} = k \{Q, \tilde{z}^2\}.$$

Thus, by the important formula (3.7) we have finally

$$(4.13) \quad Xq' = -2k \frac{P \cdot \tilde{z}}{P^2}$$

The absence of component on the direction P is due to the fact that now, G in (3.1) is constant.

Since the right-hand side of (4.12) lays on the orbital plane, we can require that

$$x' - q' \text{ lays on the orbital plane}$$

(This is stronger than the requirement permitted by (3.9)).

So we postulate a relation of the following form

$$(4.14) \quad x' = q' + \frac{P \cdot p}{P^2} (\varphi \tilde{z} + \psi \tilde{y})$$

and, by exchange of particles

$$(4.15) \quad x = q - \frac{P \cdot p'}{P^2} (\varphi' \tilde{z} + \psi' \tilde{y})$$

where the scalar coefficients φ , ψ , φ' , ψ' are to be determined. The constant factors $P \cdot p/P^2$ and $P \cdot p'/P^2$ are put in order to make as simple as possible the calculations with θ and θ' . Provided φ , ψ , φ' , ψ' will be bounded, these factors will also provide a satisfactory potential theory limit ($P^2 \rightarrow \infty$).

We are not concerned with finding a large class of solutions, but rather we look for a plausible and tractable model.

So we assume that φ , ψ (resp. φ' , ψ') are functions of θ only (resp. θ' only). This apparently arbitrary requirement will be legitimated *a posteriori* by consideration of the equal-time surface.

Let us solve (4.12). We set

$$(4.16) \quad \dot{\varphi} = d\varphi/d\theta \quad \dot{\psi} = d\psi/d\theta$$

Insert (4.14) and (4.13) into (1.1), or equivalently (4.12). Take (3.4) and (4.7) into account.

This implies the vanishing of a combination of \tilde{z}^{α} and \tilde{y}^{α} . Identification of the coefficients to zero provides finally

$$(4.17) \quad \varphi + \dot{\psi} = 0$$

$$(4.18) \quad -2kP \cdot z/P^2 + (P \cdot p/P^2)(\dot{\varphi} - 2k\psi) = 0$$

Fortunately φ gets defined by (4.17). Putting this value into (4.18) one finds simply

$$(4.19) \quad \ddot{\psi} + 2k\psi = -2k\theta$$

Note that for the free case $k = 0$ eq. (4.19) becomes completely trivial and one retrieves the free-particle solution $x' = q'$ by a suitable choice of the integration constants.

On the contrary, for $k \neq 0$, we first note the particular obvious solution

$$(4.20) \quad \psi = -\theta, \quad \varphi = 1.$$

It corresponds to

$$(4.21) \quad x' = q' + (P \cdot p/P^2)\tilde{z} - (P \cdot z/P^2)\tilde{y}$$

By use of the identity

$$(4.22) \quad P \cdot p z - P \cdot z p = P \cdot p z - P \cdot \tilde{z} \tilde{y}$$

eq. (4.21) takes the form

$$(4.23) \quad x' = q' + P \cdot (p \wedge z)/P^2.$$

This partial integral of X has been mentioned in ref. [12].

This solution does not depend on k and has no physical meaning. But the general solution of (4.19) is the sum of (4.20) and the general solution of the homogeneous equation $\ddot{\psi} + 2k\psi = 0$.

Finally the general solution of (4.17) (4.19) is

$$(4.24) \quad \varphi = 1 - a\omega \cos(\omega\theta + \alpha)$$

$$(4.25) \quad \psi = a \sin(\omega\theta + \alpha) - \theta$$

with $\omega = \sqrt{2k}$, a and $\alpha = \text{const.}$

The solution which makes $\varphi(0)$ and $\psi(0)$ to vanish is

$$(4.26) \quad \varphi = 1 - \cos \omega\theta$$

$$(4.27) \quad \psi = \omega^{-1} \sin \omega\theta - \theta.$$

We have obviously then

$$(4.28) \quad \dot{\varphi}_{(0)} = \dot{\psi}_{(0)} = \varphi_{(0)} = \psi_{(0)} = 0$$

By symmetry under particle exchange we have analogous formulae for φ' and ψ' .

The solution (4.26) (4.27), to be inserted into (4.14), seems to be the most convenient. From now on we shall consider only this solution and discuss the system so defined.

Indeed from (4.26) (4.27) we have $x' \rightarrow q'$ when $k \rightarrow 0$ which is reasonable since the positions must become canonical variables when the coupling constant vanishes.

Moreover, in the general case $k \neq 0$, x' coincide with q' (resp. x , q) on the surface

$$(4.29) \quad (\Sigma) \equiv \mathbf{P} \cdot z = 0.$$

In fact, in so far as phase space is restricted by (2.12), (4.29) is equivalent to

$$\theta = 0 \quad \text{or} \quad \theta' = 0.$$

It will turn out below that (Σ) , which is \mathcal{P} -invariant and invariant under particle exchange, is the most convenient Cauchy surface in phase-space.

By the Cauchy-Kowalewski theorem, one could prove that (4.26) (4.27) gives the *unique* solution of (1.1) which makes x' to coincide with q' on (Σ) . Since we have taken $x - q$ and $x' - q'$ in the orbital plane, $x - x'$ will differ from z only by a vector of this plane which is orthogonal to \mathbf{P} .

Thus (4.29) is equivalent to

$$(4.30) \quad (\Sigma) \equiv \mathbf{P} \cdot r = 0.$$

In any rest-frame, *i. e.* a frame of reference which has its temporal direction parallel to \mathbf{P} , the condition (4.30) reduces to $r^0 = 0$, in order words $t = t'$.

Thus (Σ) appears as the set of the points of phase-space which exhibit equal times with respect to the rest-frames. Let us call (Σ) the *Equal Time Surface*.

Note that (Σ) plays a role in the non relativistic limit.

A look at (4.26) (4.27) shows that φ and ψ are of second order in θ .

Thus $x - q$ and $x' - q'$ are of second order in θ . This has the following consequence:

We know that v and v' are defined by [6]

$$(4.31) \quad v = \{x, H\} \quad v' = \{x', H'\}.$$

Then one finds that v (resp. v') reduce to p (resp. p') on (Σ) .

Similarly it turns out that

$$(4.32) \quad \{x^\alpha, x^\beta\} \quad , \quad \{x^\alpha, x^{\beta'}\} \quad , \quad \{x^{\alpha'}, x^{\beta'}\}$$

are of the 1st order in θ .

Thus the equal-time value of the Poisson brackets (4.32) is zero (*i. e.* they vanish on (Σ)).

Discussion of the motion

The formulae (4.26) (4.27), completed by particle exchange, inserted into (4.14) (4.15) define the positions.

Then v and v' can be computed easily by (4.31). We do not need to write their explicit expressions here, but they have the form

$$(4.33) \quad v = p + 0(\theta) \quad , \quad v' = p' + 0(\theta)$$

where $+ 0(\theta)$ means: « up to a phase-space function vanishing on (Σ) ». Eq. (4.14) (4.15) (4.33) define a change of variables which is invertible. The evolution of x and x' is determined since the evolution of all the canonical variables has been previously determined.

Since x and x' have been taken satisfying eq. (1.1) it would automatically turn out of explicit calculations that x depends on τ only (resp. x' , τ'). The same holds for v and $\xi = \{v, H\}$ (resp. v' and $\xi' = \{v', H'\}$). So we finally recover the world-lines.

In order to go into more details let us consider the relative co-ordinate. From (4.14) (4.15) we have

$$(4.34) \quad r = z + \frac{P \cdot p}{P^2} (\varphi \tilde{z} + \psi \tilde{y}) + \frac{P \cdot p'}{P^2} (\varphi' \tilde{z} + \psi' \tilde{y})$$

For a given motion $P \cdot p$, $P \cdot p'$, P^2 are constant, \tilde{z} and \tilde{y} stay bounded. φ , ψ , φ' , ψ' are given by (4.26) (4.27) and particle exchange. Thus, up to some terms which are periodic and bounded, we have

$$\begin{aligned} \psi &= -\theta + \dots \quad , \quad \psi' = -\theta' + \dots \\ r &= z + (P \cdot p / P^2) \theta \tilde{z} + (P \cdot p' / P^2) \theta' \tilde{z} + \dots \end{aligned}$$

But from (3.3) we have finally:

$$(4.35) \quad r = z + \text{periodic bounded terms.}$$

Since \tilde{z} is periodic and bounded in its motion, application of Π to (4.35)

shows that \tilde{r} itself is also periodic and remains bounded in the motion. This is characteristic of a *bound state*.

Rest frame description

So far we have used the general covariant formalism and we have treated τ and τ' independently. But, for a given motion with total linear momentum P we can consider, in space-time, the family of three-planes orthogonal to P . These three-planes intersect both world-lines. Among all the possible configurations x, x' , this slicing selects the configurations satisfying $P \cdot r = 0$, that is to say the *equal-time configurations*. The corresponding points in phase-space belong to (Σ) . The rest-frame description of the system consists in considering only the sequence of its equal-time configurations. This introduces a relation between τ and τ' . Whereas the history of the system in $(M_4 \times M_4)$ defines a 2-surface parametrized by τ and τ' (the « world-surface ») we now draw a curve on this surface by picking up a one-parameter set of couples x, x' .

The induced relation between τ and τ' is easy to find, by (4.30) (4.29) and (3.6) we have simply

$$(4.36) \quad P \cdot p \tau = P \cdot p' \tau' + \text{Const.}$$

So the parameters are equated, up to their « weights » and an additional constant.

The evolution of the equal-time configurations is described with the help of a single parameter.

The most natural is to use the « weighted average »

$$(4.37) \quad \bar{\tau} = \frac{1}{2} \left(\frac{P \cdot p}{|P|} \tau + \frac{P \cdot p'}{|P|} \tau' \right)$$

Returning to eq. (2.16) we see that in the equal-time description $\tilde{\Gamma}$ is constant.

Thus the center of mass (2.14) moves along a line parallel to P . Now q and q' can be replaced by x and x' since they coincide on (Σ) . Eq. (3.8) shows that $Q \cdot P$ varies linearly in $\bar{\tau}$. Since we are on (Σ) , Q is just $\frac{1}{2}(x + x')$.

Thus, in the rest-frame $Q \cdot P = |P| Q^0$

$$(4.38) \quad Q^0 = \frac{1}{2}(t + t') = \bar{\tau} + \text{Const.}$$

Since $r \cdot P$ remains now equal to zero, we only have to deal with \tilde{r} . But it now *coincide* with \tilde{z} .

It becomes easy to express $\tau + \tau'$ linearly in term of $\bar{\tau}$ by (4.36) (4.37) and to inject the result into the equation of relative motion (4.9) where \tilde{z} can now be replaced by \tilde{r} since we work at equal times.

Consider (2.15), replace q by x , z by r , then we have in the rest-frame

$$x^i = \Gamma^i + \left(\frac{1}{2} - \frac{y \cdot P}{P^2} \right) r^i$$

So we see that the particles move elliptically around their center of mass which moves uniformly.

V. CONCLUDING REMARKS

We are now provided with a completely solved model. Thus it will be possible to refer to it when considering more elaborated systems. Our opinion is that the use of equal-time conditions on (Σ) must be systematically employed in the future.

It seems that now, trying to construct a relativistic generalization of coulombian potential will be possible also.

We do not discuss here the quantization of our oscillator since we have already done it (in a naive way) in a previous work [6].

The detailed classical study presented here will probably help to illuminate the quantum mechanical work started there.

At this stage we already have some principles and tools for the making of a potential applicable to the description of quarks binding, since we know more about the classical system corresponding to coupled wave equations [16].

In so far as quantization is considered, we are aware of the fact that a relativistic treatment should account for particle creation.

However the 2-body systems are relevant, not only as providing an alternative to the Bethe-Salpeter or to the Quasi-Potential approach, but also because, when allowing creation or annihilation of particles it is necessary first to have an idea of what is an N-particle state. After all why should the particle be created on free states only?

If, as we believe, they are created on interacting states, it is of interest to construct explicit examples of N interacting particles. At least we had to begin with $N = 2$.

Our present method of quantization is only semi-relativistic, since the motion of the center of mass is first separated before quantization is performed, so what is actually quantized is a reduced system. This is the popular method widely employed for B.-S. equation and in the quasi-potential approach. A more accurate treatment will be needed some day for instance if one envisions to perform the relativistic construction of a sort of Fock space out of all the different N particle spaces (Second quantization without fields!).

To be a little bit less ambitious for the moment, let us say that it is now

reasonable to start with a completely relativistic quantization of 2-body systems, at least for a simple model like the oscillator.

So, the open question is to consider also wave functions which do not correspond to a fixed value of linear momentum and to define a satisfactory invariant scalar product.

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