ANNALES DE L'I. H. P., SECTION A

K. GAWEDZKI

Supersymmetries-mathematics of supergeometry

Annales de l'I. H. P., section A, tome 27, nº 4 (1977), p. 335-366
<http://www.numdam.org/item?id=AIHPA_1977__27_4_335_0>

Ann. Inst. Henri Poincaré,
Vol. XXVII, n° 4, 1977, p. 335-366.

Section A :
Physique théorique.

Supersymmetries-mathematics of supergeometry

by

K. GAWEDZKI
Department of Mathematical Methods of Physics,
Warsaw University, Hoza 74, 00-682 Warsaw, Poland

ABSTRACT. — Geometric backgrounds of Arnowitt-Nath-Zumino's supergauge extensions of general relativity are studied. A rigorous theory of supermanifolds and of their geometry is proposed. Classification of non-diffeomorphic supermanifolds is reduced to that of non-isomorphic vector bundles.

1. INTRODUCTION

Supersymmetries constitute a new kind of physical symmetries. The basic properties which distinguish them from usual symmetries are the following:

i) they mix fermions and bosons,

ii) they constitute a kind of internal symmetries combining non-trivially with kinematic (Poincaré or conformal) ones,

iii) they are infinitesimal symmetries which seem to have no global counterpart, at least in the traditional sense.

For the first time they were introduced by Volkov-Akulov [*17*] and independently by Wess-Zumino [*18*]. Similarly to the case of traditional symmetries there are two mathematical aspects of supersymmetries: an algebraic and a geometric one.

As far as algebra is concerned it appears that the theory of Lie algebras usually used to describe infinitesimal symmetries is not sufficient. Instead a generalized theory of Z_2-graded Lie algebras must be used. Such a theory has been developed in the recent years to a quite satisfactory level, at least as far as applications to physics are concerned.

As for geometry not much was established on a rigorous basis up to now. However there existed a strong believe that as non-trivially coupled to Poincaré symmetries, which are of geometric nature, supersymmetries must also have a geometric character. Heuristic notions of a superspace —space of commuting and anticommuting parameters (x^μ, θ^α), its transformations and functions $\varphi(x^\mu, \theta^\alpha)$ on it (superfields) were introduced and extensively used—see *e. g.* [*4*, *15*, *16*]. They proved so useful that many attempts were made to formulate these notions rigorously [*5*, *9*, *14*] by using after Berezin [*2*] Grassmann algebras as a supply of anticommuting elements. However most of these attempts were not quite satisfactory as they led to the necessity to define commuting parameters x^μ not as usual numbers but as even elements of a Grassmann algebra. Also the problem what was the rigorous meaning of the formal procedure (widely used in the literature) of reintroduction of a Lie algebra by multiplying generators of the Z_2-graded Lie algebra by anticommuting numbers caused a big confusion. From the attempts mentioned above that of [*9*] distinguishes itself as the one free of these handicaps, at least to some point, and was our inspiration.

In the paper we give a series of definitions of main notions of supergeometry, the first one being that of a superspace itself. They may seem abstract but in our opinion the same could be said about the rigorous definition of a distribution being so distant from that of a function. Here the situation resembles the one existing in the theory of distributions in another aspect too. Working formally with the heuristic notion of a superspace, considered as a set of commuting and anticommuting parameters, we can proceed quite far if we are cautious enough. Similarly one can obtain (formally) many results, treating a distribution as a kind of function, so far one remembers that some cautiousness is needed (*e. g.* that multiplication of distributions is not allowed).

Section 2 comprises the definition of a superspace and of its diffeomorphisms and some basic facts about these notions. Since in the latest studies of supersymmetric field theories the heuristic geometry of a superspace was given a physical significance in the models unifying various gauge and gravitational fields [*1*, *11*, *19*] the need for a rigorous theory of superspaces seemed more pressing than ever. With the application just mentioned in mind we study in detail the supermanifolds, *i. e.* the objects looking piecewise like trivial superspaces. With the rigorous meaning we put behind the latter property (other possibilities cannot be excluded *a priori*) we find these objects to be nothing more than superspaces (except may-be some patological cases). The study of this problem is the topic of Section 3. Section 4 introduces further notions of supergeometry: vector fields, exterior forms, connections. We end the paper with Section 5 in which the notions of geometric actions of groups and (Z_2-graded) Lie algebras on a superspace are defined. We cite the superfield representation of the Volkov-

Akulov-Wess-Zumino Z_2-graded Lie algebra as an example of such an action. We also state one of two natural interpretations of what the Lie algebra spanned by Z_2-graded-Lie-algebra generators multiplied by « anti-commuting numbers » is. The aim is to clarify the situation largely confused up to now. The lengthest proofs, mainly those of results comprised in Sections 2 and 3, are omitted in the main text and are stated in Appendix.

The purpose of the paper is twofold. First we hope that introducing some rigour into heuristic theory of supersymmetries will allow for better understanding of this branch of field theory which, if even proves to provide no realistic model, creates possibility of studying phenomena of new types (unification of gravitation and spinor fields, drastic renormalizability improvements) both in classical and in quantum field theories and because of this only is worth-while. Secondly we treat the present paper as an introductory step to further studies of geometry of superfield theory. A geometric lagrangean formalism for superfields (something patterned after recent formulations [*7*, *8*] for standard fields) comprising a supersymmetric version of Noether theorem would be the first goal here.

Already when the paper was completed the author learnt about the reference [*20*] where a supermanifold had been defined in terms of a sheaf of local superfields. Although this definition is basically equivalent to the one we give [*20*], comprises no structural results of the type of our Theorem 1, developing instead a theory of supergroups.

The author is indebted to Professor K. Maurin for constant encouragement during the work. Special thanks are due to Professors H. Borchers and H. Reeh and Dr. W. Garber for many helpful talks accomplished during the author's stay at Institute of Theoretical Physics, University of Göttingen, where a part of the work was done. The author is also very grateful to Dr. R. Kotecky for helpful discussions and to Professor S. P. Novikov for attracting his attension to [*20*].

2. SUPERSPACES AND THEIR DIFFEOMORPHISMS

In physical literature [*4*, *15*, *16*] a superspace has been described as a space of commuting and anticommuting parameters, say $x^\mu-s$ and $\theta^\alpha-s$, $\mu = 0, 1, \ldots, n-1$, $\alpha = 1, \ldots, m$. Functions on the superspace $\varphi(x^\mu, \theta^\alpha)$ by formal expansion into Taylor series in $\theta^\alpha-s$ were written alternatively as

$$\varphi_0(x^\mu) + \varphi_{1,\alpha_1}(x^\mu)\theta^{\alpha_1} + \varphi_{2,\alpha_1\alpha_2}(x^\mu)\theta^{\alpha_1}\theta^{\alpha_2} + \ldots + \varphi_{m,\alpha_1,,,\alpha_m}(x^\mu)\theta^{\alpha_1}\ldots\theta^{\alpha_m} \quad (1)$$

producing a system of functions of $x-{}^\mu s$ $\varphi_{k,\alpha_1\ldots\alpha_k}(x^\mu)$ antisymmetric in $\alpha_1, \ldots, \alpha_k$. Our rigorous approach will be based on viewing φ as a function of $x^\mu-s$ with values in the Grassmann algebra with generators θ^α. (1) will give decomposition of $\varphi(x^\mu)$ in the natural basis of the Grassmann algebra.

In this we follow [9], our improvement being mainly notational. The algebra of Grassmann number valued functions of $x^\mu - s$ will be considered the basic object and geometry of a superspace will be patterned on the construction of the usual geometry in terms of operations on the algebra of real valued functions [12].

Let V be a smooth m-dimensional (real or complex) vector bundle over an n-dimensional Hausdorff base space M, m, $n < \infty$. For the rest of the paper we shall keep m and n fixed. Let

$$\Lambda V = \bigoplus_{i=0}^{m} \Lambda^i V$$

be the corresponding Grassmann bundle.

DEFINITION 1. — The pair $(V, \Lambda V)$ will be called a superspace.

Let $\Gamma_0(\Lambda V)$ denote the space of smooth sections of ΛV with compact support. $\Gamma_0(\Lambda V)$ is an algebra under point-wise exterior multiplication of sections and is a Z_2-graded vector space:

$$\Gamma_0(\Lambda V) = \Gamma_0(\Lambda_0 V) \oplus \Gamma_0(\Lambda_1 V), \tag{2}$$

where

$$\Lambda_0 V := \bigoplus_i \Lambda^{2i} V, \qquad \Lambda_1 V := \bigoplus_i \Lambda^{2i+1} V.$$

Both structures are compatible in the sense that if $\varphi_r \in \Lambda_r V$, $\psi_k \in \Lambda_k V$, $r, k \in Z_2 = \{ 0, 1 \}$, then $\varphi_r \psi_k \in \Lambda_{r+k} V$ (we shall omit the usual exterior multiplication mark Λ). We summarize this by saying that $\Gamma_0(\Lambda V)$ is a Z_2-graded algebra. This algebra is associative and additionally graded commutative *i. e.*

$$\varphi_r \psi_k = (-1)^{rk} \psi_k \varphi_r .$$

Let $(V, \Lambda V)$ and $(W, \Lambda W)$ be two superspaces.

DEFINITION 2. — Let T: $\Gamma_0(\Lambda V) \to \Gamma_0(\Lambda W)$ be an isomorphism of the Z_2-graded algebras, *i. e.* a linear isomorphism preserving grading and such that

$$T(\varphi\psi) = (T\varphi)(T\psi). \tag{3}$$

T will be called a diffeomorphism of the superspace $(V, \Lambda V)$ onto the superspace $(W, \Lambda W)$.

As we mentioned before the last definition is inspired by the following result of the standard geometry:

PROPOSITION 1. — Let M, N be smooth Hausdorff finite dimensional manifolds. Let

$$t : C_0^\infty(M) \to C_0^\infty(N)$$

be a linear isomorphism such that

$$t(fg) = (tf)(tg). \tag{4}$$

($C_0^\infty(M)$ denotes the space of smooth (real or complex) functions on M with compact support). Then there exists a unique diffeomorphism

$$\underline{t} : M \to N$$

such that

$$tf = f \circ \underline{t}^{-1}. \tag{5}$$

Let as before $(V, \Lambda V)$ and $(W, \Lambda W)$ be two superspaces. Suppose that M and N are the base spaces of V and W respectively. For T: $\Gamma_0(\Lambda V) \to \Gamma_0(\Lambda W)$ being a diffeomorphism of $(V, \Lambda V)$ onto $(W, \Lambda W)$ and for $\varphi \in \Gamma_0(\Lambda V)$ write

$$T\varphi = \sum_{k=0}^{m} T_k\varphi, \tag{6}$$

where $T_k\varphi \in \Gamma_0(\Lambda^k V)$. Let

$$t : C_0^\infty(M) \to C_0^\infty(N)$$

be defined by

$$tf := T_0 f. \tag{7}$$

PROPOSITION 2. — t: $C_0^\infty(M) \to C_0^\infty(N)$ is a linear isomorphism such that (4) holds.

Thus in virtue of Propositions 1 and 2 to each diffeomorphism T of $(V, \Lambda V)$ onto $(W, \Lambda W)$ we can assign a diffeomorphism $\underline{t}$ of M onto N.

PROPOSITION 3. — Let U be an open subset of M. Then

$$T(\Gamma_0(\Lambda V \restriction_U)) = \Gamma_0(\Lambda W \restriction_{\underline{t}(U)}). \tag{8}$$

Moreover

$$\operatorname{supp}(T\varphi) = \underline{t}\,(\operatorname{supp}\varphi). \tag{9}$$

Proposition 3 is basic for our study of possibility of introducing a notion of a supermanifold. A supermanifold should be an object which looks « piece-wise » like a superspace. To give a rigorous meaning of this very intuitive notion we must say first what is a piece of a superspace. This should be a notion invariant under diffeomorphisms of the superspace. In standard geometry a piece of a vector space is taken to be its open subset. If we work with the algebra of smooth functions with compact support on the vector space this corresponds to choosing ideals of functions with supports in a given open subset. Somewhat surprisingly Proposition 3 shows that in the case of supergeometry ideals $\Gamma_0(\Lambda V \restriction_U)$ are good substitutes. In Section 3 we shall show how one could glue together pieces like that to form a supermanifold. Detailed analysis shows however that, with possible exception of patologies, we do not go far beyond the case of superspaces $(V, \Lambda V)$. The conclusion is that, with the natural notion of a piece of a superspace we adopt, the postulate of full invariance under general diffeomorphisms of superspaces proposed by Nath-Arnowitt in [*11*] produces no extra-global (super-)geometric effects beyond that introduced already by classical general relativity, where analogous postulate yields possibility of occurrence of various non-diffeomorphic types of manifolds and bundles.

Before we pass to a detailed study of this problem in Section 3 we show that each diffeomorphism T of $(V, \Lambda V)$ onto $(W, \Lambda W)$ induces also a bundle isomorphism $\underline{\tau} : V \to W$ projecting down to base spaces to $\underline{t}$.

Let us notice that if $\varphi \in \Gamma_0(V)$ and $\tau : \Gamma_0(V) \to \Gamma_0(W)$ is defined by (compare (6))

$$\tau\varphi := T_1\varphi, \tag{10}$$

then τ is a linear isomorphism. Indeed. If $T' = T^{-1}$ then

$$\varphi = (T \circ T')\varphi = T\big(T'_1\varphi + T'_3\varphi + \ldots\big) = T_1 \circ T'_1\varphi + \psi,$$

where $\psi \in \Gamma_0\left(\bigoplus_{i=3}^{m} \Lambda^i V\right)$. Hence $\tau \circ \tau' = 1$ and similarly $\tau' \circ \tau = 1$.

PROPOSITION 4. — There exists a unique bundle isomorphism $\underline{\tau} : V \to W$ projecting down to base spaces to $\underline{t} : M \to N$ such that for $\varphi \in \Gamma_0(V)$

$$\tau\varphi = \underline{\tau} \circ \varphi \circ \underline{t}^{-1}. \tag{11}$$

PROPOSITION 5. — Assignments

$$T \mapsto \underline{\tau} \mapsto \underline{t}$$

have the following functorial properties:

if $T = 1$ then $\underline{\tau} = 1$ and $\underline{t} = 1$,

if $T' = T^{-1}$ then $\underline{\tau}' = \underline{\tau}^{-1}$, $\underline{t}' = \underline{t}^{-1}$,

if $T : \Gamma_0(\Lambda V) \to \Gamma_0(\Lambda W)$ and $T' : \Gamma_0(\Lambda W) \to \Gamma_0(\Lambda U)$ where $(U, \Lambda U)$ is another superspace and $T'' = T' \circ T$ then $\underline{\tau}'' = \underline{\tau}' \circ \underline{\tau}$, $\underline{t}'' = \underline{t}' \circ \underline{t}$,

if U is an open subset of M and $T' : \Gamma_0(\Lambda V \restriction_U) \to \Gamma_0(\Lambda W \restriction_{t(U)})$ is the restriction of T then $\underline{\tau}' : V \restriction_U \to W \restriction_{\underline{t}(U)}$ is the restriction of $\underline{\tau}$ and $\underline{t}' : U \to \underline{t}(U)$ is the restriction of $\underline{t}$.

Proof. — Obvious.

3. DO THERE EXIST SUPERMANIFOLDS?

Let $(V, \Lambda V)$ be a superspace and let M denote the base space of V. Let $F := \Gamma_0(\Lambda V)$. Trivializations of V and maps of M over open subsets O_a forming a covering of M define a family T_a of isomorphisms of Z_2-graded algebras

$$T_a : \Gamma_0(\Lambda V \restriction_{O_a}) \equiv F_a \to C_0(\overline{O}_a, \Lambda K^m), \tag{12}$$

where $\overline{O}_a$ are open subsets in R^n and $K = R$ or C. The set $\{ (F_a, T_a) \}$ has the following properties:

i) $\sum_a F_a = F$,

ii) for each pair (a_1, a_2) there exists an open subset $\overline{O}_{a_1a_2} \subset \overline{O}_{a_1}$ such that

$$T_{a_1}(F_{a_1} \cap F_{a_2}) = C_0^\infty(\overline{O}_{a_1a_2}, \Lambda K^m)$$

(we put $C_0^\infty(\varnothing, \Lambda K^m) = \{0\}$),

iii) for each pair (x_1, x_2), $x_1 \in \overline{O}_{a_1} \backslash \overline{O}_{a_1a_2}$, $x_2 \in \overline{O}_{a_2} \backslash \overline{O}_{a_2a_1}$ there exist open neighborhoods U_{x_1} of x_1 and U_{x_2} of x_2 such that

$$T_{a_1}^{-1}(C_0^\infty(U_{x_1}, \Lambda K^m)) \cap T_{a_2}^{-1}(C_0^\infty(U_{x_2}, \Lambda K^m)) = \{0\}.$$

We summarize this in an intuitive statement that the superspace $(V, \Lambda V)$ looks locally like trivial superspaces $(\overline{O}_a \times K^m, \overline{O}_a \times \Lambda K^m)$.

DEFINITION 3. — Let F be a Z_2-graded, graded commutative, associative algebra. Set $A = \{ (F_a, T_a) \}$ will be called an atlas for F if F_a are Z_2-graded ideals in F and T_a are isomorphisms of Z_2-graded algebras

$$T_a : F_a \to C_0^\infty(\overline{O}_a, \Lambda K^m),$$

such that *i)*, *ii)* and *iii)* stated above hold. Two atlases A and A′ for F will be called compatible if $A \cup A'$ is an atlas for F. An atlas will be called complete if it contains each atlas compatible with it. If A is a complete atlas for F, (F, A) will be called a supermanifold.

REMARK. — We notice that if $F_{a_1} \cap F_{a_2} \neq \{ O \}$ then

$$T_{a_2a_1} := T_{a_2} \cdot T_{a_1}^{-1} : C_0^\infty(\overline{O}_{a_1a_2}, \Lambda K^m) \to C_0^\infty(\overline{O}_{a_2a_1}, \Lambda K^m) \qquad (13)$$

is a diffeomorphism of trivial superspaces $(\overline{O}_{a_1a_2} \times K^m, \overline{O}_{a_1a_2} \times \Lambda K^m)$ and $(\overline{O}_{a_2a_1} \times K^m, \overline{O}_{a_2a_1} \times \Lambda K^m)$ (we identify $\Gamma_0(\overline{O} \times K^m)$ with $C_0^\infty(\overline{O}, \Lambda K^m)$).

Obviously Definition 3 has been patterned after the definition of a Hausdorff manifold in terms of its algebra of smooth functions with compact support. As the latter is natural only in the case of Hausdorff manifolds the separation axion *iii)* is included. However we must show that in our case the notion of compatibility of atlases has the usual properties which render all compatible atlases equally good. This is not immediate. The results of Proposition 3 enter crutially at this point.

PROPOSITION 6. — 1. Compatibility of atlases for F is an equivalence relation. 2. Each atlas for F is contained in a unique complete one.

Let now $A = \{ (F_a . T_a) \}$ be a complete atlas for F. Let us consider disjoint unions

$$\bigvee \overline{O}_a \quad \text{and} \quad \bigvee \overline{O}_a \times K^m$$

with relations R and R′ respectively defined by

$$x_1 \overset{R_1}{\sim} x_2 \Leftrightarrow x_1 \in \overline{O}_{a_1 a_2},\ x_2 \in \overline{O}_{a_2 a_1} \quad \text{and} \quad x_2 = \underline{t}_{a_2 a_1}(x_1), \tag{14}$$

$$(x_1, y_1) \overset{R_2}{\sim} (x_2, y_2) \Leftrightarrow x_1 \in \overline{O}_{a_1 a_2},\ x_2 \in \overline{O}_{a_2 a_1} \quad \text{and} \quad (x_2, y_2) = \underline{\tau}_{a_2 a_1}(x_1, y_1), \tag{15}$$

where $\underline{t}_{a_2 a_1}$ and $\underline{\tau}_{a_2 a_1}$ correspond to superspace diffeomorphism $T_{a_2 a_1}$ by the assignments of Proposition 5. R and R′ are equivalence relations as follows easily from Proposition 5. Let

$$M := \bigvee \overline{O}_a / R, \tag{16}$$

$$V := \bigvee \overline{O}_a \times K^m / R'. \tag{17}$$

Let $\pi : V \to M$ be given by

$$\pi([(x, y)]_{R'}) := [x]_R. \tag{18}$$

There exist unique structures

of a smooth Hausdorff n-dimensional manifold on M and

of an m-dimensional smooth vector bundle with base M and bundle projection π over V

such that canonical injections $\underline{t}_a^{-1}$ of $\overline{O}_a$ into M are diffeomorphisms onto open sets O_a in M and canonical injections $\underline{\tau}_a^{-1}$ are bundle isomorphisms of $\overline{O}_a \times K^m$ onto $V \restriction_{O_a}$. M, V and $(V, \Lambda V)$ will be called the underlying manifold, bundle and superspace of the supermanifold (F, A) respectively.

The main result of this section is that algebras $\Gamma_0(\Lambda V)$ are (up to isomorphism) the only ones with atlases. Less precisely but more intuitively one could say that superspaces $(V, \Lambda V)$ are the most general objects we can obtain by gluing together trivial superspaces piece-wise.

THEOREM 1. — Let F be a Z_2-graded, graded commutative, associative algebra with an atlas A. Let M and V be defined as above. Suppose that M is paracompact (what is the case if A is countable). Then there exists an isomorphism of Z_2-graded algebras

$$I : F \to \Gamma_0(\Lambda V).$$

REMARK. — Let A_1 be the complete atlas for F comprising A. We could restate Theorem 1 in the shorter form:

the supermanifold (F, A_1) is diffeomorphic to its underlying superspace $(V, \Lambda V)$ whenever its underlying manifold M is paracompact.

Because of this result of negative character we shall not use supermanifolds in the further parts of the paper restricting our considerations to superspaces.

Let $(W, \Lambda W)$ and $(U, \Lambda U)$ be two superspaces and let N and P denote the base spaces of W and U respectively. Let $T : \Gamma_0(\Lambda W) \to \Gamma_0(\Lambda U)$ be a diffeomorphism of $(W, \Lambda W)$ onto $(U, \Lambda U)$. We shall say that T is of class $k\left(k = 1, \ldots, E\left[\frac{m}{2}\right] + 1\right)$ if for $f \in C_0^\infty(N)$, $\varphi \in \Gamma_0(W)$

$$Tf = T_0 f + T_{2k} f + T_{2k+2} f + \ldots, \tag{19}$$

$$T\varphi = T_1\varphi + T_{2k+1}\varphi + T_{2k+3}\varphi + \ldots, \tag{20}$$

where $T_j f$, $T_j\varphi \in \Gamma_0(\Lambda^j V)$. It is easy to check that if T is of class k then restrictions of T and T^{-1} are also of class k, and that if T, T′ are of class k so is $T \circ T'$ when defined.

We shall say that an atlas $A = \{ (F_a, T_a) \}$ for F is of class k if for each a_1, a_2 such that $\bar{O}_{a_1 a_2} \neq \varnothing$ $T_{a_2 a_1}$ is of class k.

The crutial point in proof of Theorem 1 is

PROPOSITION 7. — Suppose that A_k is an atlas of class k for $F\left(k = 1, \ldots, E\left[\frac{m}{2}\right]\right)$. Then there exists an atlas A_{k+1} for F of class $k + 1$ compatible with A_k.

As each atlas is of class 1 this shows that there exists an atlas $A' = \{ (F_a, T_a) \}$ for F of class $E\left[\frac{m}{2}\right] + 1$ compatible with the original atlas A. Define for $\psi_a \in F_a$

$$I_a\psi_a := \underline{\tau}_a^{-1} \cdot (1 \oplus T_a\psi_a) \cdot \underline{t}_a. \tag{21}$$

Now Theorem 1 follows since we have the following

PROPOSITION 8. — Define for

$$\psi \in F \quad , \quad \psi = \sum_{i=1}^{k} \psi_{a_i} \quad , \quad \psi_{a_i} \in F_{a_i} \quad , \quad I\psi \in \Gamma_0(\Lambda V)$$

by

$$I\psi := \sum_{i=1}^{k} I_{a_i}\psi_{a_i}. \tag{22}$$

Then $I\psi$ depends on ψ only (not on its decomposition) and

$$F \ni \varphi \mapsto I\,\psi \in \Gamma_0(\Lambda V)$$

is an isomorphism of Z_2-graded Lie algebras.

4. SUPERSPACE TENSOR CALCULUS

As we mentioned in Introduction geometry of superspaces plays an important role in physical applications, where heuristic notions of vector fields, exterior forms, tensors, connections were introduced [*4*, *15*, *16*]. We start the section defining a vector field on a superspace $(V, \Lambda V)$ to be an (anti-)derivation of the algebra $\Gamma_0(\Lambda V)$. The latter was already used in [*2*, *9*, *13*]. However putting stress on its geometric character will enable us introduction of other geometric notions and creation of a rigorous basis for the supersymmetric generalizations of general relativity proposed in [*11*] and [*19*].

Let $(V, \Lambda V)$ be a superspace.

DEFINITION 4. — A linear mapping

$$X : \Gamma_0(\Lambda V) \to \Gamma_0(\Lambda V)$$

preserving Z_2-grading and such that

$$X(\varphi\psi) = (X\varphi)\psi + \varphi(X\psi) \tag{23}$$

will be called an even vector field on $(V, \Lambda V)$.

A linear mapping

$$Y : \Gamma_0(\Lambda V) \to \Gamma_0(\Lambda V)$$

changing Z_2-grading by 1 (*i. e.* $Y(\Gamma_0(\Lambda_r V)) \subset \Gamma_0(\Lambda_{r+1} V)$, $r \in Z_2$) such that

$$Y(\varphi\psi) = (Y\varphi)\psi + (G\varphi)(Y\psi), \tag{24}$$

where $G : \Gamma_0(\Lambda V) \to \Gamma_0(\Lambda V)$ is linear, $G\restriction_{\Lambda_r V} = (-1)^r$, will be called an odd vector field on $(V, \Lambda V)$.

A linear mapping

$$Z : \Gamma_0(\Lambda V) \to \Gamma_0(\Lambda V)$$

such that $Z = X + Y$, where X is an even and Y an odd vector fields on $(V, \Lambda V)$ will be called a vector field on $(V, \Lambda V)$.

All vector fields on $(V, \Lambda V)$ form a Z_2-graded vector space

$$\mathscr{V}(V, \Lambda V) = \mathscr{V}_0(V, \Lambda V) \oplus \mathscr{V}_1(V, \Lambda V)$$

composed of even and odd vector fields subspaces. One can introduce a bilinear bracket operation $[\ldots]$ in $\mathscr{V}(V, \Lambda V)$. Namely for $Z_r, Z'_r \in \mathscr{V}_r(V, \Lambda V)$, $r \in Z_2$, put

$$[Z_r, Z'_s] = Z_r Z'_s - (-1)^{rs} Z'_s Z_r. \tag{25}$$

Besides turning $\mathscr{V}(V, \Lambda V)$ into a Z_2-graded algebra this operation fulfils

$$[Z_r, Z'_s] + (-1)^{rs}[Z'_s, Z_r] = 0, \tag{26}$$

$$(-1)^{rt}[[Z_r, Z'_s], Z''_t] + (-1)^{sr}[[Z'_s, Z''_t], Z_r] + (-1)^{ts}[[Z''_t, Z_r], Z'_s] = 0, \tag{27}$$

that is it turns $\mathscr{V}(V, \Lambda V)$ into a Z_2-graded Lie algebra. Note also that vector fields on $(V, \Lambda V)$ can be multiplied by elements of $\Gamma(\Lambda V)$ (*i. e.* by smooth sections of ΛV).

Let $(W, \Lambda W)$ be another superspace and

$$T : \Gamma_0(\Lambda V) \to \Gamma_0(\Lambda W)$$

a diffeomorphism of $(V, \Lambda V)$ onto $(W, \Lambda W)$. If Z is a vector field on $(V, \Lambda V)$ then

$$T_* Z = T \circ Z \circ T^{-1} \tag{28}$$

is a vector field on $(W, \Lambda W)$. Moreover

$$T_* : \mathscr{V}(V, \Lambda V) \to \mathscr{V}(W, \Lambda W)$$

is an isomorphism of Z_2-graded Lie algebras.

From (23) and (24) we see easily that if φ and ψ have disjoint supports then $(X\varphi)\psi$ and $(Y\varphi)\psi$ vanish, which shows that vector fields on $(V, \Lambda V)$ do not increase supports.

Let M be the base space of V and let U be an open subset of M'. Then for each vector field Z on $(V, \Lambda V)$ we can define its restriction to $(V \upharpoonright_U, \Lambda V \upharpoonright_U)$. Moreover if $\varphi \in \Gamma_0(\Lambda V)$ then $(Z\varphi)\upharpoonright_U$ depends only on $\varphi \upharpoonright_U$. Thus we can extend Z to a linear transformation of $\Gamma(\Lambda V)$ by putting for $\varphi \in \Gamma(\Lambda V)$

$$(Z\varphi)\upharpoonright_U = (Z(h\varphi))\upharpoonright_U$$

for $h \in C_0^\infty(M)$, $U = \operatorname{Int}(h^{-1}(\{1\}))$. We shall often use this extension.

Let $V = \overline{O} \times K^m$, where O is an open subset of R^n. Denote by $(x^1, \ldots, x^n)$ the canonical coordinate chart of $\overline{O}$ and by $(\theta^1, \ldots, \theta^m)$ the canonical basis in K^m. Then

$$\frac{\partial}{\partial x^\mu}, \ \mu = 0, \ldots, n-1, \qquad \text{and} \qquad \frac{\partial}{\partial \theta^a}, \ a = 1, \ldots, m,$$

are respectively even and odd vector fields on $(V, \Lambda V)$. Here

$$\frac{\partial}{\partial \theta^a} \varphi = \theta_a^* \lrcorner \varphi, \tag{29}$$

where $(\theta_1^*, \ldots, \theta_m^*)$ form the dual basis in $(K^m)^*$ and $\lrcorner$ denotes internal multiplication (contraction).

It is rather well known that the fields $\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial \theta^a}$ form a basis of (anti-)derivations of the algebra of Grassmann-number-valued functions on $\overline{O}$ [*2*, *9*]. We state a result of this kind together with its proof for the sake of completeness and also because of slightly different contain.

PROPOSITION 9. — Any vector field on $(\overline{O} \times K^m, \overline{O} \times \Lambda K^m)$ is of the form

$$\sum_\mu X^\mu \frac{\partial}{\partial x^\mu} + \sum_a \Theta^a \frac{\partial}{\partial \theta^a} \tag{30}$$

where X^μ, $\Theta^a \in C^\infty(\overline{O}, \Lambda K^m) (\cong \Gamma(\overline{O} \times \Lambda K^m))$.

Proof. — Suppose that X is an even vector field on $(\bar{O} \times K^m, \bar{O} \times \Lambda K^m)$. Define

$$X^\mu \in C^\infty(\bar{O}, \Lambda K^m)$$

by

$$X^\mu = X(x^\mu). \tag{31}$$

Now

$$X' := X - \sum_\mu X^\mu \frac{\partial}{\partial x^\mu} \tag{32}$$

is again an even vector field and

$$(X'_\varphi)(x_0) = 0 \qquad \text{whenever} \qquad \varphi(x_0) = 0. \tag{33}$$

Indeed. Suppose that $\varphi(x_0) = 0$. Then $\varphi(x) = \sum_\mu (x^\mu - x_0^\mu)\varphi_\mu$ in a neighborhood of x_0 and

$$(X'_\varphi)(x_0) = \sum_\mu (X(x^\mu - x_0^\mu))(x_0)\varphi_\mu(x_0) - \sum_\mu X^\mu(x_0)\frac{\partial\varphi}{\partial x^\mu}(x_0) = 0 \tag{34}$$

So

$$(X'_\varphi)(x_0) = M_{x_0}\cdot\varphi(x_0),$$

where M_{x_0} is a derivation of ΛK^m preserving parity. Thus

$$M_{x_0} = \sum_a \Theta^a(x_0)\frac{\partial}{\partial\theta^a},$$

where

$$\Theta^a(x_0) = M_{x_0}(\theta^a) \in \Lambda_1 K^m.$$

For odd vector fields the proof proceeds in full analogy. ■

Let $\mathscr{V} = \mathscr{V}_0 \oplus \mathscr{V}_1$ be a vector space. We shall say that $\mathscr{V}$ is a $\Gamma(\Lambda V)$-module if we are given a bilinear operation

$$\Gamma(\Lambda V) \times \mathscr{V} \ni (\varphi, v) \mapsto \varphi v \in \mathscr{V}$$

mapping $\Gamma(\Lambda_r V) \times \mathscr{V}_s$ into $\mathscr{V}_{r+s}$, $r, s \in Z_2$, and such that

$$(\varphi\psi)v = \varphi(\psi v) \tag{35}$$

and

$$1v = v. \tag{36}$$

$\mathscr{V}(V, \Lambda V)$ with the natural multiplication by elements of $\Gamma(\Lambda V)$ is a $\Gamma(\Lambda V)$-module.

If $\mathscr{V} = \mathscr{V}_0 \oplus \mathscr{V}_1$ is a $\Gamma(\Lambda V)$-module then by $\mathscr{V}^* = \mathscr{V}_0^* \oplus \mathscr{V}_1^*$ we shall denote a $\Gamma(\Lambda V)$-module obtain the following way. $\mathscr{V}^*$ is composed of linear transformations η of $\mathscr{V}$ into $\Gamma(\Lambda V)$ such that

$$\langle \varphi v \mid \eta \rangle = \varphi \langle v \mid \eta \rangle. \tag{37}$$

$\mathscr{V}_0^*$ comprises $\eta - s$ preserving Z_2-grading,
$\mathscr{V}_1^*$ those changing it by 1.

The result of multiplication of $\eta \in \mathscr{V}^*$ by $\varphi \in \Gamma(\Lambda V)$ is given by

$$\langle v \mid \varphi\eta \rangle := \varphi \langle v \mid \eta \rangle \qquad \text{if} \quad v \in \mathscr{V}_0, \tag{38}$$

$$\langle v \mid \varphi\eta \rangle := (G\varphi) \langle v \mid \eta \rangle \qquad \text{if} \quad v \in \mathscr{V}_1. \tag{39}$$

Elements of $\mathscr{V}^*_{(0,1)}(V, \Lambda V)$ will be called (even, odd) 1-forms on $(V, \Lambda V)$. We observe that if η is a 1-form on $(V, \Lambda V)$ then for $f \in C_0^\infty(M)$, $v \in \mathscr{V}(V, \Lambda V)$ $\langle v \mid \eta \rangle = \langle fv \mid \eta \rangle$ on $f^{-1}(\{1\})$. Using this we can easily define a restriction of η to $(V\restriction_U, \Lambda V\restriction_U)$ if U is an open subset of M.

Let $V = \bar{O} \times K^m$, $\bar{O}$ – an open subset in R^n. Then

$$dx^\mu,\ \mu = 0, \dots, n-1, \qquad \text{and} \qquad d\theta^a,\ a = 1, \dots, m,$$

are even and odd 1-forms respectively if they are defined by

$$\begin{aligned} \left\langle \frac{\partial}{\partial x^\nu} \middle| dx^\mu \right\rangle &= \delta^\mu_\nu \quad , \quad \left\langle \frac{\partial}{\partial \theta^b} \middle| dx^\mu \right\rangle = 0, \\ \left\langle \frac{\partial}{\partial x^\nu} \middle| d\theta^a \right\rangle &= 0 \quad , \quad \left\langle \frac{\partial}{\partial \theta^b} \middle| d\theta^a \right\rangle = \delta^a_b \end{aligned} \tag{40}$$

As follows immediately from Proposition 9 any 1-form η on $(\bar{O} \times K^m, \bar{O} \times \Lambda K^m)$ is given by

$$= \sum_\mu X_\mu dx^\mu + \sum_a \Theta_a d\theta^a, \tag{41}$$

where

$$X_\mu,\ \Theta_a \in C^\infty(\bar{O}, \Lambda K^m).$$

Let $\mathscr{V}^i = \mathscr{V}^i{}_0 \oplus \mathscr{V}^i{}_1$, $i = 1, \dots, k$, be a family of $\Gamma(\Lambda V)$-modules. Define a $\Gamma(\Lambda V)$-module

$$\bigotimes_{i=1}^{k} \mathscr{V}^i \equiv \mathscr{V}^1 \otimes \dots \otimes \mathscr{V}^k$$

the following way.

As a vector space

$$\bigotimes_{i=1}^{k} \mathscr{V}^i := \bigotimes_{i=1}^{k} \mathscr{V}^i / N, \tag{42}$$

where $\bigotimes_{i=1}^{k} \mathscr{V}^i$ is the vector space tensor product and N is its subspace spanned by elements

$$\varphi_r v^1_{r_1} \otimes v^2_{r_2} \otimes \dots \otimes v^k_{r_k} - (-1)^{r(r_1 + \dots + r_{j-1})} v^1_{r_1} \otimes \dots \\ \otimes v^{j-1}_{r_{j-1}} \otimes \left(\varphi_r v^j_{r_j}\right) \otimes v^{j+1}_{r_{j+1}} \otimes \dots \otimes v^k_{r_k} \tag{43}$$

where $\varphi_r \in \Gamma(\Lambda_r V)$, $v^i_{r_i} \in \mathscr{V}^i_{r_i}$, $r, r_i \in Z_2$.

Let $G : \bigotimes_{i=1}^{k} \mathscr{V}^i \to \bigotimes_{i=1}^{k} \mathscr{V}^i$ be linear involution,

$$G\left(v^1_{r_1} \otimes \dots \otimes v^k_{r_k}\right) := (-1)^{r_1 + \dots + r_k} v^1_{r_1} \otimes \dots \otimes v^k_{r_k} \tag{44}$$

We shall not mix this operation with G: $\Gamma(\Lambda V) \to \Gamma(\Lambda V)$. As G leaves N invariant it defines an involution of $\bigotimes_{i=1}^{k} \mathscr{V}^i$ which we shall also denote by G. The decomposition into the 0 and 1 eigenvalue subspaces of G gives

$$\overset{k}{\underset{i=1}{\circledast}} \mathscr{V}^i = \overset{k}{\underset{i=1}{\circledast}}{}_0 \mathscr{V}^i + \overset{k}{\underset{i=1}{\circledast}}{}_1 \mathscr{V}^i. \tag{45}$$

For $v^1 \circledast \ldots \circledast v^k := v^1 \otimes \ldots \otimes v^k + N$ and $\varphi \in \Gamma(\Lambda V)$ we put

$$\varphi(v^1 \circledast \ldots \circledast v^k) = (\varphi v^1) \circledast v^2 \circledast \ldots \circledast v^k. \tag{46}$$

Now we can define $\Gamma(\Lambda V)$-module of tensor fields r-times contravariant and p-times covariant over $(V, \Lambda V)$:

$$\mathscr{V}^r_p(V, \Lambda V) := \underbrace{\mathscr{V}(V, \Lambda V) \circledast \ldots \circledast \mathscr{V}(V, \Lambda V)}_{r\text{-times}} \circledast \underbrace{\mathscr{V}^*(V, \Lambda V) \circledast \ldots \circledast \mathscr{V}^*(V, \Lambda V)}_{p\text{-times}}. \tag{47}$$

Note that restriction of vector fields and 1-forms on $(V, \Lambda V)$ to $(V \restriction_U, \Lambda V \restriction_U)$ induces restrictions of tensor fields of arbitrary variance.

If $\mathscr{V}$ is a $\Gamma(\Lambda V)$-module denote $\underbrace{\mathscr{V} \circledast \ldots \circledast \mathscr{V}}_{k\text{-times}}$ by $\overset{k}{\circledast} \mathscr{V}$. Let for $1 \leqslant i < j \leqslant k$

$$P_{ij} : \overset{k}{\circledast} \mathscr{V} \to \overset{k}{\circledast} \mathscr{V}$$

be a linear isomorphism defined by

$$\begin{aligned} P_{ij}(v^1_{r_1} \circledast \ldots \otimes v^i_{r_i} \circledast \ldots \circledast v^j_{r_j} \circledast \ldots \circledast v^k_{r_k}) : \\ = (-1)^\sigma (v^1_{r_1} \circledast \ldots \circledast v^j_{j} \circledast \ldots \circledast v^i_{r_i} \circledast \ldots \circledast v^k_{r_k}), \end{aligned} \tag{48}$$

where $v^l_{r_l} \in \mathscr{V}_{r_l}$, $r_l \in Z_2$, and σ is the number of transpositions of odd neighboring elements (here $v^l_{r_l} - s$) necessary to restore the same order on both sides of the equality. Here $\sigma = r_i r_j + (r_i + r_j)(r_{i+1} + \ldots + r_{j-1})$. This convention will hold whenever σ appears in the formulae. One can check that P_{ij} is well defined by (48) and that it commutes with multiplication by elements of $\Gamma(\Lambda V)$.

The subspace of $\overset{k}{\circledast} \mathscr{V}$ composed of tensors invariant (changing sign) under all P_{ij} forms a $\Gamma(\Lambda V)$-module which we shall denote by

$$_s\overset{k}{\circledast} \mathscr{V} \qquad (\Lambda^k \mathscr{V})$$

and call the k-fold symmetric (antisymmetric) tensor product of the $\Gamma(\Lambda V)$-module $\mathscr{V}$.

In particular elements of $\Lambda^k \mathscr{V}^*(V, \Lambda V)$ will be called k-forms on $(V, \Lambda V)$. Elements of $\Lambda^k \mathscr{V}$ are linear combinations of elements of the form

$$v^1_{r_1} \wedge \ldots \wedge v^k_{r_k} := \sum_\pi (-1)^{\sigma(\pi) - \sigma} v^{\pi(1)}_{r_{\pi(1)}} \wedge \ldots \wedge v^{\pi(k)}_{r_{\pi(k)}}, \tag{49}$$

where the sum runs over all permutations of $\{1, \ldots, k\}$ and $\sigma(\pi)$ is the number of transpositions in π.

Covariant tensors in $\mathscr{V}^0_k(V, \Lambda V)$ define k-linear forms mapping $\overset{k}{\times}\, \mathscr{V}(V, \Lambda V)$ into $\Gamma(\Lambda V)$ by

$$\langle Z^1_{r_1}, \ldots, Z^k_{r_k} \mid \eta^1_{s_1} \otimes \ldots \otimes \eta^k_{s_k} \rangle := (-1)^\sigma \langle Z^1_{r_1} \mid \eta^1_{s_1} \rangle \ldots \langle Z^k_{r_k} \mid \eta^k_{s_k} \rangle, \tag{50}$$

where $Z^i_{r_i} \in \mathscr{V}_{r_i}(V, \Lambda V)$, $\eta^j_{s_j} \in \mathscr{V}^*_{s_j}(V, \Lambda V)$ (here σ counts the number of transpositions of odd $Z^i_{r_i}$ among themselves and with odd $\eta^j_{s_j}$).

If ω is a k-linear form obtained this way then

$$\langle Z^1_{r_1}, \ldots, \varphi_r Z^i_{r_i}, \ldots, Z^k_{r_k} \mid \omega \rangle = (-1)^\sigma \varphi_r \langle Z^1_{r_1}, \ldots, Z^k_{r_k} \mid \omega \rangle \tag{51}$$

for $\varphi_r \in \Gamma(\Lambda_r V)$. Conversely. After some easy algebra one shows that any k-linear form ω on $\overset{k}{\times}\, \mathscr{V}(V, \Lambda V)$ with values in $\Gamma(\Lambda V)$ satisfying (51) is defined by a unique tensor field in $\mathscr{V}^0_k(V, \Lambda V)$. This field is in $\Lambda^k \mathscr{V}^*(V, \Lambda V)$ if and only if

$$\begin{aligned} &\langle Z^1_{r_1}, \ldots, Z^i_{r_i}, \ldots, Z^j_{r_j}, \ldots, Z^k_{r_k} \mid \omega \rangle \\ &\qquad = -(-1^\sigma) \langle Z^1_{r_1}, \ldots, Z^j_{r_j}, \ldots, Z^i_{r_i}, \ldots, Z^k_{r_k} \mid \omega \rangle. \end{aligned} \tag{52}$$

It is even (odd) if and only if

$$\langle Z^1_{r_1}, \ldots, Z^k_{r_k} \mid \omega \rangle \in \Gamma(\Lambda_s V),$$

where $s = r_1 + \ldots + r_k$ $(s = r_1 + \ldots + r_k + 1)$. From now on we shall identify covariant tensors on $(V, \Lambda V)$ with such forms.

Given a k-form ω on $(V, \Lambda V)$ one can define its exterior derivative $d\omega$ as a $(k+1)$-form such that

$$\begin{aligned} \langle Z^0_{r_0}, \ldots, Z^k_{r_k} \mid d\omega \rangle = &\sum_{i=0}^{k} (-1)^{i+\sigma} Z^i_{r_i}(\langle Z^0_{r_0}, \underset{\widehat{i}}{\ldots}, Z^k_{r_k} \mid \omega \rangle) \\ &+ \sum_{\substack{i,j=0 \\ i<j}}^{k} (-1)^{i+j+\sigma} \langle [Z^i_{r_i}, Z^j_{r_j}], Z^0_{r_0}, \underset{\widehat{i}\,\widehat{j}}{\ldots}, Z^k_{r_k} \mid \omega \rangle. \end{aligned} \tag{53}$$

Checking that $(k+1)$-linear form $d\omega$ fulfils (51) and (52) is straightforward and thus $d\omega$ is well defined. If ω is even (odd) then so is $d\omega$. It is also straightforward that $dd\omega = 0$. If, in particular, $\varphi \in \Gamma(\Lambda V)(= \Lambda^0 \mathscr{V}^*(V, \Lambda V))$ then $d\varphi$ is 1-form on $(V, \Lambda V)$ such that

$$\langle Z \mid d\varphi \rangle = Z(\varphi) \tag{54}$$

for $Z \in \mathscr{V}(V, \Lambda V)$.

If T is a diffeomorphism of two superspaces $(V, \Lambda V)$ and $(W, \Lambda W)$ and ω is an element of $\mathscr{V}^0_k(V, \Lambda V)$ then we define $T^*\omega \in \mathscr{V}^0_k(W, \Lambda W)$ by

$$\langle Z_1, \ldots, Z_k \mid T^*\omega \rangle = T(\langle T_*^{-1} Z_1, \ldots, T_*^{-1} Z_k \mid \omega \rangle), \tag{55}$$

where $Z_i \in \mathscr{V}_i(W, \Lambda W)$.

It is again straightforward that if ω is a k-form then $T^*\omega$ is also a k-form and

$$d(T^*\omega) = T^*(d\omega). \tag{56}$$

If $V = \overline{O} \times K^m$ then any k-form ω on $(V, \Lambda V)$ can be written as

$$\sum_{\substack{\mu=(\mu_1,\ldots,\mu_i)\\ \mu_1<\ldots<\mu_i\\ a=(a_1,\ldots,a_j)\\ a_1\leqslant\ldots\leqslant a_j\\ i+j=k}} I_{\mu a} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_i} \wedge d\theta^{a_1} \wedge \ldots \wedge d\theta^{a_j} \tag{57}$$

where $I_{\mu a} \in C^\infty(\overline{O}, \Lambda K^m)$. Now

$$d\omega = \sum_{\mu,a,\mu_0} \left(\frac{\partial}{\partial x^{\mu_0}} I_{\mu a}\right) dx^{\mu_0} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_i} \wedge d\theta^{a_1} \wedge \ldots \wedge d\theta^{a_j}$$
$$+ \sum_{\mu,a,a_0} (-1)^i G\left(\frac{\partial}{\partial \theta^{a_0}} I_{\mu a}\right) dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_i} \wedge d\theta^{a_0} \wedge d\theta^{a_1} \wedge \ldots \wedge d\theta^{a_j}. \tag{58}$$

We shall not consider an integration theory on superspaces here. Let us mention only that n-forms with compact support ($n = \dim M$) are natural objects to be integrated but there are also other possibilities corresponding to integration of kind of volume elements [*2*, *13*]. Integration of forms will play a major role in geometric lagrangean formulation of classical superfield theory in analogy to multisymplectic formulations of conventional field theory of [*7*, *8*].

Another notion we can introduce in analogy to usual geometry is that of a connection. A family $(\nabla_Z)_{Z\in\mathscr{V}(V,\Lambda V)}$ of linear mappings

$$\nabla_Z : \mathscr{V}(V, \Lambda V) \to \mathscr{V}(V, \Lambda V)$$

such that for $\psi \in \Gamma(\Lambda V)$

$$\nabla_{\psi Z} = \psi \nabla_Z \tag{59}$$

and for $Z_r \in \mathscr{V}_r(V, \Lambda V)$

$$\nabla_{Z_r}(\varphi X) = (Z_r\varphi)X + (G^r\varphi)(\nabla_{Z_r}X) \quad , \quad \nabla_{Z_r}(\mathscr{V}_s(V, \Lambda V)) \subset \mathscr{V}_{r+s}(V, \Lambda V) \tag{60}$$

will be called a connection on $(V, \Lambda V)$. For $Z, Y \in \mathscr{V}(V, \Lambda V)$ $\nabla_Z Y$ will be called covariant derivative of Y along Z.

If $V = \overline{O} \times K^m$ then the connection is given by

$$\Gamma^\lambda_{\nu\mu}, \Gamma^c_{b\mu}, \Gamma^c_{\nu a}, \Gamma^\lambda_{\beta a} \in C^\infty(\overline{O}, \Lambda_0 K^m)$$

and

$$\Gamma^c_{\nu\mu}, \Gamma^\lambda_{b\mu}, \Gamma^\lambda_{\nu a}, \Gamma^c_{ba} \in C^\infty(\overline{O}, \Lambda_1 K^m),$$

where (summation convention!)

$$\nabla_{\frac{\partial}{\partial x^\nu}} \frac{\partial}{\partial x^\mu} = \Gamma^\lambda_{\nu\mu} \frac{\partial}{\partial x^\lambda} + \Gamma^c_{\nu\mu} \frac{\partial}{\partial \theta^c},$$

$$\nabla_{\frac{\partial}{\partial\theta^b}} \frac{\partial}{\partial x^\mu} = \Gamma^\lambda_{b\mu} \frac{\partial}{\partial x^\lambda} + \Gamma^c_{b\mu} \frac{\partial}{\partial \theta^c},$$

$$\nabla_{\frac{\partial}{\partial x^\nu}} \frac{\partial}{\partial \theta^a} = \Gamma^\lambda_{\nu a} \frac{\partial}{\partial x^\lambda} + \Gamma^c_{\nu a} \frac{\partial}{\partial \theta^c},$$

$$\nabla_{\frac{\partial}{\partial\theta^b}} \frac{\partial}{\partial \theta^a} = \Gamma^\lambda_{ba} \frac{\partial}{\partial x^\lambda} + \Gamma^c_{ba} \frac{\partial}{\partial \theta^c}.$$

As an example of the structures defined above consider a 2-covariant symmetric tensor field g on $(V, \Lambda V)$. It can be identified with a bilinear form on $\mathscr{V}(V, \Lambda V) \times \mathscr{V}(V, \Lambda V)$ taking values in $\Gamma(\Lambda V)$ and possessing the following properties: for $Z, Z'_r \in \mathscr{V}_r(V, \Lambda V)$, $\psi \in \Gamma(\Lambda V)$

$$\langle Z_r, Z'_s \mid g \rangle \in \Gamma(\Lambda_{r+s} V), \tag{62}$$

$$\langle Z_r, Z'_s \mid g \rangle = (-1)^{rs} \langle Z'_s, Z_r \mid g \rangle, \tag{63}$$

$$\langle \psi Z_r, Z'_s \mid g \rangle = \psi \langle Z_r, Z'_s \mid g \rangle. \tag{64}$$

Note that any vector field Z on $(V, \Lambda V)$ defines a vector field Z_0 on the base space of V, M by

$$Zf = Z_0 f + Z_1 f + \ldots, \tag{65}$$

where $f \in C^\infty_0(M)$, $Z_i f \in \Gamma(\Lambda^i V)$. Suppose that $\mathscr{X}_1$, $\mathscr{X}_2$ are two vector fields on M with compact support. Take any even vector fields Z_1, Z_2 on $(V, \Lambda V)$ such that $\mathscr{X}_i = (Z_i)_0$, $i = 1, 2$. This can be always done by introducing a connection in V over a neighborhood of $\bigcup_i$ suppt Z_i and taking covariant derivatives of sections along $\mathscr{X}_i$ as Z_i. Now the component $\langle Z_1, Z_2 \mid g \rangle_0$ of $\langle Z_1, Z_2 \mid g \rangle$ in $\Gamma(\Lambda^0 V) = C^\infty(M)$ does not depend on the choice of Z_i (this is easily checked locally). If

$$\langle \mathscr{X}_1, \mathscr{X}_2 \mid g_0 \rangle := \langle Z_1, Z_2 \mid g \rangle_0 \tag{66}$$

then we easily see that g_0 is a bilinear symmetric form on vector fields with compact support on M and that for $f \in C^\infty_0(M)$

$$\langle f \mathscr{X}_1, \mathscr{X}_2 \mid g_0 \rangle = f \langle \mathscr{X}_1, \mathscr{X}_2 \mid g_0 \rangle. \tag{67}$$

Hence g_0 defines a 2-covariant symmetric tensor field on M. We shall say that g is non-degenerate, (pseudo-)Riemannian (with given signature) if these statements hold for g_0.

If $V = \bar{O} \times K^m$ then g is given by (summation convention!)

$$g = \frac{1}{2} g_{\mu\nu} dx^\mu \otimes dx^\nu + (G g_{\mu a}) dx^\mu \otimes d\theta^a + (G g_{\mu a}) d\theta^a \otimes dx^\mu - \frac{1}{2} g_{ab} d\theta^a \otimes d\theta^b, \tag{68}$$

where $g_{\mu\nu} = g_{\nu\mu}$, $g_{ab} = - g_{ba} \in C^\infty(\bar{O}, \Lambda_0 K^m)$ and $g_{\mu a} \in C^\infty(\bar{O}, \Lambda_1 K^m)$.

Similarly to the case of usual geometry to each (pseudo-)Riemannian tensor field on a superspace one can assign a superspace connection [*11*].

This gave rise to a superspace generalization of general relativity proposed in [11]. Einstein equations for the superspace case comprise a system of equations for component fields of g, including Einstein equation for g_0 with energy-momentum tensor for other components of g on the right-hand side (see [11] for some details). Thus superspace geometry groups a system of classical non-trivially coupled fields, including gravitational field and spinor fields, into a superfield. However once we have the equations of motion of the fields we can forget about supergeometry because, as follows from the discussion in Section 3, no new global geometric effects are introduced by gluing together local superfields.

5. SYMMETRIES OF A SUPERSPACE, SUPERSYMMETRIES

Superspaces appeared in physical literature as a mean to describe structures which appeared in connection with a new type of physical symmetries, so-called supersymmetries. Here we reverse the historical order introducing superspaces first and identifying supersymmetries as some geometric symmetries of a trivial superspace.

Let G_0 be a Lie group.

DEFINITION 5. — We shall say that G_0 acts on the superspace $(V, \Lambda V)$ if we are given a homomorphism of G_0 into the group of all diffeomorphisms of $(V, \Lambda V)$ such that for each $\varphi \in \Gamma_0(\Lambda V)$

$$G_0 \times M \ni (g, m) \mapsto (g\varphi)(m) \in \Lambda V \tag{69}$$

is smooth. Here, as usually, M denotes the base space of V and g stands for the image of φ under the diffeomorphism of $(V, \Lambda V)$ assigned to $g \in G_0$.

An action of G_0 on $(V, \Lambda V)$ induces (by the assignments of Proposition 5) smooth actions of G_0 on V by bundle isomorphisms and on M by diffeomorphisms. We shall call them underlying actions.

Let $\mathscr{X}$ be a Z_2-graded Lie algebra.

DEFINITION 6. — We shall say that $\mathscr{X}$ acts on $(V, \Lambda V)$ if we are given a Z_2-graded-Lie-algebra homomorphism of $\mathscr{X}$ into $\mathscr{V}(V, \Lambda V)$. In particular if $\mathscr{X}$ is a Lie algebra we shall say that it acts on $(V, \Lambda V)$ if we are given a Lie-algebra homomorphism of $\mathscr{X}_0$ into $\mathscr{V}_0(V, \Lambda V)$.

If G_0 acts on $(V, \Lambda V)$ then by infinitesimalization we obtain the action of its Lie algebra $\mathscr{X}_0$ on $(V, \Lambda V)$:

$$\mathscr{X}\varphi : = \frac{d}{dt}\Big|_{t=0} \exp (t\mathscr{X})\varphi, \tag{70}$$

$\mathscr{X} \in \mathscr{X}_0$, $\varphi \in \Gamma_0(\Lambda V)$. Here $\mathscr{X}\varphi$ denotes the image of φ under the even vector field on $(V, \Lambda V)$ assigned to $\mathscr{X}$.

For an even vector field X on (V, ΛV) denote by X_0 the underlying vector field on M (compare (65)). The assignment

$$\mathscr{V}(V, \Lambda V) \ni X \mapsto X_0 \in \mathscr{V}(M),$$

where $\mathscr{V}(M)$ denotes the Lie algebra of vector fields on M, is a homomorphism of Lie algebras. Thus any action of a Lie algebra $\mathscr{Z}_0$ on $(V, \Lambda\mathscr{V})$ induces an underlying action of $\mathscr{Z}_0$ on M.

The problem appears as to when the action of a Lie algebra on (V, ΛV) comes from infinitesimalization of a group action. We conjecture that this is the case if and only if the underlying action on M takes values in complete vector fields on M.

Consider a Z_2-graded Lie algebra $\mathscr{Z} = \mathscr{Z}_0 + \mathscr{Z}_1$ acting on the superspace (M × E, M × ΛE), where M is an n-dimensional manifold and E an m-dimensional (real or complex) vector space. Suppose moreover that the action of vector fields on (M × E, M × ΛE) assigned to elements of $\mathscr{Z}$ leaves constant maps from $C^\infty(M, \Lambda E)$ constant. Then we can define an action of elements of $\mathscr{Z}$ on elements of ΛE. Consider the vector space

$$\tilde{\mathscr{Z}} := \Lambda_0 E \otimes \mathscr{Z}_0 + \Lambda_1 E \otimes \mathscr{Z}_1. \tag{71}$$

Define a bilinear operation [.,.] on $\tilde{\mathscr{Z}}$ by

$$\begin{aligned}[e_1 \otimes Z_1, e_2 \otimes Z_2] &= (-1)^\sigma(e_1 e_2) \otimes [Z_1, Z_2] \\ &\quad + (-1)^\sigma(e_1(Z_1 e_2)) \otimes Z_2 - (-1)^\sigma(e_2(Z_2 e_1)) \otimes Z_1,\end{aligned} \tag{72}$$

where e_1, e_2 belong to $\Lambda_0 E$ or $\Lambda_1 E$ and Z_1, Z_2 to $\mathscr{Z}_0$ or $\mathscr{Z}_1$. Here σ in $(-1)^\sigma$ in front of a term on the right-hand side counts, as usually, the number of transpositions of neighboring odd elements necessary to restore the order of e_1, e_2, Z_1, Z_2 in which these elements appear on the left hand side. Define also an action of $\tilde{\mathscr{Z}}$ on $C^\infty(M, \Lambda E)$ by

$$(e \otimes Z)\varphi := e(Z\varphi). \tag{73}$$

Proposition 10. — $\tilde{\mathscr{Z}}$ with the [.,.] operation forms a Lie algebra whose action on (M × E, M × ΛE) is defined by (73).

Proof. — Direct inspection. ■

Now take for $\mathscr{Z} = \mathscr{Z}_0 \oplus \mathscr{Z}_1$ the Z_2-graded Lie algebra introduced by Volkov-Akulov and by Wess-Zumino [*17*, *18*]. In this case $\mathscr{Z}_0$ is the Lie algebra of Poincaré group (generators P^λ, $J^{\mu\nu} = -J^{\nu\mu}$) and $\mathscr{Z}_1$ is the charge conjugation invariant part of the 4-dimensional complex space $\mathscr{Z}_1^{\mathbb{C}}$ of Dirac spinors (generators Q_a, $\overline{Q}_{\dot a}$ in the Van der Waerden representation [*10*], $a = 1, 2$, $\dot a = \dot 1, \dot 2$). Charge conjugation is the antilinear involution interchanging Q_a and $\overline{Q}_{\dot a}$. The non-vanishing (complexified) brackets between the generators of $\mathscr{Z}^{\mathbb{C}}$ are given by (summation convention!)

$$[J^{\mu\nu}, J^{\rho\delta}] = g^{\mu\delta}J^{\nu\rho} - g^{\mu\rho}J^{\nu\delta} + g^{\nu\rho}J^{\mu\delta} - g^{\nu\delta}J^{\mu\rho}, \tag{74}$$

$$[J^{\mu\nu}, P^{\lambda}] = g^{\nu\lambda}P^{\mu} - g^{\mu\lambda}P^{\nu} \text{ (standard Poincaré algebra relations)}, \quad (75)$$

$$[J^{\mu\nu}, Q_a] = -\frac{1}{2}(\sigma^{\mu\nu})^b{}_a Q_b, \quad (76)$$

$$[J^{\mu\nu}, \overline{Q}_{\dot a}] = -\frac{1}{2}(\sigma^{\mu\nu})^{\dot b}{}_{\dot a}\overline{Q}_{\dot b} \text{ (Lorentz rotation of Dirac spinors)}, \quad (77)$$

$$[Q_a, \overline{Q}_{\dot b}] = (\sigma_\mu)_{a\dot b}P^{\mu}, \quad (78)$$

where $g^{\mu\nu} = 0$ if $\mu \neq \nu$, $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$. Here $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]_-$. $\gamma^\mu : \mathscr{L}_1^{\mathbb{C}} \to \mathscr{L}_1^{\mathbb{C}}$ and have the following matrix elements (Dirac matrices) in the basis $(Q_a, \overline{Q}_{\dot a})$:

$$(\gamma^\mu)^a{}_b = 0, \qquad (\gamma^\mu)^a{}_{\dot b} = -(\sigma^\mu)^a{}_{\dot b},$$
$$(\gamma^\mu)^{\dot a}{}_b = (\sigma^\mu)_b{}^{\dot a}, \qquad (\gamma^\mu)^{\dot a}{}_{\dot b} = 0. \quad (79)$$

$(\sigma^\mu)_{a\dot b}$ are elements of Pauli matrices and undotted and dotted indices are raised and lowered by means of antisymmetric matrices

$$(\varepsilon_{ab}) = (\varepsilon^{ab}) \quad , \quad (\varepsilon_{\dot a\dot b}) = (\varepsilon^{\dot a\dot b}) \quad , \quad \varepsilon_{12} = \varepsilon_{\dot 1\dot 2} = 1,$$

e. g.

$$(\sigma^\mu)^a{}_{\dot b} = \varepsilon^{ac}(\sigma^\mu)_{c\dot b}.$$

Let M be the Minkowski space. Let $E := (\mathscr{L}_1^{\mathbb{C}})^*$. E is spanned by elements of the basis $(\theta^a, \overline{\theta}^{\dot a})$, $a = 1, 2$, $\dot a = \dot 1, \dot 2$, dual to $(Q_a, \overline{Q}_{\dot a})$. $\mathscr{L}$ acts on $(M \times E, M \times \Lambda E)$, the action being given by

$$P^\nu\varphi = -\frac{\partial\varphi}{\partial x_\nu}, \quad (80)$$

$$J^{\lambda\varkappa}\varphi = x^\lambda\frac{\partial\varphi}{\partial x_\varkappa} - x^\varkappa\frac{\partial\varphi}{\partial x_\lambda} + \frac{1}{2}(\sigma^{\lambda\varkappa})^a{}_b\theta^b\frac{\partial\varphi}{\partial\theta^a} + \frac{1}{2}(\sigma^{\lambda\varkappa})^{\dot a}{}_{\dot b}\overline{\theta}^{\dot b}\frac{\partial\varphi}{\partial\overline{\theta}^{\dot a}}, \quad (81)$$

$$Q_a\varphi = \frac{\partial\varphi}{\partial\theta^a} - \frac{1}{2}(\sigma^\nu)_{a\dot b}\overline{\theta}^{\dot b}\frac{\partial\varphi}{\partial x^\nu}, \quad (82)$$

$$\overline{Q}_{\dot a}\varphi = \frac{\partial\varphi}{\partial\overline{\theta}^{\dot a}} - \frac{1}{2}(\sigma^\nu)_{b\dot a}\theta^b\frac{\partial\varphi}{\partial x^\nu}. \quad (83)$$

Now $\tilde{\mathscr{L}}$ is spanned by elements

$$\alpha_\mu P^\nu, \ \alpha_{\lambda\varkappa}J^{\lambda\varkappa}, \ \varepsilon^a Q_a, \ \overline{\varepsilon}^{\dot a}\overline{Q}_{\dot a},$$

where α_μ, $\alpha_{\lambda\kappa} \in \Lambda_0 E$ and ε^a, $\overline{\varepsilon}^{\dot a} \in \Lambda_1 E$ (we have skipped the tensor product mark). Expressions like that were used in physical literature. They were described as elements of a Lie algebra over commuting and anticommuting scalars. Anticommuting ε^a, $\overline{\varepsilon}^{\dot a}$ were introduced to restore antisymmetry of the bracket just as in our construction. There was no doubt that they should have belonged to the odd part of a Grassmann algebra. Confusion reigned about what algebra should be taken [5]. We take the one in which take values functions in the space of which acts $\mathcal{L}$. This enables representing $\tilde{\mathcal{L}}$ in the same space of functions. One must stress however that it is not possible to consider $\tilde{\mathcal{L}}$ and its representations instead of $\mathcal{L}$. Their physical content is different. For example there are subspaces of $C^{\infty}(M, \Lambda E)$ invariant under the action of $\mathcal{L}$ which are not invariant under the action of $\tilde{\mathcal{L}}$. However the global transformations of the superspace corresponding to infinitesimal action of $\tilde{\mathcal{L}}$ play an important role. For example if we use such a transformation

$$B = \exp\left(-\frac{1}{2}(\sigma^{\nu})_{a\dot{b}}\theta^{a}\bar{\theta}^{\dot{b}}\frac{\partial}{\partial x^{\nu}}\right) \tag{84}$$

then the invariant condition for superfields φ (*i. e.* elements of $C^{\infty}(M, \Lambda E)$

$$\left(\frac{\partial}{\partial\bar{\theta}^{\dot{a}}} + \frac{1}{2}(\sigma^{\nu})_{b\dot{a}}\theta^{b}\frac{\partial}{\partial x^{\nu}}\right)\varphi = 0 \tag{85}$$

can be written as

$$\frac{\partial}{\partial\bar{\theta}^{\dot{a}}}(B\varphi) = 0 \tag{86}$$

and means that there are no elements with $\bar{\theta}^{\dot{a}}$ in the decomposition of $B\varphi$ into polynomials in $\theta-s$. Such transformations were extensively used as they simplify computations considerably. In superfield notation they were written as

$$(x^{\mu}, \theta^{a}, \bar{\theta}^{\dot{a}}) \mapsto \left(x^{\mu} - \frac{1}{2}(\sigma^{\mu})_{c\dot{b}}\theta^{c}\bar{\theta}^{\dot{b}}, \theta^{a}, \bar{\theta}^{\dot{a}}\right). \tag{87}$$

This notation was taken usually too seriously causing necessity to consider $x^{\mu}-s$ as even elements of a Grassmann algebra. There is no trace of difficulties of this type in our approach.

APPENDIX

PROOF OF PROPOSITION 1. — Let for $f \in C_0^\infty(M)$

$$\widehat{x}f = f(x). \tag{A.1}$$

$\widehat{x}$ is a linear, non-zero, multiplicative functional (character) on $C_0^\infty(M)$.

LEMMA 1. — Let v be a character on $C_0^\infty(M)$. Then $v = \widehat{x}$ for some $x \in M$.

PROOF OF LEMMA 1. — Suppose that for each $x \in M$ there exists $g_x \in C_0^\infty(M)$ such that $vg_x = 0$ and $g_x(x) \neq 0$. Taking $h_x \mid g_x \mid$ with suitable $h_x \in C_0^\infty(M)$ instead of g_x if necessary we can assume that $g_x \geqslant 0$. Let $f \in C_0^\infty(M)$ and let (O_i) be a finite covering of supp f such that for each i and some x_i $g_{x_i}\restriction_{O_i}$ does not vanish. Now

$$f = \sum_k g_{x_k} f\Big(\sum_i g_{x_i}\Big)^{-1} \tag{A.2}$$

and $f\Big(\sum_i g_{x_i}\Big)^{-1} \in C_0^\infty(M)$. Thus

$$vf = \sum_k (vg_{x_k})\Big(v\Big(f\Big(\sum_i g_{x_i}\Big)^{-1}\Big)\Big) = 0. \tag{A.3}$$

Hence v vanishes—a contradiction. Thus there exists x such that $(vf = 0) \Rightarrow (f(x) = 0)$. Comparing codimensions we conclude that v is proportional to $\widehat{x}$. As both are multiplicative they must be equal. $\square$

Since functions from $C_0^\infty(M)$ separate points of M (M is Hausdorff !) the correspondence between points of M and characters of $C_0^\infty(M)$ is bijective.

Let $\underline{t}$: M $\rightarrow$ N be defined by

$$\widehat{\underline{t}(x)}(f) = \widehat{x}(t^{-1}f) \quad , \quad f \in C_0^\infty(N). \tag{A.4}$$

$\underline{t}$ is bijective and hence (A.4) implies (5). Smoothness of $\underline{t}$ and $\underline{t}^{-1}$ follows easily from (5). ∎

PROOF OF PROPOSITION 2. — We start with

LEMMA 2. — $T\Big(\Gamma_0\Big(\bigoplus_{i=k}^m \Lambda^i V\Big)\Big) \subset \Gamma_0\Big(\bigoplus_{i=k}^m \Lambda^i W\Big)$.

PROOF OF LEMMA 2. — Let $\varphi_1, \ldots, \varphi_j \in \Gamma_0(V)$, $k \leqslant j \leqslant m$. Since T preserves Z_2-grading $T\varphi_1 \in \Gamma_0\Big(\bigoplus_{i=1}^m \Lambda^i W\Big)$. Thus

$$T(\varphi_1, \ldots, \varphi_j) = (T\varphi_1) \ldots (T\varphi_j) \in \Gamma_0\Big(\bigoplus_{i=j}^m \Lambda^i W\Big).$$

But $\Gamma_0\Big(\bigoplus_{i=k}^m \Lambda^i V\Big)$ is spanned by homogeneous elements $\varphi_1 \ldots \varphi_j$. $\square$

Let $T' := T^{-1}$. Then, in virtue of Lemma 2, for $f \in C_0^\infty(N)$

$$f = (T \circ T')f = T(T'_0 f + \ldots + T'_m f) = T_0 \circ T'_0 f + \psi,$$

where $\psi \in \Gamma_0\left(\bigoplus_{i=2}^{m} \Lambda^i V\right)$. Thus $t \circ t' = 1$ and similarly $t' \circ t = 1$. So t is a linear isomorphism.

Since

$$T(fg) = (Tf)(Tg) \tag{A.5}$$

and

$$\begin{aligned} T(fg) &= t(fg) + \psi_1, \\ Tf &= tf + \psi_2, \\ Tg &= tg + \psi_3, \end{aligned}$$

with $\psi_j \in \Gamma_0\left(\bigoplus_{i=2}^{m} \Lambda^i W\right)$, (A.5) yields (4). ∎

PROOF OF PROPOSITION 3.

LEMMA 3. — $T(C_0^\infty(U)) \subset \Gamma_0(\Lambda W\restriction_{\underline{t}(U)})$.

PROOF OF LEMMA 3. — Let $f \in C_0^\infty(U)$, $h \in C_0^\infty(U)$, $h\restriction_{\mathrm{supp}\, f} \equiv 1$.

$$\begin{array}{ccl} Tf & = & \displaystyle\sum_{k=0}^{E\left[\frac{m}{2}\right]} T_{2k} f \\ \Big\| & & \\ T(fh) & = & (Tf)(Th) = \displaystyle\sum_{k=0}^{E\left[\frac{m}{2}\right]} \sum_{l=0}^{k} (T_{2l} f)(T_{2k-2l} h). \end{array}$$

Hence

$$T_{2k} f = \sum_{l=0}^{k} (T_{2l} f)(T_{2k-2l} h).$$

Suppose that we have shown for $k = 0, \ldots, j \left(j = 0, \ldots, E\left[\frac{m}{2}\right] - 1\right)$ that

$$T_{2k} f \in \Gamma_0(\Lambda W\restriction_{\underline{t}(U)})$$

(this holds for $j = 0$). But

$$T_{2j+2} f = \sum_{l=0}^{j} (T_{2l} f)(T_{2j+2-2l} h) + (T_{2j+2} f)(T_0 h).$$

The first $j + 1$ terms on the right-hand side sit in $\Gamma_0(\Lambda W\restriction_{\underline{t}(U)})$ in virtue of the inductive assumption. The last one also does since $T_0 h \in C_0^\infty(\underline{t}(U))$. □

Now let $\varphi \in \Gamma_0(\Lambda V\restriction_U)$, $h \in C_0^\infty(U)$, $h\restriction_{\mathrm{supp}} \equiv 1$.

$$T\varphi = T(\varphi h) = (T\varphi)(Th) \in \Gamma_0(\Lambda W\restriction_{\underline{t}(U)}).$$

Thus $T(\Gamma_0(\Lambda V\restriction U)) \subset \Gamma_0(\Lambda W\restriction_{\underline{t}(U)})$. Since $\underline{t}^{-1}$ is the diffeomorphism of manifolds which corresponds to T^{-1}, the inverse inclusion also holds yielding (8).

As for (9) the latter argument works as well so that it is sufficient to prove that

$$\mathrm{supp}\,(T\varphi) \subset \underline{t}(\mathrm{supp}\,\varphi).$$

For $x \notin \operatorname{supp} \varphi$ there exists an open neighborhood U_x of x such that $U_x \cap \operatorname{supp} \varphi = \varnothing$. Let $\psi \in \Gamma_0(\Lambda W\restriction_{\underline{t}(U_x)})$. By (8) $\psi = T\chi$, $\chi \in \Gamma_0(\Lambda V\restriction U_x)$. Hence

$$\psi(T\varphi) = T(\chi\varphi) = 0.$$

Thus $\underline{t}(x) \notin \operatorname{supp}(T\varphi)$. ■

PROOF OF PROPOSITION 4. — Let $f \in C_0^\infty(M)$, $\varphi \in \Gamma_0(V)$. Then

$$T(f\varphi) \quad = T_1(f\varphi) + \dots$$
$$(Tf)(T\varphi) = (T_0 f + \dots)(T_1\varphi + \dots) = (tf)(\tau\varphi) + \dots$$

Hence

$$\tau(f\varphi) = (f \circ \underline{t}^{-1})\tau\varphi. \tag{A.G}$$

Thus $(\tau\varphi)(y)$, $y \in N$, depends (linearly) only on $\varphi(\underline{t}^{-1}(y))$. We put

$$(\tau\varphi)(y) = \underline{\tau}(\varphi(\underline{t}^{-1}(y))).$$

Checking that τ is smooth is standard. Now if $T' = T^{-1}$ then $\underline{\tau} \circ \underline{\tau}' = 1$ and $\underline{\tau}' \circ \underline{\tau} = 1$. Thus $\underline{\tau}$ is a bundle isomorphism. ■

PROOF OF PROPOSITION 6. — We start with two lemmas.

LEMMA 4. — Let $A = \{(F_a, T_a)\}$ and $A \cup \{(F_0, T_0)\}$ be two atlases for F. Then

$$F_0 = \sum_a F_0 \cap F_a.$$

PROOF OF LEMMA 4. — Since $T_0(F_0 \cap F_a) = C_0^\infty(\bar{O}_{0a}, \Lambda K^m) \subset C_0^\infty(\bar{O}_0, \Lambda K^m)$ we must show that $\bigcup_a \bar{O}_{0a} = \bar{O}_0$. Suppose that $x \in \bar{O}_0$, $x \notin \bigcup_a \bar{O}_{0a}$. Let $h \in C_0^\infty(\bar{O}_0)$, $h \equiv 1$ on a neighborhood U'_x of x. Let $\varphi = \varphi_{a_1} + \dots + \varphi_{a_k}$, $\varphi_{a_i} \in F_{a_i}$. Then $(T_0^{-1}h)\varphi_{a_i} \in F_0 \cap F_{a_i}$. Now

$$T_0\big((T_0^{-1}h)\varphi\big) = \sum_i T_0\big((T_0^{-1}h)\varphi_{a_i}\big) \in C_0^\infty\Big(\bigcup_a \bar{O}_{0a}, \Lambda K^m\Big).$$

Let $U_x \subset U'_x$ be an open neighborhood of x such that

$$U_x \cap \operatorname{supp}\ \big(T_0\big((T_0^{-1}h)\varphi\big)\big) = \varnothing.$$

For $k \in C_0^\infty(U_x)$

$$(T_0^{-1}k)\varphi = (T_0^{-1}k)(T_0^{-1}h)\varphi = T_0^{-1}\big(kT_0\big((T_0^{-1}h)\varphi\big)\big) = 0.$$

Taking $\varphi = T_0^{-1}h$ we see that h must vanish on a neighborhood of x — a contradiction. □

LEMMA 5. — Let $A = \{(F_a, T_a)\}$, $A \cup \{(F_0, T_0)\}$, $A \cup \{(F_1, T_1)\}$ be three atlases for F. Then

$$F_0 \cap F_1 = \sum_a F_0 \cap F_a \cap F_1.$$

PROOF OF LEMMA 5. — $F_0 \cap F_1 \supset \sum_a F_0 \cap F_a \cap F_1$.

$$T_1\Big(\sum_a F_0 \cap F_a \cap F_1\Big) = \sum_a T_1(F_0 \cap F_a \cap F_1)$$

$$= \sum_a T_{1a} \cdot T_a(F_0 \cap F_a \cap F_1) = \sum_a T_{1a}\left(C_0^\infty(\overline{O}_{a0} \cap \overline{O}_{a1}, \Lambda K^m)\right)$$

$$= \bigcup_a C_0^\infty\left(\underline{t}_{1a}(\overline{O}_{a0} \cap \overline{O}_{a1}), \Lambda K^m\right) = C_0^\infty\left(\bigcup_a \underline{t}_{1a}(\overline{O}_{a0} \cap \overline{O}_{a1}), \Lambda K^m\right) \subset C_0^\infty(\overline{O}_1, \Lambda K^m).$$

Now we shall show that

$$T_1(F_0 \cap F_1) \subset C_0^\infty\left(\bigcup_a \underline{t}_{1a}(\overline{O}_{a0} \cap \overline{O}_{a1}), \Lambda K^m\right).$$

Let $\varphi \in F_0 \cap F_1$. We shall show that for each $x \in \overline{O}_1$, $x \notin \bigcup_a t_{1a}(\overline{O}_{a1} \cap \overline{O}_{a1})$ there exist an open neighborhood U_x of x in $\overline{O}_1$ such that $f T_1 \varphi = 0$ for $f \in C_0^\infty(U_x)$. Let $h \in C_0^\infty(\overline{O}_1)$, $h \equiv 1$ on a neighborhood U'_x of x. By lemma 4

$$\varphi = T_{a_1}^{-1}\varphi_{a_1} + \ldots + T_{a_k}^{-1}\varphi_{a_k},$$

where $\varphi_{a_i} \in C_0^\infty(\overline{O}_{a_i0}, \Lambda K^m) = T_a(F_0 \cap F_a)$. Now

$$h_1 T\varphi = T_1\left((T_1^{-1}h)\varphi\right) = T_1\left((T_1^{-1}h)\left(T_{a_1}^{-1}\varphi_{a_1}\right) + \ldots + (T_1^{-1}h)\left(T_{a_k}^{-1}\varphi_{a_k}\right)\right).$$

But

$$\left(T_1^{-1}h\right)\left(T_{a_i}^{-1}\varphi_{a_i}\right) \in F_1 \cap F_{a_i} \cap F_0$$

($F_0 \cap F_{a_i}$ is also *a* Z_2-graded ideal in F). Thus

$$hT_1\varphi \in C_0^\infty\left(\bigcup_a \underline{t}_{1a}(\overline{O}_{a0} \cap \overline{O}_{a1}), \Lambda K^m\right).$$

If U_x is an open neighborhood of x which does not intersect the support of $hT_1\varphi$, $U_x \subset U'_x$, then for $f \in C_0^\infty(U_x)$

$$f T_1 \varphi = f h T_1 \varphi = 0. \ \square$$

To prove statement 1 of proposition 6 we must show that if

$$A = \{ (F_a, T_a) \}, \qquad A' = \{ (F'_b, T'_b) \}, \qquad A'' = \{ (F''_c, T''_c) \}$$

are three atlases for F, A compatible with A′ and A′ compatible with A″ then A is compatible with A″. Thus first we must prove that property *ii*) of definition 3 holds for A ∪ A″, *i. e.* that for each pair (a, c)

$$T_a(F_a \cap F_c) = C_0^\infty(\overline{O}_{ac}, \Lambda K^m),$$
$$T_c(F_a \cap F_c) = C_0^\infty(\overline{O}_{ca}, \Lambda K^m),$$

where $\overline{O}_{ac}$ is an open subset of $\overline{O}_a$ and $\overline{O}_{ca}$ an open subset of $\overline{O}_c$. But in virtue of lemma 5

$$T_a(F_a \cap F_c) = C_0^\infty\left(\bigcup_a \underline{t}_{ab}(\overline{O}_{ba} \cap \overline{O}_{bc}), \Lambda K^m\right),$$

$$T_c(F_a \cap F_c) = C_0^\infty\left(\bigcup_b \underline{t}_{cb}(\overline{O}_{bc} \cap \overline{O}_{ba}), \Lambda K^m\right).$$

Next we prove that for A ∪ A″ property *iii*) of definition 3 holds. Let $x_1 \in \overline{O}_a \backslash \overline{O}_{ac}$, $x_2 \in \overline{O}_c \backslash \overline{O}_{ca}$. There exist b_1, b_2 such that $x_1 \in \overline{O}_{ab_1}$, $x_2 \in \overline{O}_{c_2 b}$. Let

$$x'_1 := \underline{t}_{b_1 a}(x_1) \in \overline{O}_{b_1 a} \qquad \text{and} \qquad x'_2 := \underline{t}_{b_2 c}(x_2) \in \overline{O}_{b_2 c}.$$

It is sufficient to show that there exist neighborhoods $U_{x_1'}$ of x_1' in $\bar{O}_{b_1}$ and $U_{x_2'}$ of x_2' in $\bar{O}_{b_2}$ such that

$$T_{b_1}^{-1}\left(C_0^\infty\left(U_{x_1'}, \Lambda K^m\right)\right) \cap T_{b_2}^{-1}\left(C_0^\infty\left(U_{x_2'}, \Lambda K^m\right)\right) = \{0\}.$$

But this is the case provided that $x_1' \neq \underline{t}_{b_1b_2}(x_2')$, what holds since in the opposite case $x_1 = \underline{t}_{ac}(x_2)$ and $x_1 \in \bar{O}_{ac}$, $x_2 \in \bar{O}_{ca}$ — a contradiction.

To prove statement 2 of proposition 6 consider the family (A_i) of all atlases compatible with A_0. Then $A := \bigcup_i A_i$ is an atlas compatible with A_0 (proof of this proceeds in full analogy to the proof of statement 1). A is complete since any bigger atlas is compatible with A_0 and thus comprised in A. Also any other complete atlas comprising A_0 is compatible with A_0 and so comprised in A. ■

PROOF OF PROPOSITION 7. — Let $A_k = \{(F_a, T_a)\}$ be of class k. In what follows we shall write shortly $1, 2, \dots$ instead of $a_1, a_2, \dots$

$$T_{a_2a_1} \equiv T_{21}$$

fulfils (compare (19))

$$(T_{21})_{2k}(fg) = (f \circ \underline{t}_{12})((T_{21})_{2k}g) + ((T_{21})_{2k}f)(g \circ \underline{t}_{12}). \tag{A.7}$$

Hence

$$(T_{21})_{2k}f = \sum_j w_{21}^j \frac{\partial}{\partial y^j}(f \circ \underline{t}_{12}), \tag{A.8}$$

where $w_{21}^j \in C_0^\infty(\bar{O}_{21}, \Lambda^{2k}K^m)$. Let $s_{a_2a_1} \equiv s_{21}$ be the smooth section of $TM \otimes \Lambda^{2k}V$ over $O_{21} := \underline{t}_2^{-1}(\bar{O}_{21})$ $(s_{21} \in \Gamma_{O_{21}}(TM \otimes \Lambda^{2k}V))$ given by

$$s_{21} := \sum_j \left(\underline{t}_{2*}^{-1} \frac{\partial}{\partial y^j}\right) \otimes \left(\Lambda^{2k}\underline{\tau}_2^{-1} \circ \left(1 \oplus w_{21}^j\right) \circ \underline{t}_2\right). \tag{A.9}$$

LEMMA 6. $s_{32} + s_{21} = s_{31}$ on $O_1 \cap O_2 \cap O_3$ whenever this set is not empty.

PROOF OF LEMMA 6. — From $T_{32}.T_{21} = T_{31}$ we get

$$(T_{32})_{2k}(f \circ \underline{t}_{12}) + (T_{32} \circ (T_{21})_{2k}f)_{2k} = (T_{31})_{2k}f \tag{A.10}$$

for $f \in C_0^\infty(O_{12} \cap \underline{t}_{21}^{-1}(\bar{O}_{23}))$. $(T_{32}.(T_{21})_{2k}f)_{2k}$ denotes the component of $T_{32} \circ (T_{21})_{2k}f$ in $C_0^\infty)\bar{O}_{32} \cap \underline{t}_{23}^{-1}(\bar{O}_{21}), \Lambda^{2k}K^m)$. Hence

$$\sum_j w_{32}^j \frac{\partial}{\partial z^j}(f \circ \underline{t}_{13}) + \sum_l pr_2 \circ \Lambda^{2k}\underline{\tau}_{32} \circ \left(1 \oplus w_{21}^l \frac{\partial}{\partial y^l}(f \circ \underline{t}_{12})\right) \circ \underline{t}_{23} = \sum_j w_{31}^i \frac{\partial}{\partial z^i}(f \circ \underline{t}_{13}). \tag{A.11}$$

Transforming (A.11) we get

$$\sum_j w_{32}^j \frac{\partial}{\partial z^j}(f \circ \underline{t}_{13}) + \sum_j \left[\sum_l \left(\frac{\partial \underline{t}_{32}^j}{\partial y^l} \circ \underline{t}_{23}\right)\left(pr_2 \circ \Lambda^{2k}\underline{\tau}_{32} \circ \left(1 \oplus w_{21}^l\right) \circ \underline{t}_{23}\right)\right] \frac{\partial}{\partial z^j}(f \circ \underline{t}_{13})$$
$$= \sum_j w_{31}^j \frac{\partial}{\partial z^i}(f \circ \underline{t}_{13}).$$

Hence

$$w_{31}^j = w_{32}^j + \sum_l \left(\frac{\partial \underline{t}_{32}^j}{\partial y^1} \circ \underline{t}_{23}\right)\left(pr_2 \circ \Lambda^{2k}\underline{\tau}_{32} \circ \left(1 \oplus w_{21}^l\right) \circ \underline{t}_{23}\right). \tag{A.12}$$

From (A.9) and (A.12) we obtain

$$s_{31} = s_{32} + \sum_{j,l} \left(\frac{\partial t^j_{32}}{\partial y^l} \circ \underline{t}_2 \right) \left(\underline{t}^{-1}_{2*} \frac{\partial}{\partial z^j} \right) \otimes \left(\Lambda^{2k} \underline{\tau}_2^{-1} \circ \left(1 \oplus w^l_{21} \right) \circ \underline{t}_2 \right),$$

$$s_{31} = s_{32} + \sum_j \left(\underline{t}^{-1}_2 \frac{\partial}{\partial y^l} \right) \otimes \left(\Lambda^{2k} \underline{\tau}_2^{-1} \circ \left(1 \oplus w^l_{21} \right) \circ \underline{t}_2 \right) = s_{32} + s_{21}. \square$$

In virtue of Lemma 6 $(s_{a_1a_2})$ define a 1-cocycle of local sections of $TM \otimes \Lambda^{2k}V$ for the covering (O_a) of M. As the sheaf Σ of local sections of $TM \otimes \Lambda^{2k}V$ is fine (M is paracompact !) $H^1(M, \Sigma) = 0$ [6]. Hence passing at most from A_k to a compatible atlas connected with a finer covering we can assume that there exist sections $s_a \in \Gamma_{O_a}(TM \otimes \Lambda^{2k}V)$ such that

$$s_{a_1a_2} = s_{a_1} - s_{a_2}. \tag{A.13}$$

Let us define $w^j_a \in C^\infty\left(\overline{O}_a, \Lambda^{2k}K^m\right)$ by

$$s_a := \sum_j \left(\underline{t}^{-1}_{a*} \frac{\partial}{\partial x^j} \right) \otimes \left(\Lambda^{2k} \underline{\tau}_a^{-1} \circ \left(1 \oplus w^j_a \right) \circ \underline{t}_a \right). \tag{A.14}$$

Let

$$T''_a = \exp\left(- \sum_j w^j_a \frac{\partial}{\partial x^j} \right) \circ T_a. \tag{A.15}$$

As one can easily see $\exp\left(- \sum_j w^j_a \frac{\partial}{\partial x^i} \right)$ defines a diffeomorphism of class k of $\left(\overline{O}_a \times K^m, \overline{O}_a + \Lambda K^m\right)$. Indeed. $\exp\left(- \sum_j w^j_a \frac{\partial}{\partial x^j} \right)$ is an automorphism of the Z_2-graded algebra $C^\infty_0\left(\overline{O}_a, \Lambda K^m\right)$ (this follows easily from preserving of Z_2-grading by $\sum_j w^j_a \frac{\partial}{\partial x}$ and from the Leibniz rule

$$\left(\sum_j w^j_a \frac{\partial}{\partial x^j} \right) (\varphi\psi) = \left[\left(\sum_j w^j_a \frac{\partial}{\partial x^j} \right) \varphi \right] \psi + \varphi \left[\left(\sum_j w^j_a \frac{\partial}{\partial x^j} \right) \psi \right]$$

in virtue of nilpotency of this operation).

Let us notice that

$$T''_{a_2a_1} = T''_{21} = \exp\left(- \sum_l w^l_2 \frac{\partial}{\partial y^l} \right) \circ T_{21} \circ \exp\left(\sum_j w^j_1 \frac{\partial}{\partial x^j} \right). \tag{A.16}$$

For $f \in C^\infty_0(\overline{O}_{12})$ this yields

$$\begin{aligned} T''_{21} f &= f \circ \underline{t}''_{12} + \left(T''_{21}\right)_{2k} f + \ldots \exp\left(- \sum_l w^l_2 \frac{\partial}{\partial y^l} \right) \circ T_{21} \circ \exp\left(\sum_j w^j_1 \frac{\partial}{\partial x^j} \right) f \\ &= \left(1 - \sum_l w^l_2 \frac{\partial}{\partial y^l} + \ldots \right) \circ T_{21} \circ \left(1 + \sum_j w^j_1 \frac{\partial}{\partial x^j} + \ldots \right) f \\ &= f \circ \underline{t}_{12} - \sum_l w^l_2 \frac{\partial}{\partial y^l} (f \circ \underline{t}_{12}) + (T_{12})_{2k} f + \left(T_{21} \left(\sum w^j_1 \frac{\partial f}{\partial x^j} \right) \right)_{2k} + \ldots \end{aligned} \tag{A.17}$$

Thus

$$\mathrm{T}''_{21}f = f\circ \underline{t}_{12} - \sum_l w_2^l \frac{\partial}{\partial y^l}(f\circ \underline{t}_{12}) + \sum_l w_{21}^l \frac{\partial}{\partial y^l}(f\circ \underline{t}_{12})$$
$$+ \sum_l \left(\sum_j \left(\frac{\partial \underline{t}_{21}^l}{\partial x^j} \circ \underline{t}_{12} \right) \left(pr_2 \circ \Lambda^{2k} \underline{\tau}_{21} \circ \left(1 \oplus w_1^j\right) \circ \underline{t}_{12} \right) \right) \frac{\partial}{\partial y^l}(f\circ t_{12}) + \dots \quad \text{(A.18)}$$

But from (A.13), (A.14) and (A.9) we get

$$w_{21}^l = w_2^l - \sum_j \left(\frac{\partial \underline{t}_{21}^l}{\partial x^j} \circ \underline{t}_{12} \right) \left(pr_2 \circ \Lambda^{2k} \tau_{21} \circ \left(1 \oplus w_1^j\right) \circ \underline{t}_{12} \right). \quad \text{(A.19)}$$

From (A.18) and (A.19) we obtain

$$\mathrm{T}''_{21}f = f\circ \underline{t}_{12} + \left(\mathrm{T}''_{21}\right)_{2k+2}f + \dots \quad \text{(A.20)}$$

Thus we can assume that not only $\mathrm{A} = \{ (\mathrm{F}_a, \mathrm{T}_a) \}$ is of class k but that moreover $(\mathrm{T}_{a_1a_2})_{2k} = 0$ for each pair (a_1, a_2).

Now $(\mathrm{T}_{a_2a_1})_{2k+1} = (\mathrm{T}_{21})_{2k+1}$ fulfil

$$(\mathrm{T}_{21})_{2k+1}(f\varphi) = (f\circ \underline{t}_{12})((\mathrm{T}_{21})_{2k+1}\varphi) \quad \text{(A.21)}$$

for $\varphi \in \mathrm{C}_0^\infty\left(\overline{\mathrm{O}}_{12}, \mathrm{K}^m\right)$. It follows that $((\mathrm{T}_{21})_{2k+1}\varphi)(y)$ depends (linearly) only on the value of φ at $\underline{t}_{12}(y)$. Thus

$$((\mathrm{T}_{21})_{2k+1}\varphi)(y) = \mathrm{M}_{21}(y)\varphi(\underline{t}_{12}(y)). \quad \text{(A.22)}$$

where $\mathrm{M}_{21}(y) \in \mathrm{Hom}\left(\mathrm{K}^m, \Lambda^{2k+1}\mathrm{K}^m\right)$ (*i. e.* is a linear homomorphism from K^m to $\Lambda^{2k+1}\mathrm{K}^m$) and depends smoothly on y.

Let us define $r_{a_1a_2} = r_{21} \in \Gamma_{\mathrm{O}_{21}}\left(\mathrm{Hom}\,(\mathrm{V}, \Lambda^{2k+1}\mathrm{V})\right)$ by

$$r_{21}\left(\underline{\tau}_1^{-1}(x, w)\right) := \Lambda^{2k+1}\underline{\tau}_2^{-1}\left(\underline{t}_{21}(x), \mathrm{M}_{21}(\underline{t}_{21}(x))w\right), \quad \text{(A.23)}$$

$x \in \overline{\mathrm{O}}_{12}$, $w \in \mathrm{K}^m$.

LEMMA 7. $r_{32} + r_{21} = r_{31}$ on $\mathrm{O}_1 \cap \mathrm{O}_2 \cap \mathrm{O}_3$ whenever this set is not empty.

PROOF OF LEMMA 7. — From $\mathrm{T}_{32} \circ \mathrm{T}_{21} = \mathrm{T}_{31}$ we get for $\varphi \in \mathrm{C}_0^\infty\left(\overline{\mathrm{O}}_{12} \cap \underline{t}_{21}^{-1}(\overline{\mathrm{O}}_{23}), \mathrm{K}^m\right)$

$$(\mathrm{T}_{31})_{2k+1}\varphi = (\mathrm{T}_{32})_{2k+1} \circ (\mathrm{T}_{21})_1\varphi + (\mathrm{T}_{32} \circ (\mathrm{T}_{21})_{2k+1}\varphi)_{2k+1}, \quad \text{(A.24)}$$

where $(\mathrm{T}_{32} \circ (\mathrm{T}_{21})\,\varphi_{2k+1})$ denotes the component of $\mathrm{T}_{32} \circ (\mathrm{T}_{21})_{2k+1}\varphi$ in

$$\mathrm{C}_0^\infty\left(\overline{\mathrm{O}}_3 \cap \underline{t}_{23}^{-1}(\overline{\mathrm{O}}_{21}), \Lambda^{2k+1}\mathrm{K}^m\right).$$

Inserting (A.22) to (A.24) one obtains

$$\mathrm{M}_{31}(z) = \mathrm{M}_{32}(z) \circ pr_2 \circ \underline{\tau}_{21}(\underline{t}_{13}(z), .) + pr_2 \circ \Lambda^{2k+1}\underline{\tau}_{32}(\underline{t}_{23}(z), \mathrm{M}_{21}(\underline{t}_{23}(z))). \quad \text{(A.25)}$$

(A.25) and (A.23) give

$$\begin{aligned} r_{31}\left(\underline{\tau}_1^{-1}(x, w)\right) &= \Lambda^{2k+1}\underline{\tau}_3^{-1}(\underline{t}_{31}(x), \mathrm{M}_{32}(\underline{t}_{31}(x))(pr_2 \circ \underline{\tau}_{21}(x, w)) \\ &\quad + \Lambda^{2k+1}\underline{\tau}_3^{-1}(\underline{t}_{31}(x), pr_2 \circ \Lambda^{2k+1}\tau_{32}(\underline{t}_{21}(x), \mathrm{M}_{21}(\underline{t}_{21}(x))w)) \\ &= r_{32}\left(\underline{\tau}_2^{-1}(\underline{t}_{21}(x), pr_2 \circ \underline{\tau}_{21}(x, w))\right) + \Lambda^{2k+1}\underline{\tau}_2^{-1}(\underline{t}_{21}(x), \mathrm{M}_{21}(\underline{t}_{21}(x))w)) \\ &= r_{32}\left(\underline{\tau}_1^{-1}(x, w)\right) + r_{21}\left(\underline{\tau}_1^{-1}(x, w)\right). \ \square \end{aligned}$$

In virtue of lemma 7 $(r_{a_1a_2})$ define *a* 1-cocycle of local sections of Hom $(\mathrm{V}, \Lambda^{2k+1}\mathrm{V})$

for the covering (O_a) of M. Since the sheaf P of local sections of Hom $(V, \Lambda^{2k+1}V)$ is fine (M is paracompact !) $H^1(M, P) = 0$ [6]. Thus passing at most from A_k to a compatible atlas connected with a finer covering we can assume that there exist sections $r_a \in \Gamma_{O_a}$ (Hom $(V, \Lambda^{2k+1}V)$) such that

$$r_{a_1a_2} = r_{a_1} - r_{a_2}. \tag{A.26}$$

Let us define $M_a \in C^\infty(\overline{O}_a, \text{Hom}\,(K^m, \Lambda^{2k+1}K^m))$ by

$$r_a\big(\underline{\tau}_a^{-1}(x, w)\big) = :\Lambda^{2k+1}\underline{\tau}_a^{-1}(x, M_a(x)w). \tag{A.27}$$

For $M_a(x) \in \text{Hom}\,(K^m, \Lambda^{2k+1}K^m)$ let $\overline{M}_a(x) \in \text{Hom}\,(\Lambda K^m, \Lambda K^m)$ denote the extension of $\overline{M}_a(x)$ to a derivation of ΛK^m. We set

$$T'_a : = \exp\,(-\overline{M}_a) \circ T_a, \tag{A.28}$$

where we treat $\overline{M}_a$ as a mapping of $C_0^\infty(\overline{O}_a, \Lambda K^m)$ in the natural way. Since $\exp\,(-\overline{M}_a)$ is an automorphism of Z_2-graded algebra $C_0^\infty(\overline{O}_a, \Lambda K^m)$ of class k, $\{(F'_a, T'_a)\}$ is a new atlas of class k for F compatible with A_k. It is easy to see that $(T'_{a_1a_2})_{2k} = 0$. Now for $\varphi \in C_0^\infty(\overline{O}_a, K^m)$

$$\begin{aligned}
&T'_{21}\varphi = (T'_{21})\varphi + (T'_{21})_{2k+1}\varphi + \dots \\
&\quad\| \\
&\exp\,(-\overline{M}_2) \circ T_{21} \circ \exp\,(\overline{M}_1)\varphi = (1 - \overline{M}_2 + \dots) \circ T_{21} \circ (1 + \overline{M}_1 + \dots)\varphi \\
&\qquad = (T_{21})_1\varphi - M_2(T_{21})_1\varphi + (T_{21})_{2k+1}\varphi + (T_{21}(M_1\varphi))_{2k+1} + \dots
\end{aligned}$$

Hence

$$\begin{aligned}
(T'_{21}\varphi)(y) &= ((T_{21})_1\varphi)(y) - M_2(y) \circ pr_2 \circ \underline{\tau}_{21} \circ (1 \oplus \varphi) \circ \underline{t}_{12}(y) \\
&\quad + M_{21}(y) \circ \varphi \circ \underline{t}_{12}(y) + pr_2 \circ \Lambda^{2k+1}\underline{\tau}_{21} \circ (1 \oplus M_1(\underline{t}_{12}(y)) \circ \varphi) \circ \underline{t}_{12}(y) + \dots
\end{aligned} \tag{A.29}$$

But from (A.26), (A.27) and (A.23)

$$M_{21}(y)w = M_2(y) \circ pr_2 \circ \underline{\tau}_{21}(\underline{t}_{12}(y), w) - pr_2 \circ \Lambda^{2k+1}\underline{\tau}_{21}(\underline{t}_{12}(y), M_1(\underline{t}_{12}(y))w). \tag{A.30}$$

(A.29) and (A.30) give

$$T'_{21}\varphi = (T_{21})_1\varphi + (T'_{21})_{2k+3}\varphi + \dots \tag{A.40}$$

Thus $\{(F'_a, T'_a)\}$ is an atlas of class $k + 1$. ∎

PROOF OF PROPOSITION 8.

LEMMA 8. — Suppose that $\psi_{a_i} \in \bigcap_{j=1}^{p} F_{a_j}$, $i = 1, \dots, p$. Then

$$\left(\psi : = \sum_{i=1}^{p} \psi_{a_i} = 0\right) \Leftrightarrow \left(\sum_{i=1}^{p} I_{a_i}\psi_{a_i} = 0\right).$$

PROOF OF LEMMA 8. — We have $(i \mapsto a_i)$

$$\left(\sum_{i=1}^{p} \psi_i = 0\right) \Leftrightarrow \left(T_1\sum_{i=1}^{p} \psi_i = 0\right) \Leftrightarrow \left(\sum_{i=1}^{p} T_{1i}(T_i\psi_i) = 0\right).$$

But supp $(\mathrm{I}_i\psi_i) \subset \sum_{j=1}^{p} \mathrm{O}_j$. Now

$$pr_2 \circ \underline{\tau}_1 \circ (\mathrm{I}_i\psi_i) \circ \underline{t}_1^{-1} = pr_2 \circ \underline{\tau}_{1i} \circ (1 \oplus \mathrm{T}_i\psi_i) \circ \underline{t}_{i1} = \mathrm{T}_{1i}(\mathrm{T}_i\psi_i)$$

$\left(\text{we use the fact that } \mathrm{A}' = \{ (\mathrm{F}_a, \mathrm{T}_a) \} \text{ is of maximal class } \mathrm{E}\left[\frac{m}{2}\right] + 1\right)$. Thus

$$pr_2 \circ \underline{\tau}_1 \circ \left(\sum_i \mathrm{I}_i\psi_i\right) \circ \underline{t}_1^{-1} = \sum_i \mathrm{T}_{1i}(\mathrm{T}_i\psi_i).$$

Hence

$$\left(\sum_i \mathrm{I}_i\psi_i = 0\right) \Leftrightarrow \left(\sum_i \mathrm{T}_{1i}(\mathrm{T}_i\psi_i) = 0\right). \ \square$$

LEMMA 9. — Suppose that $\psi_{a_i} \in \mathrm{F}_{a_i}$. Then

$$\left(\psi := \sum_{i=1}^{p} \psi_{a_i} = 0\right) \Leftrightarrow \left(\sum_{i=1}^{p} \mathrm{I}_{a_i}\psi_{a_i} = 0\right).$$

PROOF OF LEMMA 9. — $\Rightarrow$ It is sufficient to show that $\sum_i \mathrm{I}_i\psi_i$ vanishes locally. Let $m \in \mathrm{O}_{i_1} \cap \ldots \cap \mathrm{O}_{i_l}$, $m \in \mathrm{O}_j$ if $j \notin \{ i_1, \ldots, i_l \}$. Let U_m be an open neighborhood of m in $\mathrm{O}_{i_1} \cap \ldots \cap \mathrm{O}_{i_l}$ such that

$\mathrm{U}_m \cap$ supp $(\mathrm{I}_j\psi_j) = \varnothing$ if $j \notin \{ i_1, \ldots, i_l \}$. Let $\bar{h} \in \mathrm{C}_0^\infty(\mathrm{U}_m)$. Let

$$h_{i_k} := h \circ \underline{t}_{i_k}^{-1} \in \mathrm{C}_0^\infty(\overline{\mathrm{O}}_{i_k}).$$

$$\mathrm{T}_{i_k}^{-1}h_{i_k} = : h \in \bigcap_{r=1}^{l} \mathrm{F}_{i_r}$$

and does not depend on $k\left(\mathrm{A}' = \{ (\mathrm{F}_a, \mathrm{T}_a) \} \text{ is of class } \mathrm{E}\left[\frac{m}{2}\right] + 1\right)$. Now $h\psi_j = 0$ for $j \notin \{ i_1, \ldots, i_l \}$. Indeed. Let $f \in \mathrm{C}_0^\infty(\overline{\mathrm{O}}_j)$, $g \in \mathrm{C}_0^\infty(\overline{\mathrm{O}}_{i_1})$, $f \equiv 1$ on supp $(\mathrm{T}_j\psi_j)$, $g \equiv 1$ on supp h_{i_1}. $h\psi_j \in \mathrm{F}_{i_j} \cap \mathrm{F}_j$ and

$$\mathrm{T}_{i_1}(h\psi_j) = h_{i_1}\mathrm{T}_{i_1}(\mathrm{T}_{i_1}^{-1}(g)\psi_j).$$

Hence

$$\text{supp } (\mathrm{T}_{i_1}(h\psi_j)) \subset \text{supp } h_{i_1}$$

Furthermore

$$\mathrm{T}_{i_1}(h\psi_j) = \mathrm{T}_{i_1 j}(\mathrm{T}_j(h\psi_j))$$

and

$$\mathrm{T}_j(h\psi_j) = \left(\mathrm{T}_j\left(h\mathrm{T}_j^{-1}f\right)\right)(\mathrm{T}_j\psi_j).$$

So

$$\text{supp } (\mathrm{T}_j(h\psi_j)) \subset \text{supp } (\mathrm{T}_j\psi_j)$$

and consequently

$$\text{supp } (\mathrm{T}_1(h\psi_j)) \subset \underline{t}_{ji_1}^{-1} (\text{supp } (\mathrm{T}_j\psi_j)).$$

Finally

$$\text{supp } (\mathrm{T}_1(h\psi_j)) \subset \text{supp } (h_{i_1}) \cap \underline{t}_{ji_1}^{-1} (\text{supp } (\mathrm{T}_j\psi_j)) = \underline{t}_1(\mathrm{U}_m \cap \text{supp } (\mathrm{I}_j\psi_j) \cap \mathrm{O}_1) = \varnothing \cdot$$

Hence

$$0 = h\psi = h\sum_i \psi_i = \sum_{k=1}^{l} h_{i_k}$$

and

$$h\psi_{i_k} \in \bigcap_{r=1}^{l} F_{i_r}.$$

But

$$I_{i_k}(h\psi_{i_k}) = \bar{h} I_{i_k}\psi_{i_k}.$$

Thus

$$\bar{h}\sum_{k=1}^{l} I_{i_k}\psi_{i_k} = 0.$$

But the sections $I_j\psi_j$ are supported outside U_m if $j \notin \{i_1, \ldots, i_l\}$ and $\bar{h}I_j\psi_j = 0$. Hence

$$\bar{h}\sum_{i=1}^{p} I_i\psi_i = 0.$$

$\Leftarrow$ Take for each $m \in \bigcup_{i=1}^{p} O_i$ U_m as above and W_m to be another open neighborhood of m with compact closure sitting inside U_m. Choose a finite covering $(W_{m_s})_{s=1}^{S}$ of $\bigcup_{i=1}^{p}$ suppt $(I_i\psi_i)$. For given m_s choose $\bar{h}_s \in C^{\infty}(U_{m_s})$, $\bar{h}_s \equiv 1$ on W_{m_s}, and define $h_s \in F$ as above. Now

$$0 = \bar{h}_s\sum_{i=1}^{p} I_i\psi_i = \sum_{k=1}^{l} I_{i_k}(h_s\psi_{i_k}).$$

By Lemma 8 $\sum_{k=1}^{l} h_s\psi_{i_k} = 0.$

By changing ψ_{i_k} to $\psi_{i_k} - h_s\psi_{i_k}$ we do not change ψ and diminish supports of $I_{i_k}\psi_{i_k}$ at least by W_{m_s}. Repeating this procedure S times we get

$$\psi = \sum_{i=1}^{p} \psi_i,\ \psi_i \in F_i \qquad \text{and} \qquad \sum_{i=1}^{p} \text{supp } (I_i\psi_i) = \varnothing.$$

This means however that $\psi_i = 0$ for each i and consequently that $\psi = 0$. □

Now I is well defined and has zero kernel. Since it maps onto $\Gamma_0(\Lambda V)$ the proof of proposition 8 is completed. ■

REFERENCES

[1] R. Arnowitt, P. Nath, B. Zumino, *Phys. Lett.*, t. **56** B, 1975, p. 81-84.
[2] F. A. Berezin, *Metod Vtorichnogo Kvantovaniya*, Moscow: Nauka, 1965.
[3] L. Corwin, Y. Ne'eman, S. Sternberg, *Rev. Mod. Phys.*, t. **47**, 1975, p. 573-603.

[4] S. FERRARA, J. WESS, B. ZUMINO, *Phys. Lett.*, t. **51** B, 1974, p. 239-241.
[5] P. GODDARD, *Nucl. Phys.*, t. B **88**, 1975, p. 429-441.
[6] F. HIRZEBRUCH, *Topological Methods in Algebraic Geometry*, Berlin, Heidelberg, New York: Springer, 1966.
[7] J. KIJOWSKI, *Commun. math. Phys.*, t. **30**, 1973, p. 99-128.
[8] J. KIJOWSKI, W. SZCZYRBA, *Commun. math. Phys.*, t. **46**, 1976, p. 183-206.
[9] R. KOTECKY, *Rep. Math. Phys.*, t. **7**, 1975, p. 457-469.
[10] L. D. LANDAU, E. M. LIFSHIC, *Quantum Mechanics. Non-relativistic Theory*, London: Pergamon Press, 1958.
[11] P. NATH, R. ARNOWITT, *Phys. Lett.*, t. **56** B, 1975, p. 177-180.
[12] E. NELSON, *Tensor Analysis*, Princeton : Princeton University Press, 1967.
[13] W. F. PAHOMOV, *Matematicheskie Zametki*, t. **16**, 1974, p. 65-74.
[14] W. RÜHL, B. C. YUNN, Trier-Kaiserlautern University *preprint*, 1974.
[15] A. SALAM, J. STRATHDEE, *Nucl. Phys.*, t. B **76**, 1974, p. 477-482.
[16] A. SALAM, J. STRATHDEE, *Phys. Rev.*, t. D **11**, 1975, p. 1521-1535.
[17] D. V. VOLKOV, V. P. AKULOV, *Phys. Lett.*, t. **46** B, 1973, p. 109-110.
[18] J. WESS, B. ZUMINO, *Nucl. Phys.*, t. B **70**, 1974, p. 39-50.
[19] B. ZUMINO, TH 2120, CERN, *preprint.*
[20] F. A. BEREZIN, D. A. LEYTES, *Dokl. Akad. Nauk SSSR*, 224, 1975, p. 505-508.

(Manuscrit reçu le 6 septembre 1976)