ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 27, nº 3 (1977), p. 319-334

<http://www.numdam.org/item?id=AIHPA_1977__27_3_319_0>

Ann. Inst. Henri Poincaré,
Vol. XXVII, n° 3, 1977, p. 319-334.

Section A :
Physique théorique.

Quantum electrodynamics with one degree of freedom for the photon field

by

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ABSTRACT. — An electron-positron-field ψ interacts with one degree of freedom of the photon field, which is characterized by a classical electromagnetic potential A_μ. For the Hamiltonian we make the ansatz

$$B = B_0 + \frac{1}{2}(p^2 + \omega^2 q^2) + \int : \bar{\psi}\gamma^\mu A_\mu \psi : q ,$$

where B_0 is the free Hamiltonian of electrons and positrons and p and q are the canonical variables of a quantummechanical harmonic oscillator.

In the case of a purely electric potential the Hamiltonian is a densely defined symmetric operator after an infinite oscillator frequency renormalization, but unbounded from below, because the Coulomb-interaction is neglected. To stabilize the system a photon selfinteraction of the form bq^4 is added. For the so modified model several properties are established, among them selfadjointness and positivity of the Hamiltonian, existence of a groundstate and of asymptotic Fermi fields.

The case of a nonvanishing magnetic field is only partly solved. After an infinite wavefunction renormalization the Hamiltonian is defined as a quadratic form, less singular than the corresponding form in free Fock-space. By reducing the number of space-time-dimensions from four to three the Hamiltonian can be defined as a positive selfadjoint operator in a Hilbert space, where the representation of the canonical anticommutation relations is inequivalent to the Fock representation and where the canonical commutation relations are not realized as operator relations.

1. INTRODUCTION

Quantum electrodynamics have made predictions which are in very good agreement with experiment. Take for example the measurement of the magnetic momentum of the electron or the Lamb-shift in the energy spectrum of the hydrogen atom. But until now nobody has succeeded in defining quantum electrodynamics beyond formal perturbation theory. In general the construction of a nontrivial model of relativistic quantum field theory is an unsolved problem.

For isolating the many difficulties, which arise at the construction of a relativistic quantum field theory, one has studied simplified models, for example the Lee-model, the Nelson-model, the $P(\varphi)$- and Yukawa-models in two dimensions and the φ^4-model in three dimensions.

Another well known simplification of quantum field theory is the external field model. In quantum electrodynamics one replaces the photon field by a classical electromagnetic potential, which is not influenced by the electron-positron-field. The external field model can be completely reduced to a classical problem and is solved in principle.

The reason for the simplicity of this model is the absence of any retroaction from the electron-positron-field to the external field. Obviously, this fact restricts the utility of the model for the construction of an interacting theory.

In the following a model shall be investigated, in which the external field is affected by a very simple retroaction. It gets the degree of freedom of a quantummechanical harmonic oscillator. The Hamiltonian shall have the following form:

$$B = B_0 + \frac{1}{2}(p^2 + \omega^2 q^2) + qe \int d^3x : \overline{\psi(x)}\gamma^\mu A_\mu(x)\psi(x) :$$

(B_0 free Hamiltonian of electrons and positrons, p, q canonical variables of the harmonic oscillator, ψ electron-positron-field, A_μ time independent classical electromagnetic potential).

As in the external field case [3] [4] [5] the interaction term is less singular for a purely electric potential than in the general case. The divergences arising in the electric case correspond to the divergences in the Y_2-model and can be removed by an infinite frequency renormalization. Contrary to Y_2 the renormalized Hamiltonian is not bounded from below. The physical reason is the neglection of the Coulomb-interaction. To get a lower bound for the energy a photon selfinteraction $P(q)$ with a polynom P can be added. A term bq^4 is sufficient for this purpose and in a certain sense also minimal. In this way one gets a positive selfadjoint Hamiltonian B in free Fock-space. B has a groundstate and an absolut continuous spectrum over the electron mass. The asymptotic Fermi fields exist.

The case of a nonvanishing magnetic part of A_μ is essentially more complicated. The degree of ultraviolett divergences is in general two degrees higher. Also after an infinite wavefunction renormalization the Hamiltonian can be defined only as a quadratic form. Probably one has to give up the CAR after an infinite field strength renormalization (the CCR already are not realized as operator relations). The involved problems seem to be unsolvable by the used methods.

In three dimensions the problem is easier to handle. The degrees of divergences correspond to Y_3. The used methods allow a discussion of the model in almost the same way as in the electric case in four dimensions. The field algebra of the interacting fields differs from the free field algebra, because the CCR are not fulfilled; the representation of the CAR is inequivalent to the Fock representation.

The used methods are a combination of the Hamiltonian method of constructive quantum field theory as applied especially in the Y_2-model [*6*] [*7*] with the theory of quasifree representations of the CAR [*8*] [*9*], which permits to solve elegantly the external field problem [*3*].

An unitary approximation $U(q)$ to the waveoperator for the Dirac-equation with external potential qA_μ generates an automorphism of the Fermi field algebra. The interaction term is regularized by this transformation.

In the electric case this automorphism can be implemented in Fock-space by an unitary operator $\mathcal{U}(q)$, so that also the kinetic term $\frac{1}{2}p^2$ of the photon energy can be transformed. One gets a decomposition of the Hamiltonian

$$B = \mathcal{U}(q)(C_0 + V)\mathcal{U}(q)^{-1}, \tag{1}$$

where C_0 is the free Hamiltonian plus the anharmonic term bq^4 and V is bounded relative to C_0 with a bound < 1 in the sense of quadratic forms.

In the case of a nonvanishing magnetic field this automorphism is not unitarily implementable. We show, how in spite of this the oscillator momentum p and renormalized powers $:p^n:$ can be transformed. But in the arising decomposition (1) V is bounded relative to C_0 only if the dimension of space-time is reduced from four to three.

The existence of a groundstate is proved by applying the methods of Glimm and Jaffe [*6*] to the decomposition (1). For suitable values of the parameters e, b, ω, m we get uniqueness of the vacuum and existence and uniqueness of the one-photon-state if $\omega < 2m$ (stability of the photon) with help of analytic perturbation theory. The problem of asymptotic Fermi fields can be reduced to the Cook-criterion for the Dirac-equation with external potential [*10*].

This work is a comprehended version of the authors thesis [*1*]. An extensive use of the results of [*2*] is made.

2. NOTATION

Let H be the Hilbert space $H = \mathscr{L}_2(\mathbb{R}^3, \mathbb{C}^4)$ in which the Dirac-equation for a particle of mass m and charge e in an electromagnetic potential A_μ is given by

$$i\partial_t f(x, t) = \left(\vec{\alpha}.\left(\frac{1}{i}\vec{\nabla} - e\vec{A}\right) + eA_0 + \beta m\right)f(x, t)$$

$(f_t : x \to f(x, t) \in H,\ \alpha_i = \alpha_i^* \in \mathscr{B}(\mathbb{C}^4),\ \{\alpha_i, \alpha_j\} = 2\delta_{ij},\ i, j = 0, \ldots, 3,\ \alpha_0 = \beta)$.

Let h_0 be the free Dirac operator in H:

$$(h_0 f)(x) = \left(\vec{\alpha}.\frac{1}{i}\vec{\nabla} + \beta m\right)f(x)$$

h_0 is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ [*11*].

Let j be the potential term

$$(jf)(x) = e(A_0(x) - \vec{\alpha}.\vec{A}(x))f(x)$$

j is bounded, if A_μ, $\mu = 0, \ldots 3, \in \mathscr{L}_\infty(\mathbb{R}^3)$:

$$\| j \| \leq \sum_\mu e \| A_\mu \|_\infty$$

Then the operator

$$h = h_0 + j$$

is selfadjoint on $D(h_0)$. The Dirac-equation is solved by

$$f_t = e^{iht}f_0$$

Let $F : f \to \tilde{f}$ be the Fourier transform, taken as an unitary operator in H :

$$\tilde{f}(k) = (2\pi)^{-3/2}\int d^3x e^{-ikx}f(x)$$

F transforms h_0 into a multiplication operator:

$$(Fh_0 f)(k) = \omega(k)(P_+(k) - P_-(k))(Ff)(k)$$

with $P_\pm(k) = \frac{1}{2}\left(1 \pm \frac{\vec{\alpha}.k + m}{\omega(k)}\right)$ and $\omega(k) = (k^2 + m^2)^{1/2}$.

The projectors P_+ and $P_- = 1 - P_+$ are the projectors on the positive, respectively negative part of the spectrum of h_0.

The Dirac fields $\psi(f)$, $f \in H$, formally defined by

$$\psi(f) = \sum_{\alpha=1}^{4} \int d^3x \overline{f_\alpha(x)}\psi_\alpha(x)$$

and therefore antilinear in f, fulfill the canonical anticommutation relations, which can be described in the form

$$(\psi(f) + \psi(g)^*)^2 = (f, g).$$

Therefore they span the Cliffordalgebra $\mathscr{C}_0(\mathrm{H})$ over H with an involution and an unique C*-norm with

$$\| \psi(f) \| = \| f \|.$$

$\mathscr{C}(\mathrm{H})$ denotes the C*-completion of $\mathscr{C}_0(\mathrm{H})$. In the following we make an extensive use of the notations and results of [2]. So ω_{P_+} denotes the gauge invariant quasifree state with the two-point function

$$\omega_{\mathrm{P}_+}(\psi(f)\psi(g)^*) = (f, \mathrm{P}_+ g),$$

which is clearly the Fock vacuum. $(\mathscr{H}, \pi, \Omega)$ is the GNS-construction to ω_{P_+} and therefore the Fock representation with vacuum Ω. The one-particle-space is $\mathscr{H}_1 = \mathrm{P}_+\mathrm{H} \oplus \overline{\mathrm{P}_-\mathrm{H}}$, where $^-$ denotes the conjugate Hilbert space. $\mathscr{H}$ is the antisymmetric tensor space over $\mathscr{H}_1$:

$$\mathscr{H} = \Lambda\mathscr{H}_1 = \bigoplus_{n=0} \Lambda_n \mathscr{H}_1$$

Let I denote the identical mapping from H into $\mathscr{H}_1$. Then we can define the annihilation operators $a(g)$, $g \in \mathscr{H}_1$, by

$$a(\mathrm{I}f) = \psi(\mathrm{P}_+ f) + \psi(\mathrm{P}_- f)^*, \qquad f \in \mathrm{H}.$$

As usual, for $p \geq 1$, $\mathrm{C}_p(\mathrm{H})$ denotes the trace ideal $\{ \mathrm{A} \in \mathscr{B}(\mathrm{H}) \mid \mathrm{Tr} \mid \mathrm{A} \mid^p < \infty \}$. Especially C_2 is the Hilbert-Schmidt class and C_1 the trace class.

To each operator $\mathrm{A} \in \mathscr{B}(\mathrm{H})$ there exists an unique bilinear form $d\Gamma_0(\mathrm{A})$ on $\mathscr{C}_0(\mathrm{H})\Omega \times \mathscr{C}_0(\mathrm{H})\Omega$ with the property

$$[d\Gamma_0(\mathrm{A}), \psi(f)^*] = \psi(\mathrm{A}f)^*$$

(in the sense of bilinear forms on $\overline{\mathscr{C}_0(\mathrm{H})\Omega} \times \mathscr{C}_0(\mathrm{H})\Omega$).

For an operator A in H which commutes with P_+ the corresponding derivation of $\mathscr{C}_0(\mathrm{H})$ with $\psi(f)^* \to \psi(\mathrm{A}f)^*$ annihilates ω_{P_+}. Therefore $d\Gamma_0(\mathrm{A})$ can be extended to a closed operator $d\Gamma(\mathrm{A})$, and we have:

$$d\Gamma(\mathrm{A}) = d\Lambda(\mathrm{I}(\mathrm{AP}_+ - \mathrm{A}^*\mathrm{P}_-)\mathrm{I}^{-1}),$$

where $d\Lambda$ denotes the operation

$$d\Lambda(\mathrm{B}) = \bigoplus_{n=0} d\Lambda_n(\mathrm{B}) = \bigoplus_{n=0} \sum_{k=0}^{n-1} 1^{\otimes k} \otimes \mathrm{B} \otimes 1^{\otimes n-k-1}$$

for a closed operator B in H.

Now the free Hamiltonian B_0 in the Fermi-Fock-space $\mathscr{H}$ is defined as

$$\mathrm{B}_0 = d\Gamma(h_0) = d\Lambda(\mathrm{I} \mid h_0 \mid \mathrm{I}^{-1}).$$

The external field term is the bilinear form $d\Gamma_0(j)$.

To include the degree of freedom of the photon field, we define as our free Fock-space

$$\mathcal{F} = \mathcal{L}_2(\mathbb{R}, \mathcal{H}).$$

The oscillator variables p and q are introduced as usual by

$$(q\phi)(\lambda) = \lambda\phi(\lambda), \qquad D(q) = \left\{ X \in \mathcal{F} \,\middle|\, \int \lambda^2 \| X(\lambda) \|^2 d\lambda < \infty \right\}$$

$$(p\phi)(\lambda) = \frac{1}{i}\frac{d}{d\lambda}\phi(\lambda), \qquad D(p) = \left\{ X \in \mathcal{F} \,\middle|\, \int \left\| \frac{d}{d\lambda} X(\lambda) \right\|^2 d\lambda < \infty \right\}$$

Now we can define the formal Hamiltonian for the investigated model as the following bilinear form

$$B_{\text{formal}} = B_0 + q d\Gamma_0(j) + \frac{1}{2}(p^2 + \omega^2 q^2), \qquad \omega^2 > 0.$$

3. THE ELECTRIC CASE

As is well known (compare [3]) the Hamiltonian for the Dirac field in an external electromagnetic field can be defined as a selfadjoint operator in free Fock space, if and only if the magnetic part of A_μ vanishes (for the only-if-part see [1] [2] [12] and compare with [13]). Therefore we analyse the purely electric case separately.

Let $h(\lambda) = h_0 + \lambda j$, $\lambda \in \mathbb{R}$, be the Dirac Hamiltonian with external field λA_0, $A_0 \in \mathcal{L}_\infty$, in H. Let $B(\lambda)$ be a selfadjoint operator in $\mathcal{H}$ with

$$[B(\lambda), \psi(f)^*] = \psi(h(\lambda)f)^*.$$

$B(\lambda)$ exists, if A_0 and $\partial_k A_0 \in \mathcal{L}_2$, $k = 1, 2, 3$, and different choices of $B(\lambda)$ differ only by an additive constant. The Hamiltonian for the full problem should have the form

$$B = B(q) + \frac{1}{2}(p^2 + \omega^2 q^2) + F(q)$$

with a real function F. There arise the following questions:

(1) Is B densely defined?

(2) Is B bounded from below?

(3) What is the natural choice for the function F?

To answer these questions we construct $B(\lambda)$ explicitly by the methods of [2].

First we decompose the interaction term $d\Gamma_0(j)$ into a creation term $d\Gamma_0(j_C)$, $j_C = P_+ j P_-$, an annihilation term $d\Gamma_0(j_A)$, $j_A = P_- j P_+$, and a scattering term $d\Gamma_0(j_0)$, $j_0 = P_+ j P_+ + P_- j P_-$. For implementability only j_C is important, but for positivity also the properties of j_0 must be investigated.

Now let W be the unique operator from P_-H into P_+H with $[h_0, W] = j_C$. According to [3], W is a Hilbert-Schmidt-operator. Therefore the unitary operators

$$U(\lambda) = (1 - \lambda(W - W^*))(1 + \lambda^2(WW^* + W^*W))^{-1/2}$$

define unitarily implementable automorphisms

$$\psi(f)^* \to \psi(U(\lambda)f)^* = \mathcal{U}(\lambda)\psi(f)^*\mathcal{U}(\lambda)^{-1}$$

with $\mathcal{U}(\lambda)\Omega = \det(1 + \lambda^2 W^*W)^{-1/2} e^{-d\Gamma(\lambda W)}\Omega$.

$U(\lambda)$ transforms $h(\lambda)$ to

$$\hat{h}(\lambda) = U(\lambda)^{-1}h(\lambda)U(\lambda) = h_0 + \lambda j_0 + A(\lambda)$$

with a Hilbert-Schmidt-operator $A(\lambda)$.

To complete the construction we show that $j_0 \mid h_0 \mid^{-1/2}$ is a compact operator (i. e. ess spec $j_0 \mid h_0 \mid^{-1/2} = \{ 0 \}$).

The (momentum space) integral kernel of $j_0 \mid h_0 \mid^{-1/2}$ is

$$(P_+(k)P_+(k') + P_-(k)P_-(k'))(2\pi)^{-3/2} e\tilde{A}_0(k - k')\omega(k')^{-1/2} .$$

We see, that $j_0 \mid h_0 \mid^{-1/2} P_\kappa$ is a Hilbert-Schmidt-operator, where P_κ projects on energies with modulus less than $\kappa > 0$. It follows, that $j_0 \mid h_0 \mid^{-1/2}$ is compact as norm-limes of Hilbert-Schmidt-operators.

According to [2], a natural choice for $B(\lambda)$ is

$$\begin{aligned} B(\lambda) &= \mathcal{U}(\lambda) d\Gamma(\hat{h}(\lambda))\mathcal{U}(\lambda)^{-1} + E^{(1)}_{\text{ren}}(\lambda) + E^{(2)}(\lambda) \\ &= u\text{-lim}\,(B_0 + \lambda d\Gamma(j_\kappa) - \lambda^2 \delta_\kappa) \end{aligned}$$

where u-lim means strong convergence of the unitary groups and $\delta_\kappa = -\operatorname{Tr}(W_\kappa j_{\kappa_A})$ for an approximating net (j_κ) of Hilbert-Schmidt-operators. $B(\lambda)$ is bounded from below. $\lim \operatorname{Tr}(W_\kappa j_{\kappa_A}) = \infty$, therefore the renormalization $(-\lambda^2\delta_\kappa)$ is necessary. $E^{(2)}(\lambda)$ vanishes because of the invariance of j under charge conjugation. For details see [1].

After completing the construction of $B(\lambda)$ we have to investigate, how the greatest lower bound of $B(\lambda)$ depends on λ.

3.1. LEMMA. — $0 < E^{(1)}_{\text{ren}}(\lambda) < \text{const} \mid \lambda \mid^3$

Proof. — $E^{(1)}_{\text{ren}}(\lambda) = \lambda^4 \operatorname{Tr} \{ (1 + \lambda^2 WW^*)^{-1} WW^*W j_A \}$ [2].

Using $j_A = W^* \mid h_0 \mid + \mid h_0 \mid W^*$ we have

$$\begin{aligned} E^{(1)}_{\text{ren}}(\lambda) = \lambda^4 \operatorname{Tr} \{ &(1 + \lambda^2 WW^*)^{-1/2} WW^* \mid h_0 \mid WW^*(1 + \lambda^2 WW^*)^{-1/2} \\ &+ (1 + \lambda^2 W^*W)^{-1/2} W^*W \mid h_0 \mid W^*W(1 + \lambda^2 W^*W)^{-1/2} \} \end{aligned}$$

which is obviously positive. The upper bound

$$E^{(1)}_{\text{ren}}(\lambda) < \mid \lambda \mid^3 \operatorname{Tr}(W^*W j_A)$$

follows from the estimate $\| (1 + \lambda^2 WW^*)^{-1} \lambda W \| < 1$.

3.2. LEMMA. — $A(\lambda) = A_1(\lambda) + A_2(\lambda)$ with $\| A_1(\lambda) \|_1 < \text{const} \, |\lambda|^3$, $\| A_2(\lambda) \|_2^2 < \text{const} \, |\lambda|^3$.

Sketch of the proof: $A(\lambda)$ contains Hilbert-Schmidt-operators $\sim \lambda^2$ and trace class operators $\sim \lambda^3$. By decomposing the Hilbert-Schmidt-operators in a low energetic ($\omega(k) < c \, |\lambda|$) trace class operator and the remaining term we derice the lemma.

3.3. THEOREM. — The groundstate energy $g(\lambda)$ of $B_0 + \lambda d\Gamma(j_0)$ approaches $-\infty$ as $-\lambda^4$, i. e. (*a*) $g(\lambda) \geq -\text{const}\,\lambda^4$. (*b*) $g(\lambda) \leq -\text{const}\,\lambda^4$ for λ sufficiently large and a positive constant.

Proof. — (*a*) We decompose $\lambda d\Gamma(j_0)$ into a high energetic and a low energetic term. The high energetic term is small compared with B_0, because j_0 is bounded, and in the low energetic term the bounded character of fermion fields, i. e. the Pauli principle is used.

Let P_λ be the Projector in $\mathcal{H}_1$ on energies less than $2\lambda \| j_0 \|$. Let

$$C_\lambda = (1 - P_\lambda)S(1 - P_\lambda),\ D_\lambda = P_\lambda S\left(1 - \frac{1}{2}P_\lambda\right),$$
$$S = Ij_0(P_+ - P_-)I^{-1},\ \omega = Ih_0(P_+ - P_-)I^{-1}.$$

Then $S = C_\lambda + D_\lambda + D_\lambda^*$.

Let $\phi \in D(\omega^{1/2})$. It holds:

$$|(\phi, C_\lambda \phi)| = |(\omega^{1/2}\phi, \omega^{-1/2}C_\lambda\omega^{-1/2}\omega^{1/2}\phi)|$$
$$\leq (\phi, \omega\phi) \| \omega^{-1/2}C_\lambda\omega^{-1/2} \| \leq \frac{1}{2\lambda}(\phi, \omega\phi)$$

i. e. $\frac{1}{2}\omega + \lambda C_\lambda \geq 0$

By completing the square the remaining term $\frac{1}{2}\omega + \lambda(D_\lambda + D_\lambda^*)$ becomes

$$\left(\phi, \left(\frac{1}{2}\omega + \lambda(D_\lambda + D_\lambda^*)\right)\phi\right)$$
$$= \left\| \left(\left(\frac{1}{2}\omega\right)^{1/2} + \left(\frac{1}{2}\omega\right)^{-1/2}\lambda D_\lambda\right)\phi \right\|^2 - 2\lambda^2(\phi, D_\lambda^*\omega^{-1}D_\lambda\phi)$$

It follows: $\frac{1}{2}\omega + \lambda(D_\lambda + D_\lambda^*) \geq -2\lambda^2 D_\lambda^*\omega^{-1}D_\lambda$

The term on the right hand side is a trace class operator whose trace norm is bounded by a constant times λ^2. Therefore we get:

$$B_0 + \lambda d\Gamma(j_0) = d\Lambda\left(\frac{1}{2}\omega + \lambda C_\lambda\right) + d\Lambda\left(\frac{1}{2}\omega + \lambda(D_\lambda + D_\lambda^*)\right)$$
$$\geq -2\lambda^2 d\Lambda(D_\lambda^*\omega^{-1}D_\lambda) \geq -\text{const}\,\lambda^4$$

q. e. d.

b) The grounds tate of $B_0 + \lambda d\Gamma(j_0)$ is the state, in which all one-particle states with negative energy are occupied. The groundstate-energy is the sum of the negative eigenvalues of the one-particle-Hamiltonian, that is

$$\text{(1)} \quad g(\lambda) = \operatorname{Tr}(\omega + \lambda S)_- = \operatorname{Tr} P_+(h_0 + \lambda j_0)_- - \operatorname{Tr} P_-(h_0 + \lambda j_0)_+$$

($X = X_+ + X_-$ decomposition of a selfadjoint operator in positive and negative part)

Let $\varphi \in \mathscr{L}_2(\mathbb{R}^3)$ with (1) supp $\tilde{\varphi}$ compact

$$\text{(2)} \quad \int d^3x\, |\varphi(x)|^2 A_0(x) \neq 0$$

$$\text{(3)} \quad \|\varphi\|_2 = 1$$

Such φ exist for $A_0 \neq 0$. Let d be the diameter of the support of φ.

$$d := \sup_{k,k' \in \operatorname{sup} \tilde{\varphi}} |k - k'|$$

For $n \in Z^3$ let φ_n be the translated function (in momentum space):

$$\tilde{\varphi}_n(k) := \tilde{\varphi}(k - dn), \qquad k \in \mathbb{R}^3$$

We have $\varphi_n \perp \varphi_m$ for $n \neq m$. Let P_n be the projector on the subspace

$$\mathbb{C}^4 \otimes \{ z\varphi_n \mid z \in \mathbb{C} \} \quad \text{of} \quad H = \mathscr{L}_2(\mathbb{R}^3, \mathbb{C}^4) = \mathbb{C}^4 \otimes \mathscr{L}_2(\mathbb{R}^3).$$

Define P_κ, $\kappa > 0$ by $P_\kappa = \sum_{\omega(dn) \leq \kappa} P_n$

From (1) we get the following estimate:

$$g(\lambda) \leq \operatorname{Tr} P_+(h_0 + \lambda j_0)_- \leq \operatorname{Tr} P_\kappa P_+(h_0 + \lambda j_0)_-$$
$$\leq \operatorname{Tr} P_\kappa P_+(h_0 + \lambda j_0) = \operatorname{Tr} P_\kappa P_+ h_0 + \lambda \operatorname{Tr} P_\kappa P_+ j_0$$

Now

$$\operatorname{Tr} P_n P_+ h_0 = \int d^3k\, |\tilde{\varphi}_n(k)|^2 \operatorname{Tr} P_+(k)\omega(k) = 2 \int d^3k\, |\tilde{\varphi}(k)|^2 \omega(k + dn)$$

$$\leq 2 \int d^3k\, |\tilde{\varphi}(k)|^2 (m + d|n| + |k|) \leq 2(m + d|n| + \int d^3k\, |\tilde{\varphi}(k)|^2 |k|$$

and therefore

$$\operatorname{Tr} P_\kappa P_+ h_0 \leq \text{const}_1 \kappa^4 + 0(\kappa^3)$$

On the other hand:

$$\operatorname{Tr} P_n P_+ j_0 = \int d^3k \int d^3k' \overline{\tilde{\varphi}_n(k)} \tilde{\varphi}_n(k') (2\pi)^{-3/2} \tilde{A}_0(k - k') \operatorname{Tr} P_+(k) P_+(k')$$

We have $P_+(k)P_+(k') = P_+(k) - P_+(k)P_-(k')$ and $\operatorname{Tr} P_+(k) = 2$,

$$\operatorname{Tr} P_+(k)P_-(k') \leq 2|k - k'|^2 |k + k'|^{-2}$$

for $|k + k'|^2 + 4m^2 \geq |k - k'|^2$ ([3] [1])

Therefore

$$\operatorname{Tr} P_n P_+ j_0 \leq 2 \int d^3x \, |\varphi(x)|^2 A_0(x)(1 + |n|^{-2})$$

for $|n|^2 \geq \sup_{k,k' \in \operatorname{supp} \tilde{\varphi}} \dfrac{|k + k'|^2}{d^2}$, and

$$\operatorname{Tr} P_\kappa P_+ j_0 \leq \text{const}_2 \kappa^3 + 0(\kappa^2),$$

with $\text{const}_2 \neq 0$.

Therefore we get the following estimate for the groundstate energy $g(\lambda)$:

$$g(\lambda) \leq \text{const}_1 \kappa^4 + \lambda \, \text{const}_2 \kappa^3 + 0(\kappa^3) + \lambda 0(\kappa^2)$$

Using $g(\lambda) \leq -\operatorname{Tr} P_\kappa P_- h_0 - \operatorname{Tr} P_\kappa P_- \lambda j_0$ and $-\operatorname{Tr} P_\kappa P_- h_0 = \operatorname{Tr} P_\kappa P_+ h_0$, $\operatorname{Tr} P_\kappa P_- j_0 = \operatorname{Tr} P_\kappa P_+ j_0$ instead of (1), we get:

$$g(\lambda) \leq \text{const}_1 \kappa^4 - \lambda \, \text{const}_2 \kappa^3 + 0(\kappa^3) + \lambda 0(\kappa^2)$$

With $\kappa = c|\lambda|$, $c > 0$, we have

$$g(\lambda) \leq (c \, \text{const}_1 - |\text{const}_2|) \, c^3 \lambda^4 + 0(\lambda^3)$$

If we choose $c < \left| \dfrac{\text{const}_2}{\text{const}_1} \right|$, (b) follows. q. e. d.

After gaining a lot of information about the operators $B(\lambda)$ we want to study the sum

$$B = B(q) + \frac{1}{2}(p^2 + \omega^2 q^2)$$

Because of (3.1)-(3.3) this sum is unbounded from below. Therefore we add a photon selfinteraction bq^4 with a certain $b > 0$, such that

$$B(q) + \frac{1}{2}(p^2 + \omega^2 q^2) + (b - \varepsilon) q^4$$

is bounded from below for some $\varepsilon > 0$. Instead of studying B, we study the equivalent operator

$$\hat{B} = \mathscr{U}(q)^{-1} B \mathscr{U}(q) = B_0 + q d\Gamma(j_0) + d\Gamma(A(q)) + \frac{1}{2} \mathscr{U}(q)^{-1} p^2 \mathscr{U}(q) + \frac{1}{2} \omega^2 q^2 + bq^4$$

First we compute $\mathscr{U}(q)^{-1} p \mathscr{U}(q) =: \hat{p}$

(1) $$\hat{p} = p - i\mathscr{U}(q)^{-1}\mathscr{U}'(q), \qquad \mathscr{U}'(\lambda) = \frac{d}{d\lambda} \mathscr{U}(\lambda)$$

(2) $$[\mathscr{U}(\lambda)^{-1}\mathscr{U}'(\lambda), \psi(f)^*] = \psi(U(\lambda)^{-1}U'(\lambda)f)^*$$

(3) $$U(\lambda)^{-1}U'(\lambda) = (1 + \lambda^2(W^*W + WW^*))^{-1}(W^* - W)$$

(4) $$(\mathscr{U}(\lambda)\Omega, \mathscr{U}'(\lambda)\Omega) = 0$$

It follows:

$$\hat{p} = p - i d\Gamma\{(1 + q^2(W^*W + WW^*))^{-1}(W^* - W)\}$$

3.4. LEMMA. — $\hat{p}^2 - p^2$ is bounded relative to $p^2 + \mathrm{N}$ with an infinitesimal bound.

The proof is a straightforward application of [2] Prop. 4.1 (see [1]).

Let C_0 be the operator

$$\mathrm{C}_0 = \mathrm{B}_0 + \frac{1}{2}(p^2 + \omega^2 q^2) + bq^4$$

C_0 is selfadjoint and positive and has an unique groundstate [14]. $\hat{\mathrm{B}}$ is given by

$$\hat{\mathrm{B}} = \mathrm{C}_0 + qd\Gamma(j_0) + d\Gamma(\mathrm{A}(q)) + \frac{1}{2}(\hat{p}^2 - p^2) + \mathrm{E}^{(1)}_{\mathrm{ren}}(q)$$

3.5. THEOREM. — $\hat{\mathrm{B}}$ is essentially selfadjoint and bounded from below.

Proof. — $\mathrm{C}_0 + qd\Gamma(j_0)$ is essentially selfadjoint [1]. The other terms in $\hat{\mathrm{B}}$ are bounded relative to $(\mathrm{N} + p^2 + q^4)$ with an infinitesimal bound. On the other side, from 3.3 we have

$$\mathrm{C}_0 + qd\Gamma(j_0) \geq \text{const } (\mathrm{N} + p^2 + q^4) - \text{const}$$

We show:

$$(1) \qquad (\mathrm{C}_0 + qd\Gamma(j_0))^2 \geq \text{const } (\mathrm{N} + p^2 + q^4)^2 - \text{const}$$

Let $\mathrm{A} = 2(\mathrm{B}_0 + qd\Gamma(j_0)) + \omega^2 q^2 + 2bq^4$, $\mathrm{A}' = \mathrm{N} + q^4$. Then the proposition reads as follows:

$$(1)' \qquad (\mathrm{A} + p^2)^2 \geq \text{const } (\mathrm{A}' + p^2)^2 - \text{const}$$

We have

(2) $(\mathrm{A} + p^2)^2 = \mathrm{A}^2 + p^4 + \mathrm{A}p^2 + p^2\mathrm{A} = \mathrm{A}^2 + p^4 + [p, [p, \mathrm{A}]] + 2p\mathrm{A}p$

(3) $(\mathrm{A}' + p^2)^2 = \mathrm{A}'^2 + p^4 + [p, [p, \mathrm{A}']] + 2p\mathrm{A}'p$

(4) $\mathrm{A} \geq \text{const } \mathrm{A}' - \text{const}$, and since A and A′ commute:

(5) $\mathrm{A}^2 \geq \text{const } \mathrm{A}'^2 - \text{const}$

(6) $[p, [p, \mathrm{A}]] = -\, 2\omega^2 - 24bq^2$

(7) $[p, [p, \mathrm{A}']] = -\, 12q^2$

Now (1) follows. Therefore the remaining terms in $\hat{\mathrm{B}}$ are bounded relative to $\mathrm{C}_0 + qd\Gamma(j_0)$ with an infinitesimal bound. The theorem follows from [11], V. Th. 4.3.

q. e. d.

Now the construction of an essentially selfadjoint semibounded Hamiltonian is completed with the definition

$$\mathrm{B} = \mathcal{U}(q)\hat{\mathrm{B}}\mathcal{U}(q)^{-1}$$

To justify this definition we show that

$$\mathrm{B} = u\text{-lim } \mathrm{B}_\kappa$$

where the approximating net (B_κ) is constructed with a net (j_κ) of Hilbert-Schmidt-operators in H with the properties:

(1) $$\| j_\kappa \| \leq \| j \|$$

(2) $$\| W_\kappa - W \|_2 \to 0$$

(3) $$\| (j_{\kappa 0} - j_0) | h_0 |^{-1/2} \| \to 0$$

An ultraviolett-cutoff would have these properties. We see, that the resolvants of the transformed operators

$$\hat{B}_\kappa = \mathscr{U}_\kappa(q)^{-1} B_\kappa \mathscr{U}_\kappa(q)$$

converge in norm to the resolvant of $\hat{B}$ for sufficiently negative arguments (Re $z \ll 0$). This implies strong convergence of the unitary groups. With $\mathscr{U}_\kappa(q) \underset{s}{\to} \mathscr{U}(q)$ the proposition follows.

3.6. THEOREM. — B has a groundstate.

Proof. — We apply the methods of Glimm and Jaffe [6] to $\hat{B} = C_0 + V$ and get the result, that the spectrum of B-inf spec (B) in the intervall $[0, m)$ is discret (For details see [1]).

REMARK. — $\hat{B}(e) = C_0 + V(e)$ is a holomorphic family of selfadjoint operators for $| e | \leq e_0$, where e_0 depends on b. Therefore for sufficiently small e there exists a holomorphic family $\Omega(e)$ of unique (up to a phase) groundstate vectors in the vacuum sector. There exists also a holomorphic family of unique (up to a phase) one-photon-state-vectors, if $\omega_0 < 2m$, where ω_0 is the gap between the second and the first eigenvalue of the anharmonic oscillator $\frac{1}{2}(p^2 + \omega^2 q^2) + bq^4$.

3.7. THEOREM. — Let e^{itB} implement the automorphism α_t and e^{itC_0} the automorphism α_t^0 and let $w_t = \alpha_t \alpha_{-t}^0$. Assume that the electric potential A_0 decreases at infinity as $| x |^{-(1+\varepsilon)}$ for some $\varepsilon > 0$. Then there exist the asymptotic fields

$$\psi_{\substack{\text{out}\\ \text{in}}}(f) = \underset{t \to \pm\infty}{s\text{-lim}}\ w_t \psi(f)$$

Proof. — Under our assumptions on A_0 we have for $f \in \mathscr{C}_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ [10]:

$$\int_{-\infty}^{+\infty} dt \, \| j e^{ih_0 t} f \| < \infty$$

which guarantees the existence of waveoperators for the Dirac-equation. Now

$$w_t \psi(f) = \psi(f) + i \int_0^t ds \alpha_s (q \psi(j e^{-ih_0 s} f))$$

and because of $\| q(B+1)^{-1/2} \| < \infty$ we have

$$\| (w_t - w_s)\psi(f)(B+1)^{-1/2} \| \leq \text{const} \int_s^t dr \, \| \psi(je^{-ih_0 r}f) \| = \text{const} \int_s^t dr \, \| je^{-ih_0 r}f \| .$$

Therefore $w_t\psi(f)$ converges strongly on $D(B^{1/2})$ and because of uniform boundedness everywhere. An $\frac{\varepsilon}{3}$-argument [*15*] gives the convergence for all $f \in H$.

q. e. d.

3.8. PROPOSITION. — The asymptotic fields are free fields, i. e. they have the following properties (ex = in, out):

(1) $\alpha_t\psi_{ex}(f) = \psi_{ex}(e^{ih_0 t}f)$

(2) $(\psi_{ex}(f) + \psi_{ex}(g)^*)^2 = (f, g)$

(3) $(\phi_0, \psi_{ex}(f)\psi_{ex}(g)^*\phi_0) = (f, P_+g)$, where ϕ_0 is a groundstate of B.

Proof. — (1) is the well known property of wavemorphisms and (2) follows from $\underset{t\to\pm\infty}{s\text{-lim}}\, w_t\psi(f)^* = \psi_{\substack{\text{out}\\ \text{in}}}(f)^*$ and the continuity of multiplication on bounded subsets of $\mathscr{B}(\mathscr{F})$ in the strong topology. (3) follows from (1) and (2) by using positivity of energy.

4. THE MAGNETIC CASE

If the spatial part of the electromagnetic potential does not vanish, the Hamiltonian for the external field problem cannot be defined in free Fock space as a selfadjoint operator. Moreover, if we choose for each real λ a Hilbert space $\mathscr{H}_\lambda$, such that $B(\lambda)$ is a positive selfadjoint operator in $\mathscr{H}_\lambda$, then the corresponding representations π_λ are pairwise inequivalent. Therefore the oscillator momentum p is not definable as a selfadjoint operator in the Hilbert space

$$\mathscr{K} = \int^{\oplus} d\lambda \mathscr{H}_\lambda$$

Now first order perturbation theory indicates, that p is not an operator in the physical Hilbert space at sharp times. Therefore we try to define p and renormalized powers $:p^n:$ as quadratic forms in $\mathscr{K}$ with the properties

$$\begin{aligned} [:p^n:, \psi(f)] &= 0 \\ [:p^n:, q] &= -\,\mathrm{i}n :p^{n-1}: \end{aligned}, \qquad n \in \mathbb{N}, \quad f \in H, \quad :p^0: = 1, \quad :p^1: = p$$

To find $\mathscr{H}_\lambda$ we use the same unitary $U(\lambda)$ as in section 3 and define $(\mathscr{H}_\lambda, \pi_\lambda, \Omega_\lambda)$ as the GNS-construction to the state $\omega_\lambda = \omega_{U(\lambda)P_+U(\lambda)^{-1}}$. In the first step we shall show, that $B(\lambda)$ can be defined in $\mathscr{H}_\lambda$.

4.1. PROPOSITION. — There exists a positive selfadjoint operator $B(\lambda)$ in $\mathscr{H}_\lambda$ with

$$[B(\lambda), \psi(f)^*] = \psi(h(\lambda)f)^*$$

Proof. — Choose the decomposition

$$h(\lambda) = U(\lambda)h_0U(\lambda)^{-1} + (h(\lambda) - U(\lambda)h_0U(\lambda)^{-1}).$$

According to [2] Th. 6.1, 6.4 we have to show: $(v(\lambda) = \hat{h}(\lambda) - h_0)$

(1) $$\int_0^T dte^{ih_0t}v_C(\lambda)e^{-ih_0t} \in C_2\,, \qquad v_C(\lambda) = P_+v(\lambda)P_-$$

(2) $v_0(\lambda)\,|\,h_0\,|^{-1/2}$ is compact, $\qquad v_0(\lambda) = P_+v(\lambda)P_+ + P_-v(\lambda)P_-$

(2) follows from the fact that $j_0\,|\,h_0\,|^{-1/2}$ and $v(\lambda) - \lambda j_0$ are compact (see section 3).

To demonstrate (1) we prove the following lemma:

4.2. PROPOSITION. — $W \in C_4$

Proof. — W has the integral kernel (in momentum space)

$(k, k') \to (\omega(k) + \omega(k'))^{-1}P_+A(k - k')P_-(k')$,

$$A(k) = (2\pi)^{-3/2}(\tilde{A}_0 - \alpha\tilde{A})(k)$$

It holds:

$$(\omega(k) + \omega(k'))^{-1}\,\|\,P_+(k)A(k - k')P_-(k')\,\|_4 \leq \omega(k)^{-1}\,\|\,A(k - k')\,\|_4$$

With $k \to \omega(k)^{-1} \in \mathscr{L}_4$ and $k \to \|\,A(k)\,\|_4 \in \mathscr{L}_{4/3} \subset \mathscr{L}_2 \cap \mathscr{L}_1$ the proposition follows from [17], Lemma 2.1 and proof of lemma 2.3.

Now let us proceed in the proof of proposition 4.1! We have:

$$v_C(\lambda) = (1 + \lambda^2W^*W + \lambda^2WW^*)^{-1/2}$$
$$(\lambda^2[W, j_0] - \lambda^3Wj_AW)(1 + \lambda^2W^*W + \lambda^2WW^*)^{-1/2}$$

The second term is a Hilbert-Schmidt-operator, since $W \in C_4$ and j_A is bounded. The first term can be written in the form

$$W(1 + \lambda^2W^*W)^{-1/2}j_0(1 + \lambda^2W^*W)^{-1/2}$$
$$- (1 + \lambda^2WW^*)^{-1/2}j_0(1 + \lambda^2WW^*)^{-1/2}W.$$

Now $h_0^{-1}W$, $Wh_0^{-1} \in C_2$, and the operators $A \to (\text{ad } h_0)^{-1}(h_0A)$,

$$A \to (\text{ad } h_0)^{-1}(Ah_0)$$

in the Hilbert space $C_2(P_-H, P_+H)$ have norms smaller than one. With

$$i\int_0^T dte^{it\,\text{ad}\,h_0} = (e^{iT\,\text{ad}\,h_0} - 1)(\text{ad } h_0)^{-1}$$

the proposition follows.

q. e. d.

In the second step we want to define $:e^{iup}: = \sum \frac{i^n u^n}{n!} :p^n:$ as a bilinear form in $\mathcal{K} \times \mathcal{K}$. $:e^{iup}:$ should have the properties:

$$(1) \qquad :e^{iup}: \pi_\lambda \psi(f) = \pi_{\lambda+u} \psi(f) :e^{iup}:$$

$$(2) \qquad :e^{iup}: q = (q + u) :e^{iup}:$$

Let $\mathcal{U}(\lambda)$ be the unitary operator from $\mathcal{H}_0$ to $\mathcal{H}_\lambda$, which intertwines the equivalent representations π_λ and $\pi_0 \circ \mathrm{U}(\lambda)^{-1}$. Then a possible choice for $:e^{iup}:$ is

$$:e^{iup}: = \mathcal{U}(q) e^{iup} \mathcal{V}(u, q) \mathcal{U}(q)^{-1}$$

where $\mathcal{V}(u, \lambda)$ is the bilinear form, constructed in [2], Th. 5.1, which implements the automorphism $\psi(f)^* \to \psi(\mathrm{V}(u, \lambda) f)^*$ with

$$\mathrm{V}(u, \lambda) = \mathrm{U}(\lambda)^{-1} \mathrm{U}(\lambda + u).$$

We modify the expression of [2], Th. 5.1, by the finite factor

$$\det \{ (1 + \mathrm{LL}^*)^{-1/2} e^{u^2 \mathrm{W}^*\mathrm{W}/2} \}, \qquad \mathrm{L} \text{ as in [2], Th. 5.1.}$$

Then, formally, $:e^{iup}:$ differs from the unrenormalized expression e^{iup} by the infinite factor $\exp\left(\frac{1}{2} u^2 \| \mathrm{W} \|_2^2\right)$. For $:p^2:$ we get

$$:p^2: = \mathcal{U}(q) \{ :(p - id\Gamma(\mathrm{X}(q)(\mathrm{W}^* - \mathrm{W})))^2: + d\Gamma(\mathrm{X}(q)(\mathrm{W}^*\mathrm{W} - \mathrm{WW}^*)\mathrm{X}(q)) + \mathrm{Tr} \{ (1 - \mathrm{X}(q)^2) \mathrm{W}^*\mathrm{W} \} \} \mathcal{U}(q)^{-1}$$

where $\mathrm{X}(q) = (1 + q^2(\mathrm{W}^*\mathrm{W} + \mathrm{WW}^*))^{-1}$ and the double dots on the right hand side denote normal ordering with respect to the fermion fields.

Now we can define the following quadratic form as our Hamiltonian:

$$\mathrm{B} = \mathrm{B}(q) + \frac{1}{2}(:p^2: + \omega^2 q^2)$$

B is densely defined and symmetric, but perhaps not bounded from below, since formally $:p^2: = p^2 - \| \mathrm{W} \|_2^2$.

In four dimensions there seems to be no simple method to improve this situation. Therefore we analyze the model in three dimensions. Also in three dimensions the operator $\mathrm{W} = (\mathrm{ad}\, h_0)^{-1} j_{\mathrm{C}}$ is not of Hilbert-Schmidt-type. But contrary to the four dimensional case, B differs from

$$\mathcal{U}(q)\left(\mathrm{B}_0 + \frac{1}{2} p^2 + b q^4\right) \mathcal{U}(q)^{-1}$$

only by a small perturbation (in the form sense). This follows from the N_τ-estimates in [2], Prop. 4.1, with help of the following lemma and the fact, that $\mathrm{W} \, | h_0 |^{-1/2} \in \mathrm{C}_2$:

4.2. LEMMA. — $v(\lambda) = \lambda j_0 + A_1(\lambda) + A_2(\lambda)$ with

and
$$\| A_1(\lambda) \|_1 \leq \text{const} \, |\lambda|^3 \log(1 + |\lambda|)$$
$$\| A_2(\lambda) \, |h_0|^{-1/2} \|_2 \leq \text{const} \, |\lambda|^{3/2}$$

(compare Lemma 3.2)

4.3. THEOREM. — The form sum

$$B = \mathscr{U}(q) B_0 \mathscr{U}(q)^{-1} + \frac{1}{2}(: p^2 : + \omega^2 q^2) + bq^4 + E^{(1)}_{\text{ren}}(q) + d\Gamma(v(q))$$

defines an unique selfadjoint operator, bounded from below.

Using this theorem, we can derive the existence of a groundstate and of asymptotic Fermi fields with almost the same methods as in section 3. (For a more detailed discussion see [*1*].)

ACKNOWLEDGMENT

I am indebted to Prof. Roepstorff for initiating this work and for many helpful discussions.

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(Manuscrit reçu le 18 janvier 1977).