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Analyticity properties and many-particle structure in general quantum field theory. IV. Two-particle structural equations for the four, five and six-point functions

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Analyticity properties and many-particle structure in general quantum field theory IV. Two-particle structural equations for the four, five and six-point functions

by

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ABSTRACT. — The simultaneous extraction of two-particle singularities in triplets of channels is performed for the four-point and five-point functions of a scalar field and the six-point function of a pseudo-scalar field. Global structural equations involving the « two-particle irreducible » functions thus obtained are derived. An analytic interpretation of these equations is also given.

1. INTRODUCTION

In the previous paper [*3*] of this series [*1-3*], which is devoted to the non-linear program of general quantum field theory, we have shown the existence of the two-particle irreducible parts of the n-point functions of a scalar field (with respect to a *single* arbitrary channel and for any n).

These functions enjoy the algebraic and analytic primitive structure of general n-point functions, i. e. analyticity in the n-point primitive domain and Steinmann relations. As for their two-particle irreducibility, it turns out to be equivalent ([1]) with the completeness of asymptotic states in the two-particle spectral region. In this analytic approach we recall that by « two particle irreducibility of a n-point function with respect to a certain channel », we mean vanishing of its (analytic) discontinuity in this channel for momentum configurations lying under the three-particle threshold.

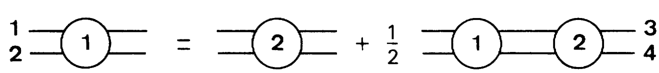
A further natural step of the program should be the introduction of n-point functions *simultaneously* two-particle irreducible (p. i.) in *several* channels. Actually the construction of a four-point function two-p. i. in all channels had been already performed by one of us in the simplest case of a pseudo-scalar field [*4*].

Here we return to this problem for the four-point and five-point functions of a scalar field and the six-point function of a pseudo-scalar field. In the four-point case we solve the problem of two-particle irreducibility in *all* channels. In the five-point and six-point case we only perform simultaneous two-particle irreducibilisation in *triplets* of channels, of the form $[\{i, j\}; \{k\} \cup \mathrm{L}]$, $[\{j, k\}; \{i\} \cup \mathrm{L}\}$ and $[\{k, i\}; \{j\} \cup \mathrm{L}]$.

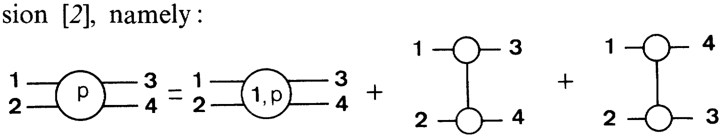
As we shall see this is however sufficient to provide us with « two-particle structural equations » which seem to be of special interest in the theoretical analysis of the $2 \to 3$ and $3 \to 3$ scattering amplitudes in the lowest energy strip of the physical region.

By « two-particle structural equation » we mean any algebraic expression of a given n-point function in terms of one-p. i. and two-p. i. terms together with various single-loop convolution products (in the sense of [*1*]).

Actually the simplest case of such a two-particle structural equation is the well-known general Bethe-Salpeter equation [*3*] [*4*]:



In order to see how more general single-loop convolution products can be generated, it is enough to replace each vertex function in the right-hand side convolution by its complete « one-particle structural » expansion [*2*], namely:



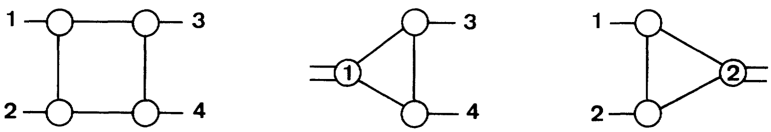
(p = 1, 2)

where 

stands for the four-point function which is one-

([1]) Up to the technical problem of C. D. D. singularities.

p. i. in all channels and p-p. i. ($p = 1, 2$) in (12 ; 34). Then among other terms the following triangle and square convolution products can be obtained :



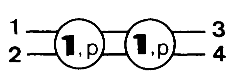
, etc...

As a consequence of the study performed in [1], each term of such a decomposition makes sense as an analytic function in the four-point primitive domain.

As a matter of fact the interest of structural equations arises from the complementarity of their algebraic aspects together with the analyticity properties of their various terms. The situation is then similar to the one encountered with dispersion relations: there the right-hand cut and left-hand cut integrals produce a decomposition of the analytic amplitude in two terms, each of which is less singular.

Here the fact that each of the various G-convolution terms occurring in structural equations is analytic in a larger domain than the original n-point functions ($n = 4, 5, 6$) will be a direct consequence of the special threshold properties of its discontinuity functions. The latter can be derived on the basis of the irreducibility of the vertex functions together with the topological structure of the graph G itself.

For instance it is a heuristic argument borrowed from perturbation theory that any Feynman amplitude occurring in the expansion of

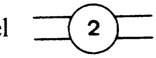


($p = 1, 2$) must have its threshold at 16 m^2 in the channel (13 ; 24). This indeed corresponds to the number of internal lines to be cut (four) in order to disconnect the corresponding Feynman diagram into two disjoint parts in this channel.

We shall see that such a threshold property can be rigorously proved with some generality in the axiomatic approach (at least here in the case of single-loop G-convolution). This is performed in Section 2 where two discontinuity formulas are given (with an appendix for technical details).

In Section 3 we recall various results previously obtained in the program. A « bubble » graphical notation is also definitely adopted, since we hope that its rigorous analytic interpretation has now become familiar to the reader.

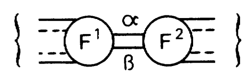
Then Sections 4, 5 and 6 are respectively devoted to the derivation of structural equations for the four, five and six-point functions, involving functions simultaneously two-p. i. with respect to various triplets of channels.

The algebraic aspect of the six-point structural equation there obtained will be very similar to the Faddeev three-body equations in potential scattering theory, with the four-point Bethe-Salpeter kernel  playing here the role of a relativistic two-body potential.

Finally the analytic aspects of these structural equations are described in Section 7 and the improved analytic structure of their various terms specified. This sets the ground for further investigations on the analytic continuation properties of the 2 $\to$ 3 and 3 $\to$ 3 scattering amplitudes.

2. MATHEMATICAL STUDY : DISCONTINUITIES OF CONVOLUTION PRODUCTS

In this section we shall deal with a mathematical problem similar to the one already solved in the first section of [3]. There we considered the convolution products H^G associated with all graphs G having two internal lines and two vertices, namely:

I  N \ I (1)

An analytic representation was then given for their « absorptive parts » in the convolution channel (I, N\I). This discontinuity formula (Theorem 1 of [3]) brought out the contribution of each vertex function together with the one due to the integration prescription.

Here we shall consider a similar problem for the absorptive parts of (1) in channels which are either subchannels of (I, N\I) or transverse channels to (I, N\I).

DEFINITION. — We say that a channel (J, N\J) is *transverse* to (I, N\I) if $J_1 = J \cap I$, $J_2 = J \cap (N\backslash I)$, $L_1 = (N\backslash J) \cap I$ and $L_2 = (N\backslash J) \cap (N\backslash I)$ are non empty. (J, N\J) is called a *subchannel* of (I, N\I) if either $J \subset I$, or $N\backslash J \subset N\backslash I$.

First let us recall that the convolution product (1) can be written under the form:

$$H^G(k_I, \hat{k}_{(I)}, \hat{k}_{(N\backslash I)}) = \int_{\mathscr{C}_k} F^1(k_I, \hat{k}_{(I)}, k_\alpha)F^2(-k_I, -k_\alpha, \hat{k}_{(N\backslash I)})[H_0^{(2)}(k_\alpha)H_0^{(2)}(k_\alpha + k_I)]^{-1}dk_\alpha . \quad (2)$$

Here we stick to the notations of our previous papers [1] [3] and refer the reader to [3], Section II.1 where they are fully specified. In particular the notion of « barycentric variables associated with a subset J » will be useful in the following. These are defined as:

$$\hat{k}_{(J)} = \left\{ \hat{k}_i = k_i - \frac{k_J}{|J|},\ i \in J\,;\ \sum_{i\in J} \hat{k}_i = 0 \right\}$$

where $|J|$ stands for the number of elements of the set J and $k_J = \sum_{i\in J} k_i$.

We shall also need to recall the notion ([2]) of « discontinuity function » $\Delta^I G^{(n)}$ of a n-point function $G^{(n)}$ with respect to a partition (I, N\I) of its arguments $(k_1, \ldots, k_n)$: it is the discontinuity of $G^{(n)}$ across the « hyperplane » $q_I = -q_{N\setminus I} = 0$. The latter is a distribution in $p_I = -p_{N\setminus I}$ having its support contained in:

$$H_m^+ \cup \overline{V_M^+} = \{ p \in \mathbb{R}^4 : p^0 = \sqrt{\vec{p}^2 + m^2} \text{ or } p^0 \geq \sqrt{\vec{p}^2 + M^2} \},$$

and depending analytically on the barycentric variables $(\hat{k}_{(I)}, \hat{k}_{(N\setminus I)})$ inside a certain primitive « flat domain » $\mathcal{D}_I$ defined on the manifold $q_I = q_{N\setminus I} = 0$. We refer the reader to [*2*], Section III.2 or [*3*], Section II.1 for a detailed account of $\mathcal{D}_I$.

2.1. Discontinuities in transverse channels

Let (J, N\J) denote a transverse channel with respect to (I, N\I); we write:

$$J = J_1 \cup J_2, \qquad J_1 = J \cap I, \qquad J_2 = J \cap (N\setminus I)$$

and:

$$N\setminus J = L_1 \cup L_2, \qquad L_1 = (N\setminus J) \cap I, \qquad L_2 = (N\setminus J) \cap (N\setminus I).$$

By $\Delta^{\alpha J_1}F^1$ (resp. $\Delta^{\underline{\alpha} J_2}F^2$) we denote the discontinuity function of F^1 (resp. F^2) in the channel

$$(\{\alpha\} \cup J_1, \{\beta\} \cup L_1) \text{ (resp. } (\{\underline{\alpha}\} \cup J_2, \{\underline{\beta}\} \cup L_2)).$$

$\Delta^{\beta J_1}F^1$ (resp. $\Delta^{\underline{\beta} J_2}F^2$) is similarly defined. We also note $p_{\alpha J_1}$ (resp. $p_{\underline{\alpha} J_2}$) for $p_\alpha + p_{J_1}$ (resp. $p_{\underline{\alpha}} + p_{J_2}$) and $k_{\underline{\beta}} = -k_\beta$ for $(k_\alpha + k_I)$.

Then we can state our first result on discontinuities:

THEOREM 1. — In its primitive domain of definition, the discontinuity function $\Delta^J H^G$ is given by:

$$\begin{aligned}
\Delta^J H^G(p_J, \hat{k}_{(J)}, \hat{k}_{(N\setminus J)}) = \\
&\int_{\mathscr{C}} \Delta^{\alpha J_1}F^1(p_{\alpha J_1}, \hat{k}_{(J_1)}, \hat{k}_{(L_1)}, k_\alpha, k_\beta)\Delta^{\underline{\alpha} J_2}F^2(p_{\underline{\alpha} J_2}, \hat{k}_{(J_2)}, \hat{k}_{(L_2)}, k_{\underline{\alpha}}, k_{\underline{\beta}}) \\
&\qquad [H_0^{(2)}(k_\alpha)H_0^{(2)}(k_\beta)]^{-1} dk_\alpha \\
&+ \int_{\mathscr{C}'} \Delta^{\beta J_1}F^1(p_{\beta J_1}, \hat{k}_{(J_1)}, \hat{k}_{(L_1)}, k_\beta, k_\alpha)\Delta^{\underline{\beta} J_2}F^2(p_{\underline{\beta} J_2}, \hat{k}_{(J_2)}, \hat{k}_{(L_2)}, k_{\underline{\beta}}, k_{\underline{\alpha}}) \\
&\qquad [H_0^{(2)}(k_\alpha)H_0^{(2)}(k_\beta)]^{-1} dk_\beta . \qquad (3)
\end{aligned}$$

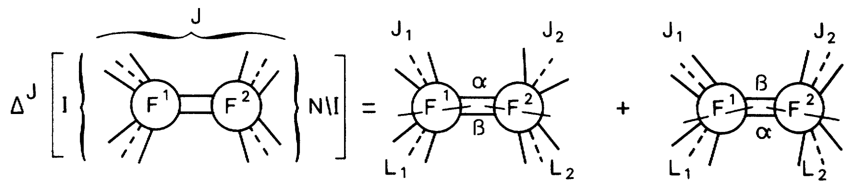
([2]) We reserve the terminology « absorptive part » for the boundary value of the « discontinuity function » on the real (i. e. not only p_I real, but also $\hat{k}_{(I)}$ and $\hat{k}_{(N\setminus I)}$ real).

Here, due to the support properties of each vertex discontinuity, the integration is taken over the following *compact* cycles:

$$\mathscr{C} = \{ k_\alpha = p_\alpha + iq_\alpha \in \mathbb{C}^4 : q_\alpha = - q_{J_1}, p_\alpha + p_{J_1} \in \operatorname{Supp} \Delta^{\alpha J_1} F_1, \\ - p_\alpha + p_{J_2} \in \operatorname{Supp} \Delta^{\alpha J_2} F^2 \}$$

$$\mathscr{C}' = \{ k_\beta \in \mathbb{C}^4 : q_\beta = - q_{J_1}, p_\beta + p_{J_1} \in \operatorname{Supp} \Delta^{\beta J_1} F^1, \\ - p_\beta + p_{J_2} \in \operatorname{Supp} \Delta^{\beta J_2} F^2 \}.$$

REMARKS. — *a*) This result can be graphically illustrated as follows:



b) Applying $p_{J_1} + p_{J_2} = p_J$, we get as a straightforward consequence of (3) the following basic relation on supports:

$$\operatorname{Supp} \Delta^J H^G \subset [\operatorname{Supp} \Delta^{\alpha J_1} F^1 + \operatorname{Supp} \Delta^{\alpha J_2} F^2] \\ \cup [\operatorname{Supp} \Delta^{\beta J_1} F^1 + \operatorname{Supp} \Delta^{\beta J_2} F^2] \quad (4)$$

which we shall use extensively in the following.

Proof. — Here we only give an outline of the proof since details can be found in the appendix.

Let us consider a couple of points k_+ and k_- separated by the face $q_J = q_{N\setminus J} = 0$, namely:

$$k_\pm = (p_J^0 \pm i\varepsilon, \vec{p}_J ; \hat{k}_{(J)} ; \hat{k}_{(N\setminus J)})$$

with $\varepsilon > 0$ and $(p_J, \hat{k}_{(J)}, \hat{k}_{(N\setminus J)})$ fixed in the primitive domain of definition of $\Delta^J H^G$.

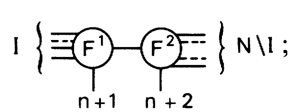
In order to study the discontinuity:

$$\Delta^J H^G(p_J, \hat{k}_{(J)}, \hat{k}_{(N\setminus J)}) = \lim_{\varepsilon \to 0} [H^G(k_+) - H^G(k_-)] \quad (5)$$

we rewrite the expression (2) of H^G under the form:

$$H^G(k_\pm) = \int_{\mathscr{C}(k_\pm)} H^T(k_\pm, k_{n+1}, k_{n+2}) \Big|_{k_{n+1} = -k_{n+2} = t} [H_0^{(2)}(t)]^{-1} dt .$$

Here H^T is the $(n + 2)$-point function associated with the tree



$\mathscr{C}(k_\pm) = \mathbb{R}^3 \times \mathscr{L}_\pm$ is a cycle with real dimension four in the space $\mathbb{C}^4$

of $t = u + iv$, with $\mathscr{L}_\pm$ some contour of the $t^0 = u^0 + iv^0$ plane threading its way from $-i\infty$ to $+i\infty$ through the singularities of the integrand. The latter are « cuts » which correspond to the « vertex partitions » ([*1*], p. 199) of the tree T. In general they are not confused and the line $\mathscr{L}_\pm$ is not pinched.

When $q_J^0 \to 0$ it is easy to see that only two couples of cuts get confused, namely:

i) $v^0 = -q_{J_1}^0$ and $v^0 = q_{J_2}^0$
ii) $v^0 = -q_{L_1}^0$ and $v^0 = q_{L_2}^0$.

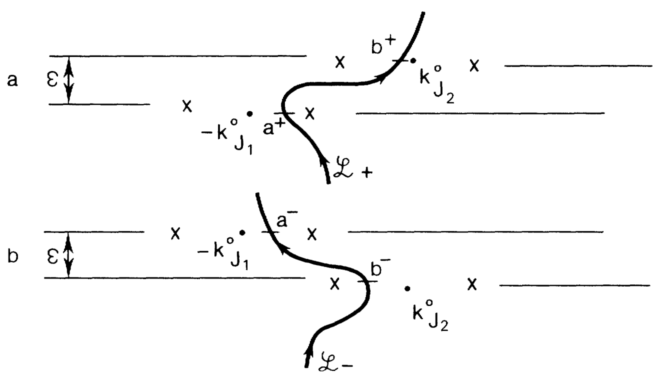


FIG. 1. — (u_0, v_0) sections

a) $q_J^0 = +\varepsilon > 0$
b) $q_J^0 = -\varepsilon < 0$

Then the situation is as shown on Fig. 1. It is also convenient to describe the two limiting contours $\mathscr{L}_\pm$ by giving (see [*1*], p. 224) their projection onto the plane (v^0, q_J^0) as in Fig. 2 (p. 315).

Now the integrand is analytic on the manifold $q_J^0 = 0$ since the partition $[J; (N\backslash J) \cup \{ n+1, n+2 \}]$ is *not* a vertex partition for T ([*1*], p. 201). Then using Stokes theorem it is not difficult to get:

$$\mathrm{H}^{\mathrm{G}}(k_+) - \mathrm{H}^{\mathrm{G}}(k_-) = \int_{\mathbb{R}^3 \times (\gamma \cup \gamma')} \mathrm{H}^{\mathrm{T}}(k_\pm, t)\big[\mathrm{H}_0^{(2)}(t)\big]^{-1} dt + o(\varepsilon).$$

Here the vanishing contribution comes from the infinite parts of the cycles $\mathscr{L}_+$ and $\mathscr{L}_-$; γ and γ' are as shown on Fig. 2.

It is then straightforward to check that in the limit $q_J^0 = 0$, the contribution of the cycle γ (resp. γ') tends to the first (resp. second) term in the right-hand side of (3). The operation on distributions which appears in the limit is a usual convolution product of distributions integrated on a compact cycle, and therefore meaningful. This ends the proof of Theorem 1.

2.2. Discontinuities in subchannels

Let (J, N\J) denote a subchannel with respect to (I, N\I) with (for instance) $J \subset I$; we write $I = J \cup L$, $L = I \cap (N\backslash J)$.

Here we make the following restrictive *assumption* (which shall be satisfied in later applications): the vertex function $F^1(\{ k_i, i \in I \}, k_\alpha, k_\beta)$ is one-particle irreducible [3] in the channels

$$[\{\alpha\} \cup J ; \{\beta\} \cup L], [\{\alpha\} \cup L ; \{\beta\} \cup J]$$

and for any $K \subset L$ ($K \neq \emptyset$, $K \neq L$), in *at least* one of the channels

$$[\{\alpha\} \cup K ; \{\beta\} \cup J \cup (L\backslash K)] \quad \text{or} \quad [\{\beta\} \cup K ; \{\alpha\} \cup J \cup (L\backslash K)].$$

Under this restriction, we can easily prove:

THEOREM 2. — At any point of the « flat » domain:

$$\{ p_J^2 < 9m^2 \} \times \{ (\hat{k}_{(J)}, \hat{k}_{(N\backslash J)}) \in \mathscr{D}_J \}$$

the discontinuity $\Delta^J H^G$ is given by:

$$\Delta^J H^G(p_J, \hat{k}_{(J)}, \hat{k}_{(N\backslash J)}) = \int_{\mathscr{C}} \Delta^J F^1(p_J, \hat{k}_{(J)}, \hat{k}_{(L)}, k_\alpha, k_\beta) F^{(2)}(-k_\alpha, -k_\beta, \hat{k}_{(N\backslash I)}) \left[H_0^{(2)}(k_\alpha) H_0^{(2)}(k_\beta) \right]^{-1} dk_\alpha \quad (6)$$

where $\mathscr{C}$ is an appropriate cycle defined below.

Proof. — When $q_J^0 \to 0$, we easily check that in the $t^0 = u^0 + iv^0$ plane the following couples of « cuts » get confused:

i) $v^0 = -q_K^0$ and $v^0 = -q_K^0 - q_J^0$, with $K \subseteq L$, K non-empty.

ii) $v^0 = 0$ and $v^0 = -q_J^0$.

But due to the above restrictions on the spectrum of F^1, when $p_J^2 < 9m^2$, the lines $\mathscr{L}_\pm$ are not pinched. Then we define $\mathscr{C}$ as any common distorsion of $\mathbb{R}^3 \times \mathscr{L}_\pm$ and the theorem just states that in such a situation « the discontinuity of the integral is the integral of the discontinuity ».

Indeed in the channel $[J ; (N\backslash J) \cup \{ n+1, n+2 \}]$ the discontinuity of the integrand H^T is obviously:

$$\Delta^J F^1 \, . \, \left[H_0^{(2)}(k_\alpha) H_0^{(2)}(k_\beta) \right]^{-1} . \, F^2 ,$$

since the latter channel *is* a vertex partition for T.

REMARK. — As easily checked, if the threshold masses are those of a pseudo-scalar theory, the theorem is still true for $p_J^2 < 16m^2$, when $| J |$ is even.

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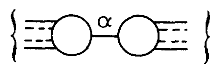
(3) Here it is understood that one-particle irreducibility in some channel (I, J) entails necessarily one-particle irreducibility in the converse channel (J, I).

3. RECALLINGS AND NOTATIONS

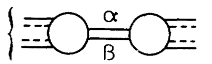
The following sections will be devoted to the central part of this paper: the derivation of « two-particle structural equations » for the four, five and six-point functions.

Since most of the work will have there to be done on the algebraic stage, we shall from now on stick to the following useful graphical « bubble » notation.

a) I  N\I stands for the tree-product

$$H^{(n_1+1)}(\{k_i, i \in I\}, k_\alpha)\,[H^{(2)}(k_\alpha)]^{-1}H^{(n_2+1)}(-k_\alpha, \{k_j, j \in N\backslash I\})$$

with $n_1 = |I|$, $n_2 = |N\backslash I|$ and $H^{(2)}$ the *complete* two-point function.

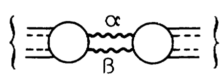
b) I  N\I stands for the *off-shell* one-loop convolution product (see (1)):

$$\int_{\mathscr{C}_k} H^{(n_1+2)}(\hat{k}_{(I)}, k_\alpha, k_\beta)H^{(n_2+2)}(-k_\alpha, -k_\beta, \hat{k}_{(N\backslash I)})[H_0^{(2)}(k_\alpha)H_0^{(2)}(k_\beta)]^{-1}dk_\alpha$$

with $H_0^{(2)}$ the *bare* ([4]) two-point function:

$$H_0^{(2)}(k) = \frac{-Z}{k^2 - m^2}$$

(Z is the « wave function renormalization constant of the field », see [*2*], p. 282).

c) I  N\I stands for the *on-shell* convolution product (see [*3*], Section II.4.3):

$$\left(\frac{2i\pi}{Z}\right)^2 \int_{\mathbb{R}^8} \hat{H}^{(n_1+2)}(\hat{k}_{(I)}, \vec{p}_\alpha, \vec{p}_\beta)\hat{H}^{(n_2+2)}(-\vec{p}_\alpha, -\vec{p}_\beta, \hat{k}_{(N\backslash I)})\delta^-(p_\alpha)\delta^-(p_\beta)$$
$$\delta(p_\alpha + p_\beta + p_I)dp_\alpha dp_\beta .$$

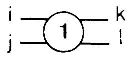
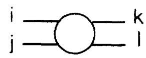
That is we associate $\frac{2i\pi}{Z}\delta^-(p_\alpha)$ with the wavy line $\overset{\alpha}{\sim\!\sim}$.

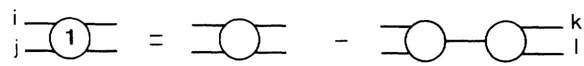
We emphasize that, in the following, graphical equations appear only as a convenient substitute for rather involved *rigorous* expressions. In particular each product of convolution involves a *convergent* four-dimen-

([4]) For the present study, it is not necessary to use the complete two-point function $H^{(2)}$ at this place, and using $H_0^{(2)}$ avoids questioning about singularities produced by possible zeros of $H^{(2)}$.

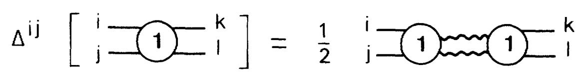
sional integral (as in regularized Feynman amplitudes): this is readily obtained ([3], Sections III and IV.1) by the use of an analytic cut-off function of the Pauli-Villars type.

Let us now review the various objects introduced in our previous papers and needed in the following ([5]):

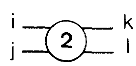
a)  is the one-particle irreducible part of the four-point function  with respect to the channel $[\{i, j\}; \{k, l\}]$, defined ([2], p. 295) by:

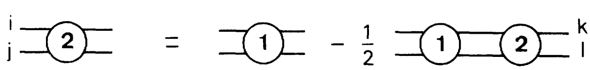
 (7)

It has been proved ([2], p. 293) that its discontinuity in $[\{i, j\}; \{k, l\}]$ vanishes for $(p_i + p_j)^2 < 4m^2$. For $4m^2 \leq (p_i + p_j)^2 < 9m^2$ it is given by ([6]):

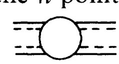
 (8)

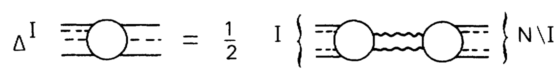
with the notations above ([2], p. 300).

b)  denotes the two-particle irreducible part of the four-point function in $[\{i, j\}; \{k, l\}]$. It is defined ([3], equation 15) as the solution of the general Bethe-Salpeter equation:

 (9)

It was proved in [3] (Theorem 2) that its discontinuity in $[\{i, j\}; \{k, l\}]$ vanishes ([6]) for $(p_i + p_j)^2 < 9m^2$.

c) We recall the two-particle « completeness relations » for the n-point functions ([2], p. 288). They express the discontinuity of  in each channel (I, N\I) when $4m^2 \leq p_{\mathrm{I}}^2 < 9m^2$:

 (10)

d) We give the basic discontinuity formula derived in [3] (Theorem 1),

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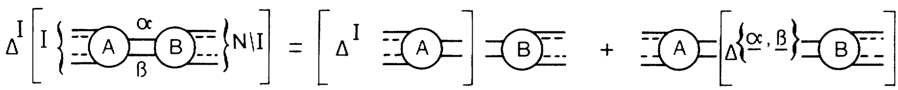
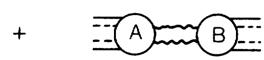
([5]) We restrict to the case of one scalar field, with a single mass m in the spectrum.
([6]) Except possibly at C. D. D. singularities [2].

which expresses the discontinuity of any one-loop convolution product in its convolution channel (I, N\I) when $4m^2 \leq p_{\mathrm{I}}^2 < 9m^2$:





(11)

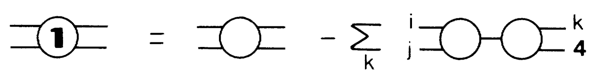
e) Finally we recall that by « one-particle (resp. two-particle) irreducibility » of a n-point function H in some channel (I, N\I) (and then also in (N\I, I)) we mean: vanishing of its (analytic) discontinuity $\Delta^{\mathrm{I}}\mathrm{H}$ in this channel for $p_{\mathrm{I}}^2 < 4m^2$ (resp. $p_{\mathrm{I}}^2 < 9m^2$).

Then we are in a position to define n-point functions ($4 \leq n \leq 6$) simultaneously irreducible in triplets of two-particle channels.

4. A STRUCTURAL EQUATION FOR THE FOUR-POINT FUNCTION

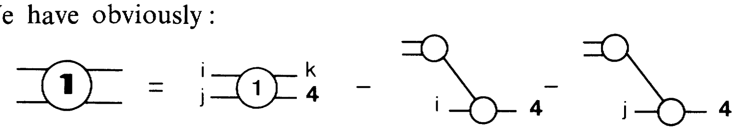
4.1. Extracting the one-particle structure

Starting from the four-point function we first extract one-particle singularities in the three two-particle channels by setting:



where the sum extends to the three circular permutations $\{i, j, k\}$ of $\{1, 2, 3\}$. As proved in [2], p. 303,  is one-particle irreducible (p. i.) in (12; 34), (13; 24) and (23; 14).

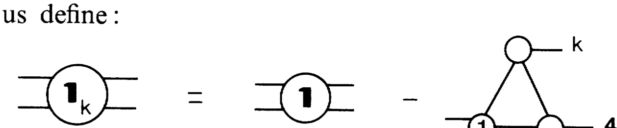
We have obviously:



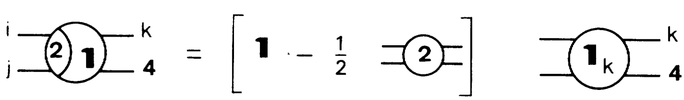
(12)

4.2. Two-particle irreducibility in one channel

Let us define:

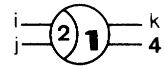


(13 *a*)

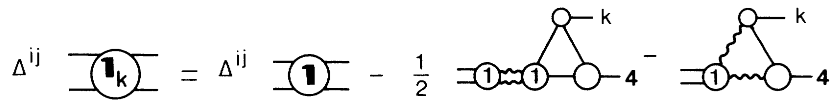


(13 *b*)

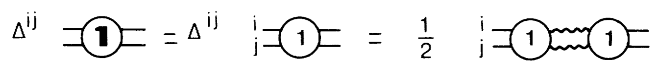
We can prove:

PROPOSITION 1. — The four-point function  is one-particle irreducible in the two channels (ik; $j4$), (kj; $i4$) and two p. i. in (ij; $k4$).

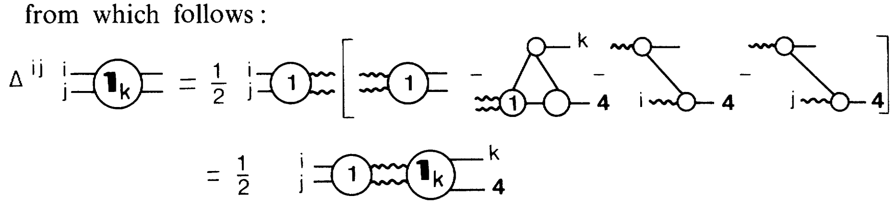
Proof. — The one-particle irreducibility is obvious. From the basic discontinuity relations (11) and (8) we further get (for $4\,m^2 \leq (p_i + p_j)^2 < 9\,m^2$):



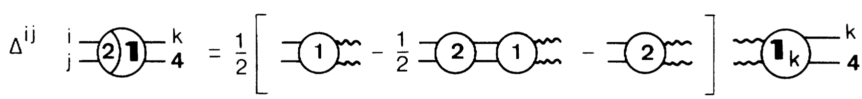
From (8) and (12) we also get:

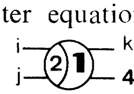


from which follows:



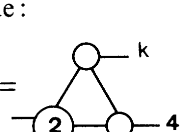
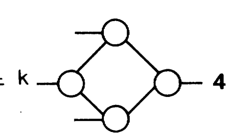
Now using (11) and inserting this result in (13 *b*), we have:



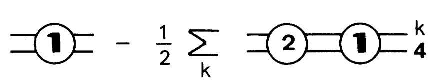
Then using the Bethe-Salpeter equation (9) achieves the proof of the two-particle irreducibility of  in (ij; $k4$).

4.3. Two-particle irreducibility in all channels

We define:

$\gamma_k =$  $\sigma_k =$ 

with $k = 1, 2, 3$ and also:

$\tau =$ 

PROPOSITION 2. — The four-point function :

$$ = \tau - \sum_k \gamma_k + \sum_k \sigma_k \tag{14}$$

is two-p. i. in the three channels (12 ; 34), (13 ; 24) and (23 ; 14).

Proof. — Since the definition of  is symmetrical with respect to the circular permutations of $\{1, 2, 3\}$, it is enough to concentrate on a given channel $(ij\,;\,k4)$.

Then the discontinuity of  will obviously vanish for $(p_i + p_j)^2 < 9m^2$ if in this region we have :

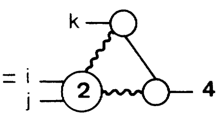
a) $\Delta^{ij}(\tau - \gamma_k) = 0.$

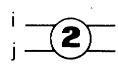
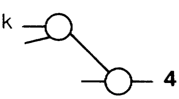
b) $\Delta^{ij}\gamma_j = \Delta^{ij}\sigma_i$

$\Delta^{ij}\gamma_i = \Delta^{ij}\sigma_j.$

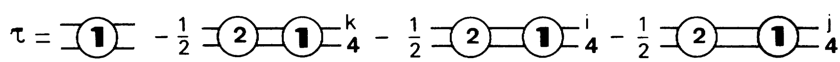
c) $\Delta^{ij}\sigma_k = 0.$

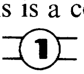
Proof of a). — As a straightforward consequence of (11) we first get :

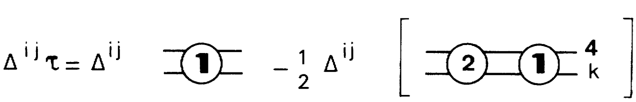
$\Delta^{ij}\ \gamma_k =$  (15)

since both discontinuities of  and  in the channel $(ij\,;\,k4)$ are vanishing when $(p_i + p_j)^2 < 9m^2$.

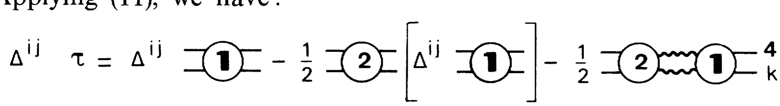
Now we write :



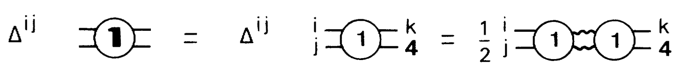
The last two terms in the right-hand side have no discontinuity in the channel $(ij\,;\,k4)$ for $(p_i + p_j)^2 < 9m^2$: this is a consequence of (4) together with the one-particle irreducibility of . So we get :



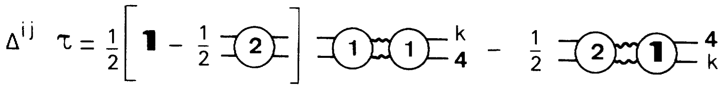
Applying (11), we have :

 (16)

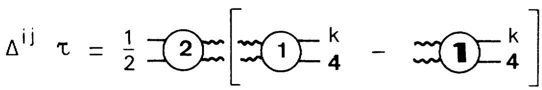
From (12) and (8), we get :



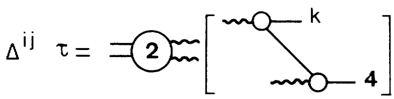
Inserting this result in (16) yields:



Apply the Bethe-Salpeter equation (9):

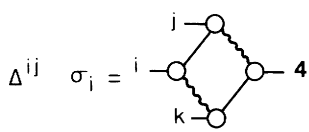


Then (12) gives:

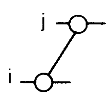
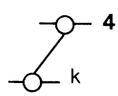


which is the desired result (to be compared with (15)).

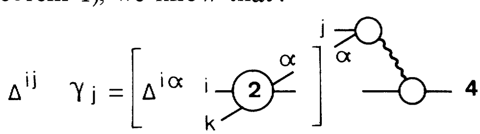
Proof of b). — From (11) we get (for $4m^2 \leq (p_i + p_j)^2 < 9m^2$):



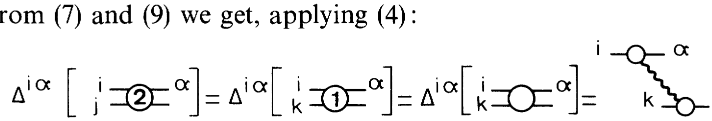
(17)

since  and  have no discontinuity on $(ij; k4)$.

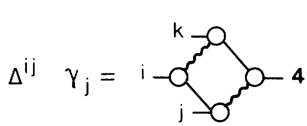
From (3) (Theorem 1), we know that:



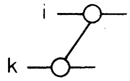
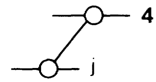
But from (7) and (9) we get, applying (4):



And finally:



which must be compared with (17).

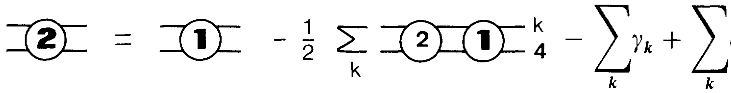
Proof of c). — It is a trivial consequence of (4) together with the special analyticity properties of  and  : $\Delta^{ij}\sigma_k$

vanishes identically everywhere.

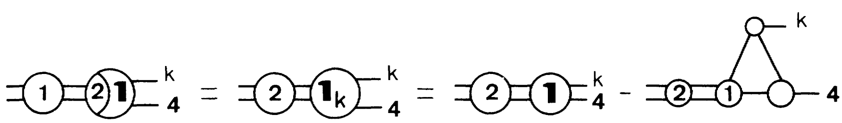
This ends the proof of Proposition 2.

4.4. A structural equation (scalar field)

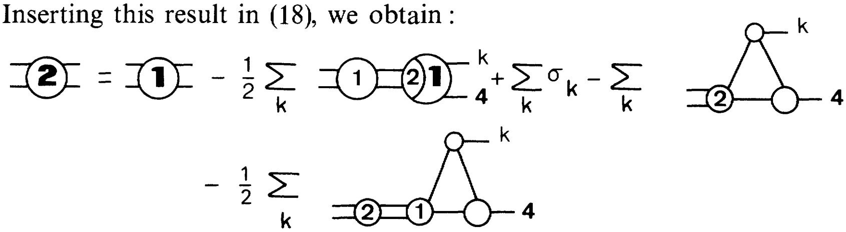
Let us rewrite the definition (14) of  under the form:


(18)

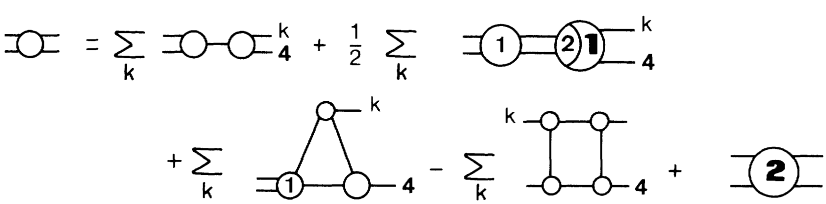
Then notice that in view of (13 b) and (9) we have ($k = 1, 2, 3$) :

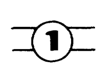


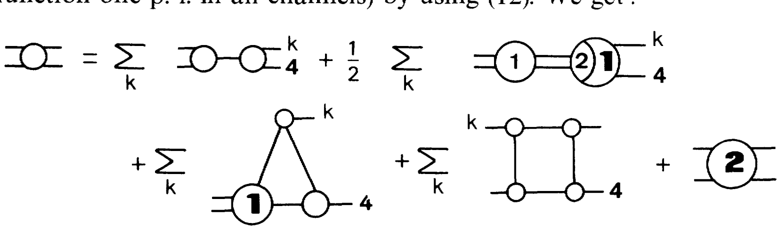
Inserting this result in (18), we obtain:



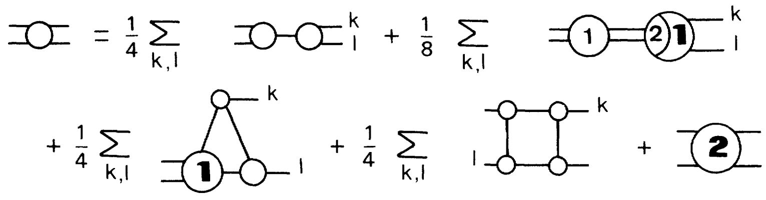
Then the Bethe-Salpeter relation (9) yields:



In the triangle graphs, we can introduce  (the four-point function one-p. i. in all channels) by using (12). We get:


(19)

Moreover we can symmetrize this result (where « 4 » still plays a special role). We obtain :



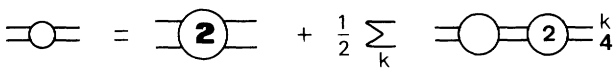
(20)

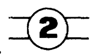
We shall call this relation a « two-particle structural equation » for the four-point function. As a matter of fact its various terms can be considered as rigorous substitutes to partial summations of the Feynman perturbative series, which only exhibit the « details » of the *two-particle* structure.

Equation (20) solves completely the problem of two-particle reductibility for the four-point function. In Section 7 we shall give a global analytic study of its various terms, showing that they exhibit analytic contributions which are less singular than the original four-point function.

4.5. A structural equation (pseudo-scalar field)

For the sake of completeness we give here the structural equation obtained in the case of an *even* theory (pseudo-scalar field) [*4*]. Then all odd n-point functions occurring in (19) identically vanish and we get:

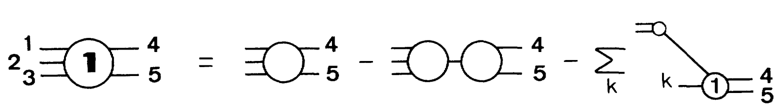


where the discontinuity of  in each of the channels (12 ; 34), (13 ; 24) and (23 ; 14) (and their converse) vanishes when the corresponding squared momentum lies below 16 m^2.

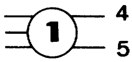
5. A STRUCTURAL EQUATION FOR THE FIVE-POINT FUNCTION

5.1. One-particle structure

We first concentrate on the definition of a function which should be one-particle irreducible in a given channel (123 ; 45) and in all subchannels, namely (12 ; 345), (13 ; 245) and (23 ; 145). This is easily done as follows. Define :

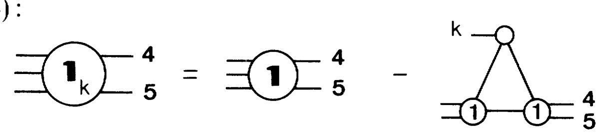


(21)

where the sum extends to all circular permutations of $\{1, 2, 3\}$. As a consequence of the analysis made in [2], p. 303, it is easily checked that  is actually one-p. i. in the above mentioned four channels.

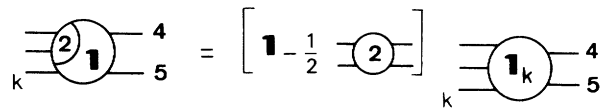
5.2. Irreducibility in one subenergy

The method is similar to the one used in Section 4.2. We first define ($k = 1, 2, 3$):

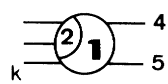


(22 *a*)

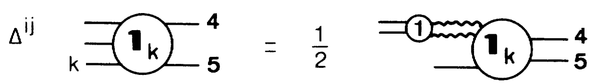
and



(22 *b*)

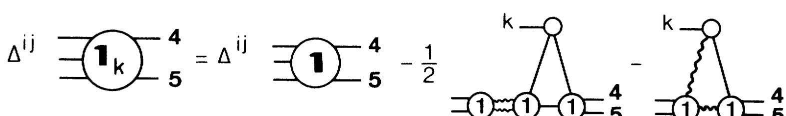
PROPOSITION 3. — The five-point function  is one-particle irreducible in the channels (123; 45), (ik; j45) and (kj; i45), and two-p. i. in the channel (ij; k45).

Proof. — The one-particle irreducibility follows from the definition. As for the two-particle irreducibility in (ij; k45) it is sufficient to prove:



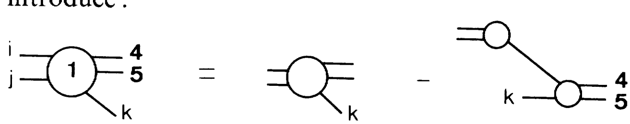
since the proof can be achieved by using the Bethe-Salpeter equation as in the proof of Proposition 1.

From (11) and (18) we get:

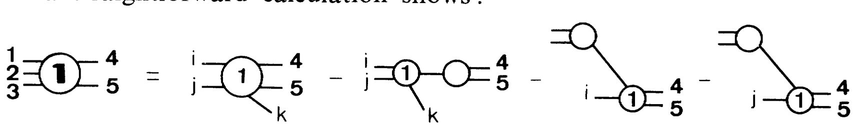


(23)

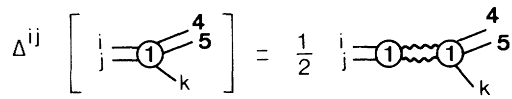
Then we introduce:



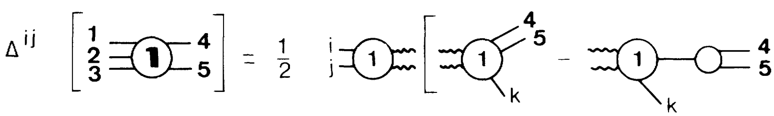
As a straightforward calculation shows:



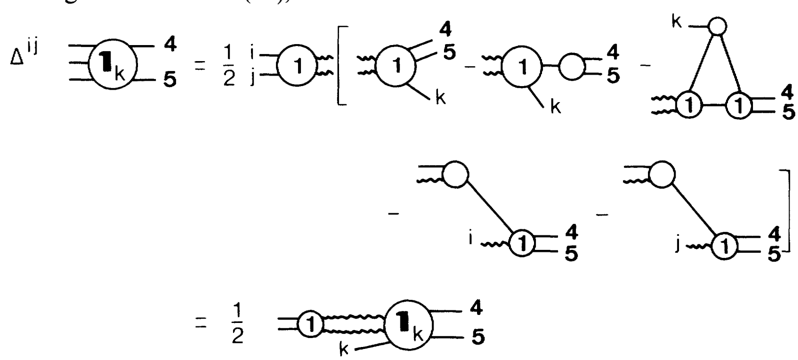
Moreover it has been shown in [2], p. 300, that:



Then (8) and (10) yield:

 (24)

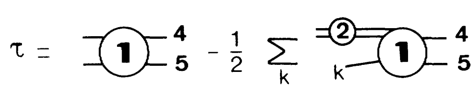
Inserting this result in (23), we obtain:



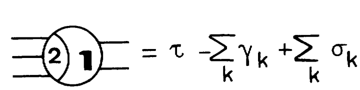
which achieves the proof of Proposition 3.

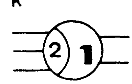
5.3. **Two-particle irreducibility in the initial triplet**

Let us define ($k = 1, 2, 3$):



and

 (25)

PROPOSITION 4. — The five-point function  is two-p. i. in the initial triplet (i. e. in the three channels (ij; $k45$), $k = 1, 2, 3$) and one-p. i. in the channel (123; 45).

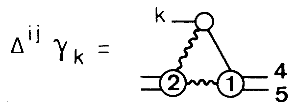
Proof. — We concentrate on the two-particle irreducibility since the one-particle irreducibility is obvious. The method is similar to the one used in Proposition 2 : it is enough to prove the result in one channel (ij ; $k45$). The same arguments than in Proposition 2 then show:

$$\Delta^{ij}\sigma_k = 0, \qquad \Delta^{ij}\gamma_j = \Delta^{ij}\sigma_i, \qquad \Delta^{ij}\gamma_i = \Delta^{ij}\sigma_j$$

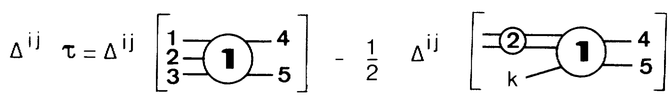
when $(p_i + p_j)^2 < 9\,m^2$. The proof of:

$$\Delta^{ij}(\tau - \gamma_k) = 0$$

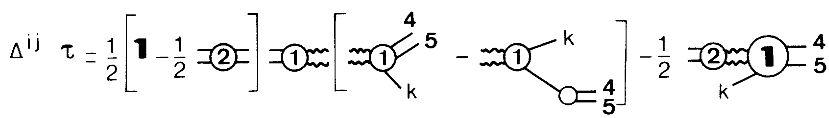
requires a separate study. Indeed we have:


(26)

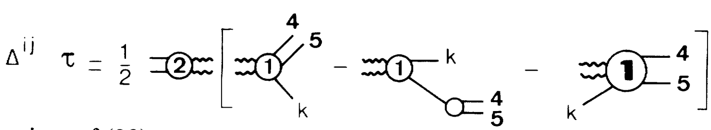
and (as a consequence of (4)):



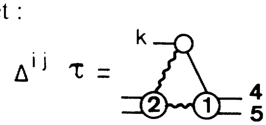
Now using (24) and (11), we obtain:



Then the Bethe-Salpeter equation (9) allows to write:



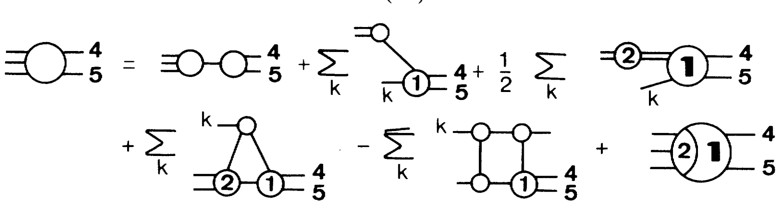
and in view of (23) we get :



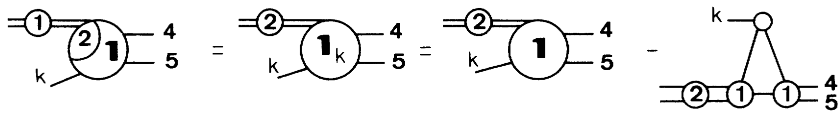
which achieves the proof of Proposition 4 (see (26)).

5.4. A structural equation

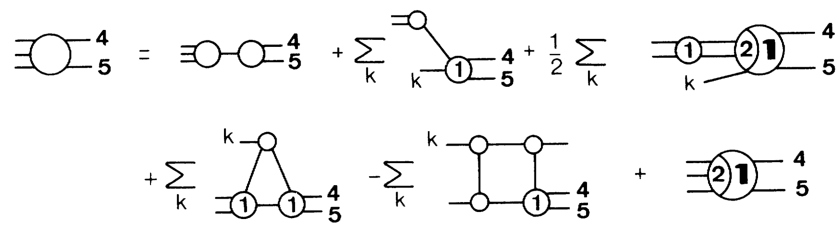
We can rewrite the definition (25) as follows:


(27)

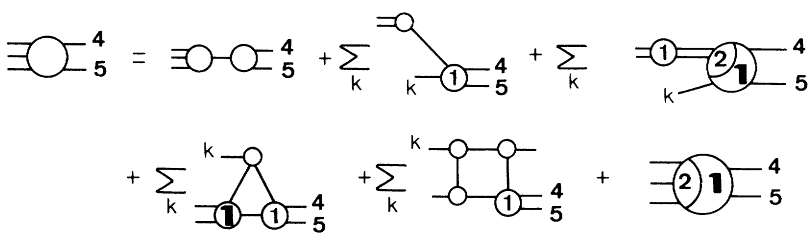
But we notice that (22 b) yields for $k = 1, 2, 3$:



Inserting this result in (27) and applying the Bethe-Salpeter equation (9), we obtain:



Inserting (12) in the triangle graphs, we obtain:



(28)

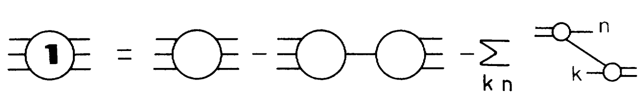
In Section 7 we shall give an analytic interpretation of this global structural equation.

6. A STRUCTURAL EQUATION FOR THE SIX-POINT FUNCTION OF A PSEUDO-SCALAR FIELD

For simplicity we shall only consider in this section the case of an even theory (pseudo-scalar field). We consider the channel $[\{1, 2, 3\}; \{4, 5, 6\}]$ and label by $\{i, j, k\}$ (resp. $\{l, m, n\}$) any circular permutation of $\{1, 2, 3\}$ (resp. $\{4, 5, 6\}$).

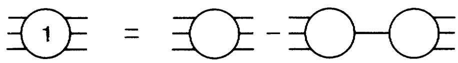
6.1. Extraction of the one-particle structure

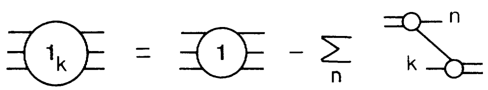
We first construct a six-point function which is one-p. i. in *all* channels. This is performed by using the procedure given in [2], p. 303, namely:

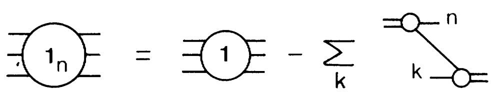


(29)

where the sum extends to circular permutations of { 1, 2, 3 } and { 4, 5, 6 }. However the following functions will be also useful:

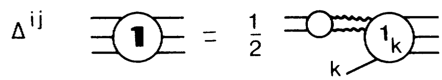

(30 a)

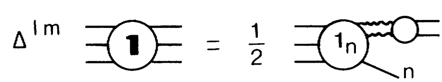

, k fixed (30 b)


, n fixed. (30 c)

In particular we can state:

Proposition 5. — In the corresponding two-particle regions, we have:

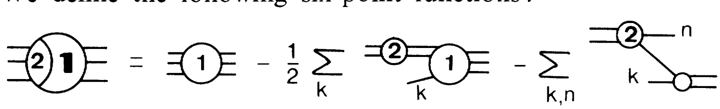

(31 a)

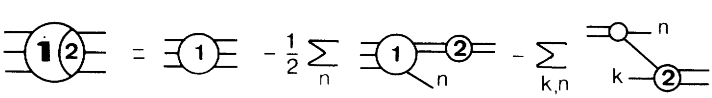

(31 b)

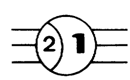
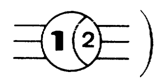
Proof. — It is a straightforward consequence of the definitions and of the completeness relations (10).

6.2. Two-particle irreducibility in the final (or initial) triplet

We define the following six-point functions:

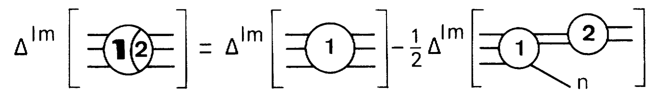

(32 a)


(32 b)

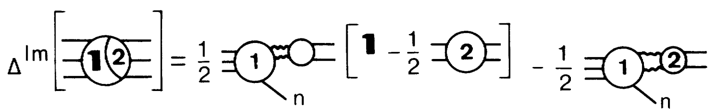
Proposition 6. —  (resp. ) is one-p. i. in all channels and two-p. i. with respect to the initial (resp. final) triplet of channels, i. e. with respect to (ij ; $k456$), (ik ; $j456$) and (kj ; $i456$) (resp. ($123l$; mn), ($123\,m$; ln) and ($123\,n$; lm)).

Proof. — For the proof of two-particle irreducibility, it is sufficient to restrict to a given triplet (say: final) and a given channel (say: (123 n, lm)).

Then it is a consequence of (4), that if $(p_l + p_m)^2 < 16\ m^2$:

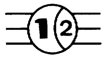
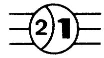


From (11) and the definition (32 b), we get:



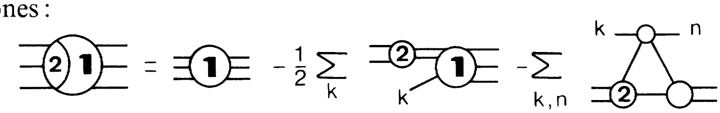
Then we apply the Bethe-Salpeter equation (9) which in the case of a pseudo-scalar field reads:

(33)

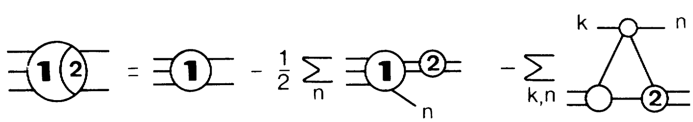
and this achieves the proof of the two-particle irreducibility. The fact that  and  are one-p. i. in all channels is straightforward from (33) and their definitions.

Finally the two following combinatorial identities will be useful in the sequel:

PROPOSITION 7. — The definitions (32) are equivalent with the fol'owing ones:



(34 a)

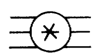


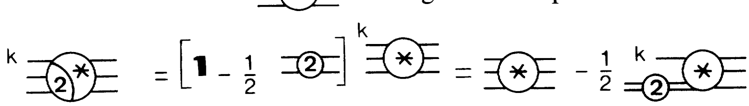
(34 b)

Proof. — Insert (29) in (32).

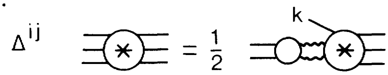
6.3. Two-particle irreducibility in one initial (final) channel

We then come to the definition of a function simultaneously two-p. i. in the final (resp. initial) triplet and in one channel of the initial (resp. final) triplet. The next lemma will be necessary:

PROPOSITION 8. — Let  be a general six-point function. Then:



is two-p. i. in the channel $(ij\,;\,k456)$ if and only if, in the corresponding two-particle region:

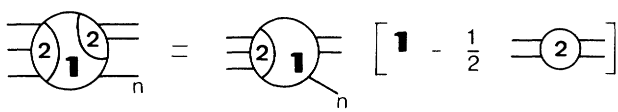


A similar result holds for any given final channel $(123\ n\,;\,lm)$.

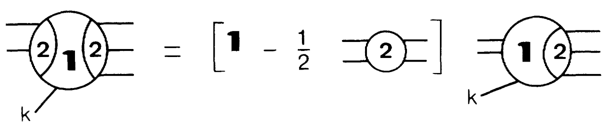
Proof. — Apply the discontinuity formula (11) and the Bethe-Salpeter equation (33).

Then we are in a position to prove:

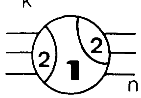
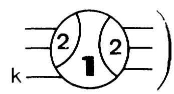
PROPOSITION 9. — Define:



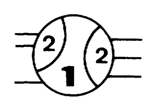
(35 *a*)



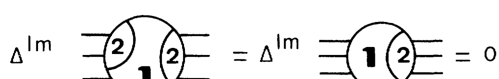
(35 *b*)

Then  (resp. ) is one-p. i. in all channels, two-p. i. in the initial (resp. final) triplet and two-p. i. in the channel $(123\ n\,;\,lm)$ (resp. $(ij\,;\,k456)$).

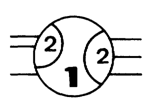
Proof. — The one-particle irreducibility in all channels is obvious.

For the two-particle irreducibility in the final triplet of 

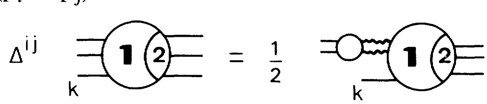
it is enough to notice that the second term in the right-hand side of (35 *b*) has no discontinuity in the two-particle region of any final two-particle channel: actually this is a consequence of Theorem 2. Then:



when $(p_l + p_m)^2 < 16\,m^2$.

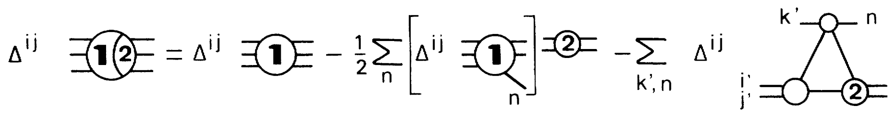
The two-particle irreducibility of  in the channel $(ij\,;\,k456)$

requires a little more attention. In view of Proposition 8 it is sufficient to prove that, if $(p_i + p_j)^2 < 16\ m^2$:

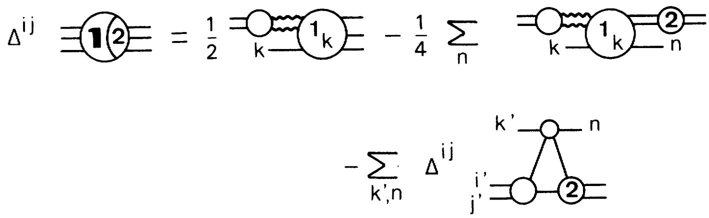


For that purpose we start from (34 *b*).

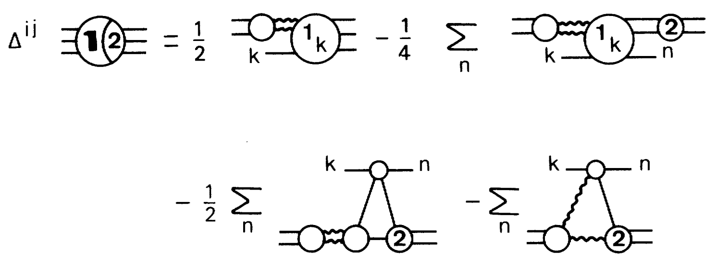
Applying Theorem 2, we first get:



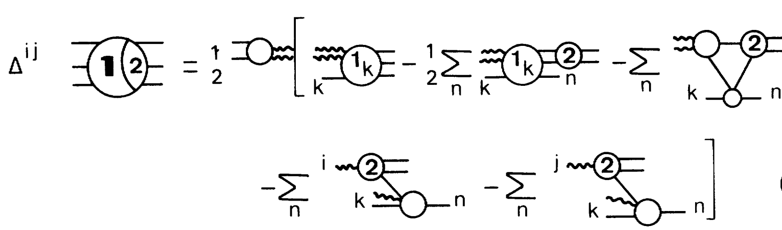
Then from Proposition 5 we can deduce that:



For the last sum in the right-hand side we apply (11) and (4). We obtain:

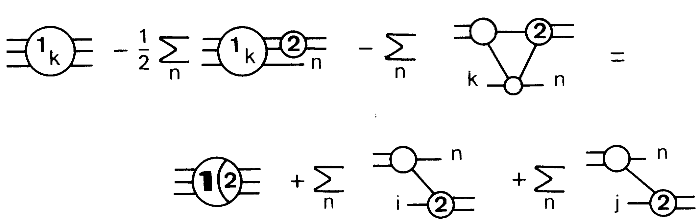


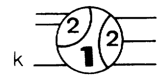
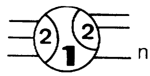
We can factorize the four-point function:



(37)

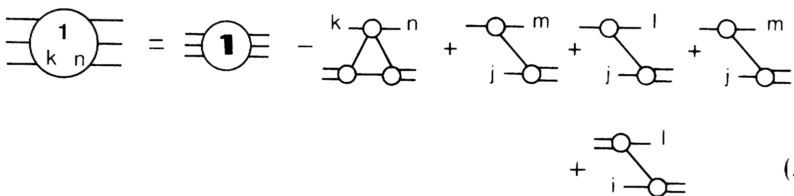
Now the following relation is easily checked, starting from (32 *b*) and (30 *b*):



Inserting this result in the last bracket of (37) then achieves the proof of (36) and therefore of the two-particle irreducibility of  in $(ij\,;\,k456)$. The argument would go similarly for the corresponding statements on . This ends the proof of Proposition 9.

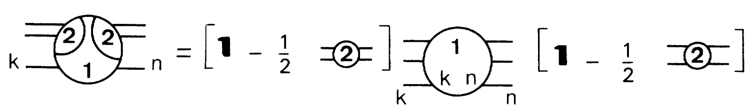
6.4. Two-particle irreducibility in one initial and one final channels

We then turn to the definition of a six-point function simultaneously two-p. i. in *one* initial channel $(ij\,;\,k456)$ and *one* final channel $(123\,n\,;\,ln)$.
For that purpose, we first set:

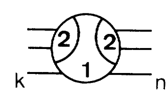


(38)

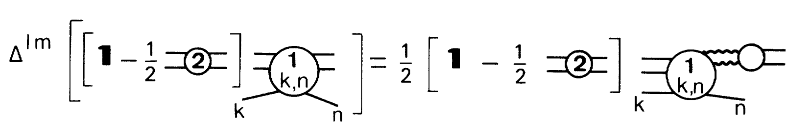
with k and n fixed.
Then we introduce the new functions:



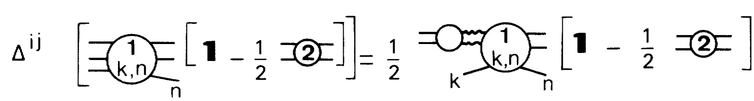
(39)

PROPOSITION 10. —  is one-p. i. in the channels $(ijl\,;\,kmn)$, $(ijm\,;\,kln)$, $(ijn\,;\,klm)$, $(jkn\,;\,ilm)$, $(ikn\,;\,jlm)$ and two-p. i. in both channels $(ij\,;\,k456)$ and $(123\,n\,;\,lm)$.

Proof. — The one-particle irreducibility property is trivial. For the proof of two-particle irreducibilities, Proposition 8 insures it is sufficient to check that, when $(p_l + p_m)^2 < 16\ m^2$:

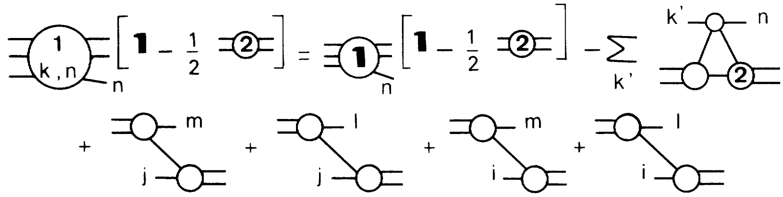


and that, when $(p_i + p_j)^2 < 16\ m^2$:

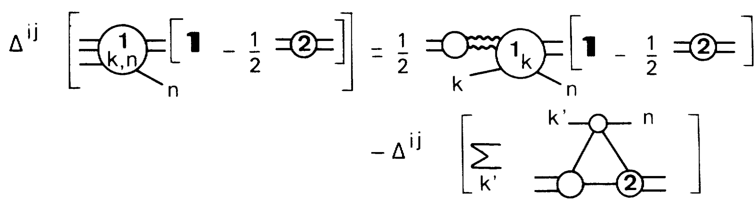


(40)

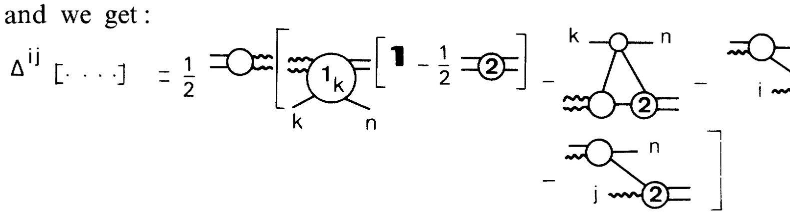
By symmetry it is sufficient to prove (40). From (38) we first get:



Applying Theorem 2 and Proposition 5 to the first term on the right-hand side, we have:

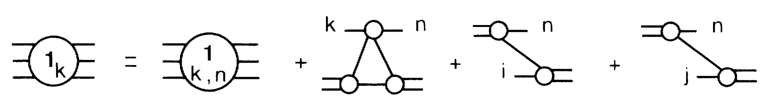


The discontinuity of the last bracket is easily computed through (4) and (11) and we get:



(41)

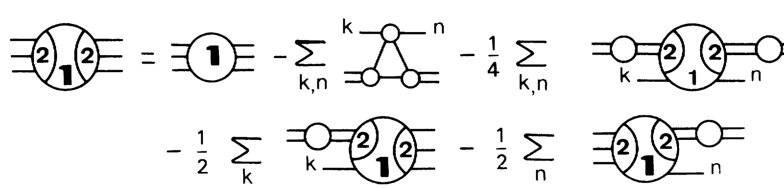
Now the following relation is easy to check from definitions (30) and (38):



And inserting this result in (41) yields (40) and achieves the proof of Proposition 10.

6.5. **The structural equation**

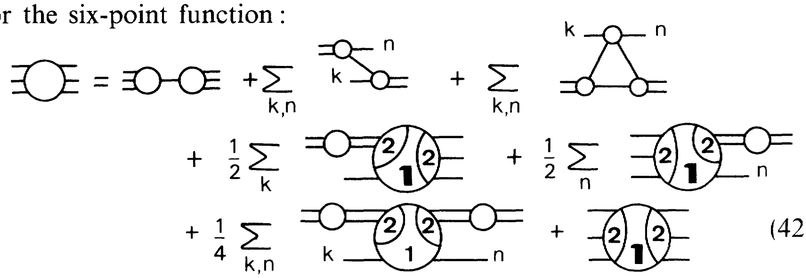
We then come to the central part of this section: the definition of a six-point function being simultaneously two-particle irreducible in any of the six channels (ij; $k456$) and (123 n; lm), k and n arbitrary. The latter is defined by means of the following equation:



(42 *a*)

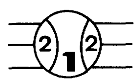
where the various six-point functions involved in the right-hand side have been defined in the previous subsections.

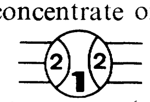
It will be useful to write (42 a) under the form of a structural equation for the six-point function:



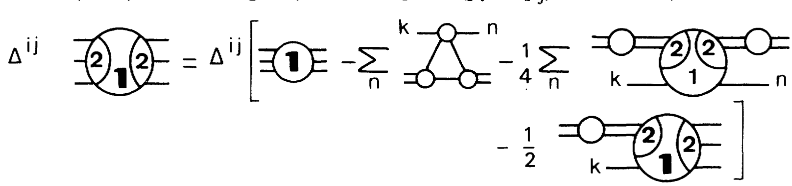
(42 b)

An analytic interpretation of this equation will be given in Section 7. At present we can prove:

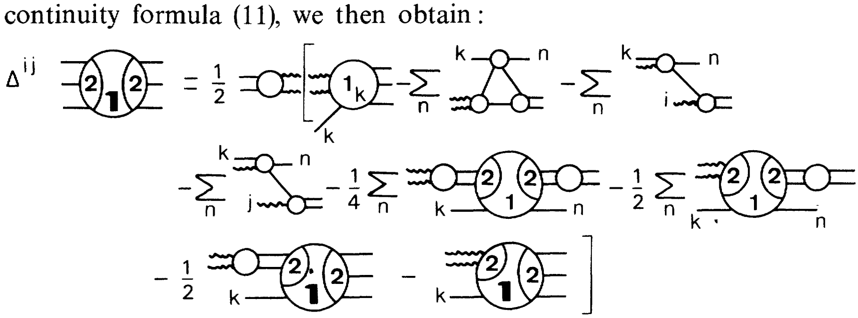
THEOREM 3.  is one-particle irreducible in all channels and two-p. i. in both initial and final triplets, i. e. in any of the channels $(ij; k456)$ $(k = 1, 2, 3)$ and $(123\,n; lm)$ $(n = 4, 5, 6)$.

Proof. — The one-particle irreducibility is obvious. As for the proof of two-particle irreducibilities it is sufficient to concentrate on *one* given initial *or* final channel: indeed the definition of  is symmetrical with respect to the circular permutations of each set $\{1, 2, 3\}$ and $\{4, 5, 6\}$. We choose for instance $(ij; k456)$.

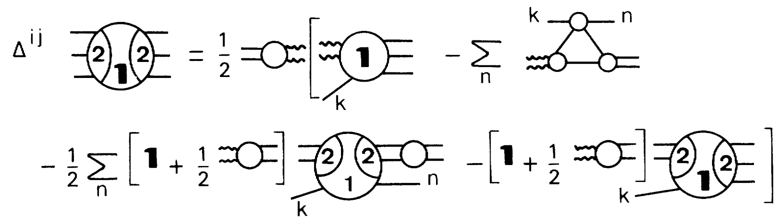
From (42 a) we then get (in the region $(p_i + p_j)^2 < 16\,m^2$):



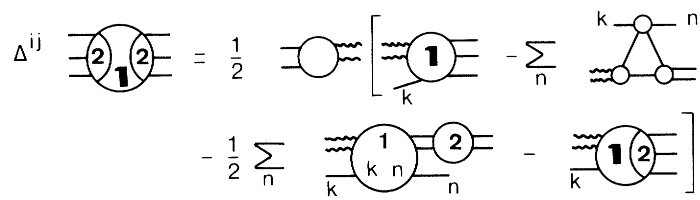
Here we have used (4), the irreducibility properties of the functions defined in previous subsections, and Theorem 2. From Proposition 5 and the discontinuity formula (11), we then obtain:



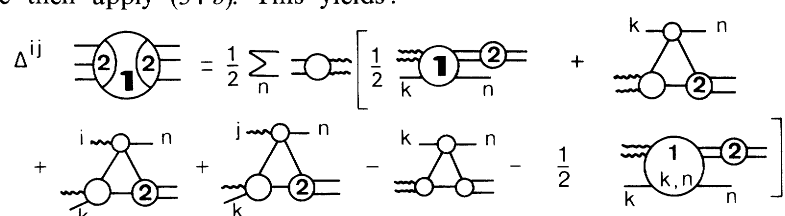
By using (29) and (30) this can be rewritten as follows:

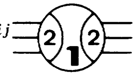


Now using the definitions (35 *b*) and (39), and taking into account the Bethe-Salpeter equation (33), this can be rewritten:



We then apply (34 *b*). This yields:



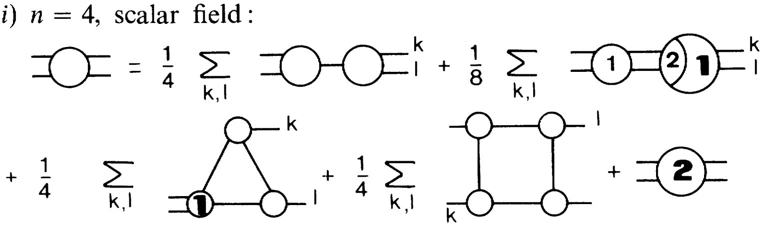
And inserting (38) in this result proves the vanishing of Δ^{ij}  in the region $(p_i + p_j)^2 < 16\, m^2$, q. e. d.

7. ANALYTIC INTERPRETATION OF STRUCTURAL EQUATIONS

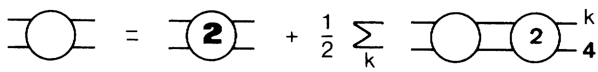
7.1. Threshold properties

Before going further let us first summarize the various two-particle structural equations derived above:

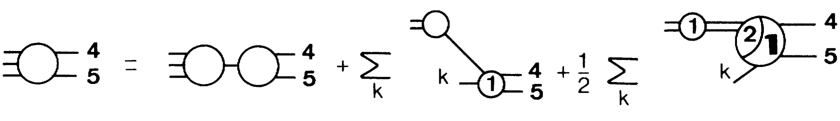
i) $n = 4$, scalar field:

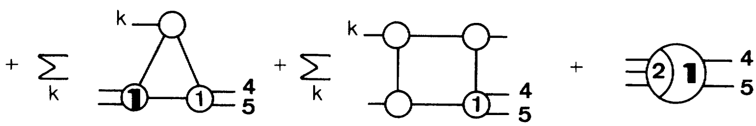
 (20)

ii) $n = 4$, pseudo-scalar field:

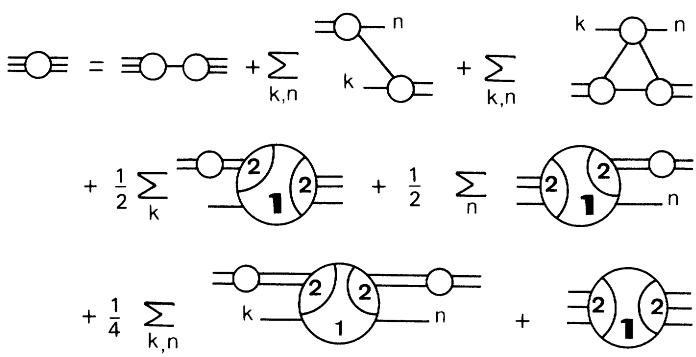


iii) $n = 5$, scalar field:

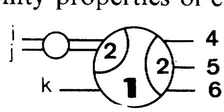
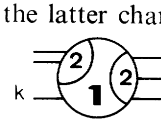


 (28)

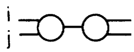
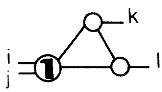
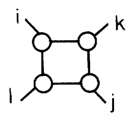
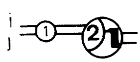
iv) $n = 6$, pseudo-scalar field:

 (42)

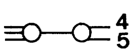
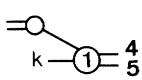
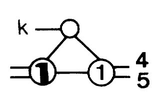
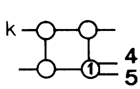
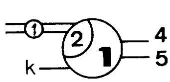
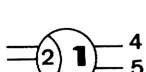
The threshold properties of the various functions involved in the right-hand side of these equations are readily obtained by applying Theorems 1 and 2 and taking into account the irreducibility properties of each vertex function.

As a typical example, let us consider:  . In the channels (ij; $k456$) and (123; 456) its thresholds are obviously the same as those of the original six-point function, namely $4\,m^2$ and $9\,m^2$. The thresholds in (jk; $i456$) and (ki; $j456$) are given by (4) since the latter channels are transverse to (ij; $k456$) : we get $16\,m^2$ since  is one-p. i. in all channels. As for ($123\ l$; mn) ($l = 4, 5, 6$), these are sub-channels of (ij; $k456$) and Theorem 2 can be applied, leading to $16\,m^2$.

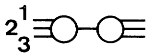
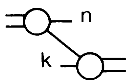
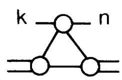
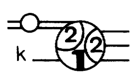
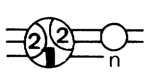
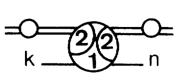
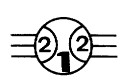
Similar results can be obtained for each individual term of (20), (28) and (42) and have been summarized on the following table

$n = 4$	$(ij\,;\,kl)$	$(ik\,;\,jl)$	$(il\,;\,jk)$
	m^2, $4\,m^2$	∞	∞
	$4\,m^2$	$9\,m^2$	$9\,m^2$
	∞	$4\,m^2$	$4\,m^2$
	$4\,m^2$	$*\,9\,m^2$	$*\,9\,m^2$
	$*\,9\,m^2$	$*\,9\,m^2$	$*\,9\,m^2$

($* = 16\,m^2$ in the pseudo-scalar case)

$n = 5$	$(ij\,;\,k45)$	$(ik\,;\,j45)$	$(jk\,;\,i45)$	$(ijk\,;\,45)$
	m^2, $4\,m^2$	m^2, $4\,m^2$	m^2, $4\,m^2$	m^2, $4\,m^2$
	m^2, $4\,m^2$	∞	∞	$4\,m^2$
	$4\,m^2$	$9\,m^2$	$9\,m^2$	$4\,m^2$
	∞	$4\,m^2$	$4\,m^2$	$4\,m^2$
	$4\,m^2$	$9\,m^2$	$9\,m^2$	$4\,m^2$
	$9\,m^2$	$9\,m^2$	$9\,m^2$	$4\,m^2$

Thresholds for the structural parts of $H^{(n)}$.

$\underline{n = 6}$	$(ij; k456)$	$(jk; i456)$	$(ik; j456)$	$(123\,l; mn)$	$(123\,m; nl)$	$(123\,n; lm)$	$(123; 456)$
	$4\,m^2$	$4\,m^2$	$4\,m^2$	$4\,m^2$	$4\,m^2$	$4\,m^2$	$m^2, 9\,m^2$
	$4\,m^2$	∞	∞	∞	∞	$4\,m^2$	∞
	$4\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$4\,m^2$	$9\,m^2$
	$4\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$9\,m^2$
	$16\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$4\,m^2$	$9\,m^2$
	$4\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$4\,m^2$	$9\,m^2$
	$16\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$16\,m^2$	$9\,m^2$

7.2. **Discussion and outlook**

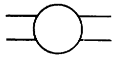
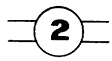
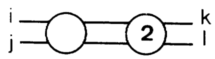
7.2.1. Global analyticity properties

It is important to recall that the determination of the analyticity domain of a given n-point function is crucially depending on the knowledge of its threshold masses in the various channels. Actually the primitive n-point domain is the union of a fixed set of large off-shell domains (the tubes) with a set of complex neighbourhoods of « coincidence regions » $\mathscr{R}^{\mathrm{I}}$ which are « bridges » between the previous domains but depend crucially on the thresholds, namely:

$$\mathscr{R}^{\mathrm{I}} = \{ p \in \mathbb{R}^{4(n-1)} : p_{\mathrm{I}}^2 < \mathrm{M}_{\mathrm{I}}^2 \} .$$

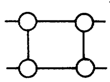
Thus if two n-point functions $\mathrm{H}^{(n)}$ and $\mathrm{H}'^{(n)}$ have the respective set of thresholds $\{ \mathrm{M}_{\mathrm{I}} \}$, $\{ \mathrm{M}'_{\mathrm{I}} \}$, $\mathrm{M}_{\mathrm{I}} < \mathrm{M}'_{\mathrm{I}}$, the corresponding primitive domains D_n and D'_n will be (star-shaped) domains in $\mathbb{C}^{4(n-1)}$ with $\mathrm{D}_n \subset \mathrm{D}'_n$ and the holomorphy envelope of D'_n will be (strictly) larger than the one of D_n.

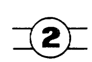
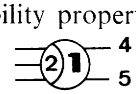
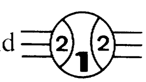
By looking back at the threshold tables given above, we can then conclude that each structural equation (20), (28) and (42) performs a *global* decomposition of the physical n-point function $\mathrm{H}^{(n)}$ in several « structural parts », each of which has *better* analyticity properties than $\mathrm{H}^{(n)}$.

Let us for instance consider the four-point pseudo-scalar case on the mass-shell $\{ k_i^2 = m^2, 1 \leq i \leq 4 \}$. We see that in the Mandelstam variables $s_{12} = (k_1 + k_2)^2$, $s_{23} = (k_2 + k_3)^2$, $s_{31} = (k_3 + k_1)^2$, the four-point function  appears in (20) as the sum of *four* analytic functions. One of them  only presents the « inelastic » cuts ($s_{12}, s_{23}, s_{31} > 16\ m^2$) as real boundary of its analyticity domain, while each of the others  presents a single two-particle cut ($s_{ij} > 4\ m^2$) and two inelastic cuts ($s_{jk}, s_{ki} > 16\ m^2$).

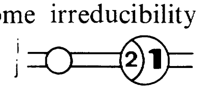
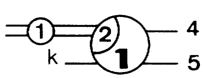
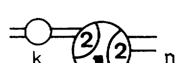
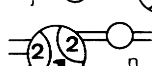
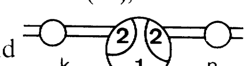
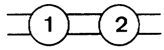
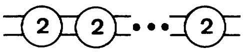
It is hoped that the classical global techniques of analytic completion (in particular the Jost-Lehmann-Dyson result [5]) should substantially improve the analyticity domain of $H^{(n)}$ ($n = 4, 5, 6$) when applied to each of its structural parts.

In particular it is a well-known fact that the obstruction to the proof of the crossing property for $2 \to n - 2$ amplitudes ($n > 4$) comes from the too low values of the thresholds there involved. Then it may be hoped that (at least for $n \leq 6$) crossing domains could be obtained for each individual term of an appropriate structural equation.

Another interesting global feature appears with square convolution products . Such a four-point function is analytic in a domain which can in principle be obtained from the knowledge of the three-point domain $D^{(3)}$ and from analysing the Landau singularities of the associated Feynman diagram. More generally it appears that three classes of contributions can be distinguished in the above derived structural equations:

i) a « leading term » which has the best irreducibility properties and involves no convolution. This is  in (20),  in (28) and  in (42). It is in the study of analyticity domains of such terms that global completion techniques should be the most powerful.

ii) « quasi-perturbative » terms: the latter are G-convolution products whose vertex functions are n_v-point functions, with $n_v < n$. These are convolution-products corresponding to trees, squares ($n = 4, 5, 6$) and triangles ($n = 5, 6$). For such terms, rather large analyticity domains and the emergence of Landau singularities may still be expected.

iii) « mixed » terms, i. e. having some irreducibility properties and involving convolutions. The various  of (20),  of (28), ,  and  of (42) belong to the latter class. On the one hand, their situation with respect to analytic completion are moderately better than those of the original n-point functions. On other hand they should also be investigated for their convolution structure, which actually suggests iterative expansion properties. Typically:  gives rise to the infinite series 

Finally let us note that as a consequence of their convolution structure, both terms of the second and third type should be also studied from the point of view of meromorphic extensions in second sheets across the two-particle cut of their respective convolution channels.

A first step in this direction had been taken in [4] on the mass shell for the four-point function of a pseudo-scalar field. The generalization of this result is at present under study.

7.2.2. Local analyticity properties

Let us now concentrate on the local interpretation of the above structural equations in the neighbourhood of the lowest energy strip of the physical region of the $2 \to 3$ and (pseudo-scalar) $3 \to 3$ scattering amplitudes, namely:

$$\mathscr{R}^{(5)} = \{ p \in \mathbb{R}^{16} : 4\, m^2 \leq (p_i + p_j)^2 < 9\, m^2 \ (i, j = 1, 2, 3) ; \quad p_k^2 = m^2,\ 1 \leqslant k \leqslant 5 \}$$

$$(\text{resp. } \mathscr{R}^{(6)} = \{ p \in \mathbb{R}^{20} : 4\, m^2 \leq (p_i + p_j)^2 < 16\, m^2 \ (i, j = 1, 2, 3), \quad 4\, m^2 \leq (p_l + p_m)^2 < 16\, m^2 \ (l, m = 1, 2, 3) ; \quad p_k^2 = m^2,\ 1 \leqslant k \leqslant 6 \}) .$$

A simple model of both geometrical situations is also provided by the four-point function in the (unphysical) region:

$$\mathscr{R}^{(4)} = \{ p_i^2 = m^2,\ i = 1, 2, 3 ; \quad 4\, m^2 \leqslant (p_i + p_j)^2 < 9\, m^2 \ (i, j = 1, 2, 3) \} .$$

It is actually in view of further applications to the mass-shell regions $\mathscr{R}^{(5)}$ and $\mathscr{R}^{(6)}$ that we have been led to write structural equations in which a triplet $\{ 1, 2, 3 \}$ plays a special role.

First recall that the local analytic study of a n-point function $\mathrm{H}^{(n)}$ in a neighbourhood of some given point p necessitates to apply the local edge-of-the-wedge theorem to any cluster of tubes exclusively separated by hyperplanes $q_{\mathrm{I}} = 0$, whose corresponding partial sum p_{I} lie below the associated threshold M_{I}.

$\mathrm{H}^{(n)}$ is then analytically extended in the intersection of a complex neighbourhood of p with the convex hull of the given cluster of tubes (7).

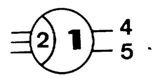
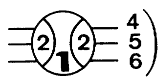
(7) Actually the intersection of the tubes with a complex neighbourhood of p should be performed before taking the convex hull; however here it can be proved that both orders are equivalent [7].

When p lies on the mass-shell, one of these clusters is distinguished as the one in which the boundary values of $H^{(n)}$ coincide with the chronological prescription. For $n = 4$ two clusters of sixteen tubes can then be found, which leads to two opposite local tubes: $q_1 + q_2 \in V^{\pm}$.

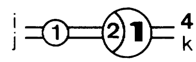
Then by using the complex Lorentz invariance of the analyticity domain of $H^{(4)}$ (or by another argument given in [6]), it has been proved that $H^{(4)}$ is analytic in a cut neighbourhood of the physical region.

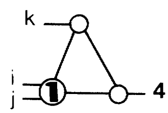
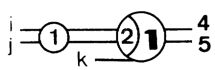
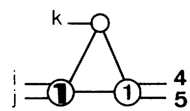
In the present context, it is clear that an equivalent property is satisfied, *mutatis mutandis*, for the respective leading terms of (20), (28) and (42). Indeed it is easy to check that above any $p \in \mathscr{R}^{(n)}$ ($n = 4, 5, 6$) the convex hulls of the involved clusters of tubes are: $q_1 + q_2 + q_3 \in V^{\pm}$.

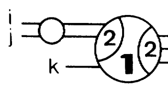
Then the same argument of complex Lorentz invariance shows that

 (resp.  . ) is locally analytic in

a cut neighbourhood of $\mathscr{R}^{(4)}$ (resp. $\mathscr{R}^{(5)}$, $\mathscr{R}^{(6)}$) with only one cut $(p_1 + p_2 + p_3)^2 = 9\, m^2 + \rho$, $\rho > 0$.

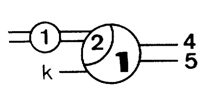
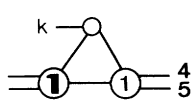
Analogous considerations show that, for $k = 1, 2, 3$, ,

, , 

and also  are locally analytic in four clusters of tubes

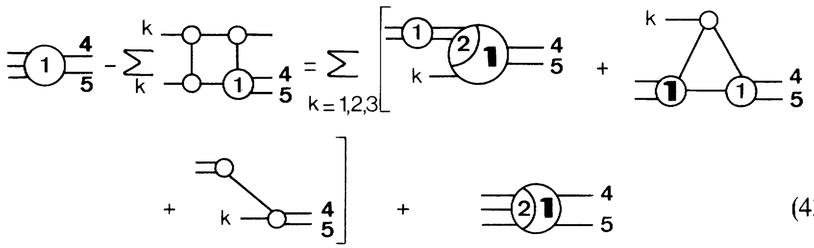
having as corresponding convex hulls: $\{ q_1 + q_2 + q_3 \in V^{\pm}, q_i + q_j \in V^{\pm} \}$. Moreover since each general n-point function is analytic in the complex Lorentz completion of the n-point primitive domain [8], the above functions can be continued to the corresponding local extended tubes in two vectors [9].

At this point it is interesting to compare these results with those obtained in [7] in the framework of the *linear program*. There it was found that in a complex neighbourhood of the $2 \rightarrow 3$ physical region, the five-point function $H^{(5)}$ can be decomposed (up to an ambiguity which is analytic in the local tubes $q_1 + q_2 + q_3 \in V^{\pm}$) as a sum of *three* auxiliary functions $F_k (k = 1, 2, 3)$, each of which respectively enjoys the local analyticity

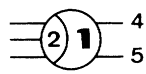
properties described above for  and 

$(k = 1, 2, 3)$.

As a matter of fact, the structural equation (28) can be rewritten :



(43)

Then it is clear that (up to  which belongs to the ambiguity) (43) provides us with a *global* decomposition of the five-point function of the left-hand side in terms of three analytic functions whose *local* analyticity properties on the mass-shall region $\mathcal{R}^{(5)}$ are exactly those of a « relativistic decomposition » in the sense of [7].

APPENDIX

COMPLEMENTS ON THE PROOF OF THEOREM 1

1. Use of Stokes Formula

We give a full account of the central formula:

$$\mathrm{H}^{\mathrm{G}}(\vec{p}_{\mathrm{J}}, p_{\mathrm{J}}^0 + i\varepsilon, \hat{k}) - \mathrm{H}^{\mathrm{G}}(\vec{p}_{\mathrm{J}}, p_{\mathrm{J}}^0 - i\varepsilon, \hat{k}) = \int_{\mathbb{R}^3} d\vec{t} \int_{\gamma\cup\gamma'} \mathrm{H}^{\mathrm{T}}(\vec{p}_{\mathrm{J}}, k_{\mathrm{J}}^0, \hat{k}, t^0, \vec{t})[\mathrm{H}_0^{(2)}(t^0, \vec{t})]^{-1} dt^0 + o(\varepsilon) \quad \text{(A-1)}$$

In the argument $\hat{k} \equiv (\hat{k}_{\mathrm{J}}, \hat{k}_{\mathrm{N.J}})$ and $\vec{p}_{\mathrm{J}}$ will be kept fixed and for simplicity we shall write:

$$f(k_{\mathrm{J}}^0, t^0, \vec{t}) = \mathrm{H}^{\mathrm{T}}(\vec{p}_{\mathrm{J}}, k_{\mathrm{J}}^0, \hat{k}, t^0, \vec{t})[\mathrm{H}_0^{(2)}(t^0, \vec{t})]^{-1}.$$

Then, due to the definition of the left-hand side of (A-1) we are led to prove that:

$$\left| \int_{\mathscr{L}_+} f(p_{\mathrm{J}}^0 + i\varepsilon, t^0, \vec{t})dt^0 - \int_{\mathscr{L}_-} f(p_{\mathrm{J}}^0 - i\varepsilon, t^0, \vec{t})dt^0 - \int_{\gamma\cup\gamma'} f(k_{\mathrm{J}}^0, t^0, \vec{t})dt^0 \right| \leq \rho(\vec{t})o'(\varepsilon) \quad \text{(A-2)}$$

for some integrable function ρ and some quantity $o'(\varepsilon)$ tending to zero with ε. Let us call $l_\pm = \{ (k_{\mathrm{J}}^0, t^0) : k_{\mathrm{J}}^0 = p_{\mathrm{J}}^0 \pm i\varepsilon, t^0 = u^0 + iv^0 \in \mathscr{L}_\pm \}$. We have to shift the contour $(l_+ - l_-)$ to $(\gamma \cup \gamma')$ in the complex space $\mathbb{C}^2_{(k_{\mathrm{J}}^0, t^0)}$: $p_{\mathrm{J}}^0 = \mathrm{Re}\, k_{\mathrm{J}}^0$ will be kept fixed throughout this shifting and the situation will thus take place in $\mathbb{R}^3_{(q_{\mathrm{J}}^0, u^0, v^0)}$ and be illustrated by Fig. 1 and 2. It is convenient to consider l_+ (resp. l_-) as the oriented « broken » line $(-i\infty, d_+c_+a_+b_+, +i\infty)$ (resp. $(-i\infty, c_-d_-b_-a_-, +i\infty)$). Note that on Fig. 1, only the upper halves of $\mathscr{L}_+$ and $\mathscr{L}_-$ are represented.

The compact cycle $\gamma = (a_+b_+a_-b_-)$ is made up of the following oriented four pieces: the two lines $(a_+b_+) \in l_+$, $(a_-b_-) \in l_-$, and the two segments $[b_+a_-]$, $[b_-a_+]$. γ' is similarly defined and the orientations of γ, γ' are indicated on Fig. 2.

We shall also use a segment $[\lambda_+\lambda_-]$, whose end-points λ_+, λ_- respectively belong to l_+, l_- (with the same value of $\mathrm{Im}\, t^0$) and will tend simultaneously to $+ i\infty$.

To prove (A-2), let us introduce a *closed* differential form $\omega_+ = fdt^0 + g_+dk_{\mathrm{J}}^0$ in the region Δ_+ which is the intersection of the analyticity domain of f with the set

$$\{ (k_{\mathrm{J}}^0, t^0) : v^0 + q_{\mathrm{J}_1}^0 > 0, v^0 - q_{\mathrm{J}_2}^0 > 0, | q_{\mathrm{J}}^0 | \leq \varepsilon \}.$$

g_+ is defined by the following formula:

$$g_+(k_{\mathrm{J}}^0, t^0, \vec{t}) = \int_{\Gamma(t_+^0, t^0)} \frac{\partial f}{\partial k_{\mathrm{J}}^0}(k_{\mathrm{J}}^0, \tau, \vec{t})d\tau.$$

Here t_+^0 denotes an arbitrary fixed point in the t^0-upper plane such that the set

$$\{ (k_{\mathrm{J}}^0, t^0) : t^0 = t_+^0, | k_{\mathrm{J}}^0 | \leq \varepsilon \}$$

belongs to Δ_+, and $\Gamma(t_+^0, t^0)$ is a path of the t^0-plane with end-points t^0, t_+^0 lying inside Δ_+ (in every section $k_{\mathrm{J}}^0 = c^{\mathrm{te}}$, $| k_{\mathrm{J}}^0 | \leq \varepsilon$). These conditions ensure that g_+ is analytic in Δ_+ and that ω_+ is closed. Now the contour $(\lambda_-a_-b_+\lambda_+)$ (made up with the above segments) encloses a two-dimensional piece of surface lying in Δ^+, so that Stokes formula can be applied to the integral of ω_+ on this contour and yields:

$$\int_{(\lambda_-a_-b_+\lambda_+)} \omega_+ = \int_{(b_+\lambda_+)-(a_-\lambda_-)} fdt^0 + \int_{[a_-b_+]+[\lambda_+\lambda_-]} \omega_+ = 0$$

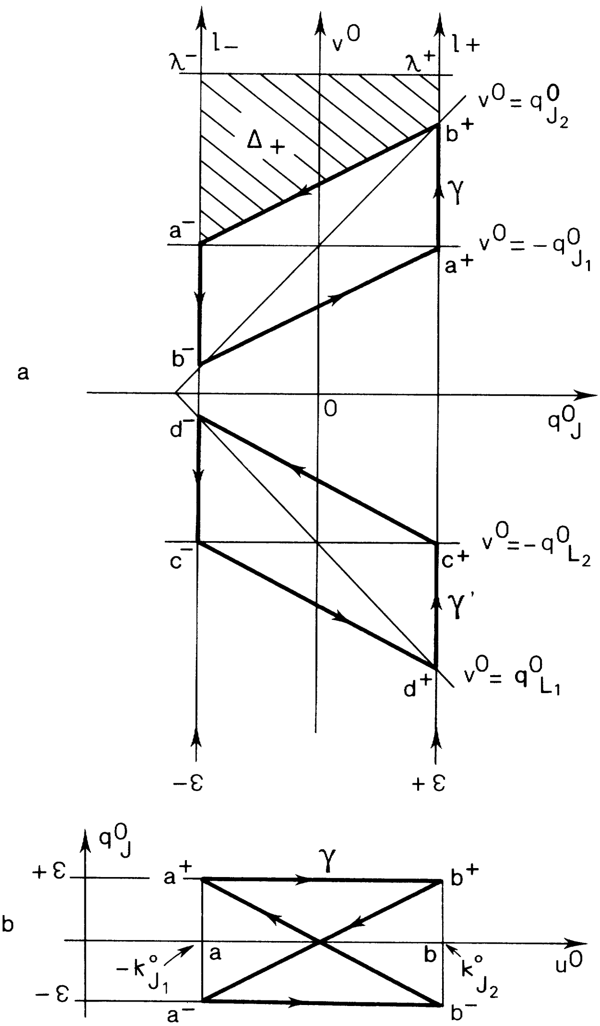


FIG. 2.

a) Projection onto the plane (q_J^0, v^0)

b) Projection onto the plane (q_J^0, u^0) (cycle γ only).

Then in the limit $\lambda_\pm \to i\infty$, we get:

$$\left| \int_{(b_+, i\infty)} f dt^0 - \int_{(a_-, i\infty)} f dt^0 - \int_{[b_+ a_-]} f dt^0 \right| = \left| \int_{[b_+ a_-]} g_+ dk_J^0 + \lim_{\lambda_\pm \to i\infty} \int_{[\lambda_- \lambda_+]} \omega_+ \right| \leq \rho_+(\vec{t})\varepsilon \qquad \text{(A-3)}$$

with ρ_+ integrable and bounded.

The inequality in (A-3) is a consequence of the uniform boundedness and $\vec{t}$-integrability of g_+ $\left(\text{entailed by the } (\vec{t}, t_0) \text{ integrability of } f \text{ and } \frac{\partial f}{\partial k_J^0}\right)$ and from the relation $dk_J^0 = C^{te} \times \varepsilon dt^0$ on $[b_+a_-]$ and $[\lambda_-\lambda_+]$.

Making a similar argument for the intermediate (resp. lower) contour $(d_-c_+a_+b_-)$ (resp. $(-i\infty, d_+c_+, -i\infty)$) with an appropriate differential form ω_0(resp. ω_-), we obtain inequalities similar to (A-3). Putting them together obviously yields (A-2), and then (A-1) through $\vec{t}$-integration.

2. Evaluation in the limit $\varepsilon \to 0$

Let us come back to the notation $k_\alpha = t$. On the four pieces of γ, four different branches of the analytic function H^T are integrated; we denote them by $H^T_{\pm\,\pm}$, with the left (resp. right) sign $\pm$ referring to the sign of $q_\alpha^0 + q_{J_1}^0$ (resp. $q_\alpha^0 - q_{J_2}^0$). In the limit $\varepsilon \to 0$, the cycle γ shrinks to a fourfold-covering of a linear segment $[a, b]$ (parallel to the p_α^0-axis) (see Fig. 2 b), and the right-hand side of (A-1) tends to:

$$\int_{\mathbb{R}^3} d\vec{p}_\alpha \int_{[a,b]} \lim\,(H^T_{++} - H^T_{+-} + H^T_{--} - H^T_{-+})(p_J, \hat{k}, p_\alpha^0 + iq_\alpha^0, \vec{p}_\alpha)dp_\alpha^0$$

Note that one can choose $a = -k_{J_1}^0$, $b = k_{J_2}^0$, and that on $[ab]$, $q_\alpha^0 (= -q_{J_1}^0 = q_{J_2}^0)$ is fixed.

But in view of the tree structure of H^T, which implies: $H^T_{\pm\,\pm} = F^1_\pm[H_0^{(2)}(k_\beta)]^{-1}F^2_\pm$, the latter integral can be rewritten:

$$\int_{\mathbb{R}^3} d\vec{p}_\alpha \int_{[ab]} \Delta^{\alpha J_1}F^1\Delta^{\alpha J_2}F^2[H_0^{(2)}(k_\alpha)H_0^{(2)}(k_\alpha + k_I)]^{-1}dp_\alpha^0\,. \tag{A-4}$$

Here we have introduced the discontinuity functions:

$$\Delta^{\alpha J_1}F^1 = \lim_{|q_\alpha^0 + q_{J_1}^0| \to 0} (F^1_+ - F^1_-)$$

$$\Delta^{\alpha J_2}F^2 = \lim_{|q_\alpha^0 - q_{J_2}^0| \to 0} (F^2_+ - F^2_-)\,.$$

We note that the support of $\Delta^{\alpha J_1}F^1$(resp. $\Delta^{\alpha J_2}F^2$) is in the set

$$\{\, p_\alpha + p_{J_1} \in \bar{V}_m^+ \,\} \text{ (resp. } \{\, p_\alpha - p_{J_2} \in \bar{V}_m^- \,\}\text{)}$$

(with $p_{J_1} + p_{J_2} = p_J \in \bar{V}_m^+$) : this entails that the integration over the set $\mathbb{R}^3 \times [ab]$ in (A-4) is restricted to a compact set. The proof of Theorem 1 is now complete in the case of continuous boundary values.

In the case when $\lim_{|q_\alpha^0 + q_{J_1}^0| \to 0} F^1_\pm$ and $\lim_{|q_\alpha^0 - q_{J_2}^0| \to 0} F^2_\pm$ are distributions, one can use the following standard argument. A regularized form $H^T_{(\varphi_1\varphi_2)} = F^1_{\varphi_1}[H_0^{(2)}]^{-1}F^2_{\varphi_2}$ of H^T is used, where $\varphi_1 \in \mathcal{D}_{(p_{J_1} + p_\alpha)}$, $\varphi_2 \in \mathcal{D}_{(p_\alpha - p_{J_2})}$($\mathcal{D}$ being the space of Schwartz test functions), and $F^j_{\varphi_j} = F^j * \varphi_j$ $(j = 1, 2)$. For every choice of φ_1, φ_2, formula (3) can be derived. Then in the limit $\varphi_j \to \delta$, both sides of (3) have a limit as distributions in p_J (the right-hand side being a well-defined *convolution* of distributions, in view of the support properties of $\Delta^{\alpha J_1}$, $\Delta^{\alpha J_2}$).

A similar argument holds for the distribution case in theorem 2.

3. The « cell version » of equation (3)

As in the case of the discontinuity formula (11) (derived in [3], Section II.3) it is possible to give a detailed « cell version » of equation (3). Indeed if $\hat{k}_J \in \mathcal{T}_{\mathcal{S}'}$, $\hat{k}_{N\setminus J} \in \mathcal{T}_{\mathcal{S}''}$ with $\mathcal{S}' \in S(J)$, $\mathcal{S}'' \in S(N\setminus J)$ (S(X) is, the set of cells of X), the associated points $\hat{k}_{J_1}$, $\hat{k}_{J_2}$, $\hat{k}_{L_1}$, $\hat{k}_{L_2}$ belong respectively to the tubes $\mathcal{T}_{\mathcal{S}'_1}$, $\mathcal{T}_{\mathcal{S}'_2}$, $\mathcal{T}_{\mathcal{S}''_1}$, $\mathcal{T}_{\mathcal{S}''_2}$. Here $\mathcal{S}'_k$ (resp. $\mathcal{S}''_k$) is the canonical image of the cell $\mathcal{S}'$ (resp. $\mathcal{S}''$) in $S(J_k)$ (resp. $S(L_k)$), ($k = 1, 2$). One easily checks that for $\hat{k}_J \times \hat{k}_{N\setminus J} \in \mathcal{C}_{\mathcal{S}'} \times \mathcal{C}_{\mathcal{S}''}$, and (for instance) $q^0_{J_2} = - q^0_{J_1} > 0$, $q^0_{L_2} = - q^0_{L_1} > 0$, (3) takes the following form:

$$\Delta^J_{\mathcal{S}',\mathcal{S}''}H^G = \int_{\mathscr{C}} \Delta^{\alpha J_1}_{\alpha\uparrow\mathcal{S}'_1,\beta\uparrow\mathcal{S}''_1} F^1 \Delta^{\underline{\alpha} J_2}_{\underline{\alpha}\downarrow\mathcal{S}'_2,\beta\downarrow\mathcal{S}''_2} F^2 [H_0^{(2)}(k_\alpha) H_0^{(2)}(k_\beta)]^{-1} dk_\alpha$$

$$+ \int_{\mathscr{C}'} (\ldots \alpha \leftrightarrow \beta \ldots) \quad \text{(A-5)}$$

In particular, this gives the precise « boundary value version » of (3) in the limit when all variables k_i become real (from the directions of $\mathcal{C}_{\mathcal{S}'} \times \mathcal{C}_{\mathcal{S}''}$).

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