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Section A :
Physique théorique.

The reconstruction of local observable algebras from the Euclidean Green's functions of relativistic quantum field theory

by

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ABSTRACT. — A general theorem is presented which says that from a sequence of Euclidean Green's functions satisfying a special form of the Osterwalder-Schrader axioms a local net of observable algebras satisfying all the Haag-Kastler axioms can be reconstructed. In the course of the proof a new sufficient condition for the bounded functions of two commuting, unbounded selfadjoint operators to commute is derived.

CONTENTS

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§ 1. INTRODUCTION, THE MAIN RESULTS

In this paper we prove new results on the reconstruction of self-adjoint quantum fields generating a local net of von Neumann algebras of observables from the Euclidean Green's (Schwinger) functions of a class of relativistic quantum field theories.

A consequence of our results is that essentially all relativistic quantum field models so far constructed or under control define Haag-Kastler theories [HK]. As an example we mention the ϕ_d^4-models, with $d=2, 3$ [GJ1, E, S, MS, FO], (and $d = 4$, assuming existence in the sense defined in [F1]).

For the sake of keeping our notations simple we consider the case of one real, scalar field $\phi(x)$, but our methods immediately extend to the case of countably many arbitrary (elementary or composite) real fields, provided their Euclidean Green's functions satisfy certain bounds described below, (which are typically valid in theories with finite field strength renormalization and a mass gap). As a possible example we mention the electromagnetic field in, say, Q. E. D. in three space-time dimensions, (assuming existence of this theory).

Let d denote the number of space-time dimensions, $x = (\vec{x}, t) \in \mathbb{R}^d$ a space-time point, $\mathcal{S}$, $\mathcal{S}'$ the Schwartz space, the space of tempered distributions, respectively.

Let $\{ S_n(x_1, \ldots, x_n) \}_{n=0}^{\infty}$ be a sequence of generalized functions satisfying all the axioms of Osterwalder and Schrader [E, OS] and, in addition,

(A) $S_1(x) = 0$, (this is *no* loss of generality; it just simplifies notations); for all $n = 2, 3, \ldots, S_n(x_1, \ldots, x_n)$ has an extension to a *tempered distribution* ($\in \mathcal{S}'(\mathbb{R}^{dn})$) which satisfies all the Osterwalder-Schrader axioms, in particular :

— symmetry under permutations of $x_1, \ldots, x_n$, and

— *physical* (Osterwalder-Schrader) *positivity* (axiom (E2) of ref. [OS], in the form stated in Section 2, (A_1)).

(B) There is some norm $\| \, . \, \|$ continuous on $\mathcal{S}(\mathbb{R}^d)$ and a Schwartz norm $| \, . \, |_{\mathcal{S}}$ on $\mathcal{S}(\mathbb{R}^{d-1})$ such that

$$\begin{aligned} \| h \otimes \chi_{[0,t]} \|^2 &\leq o(t) \, | h |_{\mathcal{S}}^2, \qquad &\text{as } t \to 0 \\ \| h \otimes \chi_{[0,t]} \|^2 &\leq \text{const.} \, | h |_{\mathcal{S}}^2 \, t, \qquad &\text{as } t \to \infty, \end{aligned} \tag{1.1}$$

and for arbitrary $f_1, \ldots, f_n$ in $\mathcal{S}(\mathbb{R}^d)$,

$$| S_n(f_1, \ldots, f_n) | \leq (n!)^{1/2} \prod_{j=1}^{n} \| f_j \| \tag{1.2}$$

REMARKS. — In the case of a real, scalar field in $d \geq 3$ space-time dimensions the norm $\| \, . \, \|$ is e. g. given by

$$\| f \|^2 = \int_0^\infty d\rho(m^2) (| f |, (-\Delta + m^2)^{-1} | f |)_{L^2},$$

where ρ is a positive measure with

$$\int_0^\infty m^{-2} d\rho(m^2) < \infty$$

In this situation inequalities (1.1) are satisfied.

Inequality (1.2) is a very special form of the distribution property, axiom (EO′), of ref. [OS]. In [F1] it has been proven that these conditions are valid in all ϕ_d^4-models, assuming only existence, (in the sense of Section 3 of [F1]).

Let $\mathscr{H}$ denote the physical Hilbert space, H the Hamiltonian and $\phi(f)$, $f \in \mathscr{S}(\mathbb{R}^d)$, the quantum field obtained from $\{ S_n \}_{n=0}^\infty$ by Osterwalder-Schrader reconstruction [GL, OS].

We prove that (A) and (B) imply that, for $f \in \mathscr{S}_{\text{real}}(\mathbb{R}^d)$,

$$\pm \phi(f) \leq | f | . (H + 1) \tag{1.3}$$

for some norm $| \, . \, |$ continuous on $\mathscr{S}(\mathbb{R}^d)$, on the quadratic form domain of H.

Our proof closely follows [F1], involving an additional idea, (a Euclidean Reeh-Schlieder type argument) first used in [SeS, McB]. This proof is presented in Section 2.

It then follows from the commutator theorem [GJ2], see also [N], that $\phi(f)$ is essentially self-adjoint on any core for H.

In Section 3 we present an abstract theorem giving sufficient conditions for the bounded functions of two unbounded self-adjoint operators to commute. This theorem yields our *main result* discussed in Section 4, which says that inequality (1.3) combined with Wightman's version of *locality implies locality for the bounded functions of the fields*

$$\{ \phi(f) : f \in \mathscr{S}_{\text{real}}(\mathbb{R}^d) \} .$$

Therefore the field ϕ determines, in a canonical way, a net of local von Neumann algebras of bounded « observables ».

Local nets of observable algebras have been a useful tool in the collision theory for massless particles [B1, B2] and in the analysis of super-selection sectors and their statistics, [DHR]. Furthermore they found a very natural application in the construction of quantum solitons [F2]. This and the fact that in constructive quantum field theory the Euclidean Green's functions are most accessible motivates this paper.

§ 2. EUCLIDEAN PROOF OF ϕ-BOUNDS

In this section we derive bounds for the quantum field, viewed as a quadratic form, in terms of the Hamiltonian.

Our starting point is a sequence $\{ S_n \}_{n=0}^\infty$ of generalized functions satisfying the Osterwalder-Schrader axioms and properties (A) and (B) of the introduction.

By the Osterwalder-Schrader reconstruction theorem the S_n's are the Euclidean Green's functions (EGF's) of a unique relativistic quantum field theory satisfying all the Wightman axioms.

We define

$$\mathbb{R}^{dn}_{+,\varepsilon} = \{ (\vec{x}_1, t_1, \dots, \vec{x}_n, t_n) : t_j \geq \varepsilon, \quad j = 1, \dots, n \}$$
$$\mathscr{S}(\mathbb{R}^{dn}_{+,\varepsilon}) = \{ f : f \in \mathscr{S}(\mathbb{R}^{dn}), \quad \text{supp } f \subseteq \mathbb{R}^{dn}_{+,\varepsilon} \}$$

and $\underline{\mathscr{S}}_{+,\varepsilon}$ the vector space of finite sequences $\underline{f} = \{ f_n \}_{n=0}^{\infty}$ with

$$f_0 \in \mathbb{C}, \qquad f_n \in \mathscr{S}(\mathbb{R}^{dn}_{+,\varepsilon}), \qquad \text{for } n = 1, 2, \dots$$

$f_n = 0$, for all $n > n_0(\underline{f})$, for some finite $n_0(\underline{f})$. Moreover $\underline{\mathscr{S}}_+ = \underline{\mathscr{S}}_{+,\varepsilon=0}$.

Let Θ denote time-reflection, $*$ the usual $*$ operation and $\times$ multiplication on the Borchers algebra.

We require Osterwalder-Schrader positivity in the somewhat stronger form :

(A_1) For $\underline{f} = \{ f_n \}_{n=0}^{\infty} \in \underline{\mathscr{S}}_+$

$$\sum_{n,m=0}^{\infty} S_{n+m}(\Theta f_n^* \times f_m) \geq 0$$

(Under certain regularity assumptions on $\{ S_n \}_{n=0}^{\infty}$ this form of Osterwalder-Schrader positivity follows from the original one proposed in [OS]; see [F1], Proposition 1.1*).

From now on (A_1) is considered to be part of condition (A) of the introduction !

By [OS] (see also [F1], Section 1) there is a mapping

$$\Phi : \underline{f} \in \underline{\mathscr{S}}_+ \to \Phi(\underline{f}) \in \mathscr{H}, \tag{2.1}$$

(where $\mathscr{H}$ is the physical (Wightman) Hilbert space with scalar product $\langle -, - \rangle$) and

$$\langle \Phi(\underline{f}), \Phi(\underline{g}) \rangle = \sum_{n,m=0}^{\infty} S_{n+m}(\Theta f_n^* \times g_m) \tag{2.2}$$

Moreover, for $t > 0$,

$$\Phi(\underline{f}_t) = e^{-tH}\Phi(\underline{f}), \tag{2.3}$$

and $\underline{f}_t$ denotes the time-translate of $\underline{f}$ by t.

Let $\underline{\mathscr{S}}^{\pi}_{+,\varepsilon}$ be the class of sequences in $\underline{\mathscr{S}}_{+,\varepsilon}$ such that, for all $\underline{f} \in \underline{\mathscr{S}}^{\pi}_{+,\varepsilon}$ and all n,

$$f_n(x_1, \dots, x_n) = f_1^{(n)}(x_1) \dots f_n^{(n)}(x_n).$$

(*) A similar version has also been proposed in Hegerfeldt, *Commun. math. Phys.*, t. **35**, 1974, p. 155.

We define $\mathscr{D}_\varepsilon$ to be the linear span of $\Phi(\underline{\mathscr{S}}^{\pi}_{+,\varepsilon})$, and

$$\mathscr{D}_0 = \bigcup_{\varepsilon > 0} \mathscr{D}_\varepsilon . \tag{2.5}$$

It follows easily from [OS] that $\mathscr{D}_0$ is dense in $\mathscr{H}$. Furthermore, by (2.3),

$$e^{-t\mathrm{H}}\mathscr{D}_\varepsilon = \mathscr{D}_{\varepsilon+t} \tag{2.6}$$

LEMMA 2.1. — $\mathscr{D}_0$ is a core for H and for H^2 and a form core for H.

Proof. — First note that $e^{-t\mathrm{H}}\mathrm{H}^n\mathscr{D}_0 \subset \mathscr{D}_0$, for all $n = 0, 1, 2, 3, \ldots, t \geqq 0$. By (2.5), (2.6) each vector in $\mathscr{D}_0$ is an analytic vector for H + 1. Therefore $\mathscr{D}_0$ is a core for H(+ 1) and, since $\mathrm{H} + 1 \geq 1$, $(\mathrm{H} + 1)\mathscr{D}_0$ is dense in $\mathscr{H}$. Thus it is a core for H + 1, i. e. $\mathscr{D}_0$ is a core for $(\mathrm{H} + 1)^2$, hence for H^2. The rest is obvious. Q. E. D.

Let $f \in \mathscr{S}(\mathbb{R}^d)$. We define

$$\mathrm{Exp}\,(f) = \{ f_n \}_{n=0}^{\infty}, \text{ with } f_0 = 1,$$

$$f_1 = f, \ldots, f_n(x_1, \ldots, x_n) = (n\,!)^{-1} \prod_{j=1}^{n} f(x_j) \tag{2.7}$$

Clearly

$$\hat{\mathrm{S}}(\mathrm{Exp}\,(f) \times \mathrm{Exp}\,(g)) = \hat{\mathrm{S}}(\mathrm{Exp}\,(f + g)), \tag{2.8}$$

where $\hat{\mathrm{S}}$ denotes symmetrization.

By property (A) and inequality (1.2) the following definition makes sense :

$$\mathrm{S}(\mathrm{Exp}\,(f)) = \sum_{n=0}^{\infty} \frac{1}{n\,!} \mathrm{S}_n(\underbrace{f \times \ldots \times f}_{n \text{ times}}) \tag{2.9}$$

This definition can be extended by linearity to elements of the form

$$\Sigma c_i \,\mathrm{Exp}\,(f_i), \qquad c_i \in \mathbb{C} .$$

From inequality (1.2) we obtain the basic estimate

$$|\,\mathrm{S}(\mathrm{Exp}\,(f))\,| \leq e^{\mathrm{const.}\,\|f\|^2} \tag{2.10}$$

where $\|\,.\,\|$ is the norm introduced in (B). Furthermore, using (1.1) and (2.10) we get, for all $h \in \mathscr{S}_{\mathrm{real}}(\mathbb{R}^{d-1})$,

$$|\,\mathrm{S}(\mathrm{Exp}\,(h \otimes \chi_{[0,t]}))\,| \leq e^{\mathrm{K}'.t} \tag{2.11}$$

where $\mathrm{K}' = \mathrm{const.}' \,|\,h\,|_{\mathscr{S}}^2$, and $|\,.\,|_{\mathscr{S}}$ has been introduced in (B).

LEMMA 2.2. — Assume (A) and (B). Then, for all N = 1, 2, 3, ..., arbitrary complex numbers $c_1, \ldots, c_{\mathrm{N}}$ and test functions $f_1, \ldots, f_{\mathrm{N}}$ in $\mathscr{S}(\mathbb{R}^d_+)$ and arbitrary $\underline{g}_1, \ldots, \underline{g}_{\mathrm{N}}$ in $\underline{\mathscr{S}}^{\pi}_{+}$, the mapping Φ extends to

$$\sum_{i=1}^{\mathrm{N}} c_i \,\mathrm{Exp}\,(f_i) \times \underline{g}_i ,$$

and

$$\Phi\left(\sum_{i=1}^{N} c_i \operatorname{Exp}(f_i) \times \underline{g}_i\right) = \sum_{i=1}^{N} c_i \Phi(\operatorname{Exp}(f_i) \times \underline{g}_i).$$

For f and g in $\mathscr{S}(\mathbb{R}^d_+)$, $\underline{g} \in \underline{\mathscr{S}}^\pi_+$,

$$\Phi(\operatorname{Exp}(f) \times \operatorname{Exp}(g) \times \underline{g}) = \Phi(\operatorname{Exp}(f+g) \times \underline{g}).$$

Proof. — This lemma is a direct consequence of (A) and the bounds (1.2) and (2.10). The details of the argument are as in the proofs of Propositions 1.2, 1.3 of [F1] and are not reproduced here. Note that the last part follows from the symmetry of the EGF's and (2.8).

Q. E. D.

REMARKS. — 1. By (2.10) — see also Proposition 1.3 of [F1] — vectors of the form $\Phi(\operatorname{Exp}(f))$, $f \in \mathscr{S}(\mathbb{R}^d_+)$, span a dense subspace of $\mathscr{H}$.

2. In ref. [F1] the existence of a Euclidean field theory associated with the EGF's $\{ S_n \}_{n=0}^{\infty}$, i. e. Nelson-Symanzik positivity, and local regularity of S_n are assumed. This is however *not* used in the proofs of Propositions 1.2-1.3 of [F1], where nothing more than (A) and (B) are required.

LEMMA 2.3. — Under the same hypotheses and for $h \in \mathscr{S}_{\text{real}}(\mathbb{R}^{d-1})$, the equation

$$\overset{\circ}{U}_t \Phi(\underline{f}) = \Phi(\operatorname{Exp}(h \otimes \chi_{[0,t]}) \times \underline{f}_t), \tag{2.12}$$

for arbitrary $\underline{f} \in \underline{\mathscr{S}}^\pi_+$, defines a symmetric semigroup on $\mathscr{D}_0$.

Proof. — By Lemma 2.2 Φ is well defined on $\operatorname{Exp}(h \otimes \chi_{[0,t]}) \times \underline{f}_t$, i. e. the r. h. s. of (2.12) is a vector in $\mathscr{H}$, for all $0 \leq t < \infty$. Using (2.12) we now get

$$\begin{aligned} \overset{\circ}{U}_t \overset{\circ}{U}_s \Phi(\underline{f}) &= \overset{\circ}{U}_t \Phi(\operatorname{Exp}(h \otimes \chi_{[0,s]}) \times \underline{f}_s) \\ &= \Phi(\operatorname{Exp}(h \otimes \chi_{[0,t]}) \times \operatorname{Exp}(h \otimes \chi_{[t,t+s]}) \times \underline{f}_{t+s}) \\ &= \Phi(\operatorname{Exp}(h \otimes \chi_{[0,t+s]}) \times \underline{f}_{t+s}) \\ &= \overset{\circ}{U}_{t+s} \Phi(\underline{f}), \end{aligned}$$

and we have applied the second part of Lemma 2.2.

Using the definitions of Φ and of the scalar product on $\mathscr{H}$, see (2.2), one checks that

$$\langle \Phi(\underline{g}), \Phi(\operatorname{Exp}(h \otimes \chi_{[0,t]}) \times \underline{f}_t) \rangle = \langle \Phi(\operatorname{Exp}(h \otimes \chi_{[0,t]}) \times \underline{g}_t), \Phi(\underline{f}) \rangle ; \tag{2.13}$$

see also [OS, F1].

From this one deduces that if $\underline{f}$ is in the kernel of Φ, i. e. $\Phi(\underline{f}) = \vec{0}$ then $\overset{\circ}{U}_t \Phi(\underline{f}) = \vec{0}$, i. e. $\overset{\circ}{U}_t$ is a densely defined, linear operator on $\mathscr{H}$. Applying (2.13) again we conclude that $\overset{\circ}{U}_t$ is symmetric, for all $0 \leq t < \infty$.

Q. E. D.

REMARK. — Using (2.10), (2.11) *and conditions* (1.1), (1.2) we easily see that

$$s\text{-}\lim_{t\to 0} \overset{\circ}{U}_t\psi = \psi, \qquad \text{for all } \psi \in \mathscr{D}_0 . \tag{2.14}$$

It can be shown (see [F1], proof of Theorem 2.1) that the conclusions of Lemma 2.3 and (2.14) imply that $\{ \overset{\circ}{U}_t \}$ has a unique self-adjoint extension $\{ U_t \}$ defined, for all $0 \leq t < \infty$, on $\mathscr{D}_0$ and satisfying the semigroup property, even in circumstances where $\{ \overset{\circ}{U}_t \}$ is not exponentially bounded.

We do however not need this result in the following, but rather prove directly that $\{ \overset{\circ}{U}_t \}$ is exponentially bounded.

Definition. — Let ψ and θ be in $\mathscr{D}_0$. For $h \in \mathscr{S}(\mathbb{R}^{d-1})$ we define the time 0-quantum field $\phi_0(h)$ as the sesquilinear form given on $\mathscr{D}_0 \times \mathscr{D}_0$ by

$$\langle \psi, \phi_0(h)\theta \rangle = \lim_{t\searrow 0} \frac{1}{t} \langle \psi, \phi(h \otimes \chi_{[0,t]})\theta \rangle \tag{2.15}$$

We must verify that this definition makes sense:

Since ψ and θ are in $\mathscr{D}_0$, there is an $\varepsilon_0 > 0$ such that ψ and θ are in $\mathscr{D}_{\varepsilon_0}$. By linearity we may then assume that $\psi = \Phi(\underline{f})$ and $\theta = \Phi(\underline{g})$, for some $\underline{f}$ and $\underline{g}$ in $\underline{\mathscr{S}}^{\pi}_{+,\varepsilon_0}$.

Using now definition (2.2) we obtain, for

$0 < t < \varepsilon_0$,

$$\frac{1}{t} \langle \psi, \phi(h \otimes \chi_{[0,t]})\theta \rangle = \sum_{n,m} \frac{1}{t} S_{n+m+1}(\Theta f_n^* \times (h \otimes \chi_{[0,t]}) \times g_m) \tag{2.16}$$

As a consequence of the fact that, for $t_1 < t_2 < \ldots < t_N$ the EGF S_N $(\vec{x}_1, t_1, \ldots, \vec{x}_N, t_N)$ is a function which is real analytic in $t_2 - t_1, \ldots, t_N - t_{N-1}$, see [GL, OS], each term in the sum on the r. h. s. of (2.16) is well defined, for $0 < t < \varepsilon_0$, and has a *limit* as $t \searrow 0$. By the definition of $\mathscr{S}^{\pi}_{+,\varepsilon_0}$, the sum on the r. h. s. of (2.16) is finite. Therefore (2.15) is meaningful.

THEOREM 2.4. — Under the previous hypotheses, and for real $h \in C_0^\infty(\mathbb{R}^{d-1})$

(1) $$\| \overset{\circ}{U}_t \| \leq e^{K.t},$$

for some finite constant K depending on h.

(2) For ψ and θ in $\mathscr{D}_0$

$$\lim_{t\searrow 0} \frac{1}{t} \langle \psi, [\overset{\circ}{U}_t - 1]\theta \rangle = - \langle \psi, [H - \phi_0(h)]\theta \rangle$$

Proof. — Since h is in $C_0^\infty(\mathbb{R}^{d-1})$ there exists some finite ξ such that $\text{supp } h \subseteq \{ x = (\vec{x}, t) ; \vec{x} = (x^1, \ldots, x^{d-1}), x^1 \geq \xi \}$.

Let $\mathscr{D}_\xi$ denote the linear span of all vectors $\Phi(\underline{f})$, where $\underline{f}$ is in $\mathscr{S}_+^\pi$, and, for all $n = 1, 2, 3, \ldots$,

$$\operatorname{supp} f_n \subseteq \{ (x_1, \ldots, x_n) : x_j^1 < \xi, \quad j = 1, \ldots, n \}.$$

We claim that $\mathscr{D}_\xi$ is *dense* in $\mathscr{H}$: This follows from the analyticity properties of the EGF's established in [OS] by a Reeh-Schlieder argument and has been noticed before and used (in the context of models) in [SeS, McB].

By the Euclidean invariance of the EGF's we have Osterwalder-Schrader positivity with respect to reflections at the plane $x^1 = \xi$. Therefore there is an inner product analogous to the one defined in (2.2) relative to $x^1 = \xi$, and the Schwarz inequality may be applied with respect to this inner product.

Using now the definition (2.12) of $\{ \overset{\circ}{U}_t \}$ and the support properties of $\operatorname{Exp}(h \otimes \chi_{[0,t]})$ and applying the Schwarz inequality with respect to the inner product relative to $x^1 = \xi$ one concludes (as in [SeS, McB]) that, for *all* $t \geq 0$ and all $\theta \in \mathscr{D}_\xi$ there is a finite constant $C(\theta)$ *independent* of t such that

$$\langle \theta, \overset{\circ}{U}_t \theta \rangle \leq C(\theta) S(\operatorname{Exp}((h + h_{(R,\xi)}) \otimes \chi_{[0,t]}))^{1/2}, \qquad (2.18)$$

where $h_{(R,\xi)}$ is the reflection of h at the plane $x^1 = \xi$; (see [SeS] for a similar inequality; Osterwalder-Schrader positivity and the Reeh-Schlieder argument replace the use of Nelson-Symanzik positivity made at this place in [F1]).

Next, by (2.11),

$$S(\operatorname{Exp}((h + h_{(R,\xi)}) \otimes \chi_{[0,t]}))^{1/2} \leq e^{K.t}, \qquad (2.19)$$

for some finite constant K depending on h. Combination of inequalities (2.18) and (2.19) and repeated use of the Schwarz inequality give

$$\langle \theta, \overset{\circ}{U}_t \theta \rangle \leq \langle \theta, \theta \rangle \lim_{n \to \infty} \langle \theta, \overset{\circ}{U}_{2^n t} \theta \rangle^{1/2^n} \leq e^{K.t} \langle \theta, \theta \rangle, \qquad \text{for all } \theta \in \mathscr{D}_\xi.$$

Since $\mathscr{D}_\xi$ is dense in H, we conclude that

$$\| \overset{\circ}{U}_t \| \leq e^{K.t}, \qquad \text{for all } t \geq 0; \qquad (2.20)$$

therefore $\{ \overset{\circ}{U}_t \}$ can be extended by continuity to a self-adjoint, exponentially bounded semigroup, denoted $\{ U_t \}$.

Let A denote the (self-adjoint) infinitesimal generator of $\{ U_t \}$, i. e. $U_t = e^{-At}$; then (2.20) yields

$$A \geq -K. \qquad (2.21)$$

The proof of (1) is now complete.

Proof of (2). — By polarization it suffices to prove (2) for $\psi = \theta \in \mathscr{D}_0$. Clearly

$$\lim_{t \searrow 0} \frac{1}{t} \langle \psi, [U_t - 1]\psi \rangle = - \langle \psi, A\psi \rangle, \qquad (2.22)$$

and the limit exists if

$$\frac{1}{t}\langle \psi, [U_t - 1]\psi \rangle$$

is bounded uniformly in $t \in (0, 1]$, (a consequence of (1) and the spectral theorem). Hence we must show this boundedness property and then identify A with $H - \phi_0(h)$. The first problem is solved as in [F1] (Section 2, proof of Theorem 2.1).

Using the definition (2.12) of U_t, inequality (1.2), the symmetry of the EGF's $S_n(x_1, \ldots, x_n)$ under permutations of the arguments (see (A)) and Taylor's theorem we show that, for $\underline{f} \in \underline{\mathcal{S}}^{\pi}_{+,\varepsilon_0}$, $\underline{g} \in \underline{\mathcal{S}}^{\pi}_{+,\varepsilon_0}$ and $0 < t < \varepsilon_0$

$$\begin{aligned}\frac{1}{t}\langle \Phi(\underline{f}), \Phi(\{\operatorname{Exp}(h \otimes \chi_{[0,t]}) - 1\} \times \underline{g}) \rangle \\ = \frac{1}{t}\langle \Phi(\underline{f}), \Phi((h \otimes \chi_{[0,t]}) \times \underline{g}) \rangle \\ + \frac{1}{2t}\langle \Phi(\underline{f}), \Phi(\operatorname{Exp}(\lambda h \otimes \chi_{[0,t]}) \times (h \otimes \chi_{[0,t]}) \\ \times (h \otimes \chi_{[0,t]}) \times \underline{g}) \rangle, \quad (2.23)\end{aligned}$$

for some $\lambda \in (0, 1)$.

This is shown essentially as in Section 2 of [F1], using the series expansion of Exp, the symmetry of $S_n(x_1, \ldots, x_n)$ under permutations of $\{1, \ldots, n\}$ and (1.1), (1.2). By (2.15)

$$\frac{1}{t}\langle \Phi(\underline{f}), \Phi((h \otimes \chi_{[0,t]}) \times \underline{g}) \rangle \to \langle \Phi(\underline{f}), \phi_0(h)\Phi(\underline{g}) \rangle, \quad (2.24)$$

as $t \searrow 0$.

Next we expand $\operatorname{Exp}(\lambda h \otimes \chi_{[0,t]})$ in the second term on the r. h. s. of (2.23) and bound each term in the series so obtained by means of inequalities (1.1) and (1.2). This yields

$$\begin{aligned}\frac{1}{t}|\langle \Phi(\underline{f}), \Phi(\operatorname{Exp}(\lambda h \otimes \chi_{[0,t]}) \times (h \otimes \chi_{[0,t]}) \\ \times (h \otimes \chi_{[0,t]}) \times \underline{g}) \rangle| \leq \text{const.} \frac{1}{t} o(t) |h|^2_{\mathcal{S}}, \quad (2.25)\end{aligned}$$

where $|\,.\,|_{\mathcal{S}}$ is as in (1.1) and const. is some finite constant depending on h, $\underline{f}$ and $\underline{g}$, but *independent* of $t \in (0, \varepsilon_0)$.

Hence the r. h. s. of (2.25) tends to 0, as $t \searrow 0$.

Given (2.22)-(2.25) the remaining arguments for the proof of (2) are essentially the same as in [F1], (proof of Theorem 2.1). There is no reason to repeat them here.

Q. E. D.

From Theorem 2.4 we immediately conclude that, for $h \in C_0^\infty(\mathbb{R}^{d-1})$ real and $\psi \in \mathscr{D}_0$

$$\pm \langle \psi, \phi_0(h)\psi \rangle \leq \langle \psi, H\psi \rangle + K \langle \psi, \psi \rangle,$$

and by (2.11), (2.19) and (2.21),

$$K \leq \text{Const.} \mid h + h_{(R,\xi)} \mid_{\mathscr{S}}^2, \tag{2.26}$$

and the r. h. s. of (2.26) is continuous on $\mathscr{D}(\mathbb{R}^{d-1})$.

By Lemma 2.1 $\mathscr{D}_0$ is a form core for H. Therefore the quadratic form $\phi_0(h)$ can be extended to the quadratic form domain Q(H) of H, and

$$\pm \langle \psi, \phi_0(h)\psi \rangle \leq \langle \psi, H\psi \rangle + K \langle \psi, \psi \rangle, \tag{2.27}$$

for all $\psi \in Q(H)$.

As Q(H) is invariant under the unitary group $\{ e^{itH} \}$ and because of (2.26), the following definition makes sense:

For $\psi \in Q(H)$ and $f \in C_0^\infty(\mathbb{R}^d)$ we define

$$\langle \psi, \phi(f)\psi \rangle = \int dt \langle e^{-itH}\psi, \phi_0(f(., t))e^{-itH}\psi \rangle \tag{2.28}$$

From the Osterwalder-Schrader reconstruction theorem one easily derives that $\phi(f)$ is a form extension of the *relativistic quantum field* reconstructed from $\{ S_n \}_{n=0}^\infty$.

COROLLARY 2.5. — There exists a (translation-invariant) norm $| . |$ continuous on $\mathscr{S}(\mathbb{R}^d)$ such that for real f

$$\pm \phi(f) \leq | f | . (H + 1),$$

in the sense of quadratic forms on Q(H).

Proof. — For f in $C_0^\infty(\mathbb{R}^d)$, define

$$\alpha_f = \text{Const.} \int dt \mid f(., t) + f_{(R,\xi)}(., t) \mid_{\mathscr{S}}^2 \tag{2.29}$$

where ξ is chosen such that supp $f \subseteq \{ x : x^1 \geq \xi \}$, and Const. is as in (2.26).

Then, by (2.26)-(2.28),

$$\pm \phi\left(\frac{1}{\alpha_f} f\right) \leq H + 1, \text{ i. e. } \pm \phi(f) \leq \alpha_f(H + 1), \tag{2.30}$$

on Q(H).

Recall that H commutes with all translations. Therefore

$$\pm \phi(f_x) \leq \alpha_f(H + 1), \tag{2.31}$$

where α_f is as in (2.29), (2.30) and f_x is the translate of f by x.

Using now a C^∞ partition of unity and applying estimates (2.29) and (2.31) we arrive at the assertion of the corollary.

Q. E. D.

We note that, in the sense of quadratic forms on e. g. the domain $\mathscr{D}_0$ introduced above or the Wightman domain for the fields,

$$i[\mathrm{H}, \phi(f)] = -\phi\left(\frac{\partial}{\partial t} f\right). \tag{2.32}$$

By [GJ2, N], Corollary 2.5 and (2.32) yield essential self-adjointness of the quantum field $\phi(f)$ on any core for H, for all $f \in \mathscr{S}_{\mathrm{real}}(\mathbb{R}^d)$.

We conclude Section 2 with an alternate version of Theorem 2.4.

THEOREM 2.4′. — Assume (A) and (instead of (B)) (B′) there are finite constants K and $p \in [1, 2]$ such that, for $f \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^p(\mathbb{R}^d)$, $\mathrm{S}(\mathrm{Exp}\,(zf))$ is analytic in z and

$$|\,\mathrm{S}(\mathrm{Exp}\,(f))\,| \leq e^{\mathrm{K}(\|f\|_1 + \|f\|_p^p)}.$$

Then Theorem 2.4 and Corollary 2.5 hold.

Proof. — For $f = h \otimes \chi_{[0,t]}$ with $h \in \mathrm{L}^1(\mathbb{R}^{d-1}) \cap \mathrm{L}^p(\mathbb{R}^{d-1})$ we get

$$|\,\mathrm{S}(\mathrm{Exp}\,(h \otimes \chi_{[0,t]}))\,| \leq e^{t.\mathrm{K}(\|h\|_1 + \|h\|_p^p)}$$

which replaces (2.11).

Furthermore, for $\underline{f}$ and $\underline{g}$ in $\mathscr{S}_+^\pi$, we derive from (B′) and the Cauchy estimate

$$\frac{1}{t}\,|\langle \Phi(\underline{f}), \Phi(\mathrm{Exp}\,(\lambda h \otimes \chi_{[0,t]}) \times (h \otimes \chi_{[0,t]}) \times (h \otimes \chi_{[0,t]}) \times \underline{g}) \rangle|$$
$$\leq \text{Const.}\,\frac{1}{t}(\,\|h \otimes \chi_{[0,t]}\|_1 + \|h \otimes \chi_{[0,t]}\|_p)^2$$

which tends to 0, as $t \searrow 0$, (since $p < 2$). This replaces (2.25). Theorem 2.4 then follows as before, and the proof of Corollary 2.5 remains unchanged, up to a redefinition of α_f; (see (2.29)).

Q. E. D.

REMARKS. — 1. Theorem 2.4′ is of some interest in applications to relativistic quantum field models.

2. It is trivial to check that condition (B′) of Theorem 2.4′ implies the (weaker) hypotheses (C1)-(C3) of Theorem 2.1 of ref. [F1].

3. A similar reformulation of Theorem 2.1 of [F1] (but with the additional assumption of Nelson-Symanzik positivity) has been proven by Glimm and Jaffe; (to appear in the Proceedings of the Cargèse Summer School, 1976. See also J. F.'s, *Erice lectures*, 1977).

§ 3. A COMMUTATOR THEOREM

In this section we discuss the following simple mathematical structure:
A separable Hilbert space $\mathscr{H}$ with scalar product $\langle -, - \rangle$, and
— a positive, self-adjoint operator H

— two closed symmetric operators A and B bounded in some sense by H and commuting weakly on some core for H^2.

We prove

THEOREM 3.1. — Assume that there is a finite constant K such that, in the sense of quadratic forms,

(1) $\pm A \leq K(H+1), \pm B \leq K(H+1),$
$\pm i[H, A] \leq K(H+1), \pm i[H, B] \leq K(H+1),$

(2) $\pm [H, [H, B]] \leq K(H+1),$

on some form core for H.

(3) For all ψ and θ in some core $\mathscr{D}_0$ for H^2 (contained in D(A) and D(B))

$$\langle A\psi, B\theta \rangle = \langle B\psi, A\theta \rangle .$$

Then A and B are selfadjoint operators, and all their bounded functions commute.

REMARK. — By [GJ2, N] essential selfadjointness of A, B on *any core* for H follows from (1). (3.1)

In the following we set $\tilde{H} \equiv H + 1$. Then

$$\tilde{H} \geq 1, \qquad \text{and} \qquad \| \tilde{H}^{-1} \| \leq 1 . \tag{3.2}$$

By [GJ2, N], (1) implies that there is a finite constant C such that

$$\| A\tilde{H}^{-1} \| \leq C, \qquad \| B\tilde{H}^{-1} \| \leq C, \tag{3.3}$$

in particular $\mathscr{D}_0$ is contained in D(A) and D(B), and (1) and (2) give

$$\| [\tilde{H}, B]\tilde{H}^{-1} \| \leq C. \tag{3.4}$$

In the following hypothesis (2) of Theorem 3.1 could always be replaced by (3.4) without changing any conclusions.

We propose to show that the resolvents of A and B commute weakly on some dense domain in $\mathscr{H}$. (From this the remaining part of Theorem 3.1 will follow at once). The central element in the proof of this is

LEMMA 3.2. — Let $z \in \mathbb{C}$ have a sufficiently large imaginary part. Then

(1) $(B + z)\mathscr{D}_0$ is in the domain $D(\tilde{H})$ of $\tilde{H}$, and $\tilde{H}(B + z)\mathscr{D}_0$ is *dense* in $\mathscr{H}$.

(2) For $z' \in \mathbb{C}$, with $\operatorname{Im} z' \neq 0$

$$\tilde{\mathscr{D}}_{z',z} \equiv \{ (A + z')(B + z)\theta : \theta \in \mathscr{D}_0 \}$$

is *dense* in $\mathscr{H}$; $\{ (B + z)\theta : \theta \in \mathscr{D}_0 \}$ is a core for A.

Proof. — (1) Let $\psi \in D(\tilde{H})$ and $\theta \in \mathscr{D}_0$. Then ψ and θ are in the domain of B, by (3.3), (and θ is in the domain of $[\tilde{H}, B]$, by (3.4)). Thus the following equation is meaningful:

$$\langle \tilde{H}\psi, (B + z)\theta \rangle = \langle (B + \bar{z})\psi, \tilde{H}\theta \rangle + \langle \psi, [\tilde{H}, B]\theta \rangle .$$

By definition of $\mathscr{D}_0$, $\tilde{H}\theta$ is in the domain of $\tilde{H}$, and therefore it is in the one of $B + z$, by (3.3). Hence

$$\langle (B + \bar{z})\psi, \tilde{H}\theta \rangle = \langle \psi, (B + z)\tilde{H}\theta \rangle ,$$

and

$$|\langle \psi, (B + z)\tilde{H}\theta \rangle| \leq \|\psi\| \, \|(B + z)\tilde{H}\theta\| \leq C \|\psi\| \, \|\tilde{H}^2\theta\|$$

By (3.4) and (3.2),

$$|\langle \psi, [\tilde{H}, B]\theta \rangle| \leq C \|\psi\| \, \|\tilde{H}\theta\| \leq C \|\psi\| \, \|\tilde{H}^2\theta\| .$$

Hence, for all $\psi \in D(H)$ and $\theta \in \mathscr{D}_0$.

$$|\langle \tilde{H}\psi, (B + z)\theta \rangle| \leq 2C \|\psi\| \, \|\tilde{H}^2\theta\| < \infty ,$$

i. e. $(B + z)\theta \in D(H^*) = D(H)$, and, as ψ varies over the dense domain D(H),

$$\tilde{H}(B + z)\theta = (B + z)\tilde{H}\theta + [\tilde{H}, B]\theta = \{ B + z + [\tilde{H}, B]\tilde{H}^{-1} \} \tilde{H}\theta \tag{3.5}$$

Since $\mathscr{D}_0$ is a core for $\tilde{H}^2$ and by (3.2), $\{ \tilde{H}\theta : \theta \in \mathscr{D}_0 \}$ is dense in $\mathscr{H}$ and is a core for $\tilde{H}$. Hence, by (3.1), it is a *core* for B. Let ψ be some vector in $\mathscr{H}$ such that

$$\langle \psi, \{ B + z + [\tilde{H}, B]\tilde{H}^{-1} \} \tilde{H}\theta \rangle = 0 ,$$

for all $\theta \in \mathscr{D}_0$.

Since by (3.4) $[\tilde{H}, B]\tilde{H}^{-1}$ is a bounded operator and $\{ \tilde{H}\theta : \theta \in \mathscr{D}_0 \}$ is a core for B, we conclude that

$$\psi \in D(B^*) = D(B) ,$$

and

$$\{ B + \bar{z} + ([\tilde{H}, B]\tilde{H}^{-1})^* \} \psi = 0 \tag{3.6}$$

By (3.4) and the triangle inequality

$$\begin{aligned} \| \{ B + \bar{z} + ([\tilde{H}, B]\tilde{H}^{-1})^* \} \psi \| &\geq \|(B + \bar{z})\psi\| - C\|\psi\| \\ &\geq |\operatorname{Im} z| \, \|\psi\| - C\|\psi\| \\ &\geq \|\psi\| , \qquad \text{for } |\operatorname{Im} z| \geq C + 1 . \end{aligned} \tag{3.7}$$

(3.6) and (3.7) imply that $\psi = \vec{0}$.

$$\text{i. e.} \qquad \{ (B + z + [\tilde{H}, B]\tilde{H}^{-1})\tilde{H}\theta : \theta \in \mathscr{D}_0 \}$$

is dense. This and (3.5) complete the proof of Lemma 3.2, (1).

Proof of (2). — By (1), $(B + z)\theta \in D(\tilde{H})$, for all $\theta \in \mathscr{D}_0$, and

$$\{ \tilde{H}(B + z)\theta : \theta \in \mathscr{D}_0 \}$$

is dense in $\mathscr{H}$, for $|\operatorname{Im} z| > C$. Therefore

$$\{ (B + z)\theta : \theta \in \mathscr{D}_0 \} = \tilde{H}^{-1} \{ \tilde{H}(B + z)\theta : \theta \in \mathscr{D}_0 \}$$

is a core for $\tilde{H}$ and hence, by (3.1), it is a core for A. Therefore, as A, is self-adjoint,

$$\{ (A + z')(B + z)\theta : \theta \in \mathscr{D}_0 \} \equiv \tilde{\mathscr{D}}_{z',z}$$

is dense in $\mathscr{H}$, for $\operatorname{Im} z' \neq 0$.

Q. E. D.

PROOF OF THEOREM 3.1. — For $\zeta \in \mathbb{C}$ with $\operatorname{Im} \zeta \neq 0$ we set

$$R_A(\zeta) = (A + \zeta)^{-1}, \qquad R_B(\zeta) = (B + \zeta)^{-1}.$$

Let $\tilde{\psi} \in \tilde{\mathcal{D}}_{\bar{z}',\bar{z}}$, $\tilde{\theta} \in \tilde{\mathcal{D}}_{z',z}$; both sets are dense by Lemma 3.2, (2), for $|\operatorname{Im} z| > C$. Then, there exist vectors ψ and θ in $\mathcal{D}_0$ such that

$$\tilde{\psi} = (A + \bar{z}')(B + \bar{z})\psi,$$
$$\tilde{\theta} = (A + z')(B + z)\theta.$$

Therefore

$$\begin{aligned}
&\langle \tilde{\psi}, R_A(z')R_B(z)\tilde{\theta} \rangle \\
&\quad = \langle (B + \bar{z})\psi, R_B(z)\tilde{\theta} \rangle \\
&\quad = \langle \psi, \tilde{\theta} \rangle = \langle \psi, (A + z')(B + z)\theta \rangle \\
&\quad = \langle (A + \bar{z}')\psi, (B + z)\theta \rangle, \text{ by (3.1) and (3.3)}, \\
&\quad = \langle (B + \bar{z})\psi, (A + z')\theta \rangle, \text{ by hypothesis (3) of Theorem 3.1}, \\
&\quad = \langle (A + \bar{z}')(B + \bar{z})\psi, \theta \rangle, \text{ by Lemma 3.2}, \\
&\quad = \langle \tilde{\psi}, \theta \rangle = \langle \tilde{\psi}, R_B(z)(B + z)\theta \rangle \\
&\quad = \langle \tilde{\psi}, R_B(z)R_A(z')(A + z')(B + z)\theta \rangle, \text{ by Lemma 3.2}, \\
&\quad = \langle \tilde{\psi}, R_B(z)R_A(z')\tilde{\theta} \rangle. \qquad (3.8)
\end{aligned}$$

Since, by Lemma 3.2, $\tilde{\psi}$ and $\tilde{\theta}$ vary in dense domains, namely $\tilde{\mathcal{D}}_{\bar{z}',\bar{z}}$, $\tilde{\mathcal{D}}_{z',z}$ we conclude from (3.8) that

$$R_A(z')R_B(z) = R_B(z)R_A(z') \text{ on } \mathcal{H}, \qquad (3.9)$$

provided $|\operatorname{Im} z| > C$, $\operatorname{Im} z' \neq 0$.

This and the holomorphy of the resolvent imply that (3.9) holds whenever z is not in the spectrum of B and z' is not in the spectrum of A.

But this yields Theorem 3.1. Q. E. D.

REMARK. — It is easy to show that under the conditions of Theorem 3.1

$$AB\theta = BA\theta, \qquad \text{for all } \theta \in D(H^2).$$

§ 4. RECONSTRUCTION OF LOCAL NETS

The harvest of the work done in Sections 2 and 3 is contained in

THEOREM 4.1. — Let $\phi(x)$ be the neutral, scalar Wightman field reconstructed from a sequence of Euclidean Green's functions obeying conditions (A) and (B) of the introduction, or (A) and (B') of Theorem 2.4'.

Then, to each open region $0 \subset \mathbb{R}^d$, there belongs (in a canonical way) a von Neumann algebra $\mathcal{R}(0)$ generated by the bounded functions of the fields $\{ \phi(f) : f \in \mathcal{S}_{\text{real}}(\mathbb{R}^d), \operatorname{supp} f \subset 0 \}$, and the net

$$\{ \mathcal{R}(0) : 0 \in \text{open regions in } \mathbb{R}^d \}$$

satisfies all the Haag-Kastler axioms.

Proof. — Let $\mathscr{D}_W$ be the Wightman domain of the field; obviously $e^{itH}H^n\mathscr{D}_W \subset \mathscr{D}_W$, for all $n = 0, 1, 2, 3, \ldots, t$ real. Moreover $\mathscr{D}_W$ contains an invariant, dense set of analytic vectors for H (which are semi-analytic for H^2). So, by Nelson's and Nussbaum's theorems $\mathscr{D}_W$ is a core for H and H^2 and a form core for H.

Set $A \equiv \phi(f)$, $B \equiv \phi(g)$, where f and g are real Schwartz space functions with supp $f \subset O_1$, supp $g \subset O_2$ and $O_1 \times O_2$ (space-like separated).

Since $\mathscr{D}_W \subset C^\infty(A) \cap C^\infty(B) \cap C^\infty(H)$, and by Corollary 2.5, (2.32) and Wightman's form of locality, all the hypotheses of Theorem 3.1 are (more than) fulfilled. Therefore all bounded functions of A and of B commute. We conclude that the von Neumann algebras $\{ \mathscr{R}(0) \}$ obey *locality.*

Among the remaining Haag-Kastler axioms the only non-obvious one is the *Reeh-Schlieder property*, i. e. weak additivity:

Let 0 be an open region in $\mathbb{R}^d$. Then, by the Reeh-Schlieder theorem the algebra of localized polynomials in the field, $\mathscr{P}(0)$, fulfills

$$\overline{\mathscr{P}(0)\Omega} = \mathscr{H}, \qquad (\Omega \text{ is the vacuum}),$$

i. e. the set of monomials

$$\left\{ \prod_{i=1}^{n} \phi(f_i)\Omega \ : n \in \mathbb{N},\ f_i \in \mathscr{S}_{\text{real}}(\mathbb{R}^d),\ \text{supp } f_i \subset 0 \right\}$$

is total in $\mathscr{H}$.

Pick such a monomial and approximate

$$\phi(f_1) \prod_{j=2}^{n} \phi(f_j)\Omega$$

by

$$A_1 \prod_{j=2}^{n} \phi(f_j)\Omega,$$

with $A_1 \in \mathscr{R}(0)$. Continuing with $\phi(f_2)$, using the boundedness of A_1, then with $\phi(f_3)$, etc. we finally get an estimate of the form

$$\left\| \left(\prod_{i=1}^{n} \phi(f_i) - \prod_{i=1}^{n} A_i \right) \Omega \right\| < \varepsilon$$

with $\varepsilon > 0$ arbitrarily small, $A_i \in \mathscr{R}(0)$, $i = , \ldots, n$.

Q. E. D.

CONCLUDING REMARKS

1) Knowing now the existence of local algebras under reasonable assumptions it is natural to ask structural questions concerning e. g.

factoriality, type, duality, etc. It appears that the type question is the easiest one, but even this one (as will be discussed elsewhere by one of us (W. D.)) can at present only be answered under much more restrictive assumptions. The fact that all generalized free fields with a mass gap satisfy the form bounds of Corollary 2.5 and the results of [G] may indicate that our form bound, Corollary 2.5, does not suffice to answer such more detailed questions.

2) Our local net $\{ \mathscr{R}(0) \}$ is however not just any such net. This is a consequence of [BW], because Theorem 4.1 gives condition IV of their Theorem 4 and its corollary.

From their work it then follows that there is another local net (in which the local algebras are constructed as intersections of « wedge » algebras) which satisfies *duality*. (Disregarding from the free field) the relation of this new net to the previous one remains mysterious, up to now.

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Note added in proof: Ed Nelson has pointed out to us that hypothesis (2) of Theorem 3.1 is unnecessary for the conclusions to be true. We thank Prof. Nelson for communicating to us this improvement.