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Kinematics of relative motion of test particles in general relativity (1)

by

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ABSTRACT. — The paper starts with a detailed mathematical study of the concept of geodesic deviation in pseudo-riemannian geometry. A generalization of this concept to geodesic deviations of a higher order is then introduced and the second geodesic deviation is investigated in some detail. A geometric interpretation of the set of generalized geodesic deviations is given and applied in general relativity to determine a covariant and local description (with a desired order of accuracy) of test motions which take place in a certain finite neighbourhood of a given world line of an observer. There is also discussed the proper time evolution of two other objects related to geodesic deviation: the space separation vector and the telescopic vector. This last name is given here to a field of null vectors along observer's world line which always point towards the same adjacent world line. The telescopic equations allow to determine the evolution of the frequency shift of electromagnetic radiation sent from and received on neighbouring world lines. On the basis of these equations also certain relations have been derived which connect the frequencies or frequency shifts with the curvature of space-time.

RÉSUMÉ. — La première partie contient une étude mathématique détaillée de la notion de déviation géodésique dans une géométrie pseudo-rieman-

(1) Cet article et le suivant [*26*] sont une version élargie d'une conférence présentée par l'auteur au Collège de France, le 2 décembre 1975. L'auteur voudrait remercier Mme Y. Choquet-Bruhat et MM. A. Lichnérowicz et M. Flato pour leur généreuse hospitalité durant son séjour à Paris et à Dijon, au cours du mois de décembre 1975.

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nienne. On introduit alors la généralisation de cette notion à la notion de déviation géodésique d'ordre supérieur et on étudie en détail la déviation géodésique du second ordre. On donne l'interprétation géométrique de l'ensemble des déviations géodésiques généralisées et on l'utilise pour déterminer, en relativité générale, la description covariante et locale (à l'ordre d'approximation qu'on veut) des mouvements des particules d'épreuve dans un voisinage fini de la ligne d'Univers d'un observateur donné. On étudie aussi l'évolution en temps propre de deux autres objets liés à la déviation géodésique : le vecteur de séparation d'espace et le vecteur télescopique. Le nom de vecteur télescopique est donné à un champ de vecteurs nuls qui commencent aux points de la ligne d'Univers de l'observateur et sont dirigés vers des points d'une autre ligne d'Univers voisine. Les équations télescopiques permettent de déterminer l'évolution du déplacement vers le rouge d'un rayonnement électromagnétique émis et reçu entre lignes d'Univers voisines. A l'aide de ces équations, on déduit aussi certaines relations entre les fréquences ou le déplacement des fréquences et la courbure de l'espace-temps.

INTRODUCTION

In the general theory of relativity certain peculiarities concerning the motion of particles are showing up. The most important one is this connected with the universality of coupling to the gravitational field. There are, however, also some other, perhaps more subtle, differences between the status of motion in general relativity and in other field theories. One of them is that from a geometric point of view the motion of a test body appears to be less interesting and renders less information about the field than the comparison of motions of two near-by small test bodies. The first motion, along a geodesic in space-time, plays here the same role as the uniform, straigh line motion in the Newtonian theory. The true intensity of the gravitational field, the Riemann curvature tensor, does not enter the equations of motion of test particles, but it does, instead, the geodesic deviation equations which were first formulated by Levi-Civita [1] (comp. also [2] and [3]). These last equations can be considered as describing the relative motion of two infinitesimally close test bodies. Thus not the free fall of one body, but at least of two of them can be used as a probe for testing the gravitational field. In a number of publications, *e. g.* [4]-[13], theoretical implications of that fact were studied and several devices for idealized experiments to determine the curvature of space-time were proposed. It also has formed a basis on which realistic gravitational waves detectors of the mechanical type have been designed (cf. [13]-[16]).

The intention of the present paper is to formulate a description of the relative motion of two not necessarily infinitesimally close test particles. The basic notion here is a set of vector fields—the generalized natural geodesic deviations determined along a chosen geodesic line Γ, *i. e.* determined in a sense locally. Then the description of a geodesic from a neighbourhood of Γ is provided in terms of a recently proven [*17*] geometric formulation of the Taylor expansion theorem.

The question of description of the relative motion of two freely falling material points has been also discussed in a paper by Hodgkinson [*18*]. There, the particles are infinitesimally close, but their relative velocity may be comparable with that of light. The approach used in [*18*] differs, however, in many respects from the one formulated here and is not at all evident how it could be extended for finite space separations. As one of the main differences it may be pointed out that in [*18*] several quantities defined on the neighbouring line must be transported back (in an approximate way) to the basic line, whereas in the present paper all quantities and the equations describing their evolution are from the beginning determined along the basic geodesic Γ.

The quantities introduced here form a sort of « kinematic moments » entering as coefficients the covariant Taylor series describing the relative position of the other particle. If one wishes to deal with an approximate expression, it is sufficient to truncate the series in a desired way and this procedure has no influence on the form of the surviving coefficients.

Let me now briefly comment the content of individual sections of this paper.

It starts with a review of some well-known properties of arbitrarily parametrized geodesic lines in a pseudo-Riemannian manifold. I have chosen to include such a review here for reference purposes, since the material included is necessary to get a deeper insight into properties inherited in a way by the generalized geodesic deviation vectors of all orders. The form of presentation accepted in section 1 has also been chosen from this point of view.

In section 2 the concept of the (first) geodesic deviation vector for arbitrarily parametrized geodesics is introduced and it is shown how by means of a constraint condition it reduces to the natural geodesic deviation which is defined as preserving the natural parametrization on adjacent geodesics (and which amounts to the usually discussed one). A big part of material presented in this section is rather known. In my opinion, however, no due stress has been laid in the past on the role of the constraint condition.

Section 3 starts with a construction which heuristically justifies the definition of the second geodesic deviation vector accepted here for the case of general parametrization. This object is then reduced by means of a constraint condition to the natural second geodesic deviation vector. This last vector is identical with the object introduced by me in [*19*]; by that time, however, the role of the constraint condition was not quite clear to me (the last question will also be discussed in a following paper [*26*] devoted to some dynamical problems connected with the relative motion). Next it is pointed out how the geometric form of the Taylor theorem implies a geometric interpretation of the second (and also of the higher) geodesic deviation. It also justifies the accepted definition and indicates how the geodesic deviations of higher order should be defined. This procedure could be carried on to an arbitrary order, if desired. The only limitation of its validity is the analycity of the connection and of geodesic lines in the considered region of space-time manifold. This program has also been performed for the third geodesic deviation. The corresponding equation will, however, not be published here as it contains a rather lengthy expression which does not introduce anything new from a general point of view. One should only mention that the equation defining in the second approximation the evolution of the relative position (in the case of natural deviations) is identical with a corresponding equation from [*18*]. These equations start to differ, however, if the third geodesic deviation is included.

The concept of the natural geodesic deviation appears to be too rigid for some applications. The requirement the general deviation to be natural is usually inconsistent with some other conditions one would like to impose additionally on these vectors. Section 4 contains a discussion of the connection between the natural geodesic deviation and another specialization of the general deviation vector—called the separation vector—on which the condition of orthogonality to the basic world line has been imposed.

In section 5 a similar discussion is performed for deviation vectors that are always null. They are called the telescopic vectors. Due to propositions proven in sections 2 and 3 these vectors can be expressed through the natural geodesic deviations. They form a convenient tool for discussion of optical effects when the two particles exchange light signals. As a natural result of this construction one has obtained here a number of relations connecting frequencies, frequency shifts and the curvature of the manifold. Some of them are connected with already known relations, derived in [*20*] and [*21*], but some are new.

Relations of that kind permit to complete the list of already existing schemes of idealized experiments aiming to determine the curvature. They are introducing some more relativistic flavour to the rather Newtonian type of constructions utilized in the past.

1. GEODESICS

Let V_n be a pseudo-Riemannian manifold endowed with a coordinate system $\{ x^\alpha \}$ valid in a region $\Omega \subset V_n$. In such a coordinate system a geodesic line Γ, parametrized by a parameter $\tau \in [a, b] = I \subset \mathbb{R}$ is then described by a set of n functions $\xi^\alpha: \mathrm{I} \to \mathbb{R}$ ($\alpha = 1, 2, \ldots, n$), where $\xi^\alpha(\tau) = x^\alpha \circ \Gamma(\tau)$. These functions must satisfy the equations

$$\mathrm{L}[\xi^\alpha] : = \frac{\mathrm{D}}{d\tau}\left(\frac{d\xi^\alpha}{d\tau}\right) - \frac{d\xi^\alpha}{d\tau}\frac{d}{d\tau}\ln\sqrt{|\, u_\alpha u^\alpha \,|} = 0, \tag{1.1}$$

where $\frac{\mathrm{D}}{d\tau}$ denotes the absolute derivative along Γ, $g_{\alpha\beta}$ are the components of the metric tensor of V_n, $u^\alpha = \frac{d\xi^\alpha}{d\tau}$ and $u_\alpha = g_{\alpha\beta}u^\beta$. It will be assumed that $u_\alpha u^\alpha \neq 0$ anywhere along Γ.

Since $u_\alpha \mathrm{L}[\xi^\alpha] \equiv 0$ is a « strong » identity (*i. e.* valid for any ξ^α) Eqs. (1.1) are not independent and together with the initial conditions

$$\xi^\alpha(\tau_0) = \xi_0^\alpha \,; \qquad \frac{d\xi^\alpha}{d\tau}(\tau_0) = u_0^\alpha \,; \tag{1.2}$$

do not determine ξ^α uniquely. This can be stated more precisely as

PROPOSITION 1.1. — If a set of n functions $\xi^\alpha: \mathrm{I} \to \mathbb{R}$, $\alpha = 1, \ldots, n$, is a solution of the system of equations (1.1), then

i) the set of composite functions $\xi^\alpha \circ f$—with an arbitrary C_2 function f: $[a, b] \to [a', b']$ such that $f' \neq 0$—is also a solution of Eqs. (1.1);

ii) any solution $\tilde{\xi}^\alpha$ of (1.1), which satisfies the same initial conditions as ξ^α, can be represented as $\tilde{\xi}^\alpha = \xi^\alpha \circ f$, where $f \in \mathrm{C}_2$ is uniquely defined by the two solutions.

The proof. — Part *i*) follows from an immediate computation.

The proof of *ii*) consists in constructing a differential equation for f. This equation must be supplemented by some initial conditions which can be taken in one of the two possible forms.

First, for $\tau = \tau_0$, ξ^α and $\tilde{\xi}^\alpha$ may fulfil the same conditions (1.2). The freedom of f is then reduced to such C_2 functions for which

$$f(\tau_0) = \tau_0; \qquad f'(\tau_0) = 1. \tag{1.3}$$

Second, ξ^α and $\tilde{\xi}^\alpha$ might describe only the same geometrical line, *i. e.* for a certain $\tilde{\tau}_0$ (not necessarily equal to τ_0)

$$\tilde{\xi}^\alpha(\tilde{\tau}_0) = \xi_0^\alpha \,; \qquad \frac{d\tilde{\xi}^\alpha}{d\tau}(\tilde{\tau}_0) = ku_0^\alpha \,; \tag{1.4}$$

where k is a nonzero and otherwise arbitrary constant. Then

$$f(\tilde{\tau}_0) = \tau_0; \qquad f'(\tilde{\tau}_0) = k. \tag{1.5}$$

The proof of part *ii*) requires yet the following.

LEMMA. — Any solution of Eqs. (1.1) which satisfies (1.2) is completely specified by a choice of a continuous function $\lambda: \mathrm{I} \to \mathbb{R}$.

[The proof of the lemma: a given solution ξ^α defines λ as

$$\lambda(\tau) = \frac{d}{d\tau} \ln \sqrt{| u_\alpha u^\alpha |}. \tag{1.6}$$

If a function $\lambda: \mathrm{I} \to \mathbb{R}$ is given, ξ^α is uniquely defined as the solution of the differential equations

$$\frac{\mathrm{D}^2 \xi^\alpha}{d\tau^2} = \lambda(\tau) \frac{d\xi^\alpha}{d\tau} \tag{1.7}$$

which satisfies (1.2) as initial conditions. For such a ξ^α (1.6) is satisfied as a « weak » identity and thus ξ^α is also a solution of (1.1)].

COROLLARY of the lemma. — There exists a one-to-one map $\mathscr{F}$ of the set of all solutions of Eqs. (1.1), corresponding to some fixed initial data in (1.2), onto the set of all continuous functions $\lambda: \mathrm{I} \to \mathbb{R}$.

Now, if ξ^α is a solution of Eqs. (1.1) fulfilling (1.2), then $\tilde{\xi}^\alpha := \xi^\alpha \circ f$ is, according to part *i*) of Prop. 1.1, also a solution. Let $\mathscr{F}(\xi^\alpha) = \lambda$, then $\mathscr{F}(\tilde{\xi}^\alpha) = \tilde{\lambda}$ is defined, as it follows from (1.6), as

$$\tilde{\lambda}(\tau) = \frac{f''(\tau)}{f'(\tau)} + f'(\tau)\lambda(f(\tau)). \tag{1.8}$$

Thus, if ξ^α and $\tilde{\xi}^\alpha$ are any two solutions of Eqs. (1.1), fulfilling respectively (1.2) or (1.4), *i. e.* $\mathscr{F}(\xi^\alpha) = \lambda$ and $\mathscr{F}(\tilde{\xi}^\alpha) = \tilde{\lambda}$ are given functions, then solving [3] (1.8) correspondingly with (1.5) or (1.3) one finds such a unique f that $\tilde{\xi}^\alpha = \xi^\alpha \circ f$ and this ends the proof.

One says that two descriptions, by ξ^α and $\tilde{\xi}^\alpha$ respectively, of a curve in a coordinate system $\{ x^\alpha \}$ are equivalent, $\xi^\alpha \sim \tilde{\xi}^\alpha$, iff there is such a function f: $[a, b] \to [a', b']$, $f \in \mathrm{C}_2$, that: *i*) $f'(\tau) \neq 0$ in all domain; *ii*) $\tilde{\xi}^\alpha = \xi^\alpha \circ f$. Prop. 1.1 states then that Eqs. (1.1) together with (1.2) uniquely determine an equivalence class of descriptions of a geodesic Γ in a coordinate system

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(3) The solution of (1.8) fulfilling (1.5) is implicitely defined by

$$\mathrm{F}(f(\tau)) = k \frac{\Phi(\tau_0)}{\tilde{\Phi}(\tau_0)} [\tilde{\mathrm{F}}(\tau) - \tilde{\mathrm{F}}(\tau_0)] + \mathrm{F}(\tau_0), \tag{*}$$

where $\Phi(\tau) = \exp \int \lambda(\tau) d\tau$; $\mathrm{F}(\tau) = \exp \int \Phi(\tau) d\tau$ (and $\tilde{\Phi}$ and $\tilde{\mathrm{F}}$ defined similarly by $\tilde{\lambda}$) are given functions. For $\tau = \tau_0$, $k = 1$ it renders the solution fulfilling (1.3). For $\lambda = \tilde{\lambda}$ the formula (*) defines a law of transformations of the parameter τ which preserve the form of (1.7) for a given λ; these transformations are parametrized by k and τ_0. For $\lambda = \tilde{\lambda} = 0$ they turn over into (1.10 *b*).

$\{ x^\alpha \}$. Each member of the class is defined by Eqs. (1.7) with a fixed function λ. Thus to find a solution of Eqs. (1.1) fulfilling (1.2) with some given initial data $\{ \xi_0^\alpha, u_0^\alpha \}$, it is sufficient to solve (1.7) with the possibly simplest choice of the function λ. Such a choice consists in taking $\lambda \equiv 0$ for any $\tau \in \mathrm{I}$. It introduces the affine parameter.

Thus any set $\{ \xi_0^\alpha, u_0^\alpha \}$ of initial data in (1.2) and the equations

$$\frac{\mathrm{D}^2 \xi^\alpha}{d\tau^2} = 0 \tag{1.9}$$

determine in a neighbourhood of the initial point $\Gamma(\tau_0)$ a unique geodesic with a unique affine parametrization. The correspondence, however, between initial data and the classes of equivalence of solutions ξ^α of (1.9) is still not a one-to-one, as two different sets of initial data might lead to equivalent solutions of (1.9), *i. e.* might render two different descriptions, characterized by two different affine parametrizations, of the same geodesic Γ. This can be restated as

PROPOSITION 1.2. — Two sets of initial data: $\{ \xi_0^\alpha, u_0^\alpha \}$ for $\tau = \tau_0$ and $\{ \xi_0^\alpha, \tilde{u}_0^\alpha \}$ for $\tau = \tilde{\tau}_0$ will lead to two equivalent solutions ξ^α and $\tilde{\xi}^\alpha = \xi^\alpha \circ f$ $(f' \neq 0)$ of (1.9) iff

$$a)\ \tilde{u}_0^\alpha = k u_0^\alpha \quad (k \neq 0) \qquad \text{and} \qquad b)\ f(\tau) = k(\tau - \tilde{\tau}_0) + \tau_0. \tag{1.10}$$

The proof is obvious. (1.10 *b*) is the solution of (1.8) for $\lambda = \tilde{\lambda} = 0$. Eq. (1.10 *a*) for an arbitrary $k \neq 0$ defines an equivalence relation between sets of initial data. One limits the freedom of choice of these data by putting on them a constraint condition which chooses one member from each class of equivalence only.

In Riemannian geometry a universal choice of this kind is *e. g.*

$$g_{\alpha\beta} u_0^\alpha u_0^\alpha = 1. \tag{1.11}$$

If a set of initial data, with $g_{\alpha\beta} u_0^\alpha u_0^\beta \neq 0$, is not a correct one, *i. e.* does not satisfy (1.11), it always can be brought to a correct one by a transformation of the form (1.10 *a*). A similar condition (with a possible change of sign) can be used in pseudo-Riemannian case, provided $u_\alpha u^\alpha \neq 0$.

Since Eqs. (1.9) admit the first integral

$$g_{\alpha\beta} u^\alpha u^\beta = \text{const} \tag{1.12}$$

(being a « weak » identity) the condition (1.11) is equivalent to

$$g_{\alpha\beta} u^\alpha u^\beta = 1 \tag{1.13}$$

This condition is thus not only assuring a one-to-one correspondence between initial data and geodesics Γ, but it is also introducing a universal, natural parametrization along all (nonnull) geodesics. This parametrization is induced by the metric structure of the manifold. In general, any condition of the type (1.12) fixes the unit of the affine parameter scale. It still leaves, as it is customary and convenient, the freedom of choice of the origin τ_0 of this scale.

In general relativity the timelike geodesics are interpreted as worldlines of freely falling material test points. The natural parameter is here measured by an ideal clock which moves with the particle and shows its proper time. The appropriate description of such a situation is then provided by Eqs. (1.9) with the constraint condition (1.11) (in case, as it is done here, the signature $+ \; - \; - \; -$ is accepted).

2. THE GEODESIC DEVIATION

Let us consider a one-parametric family of geodesics in Ω. Each member Γ_ρ of the family is labelled by a value $\rho \in [c, d] = \mathscr{I} \subset \mathbb{R}$ and the points on Γ_ρ are parametrized by $\tau \in [a, b] = \mathrm{I} \subset \mathbb{R}$. Thus in a coordinate system $\{ x^\alpha \}$ the coordinates of points belonging to any geodesic of the family are defined as $\xi^\alpha(\tau, \rho) := x^\alpha \circ \Gamma_\rho(\tau)$. It is assumed that the n functions $\xi^\alpha \colon \mathbb{R}^2 \to \mathbb{R}$ defined above are at least of class C_2. The set of points

$$\Sigma := \{ p \in V_n \mid x^\alpha(p) = \xi^\alpha(\tau, \rho); \; (\tau, \rho) \in \mathrm{I} \times \mathscr{I} \}$$

forms a two-cube in V_n. To any pair (τ, ρ) one assigns two vectors, u and r from the tangent space of V_n at a point p, with the components

$$u^\alpha(\tau, \rho) := \frac{\partial \xi^\alpha}{\partial \tau}(\tau, \rho); \qquad r^\alpha(\tau, \rho) := \frac{\partial \xi^\alpha}{\partial \rho}(\tau, \rho). \tag{2.1}$$

These vector valued functions u and r will be called here vector fields on Σ parametrized by (τ, ρ) (Since the geodesics may intersect they need not to be a restriction to Σ of any vector field in V_n). There evidently holds

$$\frac{\mathrm{D}u^\alpha}{\partial \rho} = \frac{\mathrm{D}r^\alpha}{\partial \tau}. \tag{2.2}$$

Any vector field t on Σ satisfies the Ricci identity

$$\frac{\mathrm{D}^2 t^\alpha}{\partial \rho \partial \tau} - \frac{\mathrm{D}^2 t^\alpha}{\partial \tau \partial \rho} = \mathrm{R}^\alpha{}_{\beta\gamma\delta} t^\beta r^\gamma u^\delta. \tag{2.3}$$

Let us take $t^\alpha = \dfrac{u^\alpha}{\sqrt{| u_\lambda u^\lambda |}}$. Then making use of Eqs. (1.1) which also may read as

$$\frac{\mathrm{D}}{d\tau} \frac{u^\alpha}{\sqrt{| u_\lambda u^\lambda |}} = 0, \tag{2.4}$$

we get from (2.3) the identity

$$\frac{\mathrm{D}}{d\tau}\left[\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left(\delta^\alpha_\beta - \frac{u^\alpha u_\beta}{u_\sigma u^\sigma}\right)\frac{\mathrm{D}r^\beta}{d\tau}\right] + \frac{1}{\sqrt{|u_\lambda u^\lambda|}}\mathrm{R}^\alpha{}_{\beta\gamma\delta}u^\beta r^\gamma u^\delta = 0 \quad (2.5)$$

which also, due to (2.4), can be written in the form

$$\frac{\mathrm{D}^2 r^\alpha}{d\tau^2} + \mathrm{R}^\alpha{}_{\beta\gamma\delta}u^\beta r^\gamma u^\delta = \frac{\mathrm{D}r^\alpha}{d\tau}\frac{d}{d\tau}\ln\sqrt{|u_\lambda u^\lambda|} + u^\alpha\frac{d}{d\tau}\left(\frac{u_\mu}{u_\lambda u^\lambda}\frac{\mathrm{D}r^\mu}{d\tau}\right). \quad (2.6)$$

Eqs. (2.4) and (2.6) can, in particular, be written for a fixed value of ρ, say $\rho = 0$, *e. g.* along the geodesic line Γ_0. If we repeat the procedure analogous to that above, but immersing now Γ_0 in another one-parametric family Σ of geodesics, with another field $r'^\alpha(\tau, \rho)$, we will still get the same Eqs. (2.6) for $r'^\alpha(\tau, 0)$ along Γ_0. All one-parametric families of geodesics which contain Γ_0 and for which $r'^\alpha(\tau, 0) = r^\alpha(\tau, 0)$ can be defined as equivalent along Γ_0 and the corresponding class of equivalence is called the geodesic deviation vector field along Γ_0.

One can base the definition of geodesic deviation directly on Eqs. (2.6) without any immediate appeal to Σ. Let us for this purpose assume that a single parametrized geodesic line Γ is given. We define along Γ a vector field r (determined at $p(\tau) \in \Gamma$ by $r^\alpha(\tau)$) as a solution of the differential equations (2.6) [or (2.5)] fulfilling conditions

$$r^\alpha(\tau_0) = r^\alpha_0\,; \qquad \frac{\mathrm{D}r^\alpha}{d\tau}(\tau_0) = v^\alpha_0. \quad (2.7)$$

One should take here into account that quantities like $g_{\alpha\beta}$, $\Gamma^\alpha_{\beta\gamma}$, $\mathrm{R}^\alpha{}_{\beta\gamma\delta}$, u^α, etc., enter Eqs. (2.6) being evaluated at the point $p(\tau)$ and therefore are given functions of τ. Any solution of the problem so formulated is also called the geodesic deviation vector field along Γ. This second definition is of course more general than the first.

Since Eqs. (2.6) are not independent (contracting them with u_α one gets a « strong » identity), their solution admits freedom of introducing arbitrary functions. This can be restated as in the two propositions:

PROPOSITION 2.1. — If functions r^α: $\mathrm{I} \to \mathbb{R}$ $(\alpha = 1, \ldots, n)$ are a solution of Eqs. (2.6) taken along a geodesic Γ, then $r^\alpha \circ f$, for any $f \in \mathrm{C}_1$ and $f' \neq 0$, are also a solution of (2.6) along the same Γ, but now parametrized by $f(\tau)$.

PROPOSITION 2.2. — If the set of functions r^α: $\mathrm{I} \to \mathbb{R}$ is a solution of Eqs. (2.6) taken along a geodesic Γ described by functions ξ^α, then

i) the functions $r^\alpha + \kappa u^\alpha$ (where κ: $\mathrm{I} \to \mathbb{R}$, $\kappa \in \mathrm{C}_2$ and is arbitrary) are also a solution of (2.6) along the same Γ with the same parametrization;

ii) any solution $\tilde{r}^\alpha$ of (2.6) which satisfies the same initial conditions (2.7) as r^α can be represented in the form

$$\tilde{r}^\alpha = r^\alpha + \kappa u^\alpha, \quad (2.8)$$

where $\kappa \in C_2$ is uniquely determined by r^α, $\tilde{r}^\alpha$ and the conditions:

$$\kappa(\tau_0) = \kappa'(\tau_0) = 0.$$

The proof of Prop. 2.1 and 2.1 *i*) follows from inspection. Part *ii*) requires the

LEMMA 2.1. — A solution of Eqs. (2.6) satisfying (2.7) is completely specified by a choice of a continuous function μ: I $\to \mathbb{R}$.

[The proof: A given solution r^α defines the function

$$\mu(\tau) = \frac{d}{d\tau}\left(\frac{u_\mu}{u_\lambda u^\lambda}\frac{Dr^\mu}{d\tau}\right). \tag{2.9}$$

For a given μ: I $\to \mathbb{R}$, r^α fulfilling (2.7) is defined as the unique solution of the system of differential equations

$$\frac{D^2 r^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta}u^\beta r^\gamma u^\delta = \lambda(\tau)\frac{Dr^\alpha}{d\tau} + \mu(\tau)u^\alpha \tag{2.10}$$

in which the function λ is determined by (1.6). Such r^α solves also (2.6)].

COROLLARY of the lemma: There exists a one-to-one map $\mathscr{G}$ of the set of all solutions of Eqs. (2.6), determined by fixed initial data in (2.7), onto the set of all continuous functions μ: I $\to \mathbb{R}$.

Let r^α and $\tilde{r}^\alpha$ be two solutions of (2.6) such that $\mathscr{G}(r^\alpha) = \mu$ and $\mathscr{G}(\tilde{r}^\alpha) = \tilde{\mu}$, then κ in (2.8) is defined (because of (2.9)) as the unique solution of

$$\tilde{\mu}(\tau) = \mu(\tau) + \frac{d}{d\tau}\left(\frac{d\kappa}{d\tau}(\tau) + \kappa(\tau)\lambda(\tau)\right) \tag{2.11}$$

with the initial data $\kappa(\tau_0) = \kappa'(\tau_0) = 0$; what proves Prop. 2.2.

The geometric interpretation of the fact stated in Prop. 2.2 follows at once from the procedure leading to (2.6) as an identity on Σ. Because of the freedom of transformations of the parameter: $\tau \to f(\tau, \rho)$ (where f : $\mathbb{R}^2 \to \mathbb{R}$ is a C_2 function) a, in general, different parametrization may be introduced on each geodesic Γ_ρ. In particular one can demand that the parametrization on the basic geodesic Γ_0 ($\rho = 0$) remains unchanged, but changes on those with $\rho \neq 0$ so that

$$f(\tau, 0) = \tau\,; \qquad \frac{\partial f}{\partial \rho}(\tau, 0) = \kappa(\tau)\,; \tag{2.12}$$

where κ is given. Defining then $r^\alpha(\tau, \rho)$ as in (2.1) and $\tilde{r}^\alpha(\tau, \rho)$ as

$$\tilde{r}^\alpha(\tau, \rho) := \frac{\partial \xi^\alpha(f(\tau, \rho), \rho)}{\partial \rho} \tag{2.13}$$

one easily derives that $r^\alpha(\tau, 0)$ and $\tilde{r}^\alpha(\tau, 0)$, *i. e.* both on Γ_0, satisfy the relation (2.8). The multiplicity of solutions of Eqs. (2.6), described in Prop. 2.2

can therefore be interpreted as a possibility of introducing new parametrizations on neighbouring geodesic lines, while keeping fixed the parametrization on the basic line Γ_0.

Let us return to the approach with the single geodesic Γ. If, as in general relativity, we want to parametrize Γ by the natural parameter s, then instead of (2.6) we shall have

$$\frac{D^2 r^\alpha}{ds^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta r^\gamma u^\delta = \frac{d}{ds}\left(g_{\rho\sigma} u^\rho \frac{Dr^\sigma}{ds}\right). \tag{2.14}$$

The initial value problem (2.7) for Eqs. (2.14) admits, according to Prop. 2.2 and Lemma 2.1, a whole set of solutions. Each of them is labelled by a function μ and is a unique solution of Eqs. of the type (2.10) (now with $\lambda \equiv 0$). All these solutions form an equivalence class of a relation defined by (2.8) and it is therefore sufficient to solve only the equation characterized by a possibly simplest function, chosen to be $\mu \equiv 0$. Then (2.10) reduces to

$$\frac{D^2 r^\alpha}{ds^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta r^\gamma u^\delta = 0. \tag{2.15}$$

The choice $\mu = 0$ has a simple geometric interpretation: all the geodesics adjacent to Γ are also parametrized by affine parameters (but not necessarily by the same one).

For each set of initial data in (2.7) Eqs. (2.15) have a unique solution. Two different sets of initial data might, however, lead to two equivalent solutions, as it follows from

PROPOSITION 2.3. — Two sets, $\{ r_0^\alpha, v_0^\alpha \}$ and $\{ \tilde{r}_0^\alpha, \tilde{v}_0^\alpha \}$, of initial data in (2.7) will render two solutions r^α and $\tilde{r}^\alpha$ of (2.15) equivalent in the sense of (2.8) iff

$$i) \quad \begin{aligned} \tilde{r}_0^\alpha &= r_0^\alpha + a u_0^\alpha ; \\ \tilde{v}_0^\alpha &= v_0^\alpha + b u_0^\alpha . \end{aligned} \qquad \text{Then: } ii) \quad \kappa(\tau) = b(\tau - \tau_0) + a. \tag{2.16}$$

(a and b are arbitrary constants and u_0^α are the initial data for Γ normalized by (1.11)).

The proof is straightforward and will be omitted.

Eqs. (2.16 *i*)) are establishing an equivalence relation of initial data for (2.15). A single representative from each class of equivalence is determined by constraint conditions which must be imposed in accordance with the following first integral:

$$g_{\alpha\beta} u^\alpha \frac{Dr^\beta}{ds} = \text{const} \tag{2.17}$$

of Eqs. (2.15). It is, of course, a « weak » identity. The transformation (2.16 *i*)) of initial data adds b to the constant here. To fix b it is therefore sufficient to fix the value of this constant. Usually it is done in the form

$$g_{\alpha\beta}u^{\alpha}\frac{\mathrm{D}r^{\beta}}{ds} = 0 \tag{2.18}$$

and it is sufficient to impose this condition on initial data only.

Interpretation of (2.18) follows from the Σ-definition of geodesic deviation. On each Γ_ρ one imposes the condition (1.12) in the form

$$g_{\alpha\beta}u^{\alpha}u^{\beta} = \mathrm{C}(\rho), \tag{2.19}$$

where C is a regular function of ρ. Then one gets (2.17) for Γ_0 as

$$2g_{\alpha\beta}u^{\alpha}\frac{\mathrm{D}r^{\beta}}{ds} = \frac{d\mathrm{C}}{d\rho}\bigg|_{\rho=0}.$$

If $\dfrac{d\mathrm{C}}{d\rho}\bigg|_{\rho=0} = 0$, it reduces to (2.18) which therefore is a requirement that geodesics in the « first » neighbourhood of Γ are parametrized by the same affine parameter as Γ (but the freedom of choice of its initial values is still left alone).

We call, therefore, a vector field r^{α} along a geodesics Γ the natural geodesic deviation vector iff it fulfils the constraint condition (2.18) and is a solution of Eqs. (2.15) evaluated along Γ parametrized by the natural parameter.

In general relativity the natural geodesic deviation vector along timelike geodesics parametrized by the proper time describes thus the motion of observers from a neighbourhood of Γ using ideal clocks.

The freedom represented by α in (2.16 *ii*)) can be fixed by a further constraint

$$g_{\alpha\beta}u^{\alpha}r^{\beta} = \text{const}, \tag{2.20}$$

provided (2.18) is imposed. Very often it is convenient to accept it as

$$u_{\alpha}r^{\alpha} = 0. \tag{2.21}$$

In general relativity (2.21) means that r^{α} describes the relative position in the rest frame of Γ. However, (2.21) should not be considered to be physically as obligatory as (2.20). Sometimes it is relaxed as being inconvenient. Then, however, $\sqrt{-r_{\alpha}r^{\alpha}}\rho$ cannot be interpreted as the measure of the spatial distance between two neighbouring observers.

3. THE SECOND GEODESIC DEVIATION

A solution of the equations of geodesic deviation describes, in accordance with its interpretation, only in an approximate way the behaviour of a geodesic from a neighbourhood of the basic geodesic along which the equations has been evaluated. If one wishes to improve this approximation, one should explore the possibility of generalization of the concept of geodesic deviation to higher orders. Such a generalized notion of second geodesic deviation has been introduced by the present author [*19*] for the special case of a geodesic line parametrized by the natural parameter s. Now we shall undertake a study of this notion defining it first for geodesics which are parametrized arbitrarily. This more general approach will help in better understanding of some properties which emerge even in the special case when the natural parametrization is being used.

Let us again consider the one-parametric family Σ of geodesics and let us additionally suppose that for each geodesic Γ_ρ from this family the geodesic deviation equation has been solved with some arbitrarily given initial conditions at $\tau = \tau_0$,

$$\tau^\alpha(\tau_0, \rho) = r_0^\alpha(\rho); \qquad \frac{\mathrm{D}r^\alpha}{d\tau}(\tau_0, \rho) = v_0^\alpha(\rho);$$

which are continuously parametrized by ρ. Let $r^\alpha = r^\alpha(\tau, \rho)$ be any solution of this initial value problem. Such functions r^α determine an additional to u^α vector field on the two-cube Σ. We define on Σ another field

$$w^\alpha(\tau, \rho) := \frac{\mathrm{D}r^\alpha}{d\rho}(\tau, \rho). \tag{3.1}$$

Now, to get for w^α an equation analogous to (2.6), we write the Ricci identity (2.3) with $t^\alpha = \dfrac{\mathrm{D}r^\alpha}{d\tau}(\tau, \rho)$. We apply this identity once again under the $\dfrac{\mathrm{D}}{d\tau}$-differentiation in the second term of so obtained equality and take into account Eqs. (2.6). We get

$$\frac{\mathrm{D}}{d\rho}\left[-\mathrm{R}^\alpha{}_{\beta\gamma\delta}u^\beta r^\gamma u^\delta + \frac{\mathrm{D}r^\alpha}{d\tau}\frac{d}{d\tau}\ln\sqrt{|u_\lambda u^\lambda|} + u^\alpha\frac{d}{d\tau}\left(\frac{1}{u_\lambda u^\lambda}u_\mu\frac{\mathrm{D}r^\mu}{d\tau}\right)\right]$$
$$-\frac{\mathrm{D}}{d\tau}\left[\frac{\mathrm{D}w^\alpha}{d\tau} + \mathrm{R}^\alpha{}_{\beta\gamma\delta}r^\beta r^\gamma u^\delta\right] = \mathrm{R}^\alpha{}_{\beta\gamma\delta}\frac{\mathrm{D}r^\beta}{d\tau}r^\gamma u^\delta.$$

Performing all the differentiations here, we take into account that now $\mathrm{R}^\alpha{}_{\beta\gamma\delta} = \mathrm{R}^\alpha{}_{\beta\gamma\delta}(x^\mu(\tau, \rho))$ and make use of (1.1), (2.1), (2.2), (2.3), (2.6), (3.1) and of the symmetry properties of the Riemann tensor. All this results in

$$\begin{aligned}\frac{\mathrm{D}^2w^\alpha}{d\tau^2} + \mathrm{R}^\alpha{}_{\beta\gamma\delta}u^\beta w^\gamma u^\delta &= (\mathrm{R}^\alpha{}_{\beta\gamma\delta;\varepsilon} + \mathrm{R}^\alpha{}_{\varepsilon\gamma\delta;\beta})u^\beta u^\gamma r^\delta r^\varepsilon + 4\mathrm{R}^\alpha{}_{\beta\gamma\delta}\frac{\mathrm{D}r^\beta}{d\tau}u^\gamma u^\delta \\ &+ \frac{\mathrm{D}w^\alpha}{d\tau}\frac{d}{d\tau}\ln\sqrt{|u_\lambda u^\lambda|} + 2\frac{\mathrm{D}r^\alpha}{d\tau}\frac{d}{d\tau}\left(\frac{u_\rho}{u_\lambda u^\lambda}\frac{\mathrm{D}r^\rho}{d\tau}\right) \\ &+ u^\alpha\frac{d}{d\tau}\frac{1}{u_\lambda u^\lambda}\left[\frac{\mathrm{D}r_\mu}{d\tau}\frac{\mathrm{D}r^\mu}{d\tau} + u_\rho\frac{\mathrm{D}w^\rho}{d\tau} + \mathrm{R}_{\mu\nu\rho\sigma}u^\mu r^\nu u^\rho r^\sigma - \frac{2}{u_\lambda u^\lambda}\left(u_\rho\frac{\mathrm{D}r^\rho}{d\tau}\right)^2\right].\end{aligned} \tag{3.2}$$

The set of Eqs. (2.4) [or (1.1)], (2.6) and (3.2) can, in particular, be written along a selected geodesic Γ_0 (*e. g.* labelled by $\rho = 0$) and for a selected solution $r^\alpha(\tau)$ of (2.6) for $\rho = 0$. When we immerse Γ_0 in another Σ' and extend $r^\alpha(\tau)$ to a vector field $r'^\alpha(\tau, \rho)$ (so that $r^\alpha(\tau) = r'^\alpha(\tau, 0)$) of solutions of (2.6) on the new Σ', in such a way that $w'^\alpha(\tau, 0) = w^\alpha(\tau, 0)$, we shall introduce an equivalence relation between all Σ'*s* endowed with fields of solutions of (2.6) for fixed both Γ_0 and $r^\alpha(\tau)$. The corresponding class of equivalence is called the second geodesic deviation vector field along Γ_0.

Now we pass to another, more general manner of defining the second deviation. Let us suppose that there is a single parametrized geodesic line Γ given and that along this line Eqs. (2.6) have been solved for some initial conditions (2.7). Let this solution be denoted by $r^\alpha(\tau)$ too. Then we can evaluate all coefficients in (3.2) like u^α, r^α, $g_{\alpha\beta}$, $\Gamma^\alpha_{\beta\gamma}$, $R^\alpha{}_{\beta\gamma\delta}$, etc., along Γ for this selected solution $r^\alpha(\tau)$. Eqs. (3.2) are thus turning into ordinary differential equations of the second order for $w^\alpha(\tau)$. Any solution $w^\alpha(\tau)$ of these equations, fulfilling conditions

$$w^\alpha(\tau_0) = w^\alpha_0 \, ; \qquad \frac{Dw^\alpha}{d\tau} = {}_1v^\alpha_0 \tag{3.3}$$

will be called the second geodesic deviation vector field along Γ.

The n (for $\alpha = 1, \ldots, n$) differential equations (3.2) are not independent, since contracting them with u_α we again obtain a strong identity. Therefore we have

PROPOSITION 3.1. — If functions w^α: $I \to \mathbb{R}$ ($\alpha = 1, \ldots, n$) are a solution of the second geodesic deviation equations (3.2) along a geodesic Γ, described in a coordinate system $\{ x^\alpha \}$ by functions ξ^α: $I \to \mathbb{R}$, and for a solution of Eqs. (2.5) described by functions r^α: $I \to \mathbb{R}$, then

i) the functions $w^\alpha \circ f$, for any C_2 function f such that $f' \neq 0$, are also a solution of (3.2) along the same geodesic Γ described now by $\xi^\alpha \circ f$ and for a solution of (2.5) determined by $r^\alpha \circ f$;

ii) the functions $\tilde{w}^\alpha = w^\alpha + 2\kappa \dfrac{Dr^\alpha}{d\tau} + \kappa^2 u^\alpha \ln \sqrt{| u_\lambda u^\lambda |}$ for any C_2 function κ: $I \to \mathbb{R}$, form also a solution of (3.2) along the same Γ described by ξ^α and for a solution of (2.5) determined by $\tilde{r}^\alpha = r^\alpha + \kappa u^\alpha$.

PROPOSITION 3.2. — If the set of functions w^α: $I \to \mathbb{R}$ is a solution of Eqs. (3.2) along a given geodesic Γ described by functions ξ^α and for a given solution of the first geodesic deviation equations along Γ described by functions r^α, then

i) the set of functions $w^\alpha + \psi u^\alpha$ (where ψ: $I \to \mathbb{R}$, $\psi \in C_2$, is arbitrary and $u^\alpha = \dfrac{d\xi^\alpha}{d\tau}\Big)$ is also a solution of (3.2) along the same Γ with the same parametrization and for the same r^α;

ii) any solution $\tilde{w}^\alpha$ of (3.2) fulfilling the same initial conditions (3.3) as w^α can be represented as

$$\tilde{w}^\alpha = w^\alpha + \psi u^\alpha \tag{3.4}$$

with a certain C_2 function ψ fulfilling the conditions: $\psi(\tau_0) = \psi'(\tau_0) = 0$.

The proof. — Prop. 3.1 and 3.2 *i*) follow from inspection. We also have

LEMMA 3.1. — A solution of Eqs. (3.2) satisfying (3.1) is completely specified by a choice of a continuous function $\nu: \mathrm{I} \to \mathbb{R}$.

[The proof: A given solution w^α (with ξ^α and r^α) defines the function

$$\nu(\tau) = \frac{d}{d\tau}\frac{1}{u_\lambda u^\lambda}\left[\frac{\mathrm{D}r_\mu}{d\tau}\frac{\mathrm{D}r^\mu}{d\tau} + u_\rho\frac{\mathrm{D}w^\rho}{d\tau} - \mathrm{R}_{\mu\nu\rho\sigma}u^\mu r^\nu u^\rho r^\sigma + \frac{2}{u_\lambda u^\lambda}\left(u_\rho\frac{\mathrm{D}r^\rho}{d\tau}\right)^2\right]. \tag{3.5}$$

For a given $\nu: \mathrm{I} \to \mathbb{R}$ w^α fulfilling (3.3) is defined as the unique solution of the system of differential equations

$$\frac{\mathrm{D}^2 w^\alpha}{d\tau^2} + \mathrm{R}^\alpha{}_{\beta\gamma\delta}u^\beta w^\gamma u^\delta = (\mathrm{R}^\alpha{}_{\beta\gamma\delta;\varepsilon} + \mathrm{R}^\alpha{}_{\varepsilon\gamma\delta;\beta})u^\beta u^\gamma r^\delta r^\varepsilon + 4\mathrm{R}^\alpha{}_{\beta\gamma\delta}\frac{\mathrm{D}r^\beta}{d\tau}u^\gamma r^\delta$$
$$+ \lambda(\tau)\frac{\mathrm{D}w^\alpha}{d\tau} + 2\mu(\tau)\frac{\mathrm{D}r^\alpha}{d\tau} + \nu(\tau)u^\alpha \tag{3.6}$$

where λ and μ are given by (1.6) and (2.9). Such w^α solves also (3.2)].

COROLLARY of the lemma: There exists a one-to-one map $\mathscr{H}$ of the set of all solutions of (3.2) with given initial data in (3.3) onto the set of all continuous functions $\nu: \mathrm{I} \to \mathbb{R}$.

Let w^α and $\tilde{w}^\alpha$ be two solutions of (3.2) such that $\mathscr{H}(w^\alpha) = \nu$ and $\mathscr{H}(\tilde{w}^\alpha) = \tilde{\nu}$, then ψ in (3.4) is defined [cf. (3.5)] as the unique solution of

$$\tilde{\nu}(\tau) = \nu(\tau) + \frac{d}{d\tau}\left(\frac{d\psi}{d\tau}(\tau) + \psi(\tau)\lambda(\tau)\right) \tag{3.7}$$

with the initial data $\psi(\tau_0) = \psi'(\tau_0) = 0$ and this ends the proof.

The multiplicity of solutions of Eqs. (3.2), described in Prop. 3.1 *ii*) and 3.2 can again be interpreted as a possibility of introducing in the next approximation a new arbitrary parametrization on neighbouring geodesic lines, keeping fixed the parametrization on the basic line Γ and in case of Prop. 3.2 keeping also fixed the selected solution $r^\alpha(\tau, \rho)$. To show this one should complete (2.12) by the condition

$$\frac{\partial^2 f}{\partial \rho^2}(\tau, 0) = \psi(\tau) \tag{3.8}$$

$\left(\text{and (2.13) by } \tilde{w}^\alpha := \frac{\mathrm{D}\tilde{r}^\alpha}{\partial\rho}(\tau, \rho)\right)$ and continue the argument from Section 2.

If the geodesic Γ is parametrized by the natural parameter and the selected first deviation $r^\alpha(s)$ fulfils (2.15) with the initial conditions (2.7) and the constraints (2.18), then the second geodesic deviation equations are reduced to

$$\frac{D^2 w^\alpha}{ds^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta w^\gamma u^\delta = (R^\alpha{}_{\beta\gamma\delta;\varepsilon} + R^\alpha{}_{\varepsilon\gamma\delta;\beta}) u^\beta u^\gamma r^\delta r^\varepsilon + 4R^\alpha{}_{\beta\gamma\delta} \frac{Dr^\beta}{ds} u^\gamma r^\delta + u^\alpha \frac{d}{ds} \frac{1}{u_\lambda u^\lambda} \left[\frac{Dr_\mu}{d\tau} \frac{Dr^\mu}{d\tau} + u_\rho \frac{Dw^\rho}{d\tau} - R_{\mu\nu\rho\sigma} u^\mu r^\nu u^\rho r^\sigma \right]. \quad (3.9)$$

The initial value problem (3.3) for Eqs. (3.9) admits a family of solutions each member of which is labelled by a function ν and is a unique solution of Eqs. of the type (3.6) (now with $\mu = \lambda = 0$). Since all these solutions are coupled to each other by (3.4) (which introduces an equivalence relation between solutions of (3.9)), it is sufficient to solve the simplest equation corresponding to the choice of $\nu \equiv 0$:

$$\frac{D^2 w^\alpha}{ds^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta w^\gamma u^\delta = (R^\alpha{}_{\beta\gamma\delta;\varepsilon} + R^\alpha{}_{\varepsilon\gamma\delta;\beta}) u^\mu u^\gamma r^\delta r^\varepsilon + 4R^\alpha{}_{\beta\gamma\delta} \frac{Dr^\beta}{ds} u^\gamma r^\delta. \quad (3.10)$$

This choice can again be interpreted as determining a situation in which all neighbouring geodesics up to the « second » order of neighbourhood are parametrized by affine parameters.

For each set of initial data in (3.3) Eqs. (3.10) have a unique solution. Two different sets of initial data might, however, lead to two equivalent solutions, as it follows from

PROPOSITION 3.3. — Two different sets, $\{ w_0^\alpha, {}_1v_0^\alpha \}$ and $\{ \tilde{w}_0^\alpha, {}_1\tilde{v}_0^\alpha \}$, of initial data in (3.3) will render two solutions w^α and $\tilde{w}^\alpha$ of (3.10) equivalent in the sense of (3.4) iff

$$i) \quad \begin{aligned} \tilde{w}_0^\alpha &= w_0^\alpha + c u_0^\alpha \, ; \\ {}_1\tilde{v}_0^\alpha &= {}_1v_0^\alpha + d u_0^\alpha \end{aligned} \qquad \text{Then: } ii) \ \psi(\tau) = (\tau - \tau_0) d + c \quad (3.11)$$

(c and d are arbitrary constants and u_0^α are the initial data for Γ fulfilling (1.11)).

The proof is straightforward and will be omitted.

Eqs. (3.11 i) are establishing an equivalence relation of initial data for (3.10). A single representative from each class of equivalence is determined by constraint conditions which must be imposed in accordance with the following first integral:

$$\frac{Dr_\mu}{d\tau} \frac{Dr^\mu}{d\tau} + u_\mu \frac{Dw^\mu}{d\tau} - R_{\mu\nu\rho\sigma} u^\mu r^\nu u^\rho r^\sigma = \text{const} \quad (3.12)$$

of Eqs. (3.10). It is a « weak » identity. The change (3.11 i)) of initial data adds d to the constant in (3.12). To fix d it is therefore sufficient to fix the value of this constant. It turns out that a reasonable choice is to put it equal to zero,

$$\frac{Dr_\mu}{ds}\frac{Dr^\mu}{ds} + u_\mu \frac{Dw^\mu}{ds} - R_{\mu\nu\rho\sigma}u^\mu r^\nu u^\rho r^\sigma = 0, \tag{3.13}$$

and it is sufficient to impose this condition on initial data only.

The justification and interpretation of the choice of (3.13) follows from the Σ-approach. On each geodesic Γ_ρ one imposes the condition (2.19). Differentiating it twice with respect to ρ one derives (3.12) with the constant being equal to $\left.\frac{d^2C}{d\rho^2}\right|_{\rho=0}$. The condition (3.13) should therefore be interpreted as the requirement that geodesics from the « second » neighbourhood of Γ are parametrized by the same affine parameter as Γ.

We call $w^\alpha(s)$ a natural second deviation vector iff it fulfils the constraint condition (3.13) and is a solution of Eqs. (3.10) in which Γ is a naturally parametrized geodesic and $r^\alpha(s)$—a given natural first geodesic deviation vector.

The freedom of choice of the initial point on the parameter scale in the neighbourhood of Γ can be fixed by a further constraint

$$u_\alpha w^\alpha = - \int_{\tau_0}^{\tau} \left(\frac{Dr_\mu}{d\tau}\frac{Dr^\mu}{d\tau} - R_{\mu\nu\rho\sigma}r^\mu u^\nu r^\rho u^\sigma \right) d\tau + \text{const} \tag{3.14}$$

provided (3.13) is imposed. Specification of the value of the constant here restricts the freedom brought in (3.11) by c. Let us observe that the integral here is equal to the Caratheodory action (cf. [*26*]), evaluated for a fixed solution r^α.

The second geodesic equations (3.10) (as well as (3.6) and (3.9)), in contradistinction to (2.15), are not homogeneous. Thus, even if initially for $s = s_0$ w^α and $\frac{Dw^\alpha}{ds}$ vanish, but there is a nonvanishing field r^α or $\frac{Dr^\alpha}{ds}$ along the geodesic Γ, a nonvanishing field w^α will appear for subsequent values of s (If one insists on the condition (3.13), then, of course, one must initially have at least $u_\alpha \frac{Dw^\alpha}{ds} \neq 0\Big)$.

The kinematic interpretation of the second geodesic deviation vector is implied by the geometric formulation of the Taylor theorem for curves on manifolds given in [*17*]. For our present purpose we express it in the following form:

THEOREM. — Let Λ be a curve in an n dimensional differentiable manifold V_n endowed with a symmetric affine connection analytic in a region $\Omega \subset V_n$ and let Λ be described in a coordinate system $\{ x^\alpha \}$ by equations $x^\alpha(p) = \Lambda^\alpha(\rho)$. Let further Λ be analytic in Ω, *i. e.* for any two points $p \in \Omega$, $q \in \Omega$ such that $x^\alpha(p) = \Lambda^\alpha(\rho)$, $x^\alpha(q) = \Lambda^\alpha(0)$ the series

$$\Lambda^\alpha(\rho) = \Lambda^\alpha(0) + \left(\frac{d\Lambda^\alpha}{d\rho}\right)_0 \rho + \frac{1}{2\,!}\left(\frac{d^2\Lambda^\alpha}{d\rho^2}\right)_0 \rho^2 + \dots$$

(for $\alpha = 1, \ldots, n$) are convergent, then there exists such a sequence of vectors $r_{(k)}$ ($k = 0, 1, 2, \ldots$ is numerating separate vectors), all from the tangent space $T_q(V_n)$ at q, with components $r^\alpha_{(k)}$, that the geodesic Γ_l sent from q in the direction determined by the vector $l(q) \in T_q(V_n)$ with the components

$$l^\alpha(q) = r^\alpha_{(0)} + \frac{1}{2} r^\alpha_{(1)}\rho + \ldots \frac{1}{(n+1)!} r^\alpha_{(n)}\rho^n + \ldots \tag{3.15}$$

will intersect the curve Λ at the point p for the value ρ of the affine parameter along Γ_l, provided p belongs to a certain surrounding U of q, where $U \subset \Omega$.

In [17] a general, rather complicated formula for the vector coefficients $r^\alpha_{(k)}$ corresponding to an arbitrary value of k has been given. Here we quote only a few first of them

$$\begin{gathered} r^\alpha_{(0)} = t^\alpha(q) = \left(\frac{d\Lambda^\alpha}{d\rho}\right)_0 \quad ; \quad r^\alpha_{(1)} = \left(\frac{Dt^\alpha}{d\rho}\right)_0 \quad ; \quad r^\alpha_{(2)} = \left(\frac{D^2 t^\alpha}{d\rho^2}\right)_0 ; \\ r^\alpha_{(3)} = \left(\frac{D^3 t^\alpha}{d\rho^3}\right)_0 + R^\alpha{}_{\beta\gamma\delta}(q) t^\beta(q) \left(\frac{Dt^\gamma}{d\rho}\right)_0 t^\delta(q); \end{gathered} \tag{3.16}$$

where $t^\alpha(q)$ are components of the tangent vector to Λ at q and the absolute derivatives of t^α along Λ are evaluated for $\rho = 0$, *i. e.* also at q.

Let us now take two geodesics, Γ and $\tilde{\Gamma}$, and extend them to a family Σ of geodesics, described by $x^\alpha = \xi^\alpha(\tau, \rho)$, so that $\rho = 0$ on Γ and $\rho = \tilde{\rho} = \text{const}$ on $\tilde{\Gamma}$. The curve from the theorem is then defined by $\Lambda^\alpha = \xi^\alpha(\tilde{\tau}, \rho)$ for $\tilde{\tau} = \text{const}$ and $\rho_1 < 0 \leqslant \rho \leqslant \tilde{\rho} < \rho_2$. The theorem asserts that a geodesic $\tilde{\Gamma}$ sent from the point q with the coordinates $\xi^\alpha(\tilde{\tau}, 0)$ in the direction of the vector

$$l^\alpha(\tilde{\tau}, \tilde{\rho}) = r^\alpha(\tilde{\tau}) + \frac{1}{2} w^\alpha(\tilde{\tau})\tilde{\rho}, \tag{3.17}$$

$\left[\text{where } r^\alpha(\tilde{\tau}) = \dfrac{\partial \xi^\alpha}{\partial \rho}(\tilde{\tau}, 0),\ w^\alpha(\tilde{\tau}) = \dfrac{D}{\partial \rho}\left(\dfrac{\partial \xi^\alpha}{\partial \rho}(\tilde{\tau}\ \rho)\right)\bigg|_{\rho=0}\right.$ are components of vectors from $T_q(V_n)$ defined in agreement with (2.1) and (3.1)] will almost intersect, for the affine parameter along Γ_l equal to $\tilde{\rho}$, the geodesic $\tilde{\Gamma}$ at the point p labelled by $\xi^\alpha(\tilde{\tau}, \tilde{\rho})$, missing it with an error of the order $0(\tilde{\rho}^3)$. This establishes the kinematic interpretation of the second deviation vector. Formulæ (3.16) and their generalization for an arbitrary order indicate also how one should define geodesic deviation vectors of a higher order.

Let us discuss this interpretation in the case of general relativity, where two timelike geodesics, Γ and $\tilde{\Gamma}$, parametrized by the natural parameter s are interpreted as the world lines of two freely falling test observers equipped with ideal clocks. We shall demonstrate how the knowledge of their world lines enables to formulate such initial conditions for Eqs. (2.15) and (3.10) (both taken along Γ) that their solutions $r^\alpha(s)$ and $w^\alpha(s)$ will determine the right vector (3.17) for the just described construction of the geodesic joining Γ and $\tilde{\Gamma}$. For this purpose let us take a spacelike geodesic γ_{s_0} which joins ([4]) a point $\Gamma(s_0)$ with $\tilde{\Gamma}(s_0)$, comp. fig. 1. The components of the tangent vector to γ_{s_0} at $\Gamma(s_0)$ are denoted ([5]) by r_0^α. This vector defines along γ_{s_0} an affine parameter ρ taken to be $\rho = 0$ at $\Gamma(s_0)$ and $\rho = \tilde{\rho}$ at $\tilde{\Gamma}(s_0)$ (The metrical distance between $\Gamma(s_0)$ and $\tilde{\Gamma}(s_0)$, measured along γ_{s_0}, is then equal

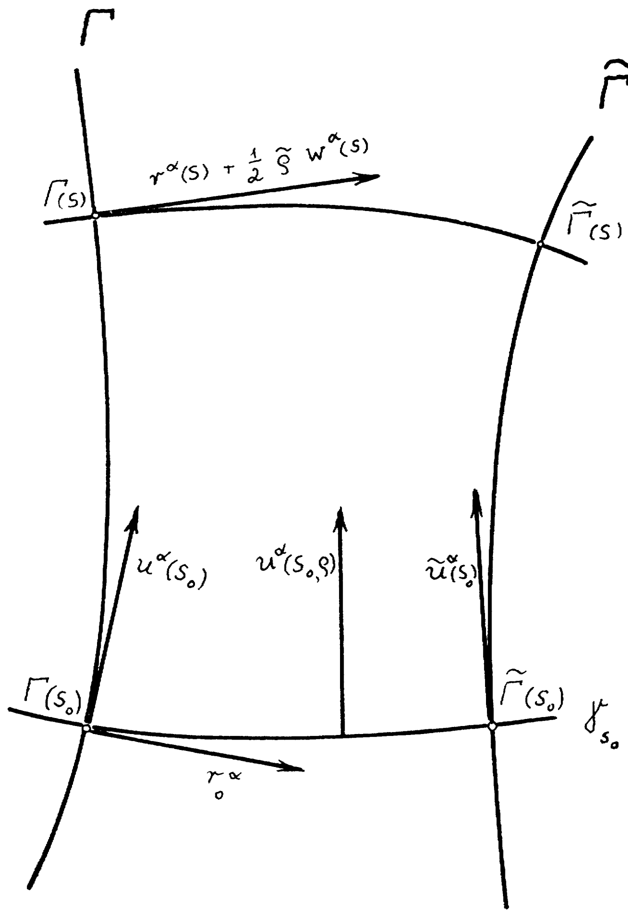

Fig. 1.

([4]) We limit our considerations to such a region of space-time in which all the constructions discussed can be performed with a unique result.

([5]) This step is not a unique one. We could also denote this vector by $r_0^\alpha + \frac{1}{2}\tilde{\rho}w_0^\alpha$, with all obvious implications.

to $\sqrt{-r_{0\alpha}r_0^\alpha} + 0(\tilde{\rho}^2)$; we could have taken a normalized r_0^α, but more convenient is to think of it as having the dimension of length and of $\tilde{\rho}$ as being dimensionless). The next step is to define along γ_{s_0} a vector field $u^\alpha(s_0, \rho)$ satisfying: $u^\alpha(s_0, 0) = u^\alpha(s_0)$ and $u^\alpha(s_0, \tilde{\rho}) = \tilde{u}^\alpha(s_0)$, where $u^\alpha(s_0)$ and $\tilde{u}^\alpha(s_0)$ are components of the tangent vectors correspondingly to Γ at $\Gamma(s_0)$ and to $\tilde{\Gamma}$ at $\tilde{\Gamma}(s_0)$. This can be done in infinitely many ways, the simplest, however, is the « linear » interpolation:

$$u^\alpha(s_0, \rho) = \frac{1}{\tilde{\rho}}[\tilde{u}^\alpha_{||}(s_0) - u^\alpha_{||}(s_0)]\rho + u^\alpha_{||}(s_0), \tag{3.18}$$

where the parallel bars $\|$ indicate that the corresponding vector has been parallely transported along γ_{s_0} from its original position to a point $\gamma_{s_0}(\rho)$ (Sending now geodesics from $\gamma_{s_0}(\rho)$, for any $\rho \in [0, \tilde{\rho}]$, in the direction of $u^\alpha(s_0, \rho)$ we can construct a two-cube Σ containing both Γ and $\tilde{\Gamma}$). Let us, in agreement with (2.2), define

$$\begin{aligned}\frac{\mathrm{D}r^\alpha}{\partial s}(s_0, \rho) :&= \frac{\mathrm{D}}{\partial \rho}\frac{u^\alpha(s_0, \rho)}{\sqrt{u^\lambda(s_0, \rho)u_\lambda(s_0, \rho)}} \\ &= \frac{1}{\tilde{\rho}\sqrt{u^\sigma(s_0, \rho)u_\sigma(s_0, \rho)}}\left(\delta^\alpha_\beta - \frac{u^\alpha(s_0, \rho)u_\beta(s_0, \rho)}{u^\lambda(s_0, \rho)u_\lambda(s_0, \rho)}\right)(\tilde{u}^\beta_{||}(s_0) - u^\beta_{||}(s_0)).\end{aligned} \tag{3.19}$$

Taking then r_0^α and $v_0^\alpha = \dfrac{\mathrm{D}r^\alpha}{ds}(s_0, 0)$ as initial data in (2.7) for Eqs. (2.15), we uniquely determine a field $r^\alpha(s)$ along Γ (It is a natural geodesic deviation vector since (2.18) is fulfilled by (3.19) automatically). A geodesic sent from the point $\Gamma(s)$ in the direction of $r^\alpha(s)$ will then, with an approximation of $0(\tilde{\rho}^2)$, intersect $\tilde{\Gamma}$ at $\tilde{\Gamma}(s)$ for the value of its affine parameter equal to $\tilde{\rho}$. Next we evaluate Eqs. (3.10) along Γ for the just determined solution $r^\alpha(s)$ of (2.15) and solve it with initial data: $w_0^\alpha = 0$; ${}_1v_0^\alpha = \dfrac{\mathrm{D}w^\alpha}{\partial s}(s_0, 0)$, taking

$$\frac{\mathrm{D}w^\alpha}{\partial s}(s_0; \rho) : = \frac{\mathrm{D}}{\partial \rho}\frac{\mathrm{D}r^\alpha}{\partial s}(s_0, \rho) + \mathrm{R}^\alpha{}_{\beta\gamma\delta}r^\beta(s_0, \rho)u^\gamma(s_0, \rho)r^\delta(s_0, \rho). \tag{3.20}$$

This agrees with (3.1) and (2.3); $r^\alpha(s_0, \rho)$ is the tangent vector to γ_{s_0} at $\gamma_{s_0}(\rho)$.

The properties of the two geodesic deviation equations and of the Taylor theorem then imply that a geodesic γ_s sent from $\Gamma(s)$ in the direction of $r^\alpha(s) + \frac{1}{2}\tilde{\rho}w^\alpha(s)$ will intersect [with an accuracy of $0(\tilde{\rho}^3)$] the geodesic $\tilde{\Gamma}$ at the point $\tilde{\Gamma}(s)$ for its affine parameter equal to $\tilde{\rho}$. The proper time intervals: s_Γ between the points $\Gamma(s)$ and $\Gamma(s_0)$ on Γ and $s_{\tilde{\Gamma}}$ between so obtained $\tilde{\Gamma}(s)$ and $\tilde{\Gamma}(s_0)$ on $\tilde{\Gamma}$—measured by two ideal clocks comoving with Γ and $\tilde{\Gamma}$—are equal to each other [modulo $0(\tilde{\rho}^3)$], as our initial data automatically satisfy the conditions (2.18) and (3.13). The geodesic distance, along γ_s, between the points $\Gamma(s)$ and $\tilde{\Gamma}(s)$ is equal to

$$\left[-\left(r^\alpha(s)+\frac{1}{2}\tilde{\rho}w^\alpha(s)\right)\left(r_\alpha(s)+\frac{1}{2}\tilde{\rho}w_\alpha(s)\right)\right]^{1/2}\tilde{\rho}+0(\tilde{\rho}^3)$$
$$=\sqrt{-r^\alpha(s)r_\alpha(s)}\tilde{\rho}-\frac{1}{2}\tilde{\rho}^2\frac{r_\alpha(s)w^\alpha(s)}{\sqrt{-r_\sigma(s)r^\sigma(s)}}+0(\tilde{\rho}^3) \tag{3.21}$$

and is a function (6) of s. This quantity, however, is not in general a measure of the spatial distance between the observers Γ and $\tilde{\Gamma}$.

Such construction of initial data at $\Gamma(s_0)$, when a geodesic $\tilde{\Gamma}$ from a neighbourhood of Γ is given, is of course not a unique one, although the simplest. Conversely, if we are given: a geodesic Γ parametrized by s; two sets of initial data for $s = s_0$, $\{ r^\alpha_0, v^\alpha_0 \}$ and $\{ w^\alpha_0, {}_1v^\alpha_0 \}$, fulfilling the constraint relations (2.18) and (3.13); and a value $\tilde{\rho}$: then solving Eqs. (2.15) with the initial conditions (2.7) we determine with an accuracy of $0(\tilde{\rho}^2)$ a unique geodesic $\tilde{\Gamma}$ in the neighbourhood of Γ. Solving then the initial value problem with conditions (3.3) for Eqs. (3.10) we improve the approximation of determining $\tilde{\Gamma}$ to the order of $0(\tilde{\rho}^3)$. Such procedure can be carried on to an arbitrary order of accuracy by introducing along Γ generalized geodesic deviations of arbitrary order.

4. THE SPACE SEPARATION VECTOR

In the construction above no use was made of conditions (2.20) and (3.14). Imposing on r^α additionally the condition (2.21) one defines the deviation vector $r^\alpha_\perp$ which can also be considered to be the projection of any other solution r^α of (2.15) on the linear subspace orthogonal to u^α :

$$r^\alpha_\perp = (\delta^\alpha_\beta - u^\alpha u_\beta)r^\beta. \tag{4.1}$$

A geodesic $\gamma_\perp$ sent from $\Gamma(s)$ in the direction of $r^\alpha_\perp(s)$ will intersect [modulo $0(\tilde{\rho}^2)$] the geodesic $\tilde{\Gamma}$ for a value $\tilde{\rho}$ of its affine parameter introduced by the initial conditions. $\gamma_\perp$ will also be orthogonal [modulo $0(\tilde{\rho}^2)$] to $\tilde{\Gamma}$. Thus $\sqrt{-r_{\perp\alpha}(s)r^\alpha_\perp(s)}\ \tilde{\rho}$ is, when neglecting terms of the order $0(\tilde{\rho}^2)$, the spatial distance between Γ and $\tilde{\Gamma}$, *i. e.* the distance measured in their mutual [modulo $0(\tilde{\rho}^2)$] rest frame. Therefore, the first natural geodesic

(6) Formula (3.21) shows thus in a direct way that although ρ is an affine parameter along any γ_s for $s_0 \leqslant s \leqslant s_1$, its choice on every γ_s is determined by a, in general, different normalization.

deviation vector $r_\perp^\alpha$, determined along Γ by Eqs. (2.15) and the conditions (2.18) and (2.21) will be called the first spatial separation vector.

A different situation arrises in the case of the second natural geodesic deviation. Here, even if we impose on solutions of Eqs. (3.10) (taken along Γ for a given $r_\perp^\alpha(s)$) the conditions (3.13) and (3.14) (putting there: const$=0$) and take initial data satisfying $u_{0\alpha}w_0^\alpha = 0$, we shall still get $u_\alpha w^\alpha \neq 0$ for subsequent values of the parameter along Γ.

We define the second separation vector s^α along a timelike geodesic Γ as a second deviation vector for which: *i)* $s^\alpha u_\alpha = 0$ at any point of Γ; *ii)* a geodesic γ_s led from $\Gamma(s)$ in the direction of $r_\perp^\alpha(s) + \frac{1}{2}\tilde{\rho} s^\alpha(s)$ will intersect $\tilde{\Gamma}$ modulo $0(\tilde{\rho}^3)$.

Assume that Γ is parametrized by s. The evolution of s^α along Γ will not be determined any more by (3.10), but by the more general Eqs. (3.9), because the geodesics γ_s will in general introduce on $\tilde{\Gamma}$ a parametrization which will not be a natural one (We limit our consideration to cases in which geodesics γ_s project the parametrization on Γ into a parametrization on $\tilde{\Gamma}$)·

Thus the evolution of the second separation vector, along Γ parametrized by s for a given $r_\perp^\alpha$, is determined by the equations

$$\begin{aligned}\frac{D^2 s^\alpha}{ds^2} + R^\alpha{}_{\beta\gamma\delta}u^\beta s^\gamma u^\delta = (R^\alpha{}_{\beta\gamma\delta;\varepsilon} + R^\alpha{}_{\varepsilon\gamma\delta;\beta})u^\beta u^\gamma r_\perp^\delta r_\perp^\varepsilon + 4R^\alpha{}_{\beta\gamma\delta}\frac{Dr_\perp^\beta}{ds}u^\gamma r_\perp^\delta \\ + u^\alpha \frac{d}{ds}\left(\frac{Dr_\perp^\mu}{ds}\frac{Dr_{\perp\mu}}{ds} + u_\lambda \frac{Ds^\lambda}{ds} - R_{\mu\nu\rho\sigma}u^\mu r_\perp^\nu u^\rho r_\perp^\sigma\right)\end{aligned} \tag{4.2}$$

(which, as we know, are dependent) and the condition

$$s^\alpha u_\alpha = 0. \tag{4.3}$$

Prop. 3.2 implies then that the initial value problem for the system (4.2) and (4.3) is well defined. To show this let us take a solution $w^\alpha(s)$ of (3.9) satisfying for $s = s_0$ the conditions: $u_0^\alpha w_{0\alpha} = 0$ and (3.13). Then due to (3.14) we must have: $s^\alpha = w^\alpha + \psi u^\alpha$, and (4.3) holds iff $\psi = -u_\alpha w^\alpha$. Thus

$$s^\alpha = (\delta^\alpha_\beta - u^\alpha u_\beta)w^\beta. \tag{4.4}$$

The geodesic distance between Γ and $\tilde{\Gamma}$ along geodesics γ_s is equal to

$$l(s) = \sqrt{-\tau_{\perp\alpha}(s)r_\perp^\alpha(s)}\left(1 + \frac{1}{2}\tilde{\rho}\frac{r_\perp^\alpha(s)w_\alpha(s)}{r_{\perp\sigma}(s)r_\perp^\sigma(s)}\right)\tilde{\rho} + 0(\tilde{\rho}^3). \tag{4.5}$$

This quantity is the spatial distance of $\tilde{\Gamma}$ in the rest frame of Γ modulo terms $0(\tilde{\rho}^3)$. It is in general not equal to the corresponding distance of Γ from $\tilde{\Gamma}$ in the rest frame of $\tilde{\Gamma}$, since in the approximation considered the geodesics γ_s are not, in general, orthogonal to $\tilde{\Gamma}$ any more.

Similarly, one can define separation vectors of higher orders. All they can be expressed by means of higher order geodesic deviation vectors.

5. THE TELESCOPIC VECTORS

At first sight the concept of the natural geodesic deviation, although provided with an intuitive geometric interpretation, may seem, from the point of view of a relativistic program of constructing space-time objects by means of propagation of light signals, not to be very interesting. This concept can, however, be easily related to objects which have a direct physical interpretation.

Let us define the first telescopic vector k^α along a timelike geodesic Γ as a first geodesic deviation vector for which: *i)* $k^\alpha k_\alpha = 0$ at any point of Γ; *ii)* a null geodesic λ_τ led from $\Gamma(\tau)$ in the direction of k^α will intersect [modulo $0(\tilde{\rho}^2)$] the same geodesic $\tilde{\Gamma}$ for any value $\tau \in \mathrm{I}$ of the parameter on Γ.

The evolution of k^α along Γ cannot be described by Eqs. (2.15) as they are not consistent with *i*). It should, however be described by means of Eqs. (2.14). Thus, when Γ is parametrized by s, the first telescopic vector along it is determined by the set of equations

$$\frac{D^2 k^\alpha}{ds^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta k^\gamma u^\delta = u^\alpha u_\beta \frac{D^2 k^\beta}{ds^2} \,; \tag{5.1}$$

$$k^\alpha k_\alpha = 0. \tag{5.2}$$

Let r^α be the first space separation vector along Γ (from now on we omit the subscript $\perp$). Then from Prop. 2.2: $k^\alpha(s) = r^\alpha(s) + \kappa(s) u^\alpha(s)$, and (5.2) turns out to be equivalent to

$$\kappa(s) = \pm \sqrt{-\, r^\alpha(s) r_\alpha(s)}. \tag{5.3}$$

Our problem has thus two solutions: k^α_+—the advanced telescopic vector (which instead of a telescope characterizes rather a photon emitting device) and k^α_-—the retarded telescopic vector, both given by

$$k^\alpha_\pm = r^\alpha \pm \sqrt{-\, r^\lambda r_\lambda}\, u^\alpha. \tag{5.4}$$

The relation (5.3) can now be also written as

$$\kappa_\pm = \pm \sqrt{-\, r^\alpha r_\alpha} = u_\alpha k^\alpha_\pm \tag{5.5}$$

to yield

$$r^\alpha = (\delta^\alpha_\beta - u^\alpha u_\beta) k^\beta_\pm. \tag{5.6}$$

Eqs. (5.1) are general geodesic deviation equations and so their solution, the vector k^α, enjoys the properties of any deviation vector. In particular, a null geodesic λ_s sent from $\Gamma(s)$ will meet $\tilde{\Gamma}$ [modulo $0(\tilde{\rho}^2)$] for the value of the affine parameter along λ_s equal to $\tilde{\rho}$. For each null geodesic λ_s going out from $\Gamma(s)$ its tangent vector k^α and with it the normalization of the corresponding affine parameter ρ on λ_s depend on s. To obtain on each λ_s a parametrization suitable for comparing the affine parameter distances along different λ_s and $\lambda_{s'}$, one ought to introduce a tangent vector $\tilde{k}^\alpha$ whose normalization is s-independent. If one choses $\tilde{k}^\alpha_\pm u_\alpha = \pm 1$, then, because of (5.5),

$$\tilde{k}^\alpha_\pm = \frac{k^\alpha_\pm}{u_\lambda k^\lambda_\pm} = u^\alpha \pm e^\alpha,$$

where

$$e^\alpha := \frac{r^\alpha}{\sqrt{-r_\mu r^\mu}} \tag{5.7}$$

is defining the spatial direction of a telescope in the rest frame of Γ. The affine parameter distance between Γ and $\tilde{\Gamma}$ is then equal, modulo $0(\tilde{\rho}^2)$, to their spatial distance.

Let us take a family of null geodesics which go out from points belonging to an arc of a timelike world line Γ and all of which intersect a second timelike world line $\tilde{\Gamma}$. In general, provided these geodesics will not focus or form caustics in the region between Γ and $\tilde{\Gamma}$, if we parametrize this family by means of the parameter s from Γ, it will introduce a new parameter τ on $\tilde{\Gamma}$, being a projection of s. In terms of the natural parameter $\tilde{s}$ on $\tilde{\Gamma}$ we have, of course: $\tau = f(\tilde{s})$, $f' \neq 0$. Let us now suppose that the observer Γ sends a light signal with a frequency ν_0 which is received at $\tilde{\Gamma}$ with a frequency ν, then, assuming that the ideal clocks along Γ and $\tilde{\Gamma}$ are properly synchronized, *i. e.* $\Delta s = \Delta\tilde{s}$, we shall get

$$\frac{\nu_0}{\nu} = \frac{df_+}{ds} \tag{5.8}$$

(comp. the upper part of fig. 2). Similarly, when Γ is receiving with a frequency $\tilde{\nu}$ a signal emitted by $\tilde{\Gamma}$ with the frequency $\tilde{\nu}_0$, one obtains

$$\frac{\tilde{\nu}}{\tilde{\nu}_0} = \frac{df_-}{ds}, \tag{5.9}$$

where the subscripts + and − indicate that the projection has been performed in each case by a different family of null geodesics. These formulæ are just other forms, convenient for our purpose, of the well-known general formula for the frequency shift (cf. [*22*], [*23*], [*3*], [*24*], [*25*]).

When Γ and $\tilde{\Gamma}$ are two timelike geodesics and $\tilde{\Gamma}$ is characterized [modulo $0(\tilde{\rho}^2)$] by the deviation vector field k^α on Γ and by the value of $\tilde{\rho}$

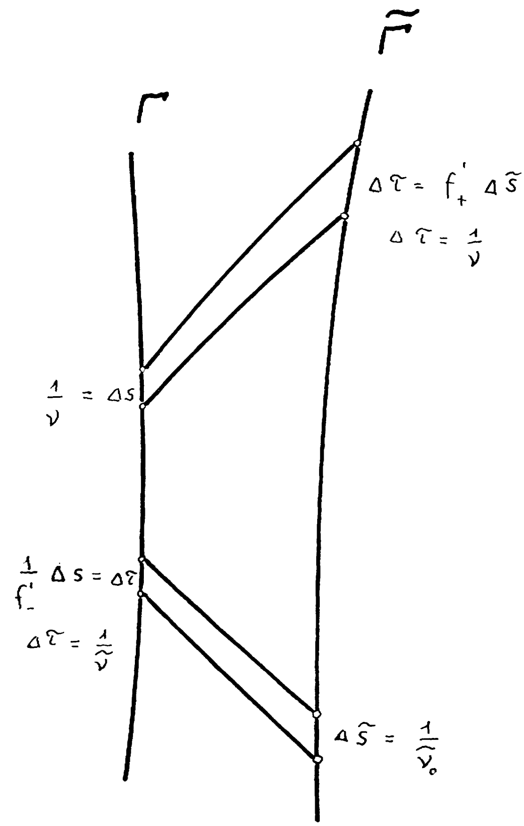

Fig. 2.

introduced by the initial conditions, we have [cf. Prop. 2.2; (2.15); (5.4)]:

$$f_{\pm}(s, \tilde{\rho}) = s + \kappa_{\pm}(s)\tilde{\rho} + 0(\tilde{\rho}^2) \tag{5.10}$$

with κ given by (5.5). Neglecting terms of the order $0(\tilde{\rho}^2)$, we obtain

$$\frac{\nu_0}{\nu} = 1 + \tilde{\rho}\frac{d\kappa_+}{ds} = 1 + \frac{d}{ds}l_1(s); \tag{5.11}$$

$$\frac{\tilde{\nu}_0}{\tilde{\nu}} = \frac{1}{1 + \tilde{\rho}\dfrac{d\kappa_-}{ds}} = 1 + \frac{d}{ds}l_1(s); \tag{5.12}$$

where $l_1(s) = \sqrt{-r_\alpha(s)r^\alpha(s)}\ \tilde{\rho}$ is the spatial distance between Γ and $\tilde{\Gamma}$ modulo $0(\tilde{\rho}^2)$. Introducing $z := \dfrac{\lambda - \lambda_0}{\lambda_0}$, one can rewrite (5.11) and (5.12) in the form

$$z = \tilde{z} = \frac{d}{ds}l_1(s). \tag{5.13}$$

This formula has been derived some time ago by Ehlers [20] as a result of certain elementary considerations. Here it is a particular consequence of Prop. 2.2 and appears as the first step of a systematic, completely covariant procedure.

We can now easily obtain the law of evolution of z along Γ in the considered approximation. Differentiating (5.13) with respect to s [cf. (2.15)] one gets

$$\frac{dz}{ds} = \frac{\tilde{\rho}}{\sqrt{-r_\lambda r^\lambda}}\left[\mathrm{R}_{\alpha\beta\gamma\delta}r^\alpha u^\beta r^\gamma u^\delta - \left(g_{\alpha\beta} - \frac{r_\alpha r_\beta}{r_\sigma r^\sigma}\right)\frac{\mathrm{D}r^\alpha}{ds}\frac{\mathrm{D}r^\beta}{ds}\right].$$

Making use of definition (5.7) one can rewrite this formula as

$$\frac{dz}{ds} = \tilde{\rho}\sqrt{-r_\lambda r^\lambda}\left[\mathrm{R}_{\mu\nu\rho\sigma}e^\mu u^\nu e^\rho u^\sigma - \frac{\mathrm{D}e_\alpha}{ds}\frac{\mathrm{D}e^\alpha}{ds}\right]. \tag{5.14}$$

This relation enables one to determine the scalar curvature of space-time at the point of observation, corresponding to the bivector $e^{[\mu}u^{\nu]} = k^{[\mu}u^{\nu]}$, by receiving light signals from a satellite and measuring the rates of change of the redshift and of the direction of the telescope as well as the distance from the satellite. In this approximation it is irrelevant whether the curvature is taken at the point of observation or of emission of the signal. Eq. (5.14) is related to a result of Bertotti (7) [21].

The second telescopic vector is defined as such a second geodesic deviation vector n^α along a timelike geodesic Γ that:

i) $$\left(k^\alpha + \frac{1}{2}\tilde{\rho}n^\alpha\right)\left(k_\alpha + \frac{1}{2}\tilde{\rho}n_\alpha\right) = 0(\tilde{\rho}^2)$$

at any point of Γ;

ii) for any $\tau \in \mathrm{I}$ a geodesic λ_τ led from $\Gamma(\tau)$ in the direction of $k^\alpha + \frac{1}{2}\tilde{\rho}n^\alpha$ intersects [modulo $0(\tilde{\rho}^3)$] the geodesic $\tilde{\Gamma}$.

In effect of this definition and of (3.2), in the case of Γ parametrized by s, the second telescopic vector is determined by the eqs.

$$\begin{aligned}\frac{\mathrm{D}^2n^\alpha}{ds^2} + \mathrm{R}^\alpha{}_{\beta\gamma\delta}u^\beta n^\gamma u^\delta = (\mathrm{R}^\alpha{}_{\beta\gamma\delta;\varepsilon} + \mathrm{R}^\alpha{}_{\varepsilon\gamma\delta;\beta})u^\beta u^\gamma k^\delta k^\varepsilon + 4\mathrm{R}^\alpha{}_{\beta\gamma\delta}\frac{\mathrm{D}k^\beta}{ds}u^\gamma k^\delta \\ + 2\frac{\mathrm{D}k^\alpha}{ds}\frac{d^2}{ds^2}(k^\rho u_\rho) + u^\alpha\frac{d}{ds}\left[\frac{\mathrm{D}k^\rho}{ds}\frac{\mathrm{D}k_\rho}{ds} + u_\rho\frac{\mathrm{D}n^\rho}{ds} - \mathrm{R}_{\mu\nu\rho\sigma}u^\mu k^\nu u^\rho k^\sigma - 2\left(\frac{d}{ds}k^\rho u_\rho\right)^2\right];\end{aligned} \tag{5.15}$$

$$k^\alpha n_\alpha = 0; \tag{5.16}$$

where k^α is a given first telescopic vector along Γ.

(7) I am very much indebted to Professor Bruno Bertotti for a discussion and for making me aware about some of the references.

According to Prop. 3.1 *ii*) and 3.2, Eqs. (5.15) have a unique solution of the initial value problem, compactible with (5.16). Let w^α be the natural second geodesic deviation vector along Γ for the given natural first geodesic deviation vector $r^\alpha = (\delta^\alpha_\beta - u^\alpha u_\beta)k^\beta$ [cf. (5.6)], then from Prop. 3.1 *ii*) and 3.2: $n^\alpha = w^\alpha + 2\dfrac{\mathrm{D}r^\alpha}{ds}k^\rho u_\rho + \psi u^\alpha$. Because of (5.16):

$$\psi = -\frac{w^\lambda k_\lambda}{u_\rho k^\rho} - 2k_\alpha\frac{\mathrm{D}r^\alpha}{ds} = -\frac{w^\lambda k_\lambda}{u_\rho k^\rho} + \frac{d}{ds}(k^\rho u_\rho)^2 \tag{5.17}$$

and

$$n^\alpha = \left(\delta^\alpha_\beta - \frac{u^\alpha k_\beta}{u_\rho k^\rho}\right)w^\beta + 2k^\lambda u_\lambda\frac{\mathrm{D}k^\alpha}{ds}. \tag{5.18}$$

These quantities can easily be expressed in terms of the first and the second natural geodesic deviations: r^α (now $= r^\alpha_\perp$) and w^α. We have

$$\psi_\pm = \mp\frac{r^\lambda w_\lambda}{\sqrt{-r_\mu r^\mu}} - \frac{d}{ds}(r^\lambda r_\lambda) - u_\lambda w^\lambda, \tag{5.19}$$

where the subscripts + and − should correspondingly refer to the advanced ($k^\alpha u_\alpha > 0$) and the retarded ($k^\alpha u_\alpha < 0$) first telescopic vector. So

$$n^\alpha_\pm = w^\alpha + 2\kappa_\pm\frac{\mathrm{D}r^\alpha}{ds} + \psi_\pm u^\alpha. \tag{5.20}$$

The affine parameter distance between Γ and $\tilde{\Gamma}$ is in this approximation not any more equal to the corresponding spatial separation distance [cf. (4.5)]

$$l_2(s) := \tilde{\rho}\sqrt{-r_\alpha r^\alpha} - \frac{1}{2}\tilde{\rho}^2\frac{r_\alpha w^\alpha}{\sqrt{-r_\lambda r^\lambda}} = l_1(s) - \frac{1}{2}\tilde{\rho}^2 e_\alpha w^\alpha, \tag{5.21}$$

but to

$$u_\alpha\left(k^\alpha_\pm + \frac{1}{2}\tilde{\rho}n^\alpha_\pm\right)\tilde{\rho} = \pm\, l_2(s) - \frac{1}{2}\tilde{\rho}^2\frac{d}{ds}(r_\alpha r^\alpha), \tag{5.22}$$

i. e. there appears a correction describing the recession of the second body between the times of emission and reception of the light signal.

To find the redshift in the approximation considered now, we must complete the expression (5.10) and write it, due to Prop. 3.2 and to (3.8), as

$$f_\pm = s + \kappa_\pm(s)\tilde{\rho} + \frac{1}{2}\psi_\pm(s)\tilde{\rho}^2 + 0(\tilde{\rho}^3). \tag{5.23}$$

Thus, neglecting terms of the order $0(\tilde{\rho}^3)$, we have according to (5.8):

$$\frac{v_0}{v} = 1 + \frac{d}{ds}l_2(s) + \frac{1}{4}\frac{d^2 l_1^2(s)}{ds^2}, \tag{5.24}$$

where the last term has been obtained by making use of (3.14) which due to (2.15) may be also written as

$$u_\alpha w^\alpha = -\frac{1}{2}\frac{d}{ds}(r_\alpha r^\alpha) + \text{const.} \tag{5.25}$$

Similarly from (5.9)

$$\frac{\tilde{\nu}}{\tilde{\nu}_0} = 1 - \frac{d}{ds} l_2(s) + \frac{1}{4}\frac{d^2 l_1^2(s)}{ds^2} \tag{5.26}$$

or, in the approximation considered,

$$\frac{\tilde{\nu}_0}{\tilde{\nu}} = 1 + \frac{d}{ds} l_2(s) - \frac{1}{4}\frac{d^2 l_1^2(s)}{ds^2} + \left(\frac{dl_1(s)}{ds}\right)^2. \tag{5.27}$$

Now, one could without any difficulty write down the formulæ corresponding to (5.13) and (5.14) in the approximation considered. There are, however, some other relations which enable one to exhibit the second order effects alone and not on the background of the first order terms.

Let us consider the situation shown on fig. 3, where the observer Γ sends

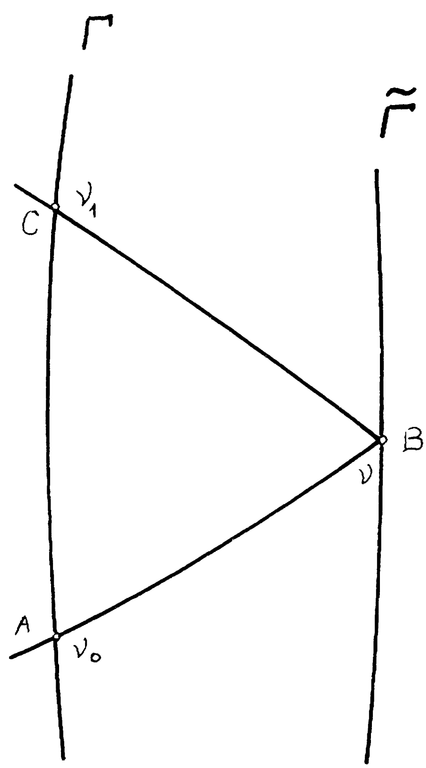

Fig. 3.

from a point A a light signal with a frequency ν_0 which is received by $\tilde{\Gamma}$ with the frequency ν and instantaneously reflected back. The reflected signal returns to Γ at C with the frequency equal to ν_1. Denoting by s_A and s_C the corresponding proper times along Γ, we have

$$\frac{\nu_0}{\nu} = 1 + \frac{d}{ds} l_2(s_A) + \frac{1}{4}\frac{d^2}{ds^2} l_1^2(s_A); \tag{5.28}$$

$$\frac{\nu_1}{\nu} = 1 - \frac{d}{ds} l_2(s_C) + \frac{1}{4}\frac{d^2}{ds^2} l_1^2(s_C). \tag{5.29}$$

Since $s_C - s_A = 2l_1(s_A) + 0(\tilde{\rho}^2)$, eliminating ν from (5.28) and (5.29) we get

$$\begin{aligned}\frac{\nu_1}{\nu_0} &= 1 - 2l_1(s_A)\frac{d^2}{ds^2} l_1(s_A) - \left(\frac{dl_1(s_A)}{ds}\right)^2 + \frac{1}{2}\frac{d^2}{ds^2} l_1^2(s_A) + 0(\tilde{\rho}^3)\\ &= 1 - l_1(s_A)\frac{d^2}{ds^2} l_1(s_A).\end{aligned}$$

Thus

$$\frac{\nu_0 - \nu_1}{\nu_0} = l_1^2\left(R_{\mu\nu\rho\sigma}e^\mu u^\nu e^\rho u^\sigma - \frac{De_\alpha}{ds}\frac{De^\alpha}{ds}\right), \tag{5.30}$$

where the r. h. side can be evaluated for s_A or s_C as well. Exactly the same formula has been derived in a different way by Bertotti *in* [9].

If we replace (5.29), because of (5.27), by

$$\frac{\nu}{\nu_1} = 1 + \frac{d}{ds} l_2(s_C) - \frac{1}{4}\frac{d^2}{ds^2} l_1^2(s_C) + \left(\frac{d}{ds} l_1(s_C)\right)^2,$$

we obtain from it and from (5.28) a new relation which also contains only the second order effect

$$\frac{\nu^2 - \nu_0\nu_1}{\nu\nu_1} = l_1^2\left(R_{\mu\nu\rho\sigma}e^\mu u^\nu e^\rho u^\sigma - \frac{De_\alpha}{ds}\frac{De^\alpha}{ds}\right). \tag{5.31}$$

The concept of telescopic vectors implies thus a number of relations between the curvature of space-time and the frequencies of light signals exchanged between two freely falling observers. These relations can be taken as a basis for idealized thought experiments meant to detect the curvature of space-time. From a realistic point of view, however, these effects seem to be beyond the reach of contemporary experimental technics. But still they seem to look more promising than effects of the mechanical type.

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