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The Legendre transformation

by

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RÉSUMÉ. — On donne une définition géométrique générale de la transformation de Legendre, suivie par des exemples dans le domaine de mécanique des particules et de thermostatique. Cette définition est basée sur les notions de la géométrie symplectique exposée brièvement dans les premières sections servant d'introduction.

DEFINITIONS OF SYMBOLS

TM	tangent bundle of a manifold M,
$\tau_M : TM \to M$	tangent bundle projection,
T_aM	tangent space at $a \in M$,
T^*M	cotangent bundle of M,
$\pi_M : T^*M \to M$	cotangent bundle projection,
ϑ_M	canonical 1-form on T^*M,
$\omega_M = d\vartheta_M$	canonical 2-form on T^*M,
$\langle v, p \rangle$	evaluation of a covector p on a vector v,
$\langle v, \mu \rangle$	evaluation of a form μ on a vector v,
d	exterior differential of forms,
$\wedge$	exterior product of vectors, covectors or forms,
Φ_M	exterior algebra of forms on M,
$\alpha^*\mu$	pullback of a form μ by a mapping α.

A general geometric definition of the Legendre transformation is given and illustrated by examples from particle dynamics and thermostatics. The definition is based on concepts of symplectic geometry reviewed in the early sections which serve as an introduction.

1. LAGRANGIAN SUBMANIFOLDS AND SYMPLECTIC DIFFEOMORPHISMS

Let P be a differential manifold. The tangent bundle of P is denoted by TP and $\tau_P : TP \to P$ is the tangent bundle projection. Let ω be a 2-form on P. The form ω is called a *symplectic form* if $d\omega = 0$ and if $\langle u \wedge w, \omega \rangle = 0$ for each $u \in TP$ such that $\tau_P(u) = \tau_P(w)$ implies $w = 0$. If ω is a symplectic form then (P, ω) is called a *symplectic manifold.*

DEFINITION 1.1. — Let (P, ω) be a symplectic manifold. A submanifold N of P such that $\omega \mid N = 0$ and $\dim P = 2 \dim N$ is called a *Lagrangian submanifold* of (P, ω) [*11*].

DEFINITION 1.2. — Let (P_1, ω_1) and (P_2, ω_2) be symplectic manifolds. A diffeomorphism $\varphi : P_1 \to P_2$ is called a *symplectic diffeomorphism* of (P_1, ω_1) onto (P_2, ω_2) if $\varphi^*\omega_2 = \omega_1$.

Let (P_1, ω_1) and (P_2, ω_2) be symplectic manifolds and let pr_1 and pr_2 denote the canonical projections of $P_2 \times P_1$ onto P_1 and P_2 respectively. The 2-form $\omega_2 \ominus \omega_1 = pr_2{}^*\omega_2 - pr_1{}^*\omega_1$ is clearly a symplectic form on $P_2 \times P_1$.

PROPOSITION 1.1. — *The graph of a symplectic diffeomorphism φ of (P_1, ω_1) onto (P_2, ω_2) is a Lagrangian submanifold of $(P_2 \times P_1, \omega_2 \ominus \omega_1)$.*

Proof. — The graph of $\varphi : P_1 \to P_2$ is the image of $(\varphi, \mathrm{I}d) : P_1 \to P_2 \times P_1$ and $(\varphi, \mathrm{I}d)^*(\omega_2 \ominus \omega_1) = \varphi^*\omega_2 - \omega_1 = 0$. Hence $(\omega_2 \ominus \omega_1) \mid \text{graph } \varphi = 0$. Also $\dim (P_2 \times P_1) = 2 \dim (\text{graph } \varphi)$. Hence graph φ is a Lagrangian submanifold of $(P_2 \times P_1, \omega_2 \ominus \omega_1)$.

The converse is also true. If the graph of a diffeomorphism $\varphi : P_1 \to P_2$ is a Lagrangian submanifold of $(P_2 \times P_1, \omega_2 \ominus \omega_1)$ then φ is a symplectic diffeomorphism of (P_1, ω_1) onto (P_2, ω_2).

2. LOCAL EXPRESSIONS

Let (P, ω) be a symplectic manifold and let (x^i, y_j), $1 \le i, j \le n$ be local coordinates of P such that $\omega = \Sigma_i dy_i \wedge dx^i$. Coordinates (x^i, y_j) are called *canonical coordinates* of (P, ω) [*1*]. Existence of canonical coordinates is guaranteed by Darboux theorem. A submanifold N of P of dimension n represented locally by $x^i = \xi^i(u^k)$, $y_j = \eta_j(u^k)$, $1 \le i, j, k \le n$ is a Lagrangian submanifold of (P, ω) if and only if

$$\omega \mid N = \Sigma_{i,j,k} \frac{\partial \eta_i}{\partial u^j} \frac{\partial \xi^i}{\partial u^k} du^j \wedge du^k = 0.$$

This condition is equivalent to $[u^i, u^j] = 0$, $1 \leq i, j \leq n$, where

$$[u^i, u^j] = \Sigma_k \left[\frac{\partial \xi^k}{\partial u^i} \frac{\partial \eta_k}{\partial u^j} - \frac{\partial \xi^k}{\partial u^j} \frac{\partial \eta_k}{\partial u^i} \right]$$

are the *Lagrange brackets* [2].

Let (x^i, y_j), $1 \leq i, j \leq n$ and (x'^i, y'_i), $1 \leq i, j \leq n$ be canonical coordinates of symplectic manifolds (P_1, ω_1) and (P_2, ω_2) respectively. The two sets of coordinates are combined into a set (x'^i, y'_j, x^k, y_l), $1 \leq i, j, k, l \leq n$ of local coordinates of $P_2 \times P_1$. Then $\omega_2 \ominus \omega_1 = \Sigma_i (dy'_i \wedge dx'^i - dy_i \wedge dx^i)$. A diffeomorphism $\varphi : P_1 \to P_2$ represented locally by $x'^i = \psi^i(x^k, y_l)$, $y'_j = \chi_j(x^k, y_l)$ is a symplectic diffeomorphism of (P_1, ω_1) onto (P_2, ω_2) if and only if

$$\omega_2 \ominus \omega_1 \mid \text{graph } \varphi = \Sigma_{i,j,k} \left[\frac{\partial \chi_i}{\partial x^j} \frac{\partial \psi^i}{\partial x^k} dx^j \wedge dx^k + \frac{\partial \chi_i}{\partial x^j} \frac{\partial \psi^i}{\partial y_k} dx^j \wedge dy_k \right.$$
$$\left. + \frac{\partial \chi_i}{\partial y_j} \frac{\partial \psi^i}{\partial x^k} dy_j \wedge dx^k + \frac{\partial \chi_i}{\partial y_j} \frac{\partial \psi^i}{\partial y_k} dy_j \wedge dy_k \right] - \Sigma_i dy_i \wedge dx^i = 0.$$

This condition is equivalent to $[x^i, x^j] = 0$, $[x^i, y_j] = \delta^i{}_j$, $[y_i, y_j] = 0$, $1 \leq i, j \leq n$.

3. SPECIAL SYMPLECTIC MANIFOLDS AND GENERATING FUNCTIONS [4]

Let Q be a manifold, let TQ denote the tangent bundle of Q and $\tau_Q : TQ \to Q$ the tangent bundle projection. The cotangent bundle of Q is denoted by T*Q and $\pi_Q : T^*Q \to Q$ is the cotangent bundle projection. The canonical 1-form ϑ_Q on T*Q is defined by

$$\langle u, \vartheta_Q \rangle = \langle T\pi_Q(u), \tau_{T^*Q}(u) \rangle \qquad \text{for each} \qquad u \in TT^*Q.$$

The canonical 2-form $\omega_Q = d\vartheta_Q$ is known to be a symplectic form. Hence (T^*Q, ω_Q) is a symplectic manifold.

Let F be a differentiable function on the manifold Q. The 1-form dF is a section $dF : Q \to T^*Q$ of the cotangent bundle. The image N of dF is a submanifold of T*Q, the mapping $\rho = \pi_Q \mid N : N \to Q$ is a diffeomorphism and $\vartheta_Q \mid N = \rho^* dF$. Hence $\omega_Q \mid N = 0$ and N is a Lagrangian submanifold of (T^*Q, ω_Q).

The above construction of Lagrangian submanifolds is generalized in the following proposition.

PROPOSITION 3.1. — *Let* K *be a submanifold of* Q *and* F *a function on* K. *The set*

$$N = \{ p \in T^*Q;\ \pi_Q(p) \in K \text{ and } \langle u, p \rangle = \langle u, dF \rangle \text{ for each } u \in TK \subset TQ \text{ such that } \tau_Q(u) = \pi_Q(p) \}$$

is a Lagrangian submanifold of (T^*Q, ω_Q).

Proof. — Using local coordinates it is easily shown that N is a submanifold of T^*Q of dimension equal to dim Q. The submanifold K is the image of N by π_Q. Let $\rho : N \to K$ be the mapping defined by the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\text{injection}} & T^*Q \\ \rho \downarrow & & \downarrow \pi_Q \\ K & \xrightarrow{\text{injection}} & Q \end{array}$$

Then $\langle u, \rho^* dF \rangle = \langle T\rho(u), dF \rangle = \langle T\rho(u), \tau_{T^*Q}(u) \rangle = \langle u, \vartheta_Q \rangle$ for each vector $u \in TN \subset TT^*Q$. Hence $\vartheta_Q \mid N = \rho^* dF$, $\omega_Q \mid N = 0$ and N is a Lagrangian submanifold of (T^*Q, ω_Q).

DEFINITION 3.1. — The function F in Proposition 3.1 is called a *generating function* of the Lagrangian submanifold N. The Lagrangian submanifold N is said to be generated by F.

There is a canonical submersion κ of $\pi_Q^{-1}(K)$ onto T^*K and the Lagrangian submanifold N is given by $N = \kappa^{-1}(dF(K))$. The Lagrangian submanifold N can also be characterized as the maximal submanifold N of T^*Q such that $\pi_Q(N) = K$ and $\vartheta_Q \mid N = \rho^* dF$, where $\rho : N \to K$ is the mapping defined in the proof of Proposition 3.1.

In many applications of symplectic geometry it is convenient to consider symplectic manifolds which are not directly cotangent bundles but are isomorphic to cotangent bundles.

DEFINITION 3.2. — Let (P, Q, π) be a differential fibration and ϑ a 1-form on P. The quadruple (P, Q, π, ϑ) is called a *special symplectic manifold* if there is a diffeomorphism $\alpha : P \to T^*Q$ such that $\pi = \pi_Q \circ \alpha$ and $\vartheta = \alpha^* \vartheta_Q$.

If the diffeomorphism α exists it is unique. If (P, Q, π, ϑ) is a special symplectic manifold then $(P, \omega) = (P, d\vartheta)$ is a symplectic manifold called the *underlying symplectic manifold* of (P, Q, π, ϑ).

If (P, Q, π, ϑ) is a special symplectic manifold, K a submanifold of Q and F a function on K then the set $N = \{ p \in P;\ \pi(p) \in K$ and $\langle u, \vartheta \rangle = \langle T\pi(u), dF \rangle$ for each $u \in TP$ such that $\tau_P(u) = p$ and $T\pi(u) \in TK \subset TD \}$ is a Lagrangian submanifold of $(P, d\vartheta)$ said to be generated with respect to (P, Q, π, ϑ) by the function F. The function F is called a generating function of N with respect to (P, Q, π, ϑ). The diffeomorphism $\alpha : P \to T^*Q$ maps the Lagrangian submanifold N onto the Lagrangian submanifold of (T^*Q, ω_Q) generated by F.

Let $(P_1, Q_1, \pi_1, \vartheta_1)$ and $(P_2, Q_2, \pi_2, \vartheta_2)$ be special symplectic manifolds and let $\vartheta_2 \ominus \vartheta_1$ denote the 1-form $pr_2^* \vartheta_2 - pr_1^* \vartheta_1$, where pr_1 and pr_2 are the canonical projections of $P_2 \times P_1$ onto P_1 and P_2 respectively.

PROPOSITION 3.2. — *The quadruple* $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \vartheta_2 \ominus \vartheta_1)$ *is a special symplectic manifold.*

Proof. — Let $\alpha_1 : P_1 \to T^*Q_1$ and $\alpha_2 : P_2 \to T^*Q_2$ be diffeomorphisms such that $\pi_1 = \pi_{Q_1} \circ \alpha_1$, $\pi_2 = \pi_{Q_2} \circ \alpha_2$, $\vartheta_1 = \alpha_1{}^*\vartheta_{Q_1}$ and $\vartheta_2 = \alpha_2{}^*\vartheta_{Q_2}$. Then the mapping

$$\alpha_{21} : P_2 \times P_1 \to T^*(Q_2 \times Q_1) = T^*Q_2 \times T^*Q_1 : (p_2, p_1) \mapsto (\alpha_2(p_2), -\alpha_1(p_1))$$

is a diffeomorphism such that

$$\pi_2 \times \pi_1 = (\pi_{Q_2} \times \pi_{Q_1}) \circ \alpha_{21} \qquad \text{and} \qquad \vartheta_2 \ominus \vartheta_1 = \alpha_{21}{}^*(\vartheta_{Q_2} \oplus \vartheta_{Q_1}).$$

The identification $T^*(Q_2 \times Q_1) = T^*Q_2 \times T^*Q_1$ implies the identification of $\vartheta_{Q_2} \oplus \vartheta_{Q_1} = pr_2{}^*\vartheta_{Q2} + pr_1{}^*\vartheta_{Q_1}$ with $\vartheta_{Q_2 \times Q_1}$. Hence $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \vartheta_2 \ominus \vartheta_1)$ is a special symplectic manifold.

If (P_1, ω_1) and (P_2, ω_2) are underlying symplectic manifolds of $(P_1, Q_1, \pi_1, \vartheta_1)$ and $(P_2, Q_2, \pi_2, \vartheta_2)$ then $(P_2 \times P_1, \omega_2 \ominus \omega_1)$ is the underlying symplectic manifold of $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \vartheta_2 \ominus \vartheta_1)$. Let φ be a symplectic diffeomorphism of (P_1, ω_1) onto (P_2, ω_2).

DEFINITION 3.3. — If the graph of the diffeomorphism $\varphi : P_1 \to P_2$ is generated with respect to the special symplectic structure $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \vartheta_2 \ominus \vartheta_1)$ by a function G on a submanifold M of $Q_2 \times Q_1$ then φ is said to be *generated* with respect to $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \vartheta_2 \ominus \vartheta_1)$ by the function G and G is called a *generating function* of φ with respect to $(P_2 \times P_1, Q_2 \times Q_1, \pi_2 \times \pi_1, \vartheta_2 \ominus \vartheta_1)$.

If N_1 is a Lagrangian submanifold of (P_1, ω_1) and φ is a symplectic diffeomorphism of (P_1, ω_1) onto (P_2, ω_2) then $N_2 = \varphi(N_1)$ is a Lagrangian submanifold of (P_2, ω_2). Let (P_1, ω_1) and (P_2, ω_2) be underlying symplectic manifolds of special symplectic manifolds $(P_1, Q_1, \pi_1, \vartheta_1)$ and $(P_2, Q_2, \pi_2, \vartheta_2)$ respectively and let N_1, φ and N_2 be generated by functions F_1, G and F_2 defined on submanifolds $K_1 \subset Q_1$, $M \subset Q_2 \times Q_1$ and $K_2 \subset Q_2$ respectively.

PROPOSITION 3.3. — *Let* K_{21} *denote the image of* N_1 *by*

$$(\pi_2 \times \pi_1) \circ (\varphi, Id) : P_1 \to Q_2 \times Q_1.$$

Then $K_{21} = \{ (q_2, q_1) \in Q_2 \times Q_1;\ q_1 \in K_1,\ (q_2, q_1) \in M$ *and* $\langle (v_2, v_1), dG \rangle + \langle v_1, dF_1 \rangle = 0$ *for each* $v_1 \in T_{q_1}K_1$ *such that* $(v_2, v_1) \in T_{(q_2,q_1)}M \subset T_{q_2}Q_2 \times T_{q_1}Q_1$ *and* $v_2 = 0 \}$.

Proof (for $P_1 = T^*Q_1$ and $P_2 = T^*Q_2$). — If $(q_2, q_1) \in K_{21}$ then $q_1 \in K_1$, $(q_2, q_1) \in M$ and there is a covector $p_1 \in N_1$ such that $\pi_1(p_1) = q_1$ and $\pi_2(\varphi(p_1)) = q_2$. It follows that $\langle v_1, p_1 \rangle = \langle v_1, dF_1 \rangle$ and

$$-\langle v_1, p_1 \rangle = \langle (v_2, v_1), dG \rangle$$

and finally $\langle (v_2, v_1), dG \rangle + \langle v_1, dF_1 \rangle = 0$ for each $v_1 \in T_{q_1}K_1$ such that $(v_2, v_1) \in T_{(q_2,q_1)}M$ and $v_2 = 0$. Conversely if $q_1 \in K_1$ and $(q_2, q_1) \in M$ then there are covectors $p_1' \in P_1$, $p_1'' \in P_1$ and $p_2'' \in P_2$ such that

$$\pi_1(p_1') = \pi_1(p_1'') = q_1,\ \pi_2(p_2'') = q_2,\ p_1' \in N_1 \qquad \text{and} \qquad p_2'' = \varphi(p_1'').$$

Consequently

$$\langle u_1, p_1' \rangle = \langle u_1, dF_1 \rangle \quad \text{for each} \quad u_1 \in T_{q_1}K_1$$

and

$$\langle w_2, p_2'' \rangle - \langle w_1, p_1'' \rangle = \langle (w_2, w_1), dG \rangle \quad \text{if} \quad (w_2, w_1) \in T_{(q_2,q_1)}M.$$

If in addition $\langle (v_2, v_1), dG \rangle + \langle v_1, dF_1 \rangle = 0$ for each $v_1 \in T_{q_1}K_1$ such that $(v_2, v_1) \in T_{(q_2,q_1)}M$ and $v_2 = 0$ then $\langle v_1, p_1' - p_1'' \rangle = 0$ for each v_1 satisfying the same conditions. It follows from a simple algebraic argument that there are covectors $p_1 \in P_1$ and $p_2 \in P_2$ such that $\pi_1(p_1) = q_1, \pi_2(p_2) = q_2$, $\langle u_1, p_1 \rangle = \langle u_1, p_1' \rangle = \langle u_1, dF_1 \rangle$ for each $u_1 \in T_{q_1}K_1$ and

$$\langle w_2, p_2 \rangle - \langle w_1, p_1 \rangle = \langle w_2, p_2'' \rangle - \langle w_1, p_1'' \rangle = \langle (w_2, w_1), dG \rangle$$

for each $(w_2, w_1) \in T_{(q_2,q_1)}M$. Hence $p_1 \in N_1$, $p_2 = \varphi(p_1)$ and $(q_2, q_1) \in K_{21}$.

The following proposition is an immediate consequence of the definition of K_{21}.

PROPOSITION 3.4. — *The submanifold* K_2 *is the set*

$$\{ q_2 \in Q_2 \; ; \; \exists_{q_1 \in K_1} (q_2, q_1) \in K_{21} \}.$$

PROPOSITION 3.5. — *If* $(q_2, q_1) \in K_{21}$, $v_1 \in T_{q_1}K_1$, $v_2 \in T_{q_2}K_2$ *and* $(v_2, v_1) \in T_{(q_2,q_1)}M$ *then* $\langle v_2, dF \rangle = \langle (v_2, v_1), dG \rangle + \langle v_1, dF_1 \rangle$.

Proof (for $P_1 = T^*Q_1$ and $P_2 = T^*Q_2$). — If $(q_2, q_1) \in K_{21}$ then there are covectors $p_1 \in P_1$ and $p_2 \in P_2$ such that $\pi_1(p_1) = q_1$, $\pi_2(p_2) = q_2$, $p_1 \in N_1$, $p_2 \in N_2$ and $p_2 = \varphi(p_1)$. It follows that $\langle u_1, p_1 \rangle = \langle u_1, dF_1 \rangle$ for each $u_1 \in T_{q_1}K_1$, $\langle u_2, p_2 \rangle = \langle u_2, dF_2 \rangle$ for each $u_2 \in T_{q_2}K_2$ and $\langle w_2, p_2 \rangle - \langle w_1, p_1 \rangle = \langle (w_2, w_1), dG \rangle$ for each $(w_2, w_1) \in T_{(q_2,q_1)}M$. Hence $\langle v_2, dF_2 \rangle = \langle (v_2, v_1), dG \rangle + \langle v_1, dF_1 \rangle$ for each $(v_2, v_1) \in T_{(q_2,q_1)}M$ such that $v_1 \in T_{q_1}K_1$ and $v_2 \in T_{q_2}K_2$.

Let G_{q_2} denote the function defined by $G_{q_2}(q_1) = G(q_2, q_1)$. Then for each $q_2 \in Q_2$ the function $G_{q_2} + F_1$ is defined on the set $\{ q_1 \in K_1 ; (q_2, q_1) \in M \}$. The following two propositions are simplified versions of Propositions 3.3 and 3.4 valid under the additional assumption that for each $q_2 \in Q_2$ the set $\{ q_1 \in K_1 ; (q_2, q_1) \in M \}$ is a submanifold of Q_1.

PROPOSITION 3.3′. — *The set* K_{21} *is the subset of* $Q_2 \times Q_1$ *such that* $(q_2, q_1) \in K_{21}$ *if and only if* q_1 *is a critical point of* $G_{q_2} + F_1$.

PROPOSITION 3.4′. — *The set* K_2 *is the subset of* Q_2 *such that* $q_2 \in K_2$ *if and only if* $G_{q_2} + F_1$ *has critical points.*

For each $q_2 \in K_2$ the set of critical points of $G_{q_2} + F_1$ is the set $\{ q_1 \in K_1 ; (q_2, q_1) \in K_{21} \}$. The following proposition holds if for each $q_2 \in K_2$ the set of critical points of $G_{q_2} + F_1$ is a connected submanifold of Q_1.

PROPOSITION 3.5′. — *The function* F_2 *defined on* K_2 *by setting* $F_2(q_2)$ *equal to the (unique) critical value of* $G_{q_2} + F_1$ *is a generating function of* N_2.

We write $F_2(q_2) = \mathrm{Stat}_{q_1}(G(q_2, q_1) + F_1(q_1))$ meaning that $F_2(q_2)$ is equal to the function $G_{q_2} + F_1$ evaluated at a point q_1 at which it is stationary, that is at a critical point, and that $F_2(q_2)$ is not defined if no critical points of $G_{q2} + F_1$ exist.

4. LOCAL EXPRESSIONS

Let (x^i), $1 \leq i \leq n$ be local coordinates of a manifold Q_1. We use coordinates (x^i, y_j), $1 \leq i, j \leq n$ of $P_1 = T^*Q_1$ such that $\vartheta_1 = \vartheta_{Q_1} = \Sigma_i y_i dx^i$. Let a Lagrangian submanifold N_1 of (P_1, ω_1) be generated by a function F_1 defined on a submanifold K_1 of Q_1. If the submanifold K_1 is described locally by equations $U^\kappa(x^i) = 0$, $1 \leq \kappa \leq k$ and if $\overline{F}_1(x^i)$ is the local expression of an arbitrary (local) continuation $\overline{F_1}$ of the function F_1 to Q_1 then the Lagrangian submanifold N_1 is described by the equation

$$\Sigma_i y_i dx^i = d(\overline{F}_1(x^i) + \Sigma_\kappa \lambda_\kappa U^\kappa(x^i))$$
$$= \Sigma_i \left(\frac{\partial \overline{F}_1}{\partial x^i} + \Sigma_\kappa \lambda_\kappa \frac{\partial U^\kappa}{\partial x^i} \right) dx^i + \Sigma_\kappa U^\kappa(x^i) d\lambda_\kappa$$

equivalent to the system

$$y_i = \frac{\partial \overline{F}_1}{\partial x^i} + \Sigma_\kappa \lambda_\kappa \frac{\partial U^\kappa}{\partial x^i}, \ 1 \leq i \leq n$$
$$U^\kappa(x^i) = 0, \ 1 \leq \kappa \leq k.$$

We note that $F_1(x^i) = \mathrm{Stat}_{(\lambda_\kappa)} [\overline{F}_1(x^i) + \Sigma_\kappa \lambda_\kappa U^\kappa(x^i)]$ is the local expression of F_1 for values of coordinates (x^i), $1 \leq i \leq n$ satisfying $U^\kappa(x^i) = 0$, $1 \leq \kappa \leq k$. In the special case of $K_1 = Q_1$ we have the equation $\Sigma_i y_i dx^i = dF_1(x^i)$ equivalent to $y_i = \dfrac{\partial F_1}{\partial x^i}$, $1 \geq i \geq n$.

Let (x'^i), $1 \leq i \leq n$ be local coordinates of a manifold Q_2 and let (x'^i, y'_j), $1 \leq i, j \leq n$ be coordinates of $P_2 = T^*Q_2$ such that $\vartheta_2 = \vartheta_{Q_2} = \Sigma_i y'_i dx'^i$. We use coordinates (x'^i, x^j), $1 \leq i, j \leq n$ for $Q_2 \times Q_1$ and coordinates (x'^i, y'_j, x^k, y_l), $1 \leq i, j, k, l \leq n$ for $P_2 \times P_1$. The local expression of the form $\vartheta_2 \ominus \vartheta_1$ is $\vartheta_2 \ominus \vartheta_1 = \Sigma_i(y'_i dx'^i - y_i dx^i)$. Let a symplectic diffeomorphism φ of (P_1, ω_1) onto (P_2, ω_2) be generated by a function G defined on a submanifold M of $Q_2 \times Q_1$. Let the submanifold M be described locally by equations $W^\mu(x'^i, x^j) = 0$, $1 \leq \mu \leq m$ and let $\overline{G}(x'^i, x^j)$ be the local expression of an arbitrary continuation $\overline{G}$ of the function G to $Q_2 \times Q_1$. An implicit description of the diffeomorphism φ is given by the equation

$\Sigma_i(y'_i dx'^i - y_i dx^i) = d(\overline{G}(x'^i, x^j) + \Sigma_\mu v_\mu W^\mu(x'^i, x^j))$ equivalent to the system

$$y'_i = \frac{\partial \overline{G}}{\partial x'^i} + \Sigma_\mu v_\mu \frac{\partial W^\mu}{\partial x'^i}, \ 1 \leq i \leq n$$

$$y_i = -\frac{\partial \overline{G}}{\partial x^i} + \Sigma_\mu v_\mu \frac{\partial W^\mu}{\partial x^i}, \ 1 \leq i \leq n$$

$$W^\mu(x'^i, x^j) = 0, \ 1 \leq \mu \leq m.$$

The local expression of G for values of coordinates (x'^i, x^j), $1 \leq i, j \leq n$ satisfying $W^\mu(x'^i, x^j) = 0$, $1 \leq \mu \leq m$ is obtained from

$$G(x'^i, x^j) = \mathrm{Stat}_{(v_\mu)}\,[\overline{G}(x'^i, x^j) + \Sigma_\mu v_\mu W^\mu(x'^i, x^j)].$$

If $M = Q_2 \times Q_1$ then we have the equation $\Sigma_i(y'_i dx'^i - y_i dx^i) = dG(x'^i, x^j)$ equivalent to $y'_i = \dfrac{\partial G}{\partial x'^i}$, $y_i = -\dfrac{\partial G}{\partial x^i}$, $1 \leq i \leq n$.

If the Lagrangian submanifold $N_2 = \varphi(N_1)$ is generated by a generating function then N_2 is described by the equation

$$\Sigma_i y'_i dx'^i = d\big(\overline{G}(x'^i, x^j) + \overline{F}_1(x^j) + \Sigma_\mu v_\mu W^\mu(x'^i, x^j) + \Sigma_\kappa \lambda_\kappa U^\kappa(x^j)\big).$$

Hence the local expression of a generating function F_2 of N_2 is

$$F_2(x'^i) = \mathrm{Stat}_{(x^j, \lambda_\kappa, v_\mu)}[\overline{G}(x'^i, x^j) + \overline{F}_1(x^j) + \Sigma_\mu v_\mu W^\mu(x'^i, x^j) + \Sigma_\kappa \lambda_\kappa U^\kappa(x^j)].$$

If $K_1 = Q_1$ and $M = Q_2 \times Q_1$ then $F_2(x'^i) = \mathrm{Stat}_{(x^j)}[G(x'^i, x^j) + F_1(x^j)]$.

The following simple example illustrates composition of generating functions. Let Q_1 and Q_2 be manifolds of dimension 2. The submanifold N_1 of P_1 described locally by equations $y_1 = 2x^1(1 - y_2)$, $x^2 = (x^1)^2$ is a Lagrangian submanifold of (P_1, ω_1). The mapping $\varphi : P_1 \to P_2$ described locally by equations $x'^1 = x^1$, $x'^2 = -y_2$, $y'_1 = y_1$, $y'_2 = x^2$ is a symplectic diffeomorphism of (P_1, ω_1) onto (P_2, ω_2). The Lagrangian submanifold N_1 is generated by a function F_1 on a submanifold K_1 of Q_1. The submanifold K_1 is described by $U(x^1, x^2) = x^2 - (x^1)^2 = 0$ and $\overline{F}_1(x^1, x^2) = (x^1)^2$ is the local expression of a continuation of F_1 to Q_1. The symplectic diffeomorphism φ is generated by a function G defined on a submanifold M of $Q_2 \times Q_1$. The submanifold M is described locally by

$$W(x'^1, x'^2, x^1, x^2) = x'^1 - x^1 = 0 \quad \text{and} \quad \overline{G}(x'^1, x'^2, x^1, x^2) = x'^2 x^2$$

is the local expression of a continuation of G to $Q_2 \times Q_1$. The Lagrangian submanifold N_2 is generated by a function F_2 defined on Q_2. The local expression of F_2 is $F_2(x'^1, x'^2) = (x'^1)^2(1 + x'^2)$. The relation

$$F_2(x'^1, x'^2) = \mathrm{Stat}_{(x^1, x^2, v, \lambda)}[\overline{G}(x'^1, x'^2, x^1, x^2) + \overline{F}_1(x^1, x^2) + vW(x'^1, x'^2, x^1, x^2) + \lambda U(x^1, x^2)]$$

is easily verified.

5. THE LEGENDRE TRANSFORMATION

Lef (P, ω) be the underlying symplectic manifold of two special symplectic manifolds (P, Q_1, π_1, ϑ_1) and (P, Q_2, π_2, ϑ_2). Lagrangian submanifolds of (P, ω) may be generated by generating functions with respect to both special symplectic structures.

DEFINITION 5.1. — The transition from the representation of Lagrangian submanifolds of (P, ω) by generating functions with respect to (P, Q_1, π_1, ϑ_1) to the representation by generating functions with respect to (P, Q_2, π_2, ϑ_2) is called the *Legendre transformation* from (P, Q_1, π_1, ϑ_1) to (P, Q_2, π_2, ϑ_2).

Let the identity mapping of P be generated with respect to ($P \times P$, $Q_2 \times Q_1$, $\pi_2 \times \pi_1$, $\vartheta_2 \ominus \vartheta_1$) by a generating function E_{21} defined on a submanifold I_{21} of $Q_2 \times Q_1$.

DEFINITION 5.2. — The function E_{21} is called a *generating function* of the Legendre transformation from (P, Q_1, π_1, ϑ_1) to (P, Q_2, π_2, ϑ_2).

If F_1 is a generating function of a Lagrangian submanifold N of (P, ω) with respect to (P, Q_1, π_1, ϑ_1) and if the special conditions assumed at the end of Section 3 hold then the Legendre transformation leads to a function F_2 satisfying $F_2(q_2) = \mathrm{Stat}_{q_1} [E_{21}(q_2, q_1) + F_1(q_1)]$.

Physicists use the term Legendre transformation also in a different sense. Let $\Delta : P \to P \times P$ denote the diagonal mapping. If the image K_{21} of N by the mapping $(\pi_2 \times \pi_1) \circ \Delta : P \to Q_2 \times Q_1$ is the graph of a mapping $\kappa_{21} : Q_1 \to Q_2$ then κ_{21} is called the Legendre transformation of Q_1 into Q_2 corresponding to N. We call K_{21} the *Legendre relation* and κ_{21} the *Legendre mapping* of Q_1 into Q_2 corresponding to N. The Legendre relation can be obtained from the generating functions F_1 and E_{21} following Proposition 3.3 or Proposition 3.3'.

6. THE LEGENDRE TRANSFORMATION OF PARTICLE DYNAMICS

Let Φ_P denote the graded algebra of differential forms on a manifold P and let Φ_{TP} be the graded algebra of forms on the tangent bundle TP of P. A linear mapping $a : \Phi_P \to \Phi_{TP} : \mu \mapsto a\mu$ is called a *derivation* of *degree r* of Φ_P into Φ_{TP} relative to τ_P if

degree $(a\mu)$ = degree $\mu + r$ and $a(\mu \wedge \nu) = a\mu \wedge \tau_P^* \nu + (-1)^{pr} \tau_P^* \mu \wedge a\nu$,
where p = degree μ.

An important property of derivations is that a derivation is completely characterized by its action on functions and 1-forms [3]. We define derivations i_T and d_T of Φ_P into Φ_{TP} of degrees -1 and 0 respectively [7], [8].

If f is a function on P then $i_T f = 0$ and if μ is a 1-form on P then $i_T\mu$ is a function on TP defined by $(i_T\mu)(u) = \langle u, \mu \rangle$ for each $u \in$ TP. The derivation d_T is defined by $d_T\mu = i_T d\mu + d i_T \mu$ for each $\mu \in \Phi_P$.

We summarize results derived in earlier publications [*6*], [*8*], [*9*], [*10*]. Let P be the cotangent bundle T*Q of a differential manifold Q. Let π denote the bundle projection $\pi_Q : P \to Q$, let ϑ be the canonical 1-form ϑ_Q on P and ω the canonical 2-form $\omega_Q = d\vartheta_Q$ on P. The tangent bundle TP together with the 2-form $d_T\omega$ form a symplectic manifold (TP, $d_T\omega$). The symplectic manifold (TP, $d_T\omega$) is the underlying symplectic manifold of two special symplectic manifolds (TP, P, τ, χ) and (TP, TQ, Tπ, λ), where τ is the tangent bundle projection $\tau_P : TP \to P$, χ is the 1-form $i_T\omega$ and λ is the 1-form $d_T\vartheta$.

Let Q be the configuration manifold of a particle system and let the dynamics of the system be represented by a Lagrangian submanifold D of (TP, $d_T\omega$) [*8*], [*9*], [*10*]. If D is generated by generating functions with respect to both special symplectic structures given above then the generating functions are related by Legendre transformations.

DEFINITION 6.1. — If the Lagrangian submanifold D representing the dynamics of a particle system is generated with respect to the special symplectic structure (TP, TQ, Tπ, λ) by a generating function L on a submanifold J of TQ then L is called a *Lagrangian* of the particle system and J is called the *Lagrangian constraint.*

DEFINITION 6.2. — If the Lagrangian submanifold D is generated with respect to the special symplectic structure (TP, P, τ, χ) by a function F on a submanifold K of P then the function H = – F is called a *Hamiltonian* of the particle system and K is called the *Hamiltonian constraint.*

DEFINITION 6.3. — The Legendre transformation from (TP, TQ, Tπ, λ) to (TP, P, τ, χ) is called the *Legendre transformation of particle dynamics* and the Legendre transformation from (TP, P, τ, χ) to (TP, TQ, Tπ, λ) is called the *inverse Legendre transformation of particle dynamics.*

PROPOSITION 6.1. — *The Legendre transformation of particle dynamics is generated by the function* E *defined on the Whitney sum*

$$I = T^*Q \times_Q TQ \subset P \times TQ$$

by $E(p, v) = - \langle v, p \rangle$.

Proof. — Let ρ be the mapping defined by the commutative diagram

$$\begin{array}{ccc} TP & \xrightarrow{\ \Delta\ } & TP \times TP \\ \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \tau \times T\pi} \\ I & \xrightarrow{\text{injection}} & P \times TQ, \end{array}$$

where Δ is the diagonal mapping. Then

$$(E \circ \rho)(w) = E(\tau(w), T\pi(w)) = - \langle T\pi(w), \tau(w) \rangle = - \langle w, \vartheta \rangle.$$

Hence $E \circ \rho = - i_T\vartheta$. Further

$$\Delta^*(\chi \ominus \lambda) = \chi - \lambda = i_T d\vartheta - d_T\vartheta = - d i_T\vartheta = d(E \circ \rho).$$

It follows that the diagonal of TP × TP is contained in the Lagrangian submanifold generated by E. The diagonal of TP × TP and the Lagrangian submanifold generated by E are closed submanifolds of TP × TP of the same dimension. If Q is connected then the Lagrangian submanifold generated by E is connected and hence equal to the diagonal of TP × TP. If Q is not connected then the same argument applies to each connected component of Q.

The proof of the following proposition is similar.

PROPOSITION 6.2. — *The inverse Legendre transformation of particle dynamics is generated by the function* E′ *on* $I' = TQ \times_Q T^*Q$ *defined by*

$$E'(v, p) = \langle v, p \rangle.$$

7. LOCAL EXPRESSIONS AND EXAMPLES

Let (x^i), $1 \leq i \leq n$ be local coordinates of Q and (x^i, y_j), $1 \leq i, j \leq n$ local coordinates of $P = T^*Q$ such that $\vartheta_Q = \Sigma_i y_i dx^i$. We use coordinates $(x^i, \dot{x}_j)$, $1 \leq i, j \leq n$ for TQ and coordinates $(x^i, y_j, \dot{x}^k, \dot{y}_l)$, $1 \leq i, j, k, l \leq n$ for TP. Functions $\dot{x}^i$ and $\dot{y}_j$ are defined by $\dot{x}^i = d_T x^i$ and $\dot{y}_j = d_T y_j$. Local expressions of the forms $d_T\omega$, λ and χ are $d_T\omega = \Sigma_i(d\dot{y}_i dx^i + dy_i d\dot{x}^i)$, $\lambda = \Sigma_i(\dot{y}_i dx^i + y_i d\dot{x}^i)$ and $\chi = \Sigma_i(\dot{y}_i dx^i - \dot{x}^i dy_i)$. Let $(x^i, \dot{x}^j, y_k)$, $1 \leq i, j, k \leq n$ be coordinates of I and also of I′. Then $E(x^i, \dot{x}^j, y_k) = - \Sigma_i y_i \dot{x}^i$ and $E'(x^i, \dot{x}^j, y_k) = \Sigma_i y_i \dot{x}^i$ are local expressions of functions E and E′.

EXAMPLE 7.1. — Let Q be the configuration manifold of a non-relativistic particle of mass m and let $V(x^i)$ be the local expression of the potential energy of the particle. The dynamics of the particle is represented by the Lagrangian submanifold D of $(TP, d_T\omega)$ defined locally by $y_i = m\dot{x}^i$ and $\dot{y}_j = - \frac{\partial V}{\partial x^j}$. The submanifold D can also be described by equations

$$\Sigma_i(\dot{y}_i dx^i + y_i d\dot{x}^i) = d\left(\frac{1}{2} m \Sigma_i(\dot{x}^i)^2 - V(x^i)\right)$$

or

$$\Sigma_i(\dot{y}_i dx^i - \dot{x}^i dy_i) = - d\left(\frac{1}{2m} \Sigma_i(y_i)^2 + V(x^i)\right).$$

Hence

$$L(x^i, \dot{x}^j) = \frac{1}{2} m\Sigma_j(\dot{x}^j)^2 - V(x^i) \qquad \text{and} \qquad H(x^i, y_j) = \frac{1}{2m}\Sigma_j(y_j)^2 + V(x^i)$$

are local expressions of a Lagrangian L and a Hamiltonian H. Relations

$$H(x^i, y_j) = \mathrm{Stat}_{(\dot{x}^k)}[\Sigma_j y_j \dot{x}^j - L(x^i, \dot{x}^k)]$$

and

$$L(x^i, \dot{x}^j) = \mathrm{Stat}_{(y_k)}[\Sigma_j y_j \dot{x}^j - H(x^i, y_k)]$$

are local expressions of the Legendre transformation and the inverse Legendre transformation.

The following example illustrates a situation slightly more general than that described in Section 6.

EXAMPLE 7.2. — Let Q be the flat space-time of special relativity, let (x^i), $0 \leq i \leq 3$ be affine coordinates of Q and let g_{ij}, $0 \leq i, j \leq 3$ be components of the constant indefinite metric tensor on Q. The dynamics of a free particle of mass m is represented by the Lagrangian submanifold D defined locally by $y_i = m(\Sigma_{k,l}g_{kl}\dot{x}^k\dot{x}^l)^{-1/2}\Sigma_j g_{ij}\dot{x}^j$, $\Sigma_{k,l}g_{kl}\dot{x}^k\dot{x}^l > 0$ and $\dot{y}_j = 0$. The definition is equivalent to: $\Sigma_i(\dot{y}_i dx^i + y_i d\dot{x}^i) = md(\Sigma_{k,l}g_{kl}\dot{x}^k\dot{x}^l)^{1/2}$, $\Sigma_{k,l}g_{kl}\dot{x}^k\dot{x}^l > 0$. Hence D is generated by a Lagrangian $L(x^i, \dot{x}^j) = m(\Sigma_{i,j}g_{ij}\dot{x}^i\dot{x}^j)^{1/2}$ defined on the open submanifold J of TQ satisfying $\Sigma_{k,l}g_{kl}\dot{x}^k\dot{x}^l > 0$. The submanifold D is not generated by a generating function with respect to (TP, P, τ, χ). The definition of D is equivalent to: there is a number $\lambda > 0$ such that $\Sigma_i(\dot{y}_i dx^i - \dot{x}^i dy_i) = -d[\lambda((\Sigma_{i,j}g^{ij}y_i y_j)^{1/2} - m)]$, where g^{ij}, $0 \leq i, j \leq 3$ are components of the contravariant metric tensor. We call the function H defined locally on P × R by $H(x^i, y_j, \lambda) = \lambda((\Sigma_{i,j}g^{ij}y_i y_j)^{1/2} - m)$ the generalized Hamiltonian of the relativistic particle. We call the submanifold K of P defined by $\Sigma_{i,j}g^{ij}y_i y_j = m$ the Hamiltonian constraint. The relation

$$m(\Sigma_{i,j}g_{ij}\dot{x}^i\dot{x}^j)^{1/2} = \mathrm{Stat}_{(y_i,\lambda>0)}[\Sigma_i y_i \dot{x}^i - \lambda((\Sigma_{k,l}g^{kl}y_k y_l)^{1/2} - m)]$$

is the local expression of a generalized version of the inverse Legendre transformation.

8. LEGENDRE TRANSFORMATIONS IN THERMOSTATICS OF IDEAL GASES

Let P be a manifold with coordinates (V, S, p, T) interpreted as the volume, the metrical entropy, the pressure and the absolute temperature respectively of one mole of an ideal gas. The manifold P together with the form

$$\omega = dV \wedge dp + dT \wedge dS$$

define a symplectic manifold (P, ω). The behaviour of the gas is gouverned by the two equations of state: $pV = RT$ and $pV^{\gamma} = K \exp \frac{S}{c_V}$, where R, γ and K are constants and $c_V = \frac{R}{\gamma - 1}$. It is easy to see that the equations of state define a Lagrangian submanifold N of (P, ω).

Let Q_1, Q_2, Q_3 and Q_4 be manifolds with coordinate systems (V, S), (V, T), (p, T) and (S, p) respectively. The mappings

$$\begin{aligned} \pi_1 &: P \to Q_1 : (V, S, p, T) \mapsto (V, S), \\ \pi_2 &: P \to Q_2 : (V, S, p, T) \mapsto (V, T), \\ \pi_3 &: P \to Q_3 : (V, S, p, T) \mapsto (p, T), \\ \pi_4 &: P \to Q_4 : (V, S, p, T) \mapsto (S, p) \end{aligned}$$

and forms

$$\begin{aligned} \vartheta_1 &= - p dV + T dS, \\ \vartheta_2 &= - p dV - S dT, \\ \vartheta_3 &= V dp - S dT, \\ \vartheta_4 &= V dp + T dS \end{aligned}$$

define special symplectic manifolds $(P, Q_1, \pi_1\ \vartheta_1)$, $(P, Q_2, \pi_2, \vartheta_2)$, $(P, Q_3, \pi_3, \vartheta_3)$ and $(P, Q_4, \pi_4, \vartheta_4)$. The Lagrangian submanifold N is generated by generating functions $F_1 = U$, $F_2 = F$, $F_3 = G$ and $F_4 = H$ with respect to the above special symplectic structures. The generating functions are given by formulæ

$$\begin{aligned} U(V, S) &= \frac{K}{\gamma - 1} V^{(1-\gamma)} \exp \frac{S}{c_V}, \\ F(V, T) &= c_V T(1 - \ln T + \ln K - \ln R) - RT \ln V, \\ G(p, T) &= c_P T(1 - \ln T - \ln R) + c_V T \ln K + RT \ln p, \end{aligned}$$

and

$$H(S, p) = \frac{\gamma}{\gamma - 1} K^{\frac{1}{\gamma}} p^{\frac{\gamma - 1}{\gamma}} \exp \frac{S}{c_p},$$

where $c_p = R + c_V$. The generating functions U, F, G and H are known as *thermodynamic potentials* and are called the *internal energy*, the *Helmholtz function*, the *Gibbs function* and the *enthalpy* respectively.

Three examples of the twelve Legendre transformations relating the four special symplectic structures are given below. The mapping $\pi_2 \times \pi_1$ maps the diagonal of $P \times P$ onto a submanifold I_{21} of $Q_2 \times Q_1$ with coordinates (V, S, T) related to the coordinates (V, S, p, T) in an obvious way. The Legendre transformation from $(P, Q_1, \pi_1, \vartheta_1)$ to $(P, Q_2, \pi_2, \vartheta_2)$ is generated by the function E_{21} defined on I_{21} by $E_{21}(V, S, T) = - TS$. The Legendre transformation from $(P, Q_1, \pi_1, \vartheta_1)$ to $(P, Q_3, \pi_3, \vartheta_3)$ is generated by the function E_{31} defined on $I_{31} = Q_3 \times Q_1$ by

$$E_{31}(V, S, p, T) = pV - TS$$

and the Legendre transformation from $(P, Q_1, \pi_1, \vartheta_1)$ to $(P, Q_4, \pi_4, \vartheta_4)$ is generated by the function E_{41} on a submanifold I_{41} of $Q_4 \times Q_1$ with coordinates (V, S, p) defined by $E_{41}(V, S, p) = pV$. Relations

$$F(V, T) = \mathrm{Stat}_S\,(U(V, S) - TS),$$
$$G(p, T) = \mathrm{Stat}_{(V,S)}\,(U(V, S) + pV - TS),$$
$$H(S, p) = \mathrm{Stat}_V\,(U(V, S) + pV)$$

are easily verified.

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