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Section A :
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Wave operators for momentum dependent long range potential

by

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ABSTRACT. — Let $H_0 = -\Delta$ in $L^2(\mathbb{R}^3)$ and $H = -\Delta + T$ where T is a pseudo-differential operator. We consider a class of operators T which correspond physically to momentum dependent long range potentials. For each such T we give a modified free propagator e^{iW_t} depending on the momentum operators and prove the existence of the modified wave operators $W_\pm = s - \lim \exp(itH)e^{-iW_t}$ as $t \to \pm\infty$. In order to reduce the complexity of the proof we impose a condition on the decrease of the interaction at large distances which, in the case of a simple potential, reduces to $|V(\vec{x})| < \text{const}\, |\vec{x}|^{-\alpha}$ with $\alpha > 1/2$.

RÉSUMÉ. — Nous considérons dans $L^2(\mathbb{R}^3)$ les Hamiltoniens $H_0 = -\Delta$ et $H = -\Delta + T$ où T est un opérateur pseudo-différentiel, ce qui correspond physiquement à des interactions dépendant de l'impulsion. Pour de tels opérateurs, nous définissons un propagateur libre modifié e^{iW_t} dépendant des impulsions et nous démontrons l'existence des opérateurs d'ondes modifiés $W_\pm = s - \lim_{t \to \pm\infty} e^{itH}e^{-iW_t}$. Afin de simplifier la démonstration, nous nous limitons à une décroissance de l'interaction qui, dans le cas d'un potentiel, se réduit à $|V(\vec{x})| < \text{const}\, |\vec{x}|^{-\alpha}$ avec $\alpha > 1/2$.

I. INTRODUCTION

In the last few years, there has been considerable interest in the problem of the existence of the generalized wave operators in scattering theory with long range potentials. One may cite here the results of Dollard [6], Amrein, Martin and Misra [3], Buslaev and Matveev [5], Alsholm [1] and of Hörmander [9]. In a recent paper we have studied the scattering problem with momentum dependent short range interactions. It seems interesting to generalize this method to momentum dependent long range interactions. For this, the proofs become more involved. Indeed one then has to replace the free evolution by a modified propagator, which requires more refined estimates to establish the convergence of the generalized wave operators. The free Hamiltonian H_0 is the unique self-adjoint extension in $L^2(\mathbb{R}^3)$ of the symmetric operator $\hat{H}_0$ defined on Schwarz space by

$$\hat{H}_0 = -\Delta$$

The interactions that we shall study are given by pseudo-differential operators of the form

$$(1) \qquad (Tf)(x) = \iint e^{i\langle x-y|\xi\rangle} a(x, \xi) f(y) dy d\xi .$$

These operators have been studied by Hörmander [8]. A particular case are differential operators with variable coefficients.

Under certain conditions on the symbol $a(x, \xi)$ it is possible to prove the existence and the asymptotic completeness of the wave operators associated with the pair of Hamiltonians H_0 and $H_0 + T$ [4]. We give here more general conditions on the symbol $a(x, \xi)$ such that the long range wave operators

$$(2) \qquad W_\pm = s - \lim_{t \to \pm\infty} e^{itH} e^{-iW_t}$$

exist for each self-adjoint extension H of $\hat{H}_0 + T$, where W_t is a pseudo-differential operator which we shall define further on. (For a detailed description of scattering by a potential the reader is referred to [2]).

In the sequel, $\mathcal{F}f$ and $\hat{f}$ denote the Fourier transform of the function f in $L^2(\mathbb{R}^3)$. We also define $\Omega = \mathbb{R}^3/\{0\}$.

We shall always assume that the symbol $a(x, \xi)$ of the pseudo-differential operator T verifies the following conditions (where the existence of the occuring partial derivatives is understood in $\mathcal{S}'$):

(C_1): *a*) The measurable function $a(.,.)$ belongs to $\mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}^3)$ and, if $\hat{a}(.,.)$ is the Fourier transform of the distribution a with respect to the first variable, we have:

$$\hat{a}(\zeta + \theta, -\theta) = \hat{\bar{a}}(\zeta + \theta, \zeta) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}^3).$$

b) a is a real function.

c) a verifies either $a(x, \xi) = a(x, -\zeta)$ or $a(-x, \xi) = a(x, \xi)$.

(C_2): There exists a compact set K_0 in $\mathbb{R}^3$ such that: for every compact set K in $\mathbb{R}^3$, there is a constant c_K and a number $1 \geqslant \eta > 1/2$ such that for all multi-indices α and β with $|\alpha + \beta| \leqslant 5$ and for every point x in $\complement K_0$ (the complement of K_0) we have:

$$\operatorname*{Sup}_{\xi \in K} |(\partial^{\alpha}{}_x \partial^{\beta}{}_{\xi} a)(x, \xi)| \leqslant c_K(1 + |x|)^{-(\eta + |\alpha|)}$$

(C_3): *a)* For every compact set K, $K \subset \Omega$, there exist constants M_K and C_K such that for each multi-index α with $\alpha \leqslant \sup\{2M_K, M_K + 4\}$:

$$\int |(\partial^{\alpha}{}_{\xi} a)(x, \xi)|^2 (1 + x^2)^{-M_K} dx \leqslant C_K, \qquad \forall \xi \in K.$$

b) For every point x in $\mathbb{R}^3$, the function $a(x, .)$ belongs to $L^1_{Loc}(\mathbb{R}^3)$.

The conditions (C_3, *a*) and (C_3, *b*) imply that T is an operator in $L^2(\mathbb{R}^3)$ with domain $\mathscr{F}^{-1}(C_0^{\infty}(\Omega))$. On this domain we can define an operator $\hat{H}$ by

$$\hat{H} = \hat{H}_0 + T.$$

One deduces from conditions (C_1, $C_3 a$ and *b*) as in ([*4*], § II) that $\hat{H}$ is a symmetric operator with equal deficiency indices, which implies that it has self-adjoint extensions. In fact, instead of condition ($C_1 c$), we could more generally assume that there exists a conjugation J of $L^2(\mathbb{R}^3)$ such that $[J, T] = 0$ on $\mathscr{F}^{-1}(C_0^{\infty}(\Omega))$.

We may also remark that, if the pseudo-differential operator T is given by a potential V, our conditions are analogous to those of [*1*] [*6*] [*5*] and more recently to condition (2-1) of Hörmander [*9*].

In the second part of this paper we briefly summarize the long range scattering theory. In the third one we prove the existence of the wave operators using the stationary phase method [*7*] and the Morse lemma [*10*].

II. SCATTERING THEORY FOR LONG RANGE INTERACTIONS

Let K_0 be as in (C_2) and let K'_0 be a compact set such that $K_0 \subset \mathring{\widehat{K'_0}}$ ($\mathring{A}$ denotes the interior of A). Choose a function χ in $\mathscr{D}(\mathbb{R}^3)$ such that

$$\begin{cases} \chi \equiv 1 & \text{on } K_0 \\ \chi \equiv 0 & \text{on } \complement K'_0 \end{cases}$$

The symbol $a(x, \xi)$ may be written as

$$a(x, \xi) = \chi(x) a(x, \xi) + (1 - \chi(x)) a(x, \xi) = a_{SR}(x, \xi) + a_{LR}(x, \xi).$$

The two symbols $a_{SR}(x, \xi)$ and $a_{LR}(x, \xi)$ have the following properties:

i) $a_{SR}(.,.)$ has support in K'_0 (with respect to the variable x) and satisfies (C_1) and (C_3).

ii) $a_{LR}(.,.)$ has support in $\overline{\complement K'_0}$ (with respect to the variable x) and satisfies (C_1) (C_2) and (C_3).

DÉFINITION 1. — *We define two functions* $W_t(.)$ *and* $r_t(.)$ *by*

$$W_t(\xi) = t\xi^2 + r_t(\xi)$$
$$r_t(\xi) = \int_{t_0}^{t} a_{LR}(2\tau\xi, \xi)d\tau$$

where t_0 *is a constant.*

The operator e^{iW_t} is defined by

$$\widehat{(e^{iW_t}f)}(\xi) = e^{iW_t(\xi)}\hat{f}(\xi), \qquad \forall f \in L^2(\mathbb{R}^3).$$

Since the function W_t is real, the modified free evolution e^{-iW_t} is a family of unitary operators. Furthermore e^{-iW_t} is a function of the momentum operators, and the latter commute with H_0, so that our theory agrees with the algebraic theory of scattering developped in [*3*] if one takes for the algebra of free observables the Von Neumann algebra generated by the momentum operators.

In order to prove the existence of the modified wave operators, it suffices to prove that the limit (2) exists on a dense subset of $L^2(\mathbb{R}^3)$ which we shall take to be

$$D(\hat{H}) = \mathscr{F}^{-1}(C_0^\infty(\Omega)) \tag{3}$$

In the sequel, we consider a self-adjoint extension H of $\hat{H}$. We shall give the proof for $t \to +\infty$; the limit $t \to -\infty$ can be dealt with by the same method. By lemma 1 of Dollard [*6*], the limit in (2) as $t \to +\infty$ exists if for some number $T_1 > 0$

$$\int_{T_1}^{+\infty} \left\| \left(H - \frac{\partial W_t}{\partial t} \right) e^{-iW_t} u \right\| dt < +\infty, \qquad \forall u \in \mathscr{F}^{-1}(C_0^\infty(\Omega)).$$

We have $\frac{\partial W_t}{\partial t}(\xi) = \xi^2 + a_{LR}(2t\xi, \xi)$. Therefore $H - \frac{\partial W_t}{\partial t}$ is a pseudo-differential operator whose symbol is

$$a_{SR}(x, \xi) + (a_{LR}(x, \xi) - a_{LR}(2t\xi, \xi)). \tag{4}$$

In the sequel we denote by T_{SR} (resp. T_{LR}) the pseudo-differential operator with symbol $a_{SR}(x, \xi)$ (resp. $a_{LR}(x, \xi) - a_{LR}(2t\xi, \xi)$). We are going to show that the following two integrals are convergent

$$\int_{T_1}^{+\infty} \|T_{SR}u_t\| dt < +\infty \quad \text{and} \quad \int_{T_1}^{+\infty} \|T_{LR}u_t\| dt < +\infty, \quad \forall u \in \mathscr{F}^{-1}(C_0^\infty(\Omega)) \tag{5}$$

where

$$u_t(x) = (2\pi)^{-3} \iint e^{i\langle x-y|\xi\rangle} e^{-iW_t(\xi)} u(y) dy d\xi.$$

This will give the proof of the existence of the wave operator W_+. The proof for W_- is similar.

III. EXISTENCE OF THE WAVE OPERATORS FOR LONG RANGE SCATTERING

For each $u \in D(\hat{H})$ we define

$$(6)\qquad \begin{cases} r_u = 1/2 \operatorname*{Inf}_{\xi \in \operatorname{Supp} \hat{u}} |2\vec{\xi}| \\ R_u = 2 \operatorname*{Sup}_{\xi \in \operatorname{Supp} \hat{u}} |2\vec{\xi}| \end{cases}$$

This definition will be useful in separating the stationary point in the integrals in (5).

3.1 Majorization of the integral $I_{LR} = \int_{T_1}^{+\infty} \| T_{LR} u_t \| \, dt$.

This integral will be handled by the stationary phase method and in order to do this we have to isolate the points where the phase is critical. We can write

$$(7)\quad I_{LR} = \int_{T_1}^{+\infty} \| T_{LR} u_t \| \, dt = \int_{T_1}^{+\infty} \left(\int_{\left|\frac{x}{t}\right| \notin]r_u, R_u[} |(T_{LR} u_t)(x)|^2 dx \right)^{1/2} dt + \int_{T_1}^{+\infty} \left(\int_{\left|\frac{x}{t}\right| \in]r_u, R_u[} |(T_{LR} u_t)(x)|^2 dx \right)^{1/2} dt .$$

From now on, we shall not indicate the dependence on u since no confusion is possible.

3.1.1 Majorization of the integral

$$I = \int^{+\infty} \left(\int_{\left|\frac{x}{t}\right| \notin]r_u, R_u[} |(T_{LR} u_t)(x)|^2 dx \right)^{1/2} dt .$$

We shall now prove the following result

THEOREM 1. — *If the conditions* (C_1) *and* (C_2), $(C_3$ *a and b) are verified, then there exists* $0 < T_1 < \infty$ *such that the function*

$$I(t) = \left(\int_{\left|\frac{x}{t}\right| \notin]r, R[} |(T_{LR} u_t)(x)|^2 dx \right)^{1/2} \qquad \textit{belongs to } L^1(T_1, +\infty) .$$

We define

$$(8)\qquad \begin{cases} \omega = |x| + |t| + 1 \\ \phi(x, t, \xi) = (\langle x | \xi \rangle - t\xi^2 - r_t(\xi))(|x| + |t| + 1)^{-1} \end{cases}$$

and

$$(9)\quad I_{LR}(x, t) = (T_{LR} u_t)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\omega\phi(x,t,\xi)} [a_{LR}(x, \xi) - a_{LR}(2t\xi, \xi)] . \hat{u}(\xi) d\xi .$$

We have the following lemma

LEMMA 1. — *Let* $\mathscr{A}_t(x) = \omega^3 \mathrm{I}_{\mathrm{LR}}(x, t)$. *There exist constants* $\mathrm{T}_2 < +\infty$ *and* $0 < \rho < +\infty$ *such that*

$$\| \mathscr{A}_t(.) \|_{\mathrm{L}^2(\complement\{x \mid tr < |x| < t\mathrm{R}\})} \leqslant \rho\,, \qquad \forall t > \mathrm{T}_2\,.$$

Proof. — Let K be a compact subset of Ω containing the support of $\hat{u}$. We first recall the general formula of integration by parts: if

$$\| (\vec{\nabla}_\xi \phi)(x, t, \xi) \| \neq 0$$

on the domain of integration in the variable ξ, and if $f(x, t, \xi)$ is a C^1 function with compact support in ξ for x and t fixed, we have

$$(10) \quad \int_{\mathbb{R}^3} f(x, t, \xi) e^{i\omega\phi(x,t,\xi)} d\xi = -\frac{i}{\omega} \int_{\mathbb{R}^3} e^{i\omega\phi(x,t,\xi)} \operatorname{div} \left(\frac{f(x, t, \xi)(\vec{\nabla}_\xi \phi)(x, t, \xi)}{\| (\vec{\nabla}_\xi \phi)(x, t, \xi) \|^2} \right) d\xi\,.$$

We can now write the integral defining $\mathrm{I}_{\mathrm{LR}}(x, t)$ in the following form

$$(11) \qquad \mathrm{I}_{\mathrm{LR}}(x, t) = (2\pi)^{-3} \left[\int f_1(x, t, \xi) e^{i\omega\phi(x,t,\xi)} d\xi - \int f_2(x, t, \xi) e^{i\omega\phi(x,t,\xi)} d\xi \right]$$

where

$$\begin{cases} f_1(x, t, \xi) = a_{\mathrm{LR}}(x, \xi) \hat{u}(\xi) \\ f_2(x, t, \xi) = a_{\mathrm{LR}}(2t\xi, \xi) \hat{u}(\xi)\,. \end{cases}$$

We now apply the formula (10) to both integrals occuring in (11). Let us prove first that there exists a number $\mathrm{T}_2 < +\infty$ and a number $d > 0$ such that for $t > \mathrm{T}_2$ and $|xt^{-1}| \notin]r, \mathrm{R}[$

$$\| (\vec{\nabla}_\xi \phi)(x, t, \xi) \| > d \qquad \text{for every } \xi \text{ in Supp } \hat{u}$$

For this we calculate $(\vec{\nabla}_\xi \phi)(x, t, \xi)$:

$$\begin{aligned} (\vec{\nabla}_\xi \phi)(x, t, \xi) &= \frac{t}{\omega} \left\{ \frac{\vec{x}}{t} - 2\vec{\xi} - \frac{1}{t} \left[2 \int_{t_0}^{t} \tau (\vec{\nabla}_x a_{\mathrm{LR}}(2\tau\xi, \xi)) d\tau - \int_{t_0}^{t} \vec{\nabla}_\xi a_{\mathrm{LR}}(2\tau\xi, \xi) d\tau \right] \right\} \\ &= \left(\frac{\vec{x}}{t} - 2\vec{\xi} - \vec{\mathrm{R}} \right) \left(t^{-1} + 1 + \left| \frac{x}{t} \right| \right)^{-1} \end{aligned}$$

Now one obtains from the triangle inequality

$$(12) \quad \| (\vec{\nabla}_\xi \phi)(x, t, \xi) \| \geqslant \left| \frac{1}{\frac{1}{t} + 1 + \left| \frac{\vec{x}}{t} \right|} \left\| \frac{\vec{x}}{t} - 2\vec{\xi} \right\| - \frac{\| \vec{\mathrm{R}} \|}{\frac{1}{t} + 1 + \left| \frac{\vec{x}}{t} \right|} \right|$$

The first term in (12) is bounded from below by a positive constant for $|xt^{-1}| \notin]r, R[$ and $\xi \in \text{Supp}\, \hat{u}$, while the second term tends to zero uniformly in x as $t \to +\infty$ by condition (C_2).

Similar estimations hold for the higher order derivatives of the phase ϕ. The derivatives with respect to ξ give terms of the form $2t(\vec{\nabla}_x a_{LR})(2t\xi, \xi)$. These terms as well as their derivatives are uniformly bounded in x and t for $\xi \in \text{Supp}\, \hat{u}$ by condition (C_2). In fact one has

$$(13) \qquad \operatorname*{Sup}_{\xi \in K} t^{|\alpha|}(\partial_x^\alpha \partial_\xi a)(2t\xi, \xi) \leqslant c_K \operatorname*{Sup}_{\xi \in K} t^{|\alpha|}(1 + t|\xi|)^{-(\eta + |\alpha|)}$$

Since $\|(\vec{\nabla}_\xi \phi)(x, t, \xi)\| > d$, the first integral in (11) is handled by using the formula (10) in the same way as in ([4], lemma 1). For the second integral in (11) we apply five times the formula (10). By taking into account also the inequality (13), one obtains for $t > T_2$

$$\left| \int f_2(x, t, \xi) e^{i\omega\phi(x,t,\xi)} \hat{u}(\xi) d\xi \right| \leqslant \frac{\delta}{\omega^5}$$

where δ is a positive constant.

Let $E[M_K]$ be the largest integer less than or equal to M_K. We have now for the family $\mathscr{A}_t(x)$

$$(14) \qquad \begin{cases} \mathscr{A}_t(x) = (\omega^{E[M_K]+1})^{-1} \displaystyle\int_{\mathbb{R}^3} e^{i\omega\phi(x,t,\xi)} b_1(x, t, \xi) d\xi \\[2ex] \qquad - \omega^3 \displaystyle\int_{\mathbb{R}^3} f_2(x, t, \xi) e^{i\omega\phi(x,t,\xi)} d\xi . \end{cases}$$

where $b_1(x, t, \xi)$ is a finite linear combination of derivatives of the symbol $a_{LR}(x, \xi)$ with respect to ξ up to the order $E[M_K] + 4$, with coefficients in a bounded subset of $C_0^\infty(\Omega)$ if $|xt^{-1}| \notin [r, R]$. By using $(1 + x^2)^{\frac{M_K}{2}} \times (1 + |x| + |t|)^{-E[M_K+1]} \leqslant 1$ in (14) we get the following estimate

$$|\mathscr{A}_t(x)| \leqslant \int_{\mathbb{R}^3} |b_1(x, t, \xi)| (1 + x^2)^{-\frac{M_K}{2}} d\xi + \frac{\delta}{\omega^2} .$$

By calculating the L^2-norm of $\mathscr{A}_t(x)$ and by using the Schwarz inequality as in the proof of lemma 1 in [4] one obtains

$$\|\mathscr{A}_t(.)\|^2_{L^2(\complement\{x | tr < |x| < tR\})} \leqslant \rho^2(t),$$

where $\rho^2(t)$ is the bounded function on $[T_2, +\infty)$ given by

$$\rho^2(t) = \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |b_1(x, t, \xi)| (1 + x^2)^{-\frac{M_K}{2}} d\xi + \frac{\delta}{\omega^2} \right)^2 dx$$

If we define $\rho = \text{Sup}_{t \in [T_2, +\infty)} \rho(t)$ we have

$$\|\mathscr{A}_t(.)\|_{L^2(\complement\{x | tr < |x| < tR\})} \leqslant \rho, \qquad \forall t \in [T_2, +\infty)$$

which proves the lemma. ■

From lemma 1 one deduces that for $T_1 \geqslant T_2$

$$I_{LR} = \int_{T_1}^{+\infty} \left(\int_{\left|\frac{x}{t}\right| \notin]r,R[} |I_{LR}(x,t)|^2 dx \right)^{1/2} dt$$
$$\leqslant \int_{T_1}^{+\infty} \left(\int_{\left|\frac{x}{t}\right| \notin]r,R[} \omega^{-6} |\mathscr{A}_t(x)|^2 dx \right)^{1/2} dt \leqslant \int_{T_1}^{+\infty} \rho(1 + |t|)^{-3} dt < +\infty$$

which proves the convergence of the integral I_{LR}.

3.1.2 Majorization of the integral

$$H_{LR} = \int^{+\infty} \left(\int_{\left|\frac{x}{t}\right| \in]r,R[} |(T_{LR}u_t)(x)|^2 dx \right)^{1/2} dt$$

The convergence of the integral H_{LR} will be demonstrated in several steps. First of all we study the determination of the point ξ_0 where the phase is stationary. Using this result we shall introduce two domains of integration:

1) A neighbourhood of the stationary point, which will give an integral denoted $H_{LR}^{(1)}$.

2) The complement of the preceeding set in Supp $\hat{u}$, which leads to an integral $H_{LR}^{(2)}$.

This decomposition will allow us to prove

THEOREM 2. — *Suppose that the conditions* (C_1) *and* (C_2), (C_3) *are verified. Then there exists a constant* $T_3 < +\infty$ *such that the function*

$$H(t) = \left(\int_{\left|\frac{x}{t}\right| \in [r,R]} |(T_{LR}u_t)(x)|^2 dx \right)^{1/2} \quad \textit{belongs to} \quad L^1(T_3, +\infty).$$

For $|xt^{-1}| \in [r, R]$ we define the function

$$(15) \quad H_{LR}(x,t) = (T_{LR}u_t)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\langle x|\xi\rangle} e^{-it\xi^2} e^{-ir_t(\xi)} . [a_{LR}(x,\xi) - a_{LR}(2t\xi,\xi)]\hat{u}(\xi)d\xi .$$

We remark that this function is identical with I_{LR} but defined on a different domain.

We set $y = xt^{-1}$ with $|y| \in [r, R]$, and we obtain from formula (15)

$$(16) \quad H_{LR}(ty,t) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{-it\langle\xi|\xi\rangle} e^{it\langle y|\xi\rangle} e^{-ir_t(\xi)} . [a_{LR}(ty,\xi) - a_{LR}(2t\xi,\xi)]\hat{u}(\xi)d\xi .$$

We wish to determine the points where the phase of this last integral is stationary. This phase is given by

$$\phi(y,t,\xi) = -\langle \xi | \xi \rangle + \langle y | \xi \rangle - t^{-1} r_t(\xi) .$$

We have the following lemma

LEMMA 2. — *Let* R *and* r *be given by* (6) *and let* $\vec{y}$ *in* $\mathbb{R}^3$ *be such that* $|\vec{y}| \in [r, R]$. *Then there exists a number* $0 < T_4 < +\infty$ *such that for* $t > T_4$ *the equation*

$$(\vec{\nabla}_\xi \phi)(x, t, \xi) = 0$$

has at most one solution $\xi_0(y, t)$ *belonging to the support of* $\hat{u}$.

Remark. — In the sequel, we denote by $B(\vec{z}, \gamma)$ the closed ball in $\mathbb{R}^3$ of center $\vec{z}$ and radius γ.

Proof. — The equation $\vec{\nabla}_\xi \phi = 0$ is equivalent to

$$\vec{y} = 2\vec{\xi} + t^{-1}\vec{\nabla}_\xi r_t(\xi). \tag{17}$$

To prove that (17) has at most one solution, we shall apply a fixed point theorem. For this, choose R_1 such that $B\left(\frac{\vec{y}}{2}, R_1\right)$ contains the support of $\hat{u}$, and let $r_1 < r/2$. We also define $\mathscr{A} = B\left(\frac{\vec{Y}}{2}, R_1\right) \cap \complement \overset{\circ}{\overline{B(0, r_1)}}$. Clearly $\operatorname{supp} \hat{u} \subset \mathscr{A}$. We have

$$\vec{\nabla}_\xi r_t(\vec{\xi}) = 2\int_{t_0}^{t} \tau \vec{\nabla}_x a_{LR}(2\tau\xi, \xi)d\tau + \int_{t_0}^{t} \vec{\nabla}_\xi a_{LR}(2\tau\xi, \xi)d\tau.$$

Since the support of $a_{LR}(x, \xi)$ in the variable x is contained in $\overline{\complement K_0}$, it follows from condition (C_2) that for $\vec{\xi} \in \mathscr{A}$

$$\| \vec{\nabla}_\xi r_t(\vec{\xi}) \| \leqslant 2\int_{t_0}^{t} \tau(1 + \tau|\xi|)^{-(\eta+1)}d\tau + \int_{t_0}^{t} (1 + \tau|\xi|)^{-\eta}d\tau$$

Hence we obtain for all $\vec{\xi} \in \mathscr{A}$ (in particular for all $\vec{\xi} \in \operatorname{Supp} \hat{u}$)

$$\| t^{-1}\vec{\nabla}_\xi r_t(\vec{\xi}) \| \leqslant ct^{-\eta}. \tag{18}$$

We now define the mapping $h_t : \mathscr{A} \to \mathbb{R}^3$ by

$$h_t(\vec{\xi}) = \frac{\vec{y}}{2} + \frac{1}{2}t^{-1}\vec{\nabla}_\xi r_t(\vec{\xi}).$$

It follows from (18) that $\| h_t(\vec{\xi}) - \vec{y}/2 \| \to 0$ as $t \to +\infty$ uniformly in $\vec{\xi} \in \mathscr{A}$. Since $|\vec{y}| \in [r, R]$, $\vec{y}/2$ belongs to the interior of $\mathscr{A}$. Hence there exists a number $T_4 < +\infty$ such that the image of h_t is contained in $\mathscr{A}$ provided that $t > T_4$.

Now

$$h_t(\xi) - h_t(\xi') = \frac{1}{2}t^{-1}\int_0^1 (\vec{\xi} - \vec{\xi}').\partial_s(\vec{\nabla}_\xi r_t)(s\xi + (1 - s)\xi')ds.$$

By using again condition (C_2), one deduces from this that

$$\| h_t(\xi) - h_t(\xi') \| < c_2 t^{-\eta}.$$

Thus for t sufficiently large h_t is a contraction of $\mathscr{A}$ into itself, and the uniqueness of the solution of (17) now follows from a fixed point theorem ([11], theorem 46). ■

To continue with the proof of Theorem 2, let $0 < \gamma < \frac{r}{4}$. It then follows from (18) that, there exists $T_5 < +\infty$ such that for all

$$\vec{\xi} \in \operatorname{Supp} \hat{u} \cap \complement B\left(\frac{y}{2}, \gamma\right)$$

and all $t > T_5$

$$\| (\vec{\nabla}_\xi \phi)(ty, \xi, t) \| \geqslant \frac{\gamma}{2}. \tag{19}$$

We shall now isolate the stationary point ξ_0. For this we pick a function $g \in \mathscr{D}(\mathbb{R}^3)$ such that

$$\begin{cases} g \equiv 1 & \text{on the ball } B\left(\frac{\vec{y}}{2}, \gamma\right) \\ g \equiv 0 & \text{on } \complement B\left(\frac{\vec{y}}{2}, 2\gamma\right). \end{cases}$$

One can then write $H_{LR}(ty, t)$ as a sum of two integrals (denoted by $H_{LR}^{(1)}$ and $H_{LR}^{(2)}$):

$$\begin{aligned} H_{LR}(ty, t) = (2\pi)^{-3} \int & g(\xi) e^{-it\langle\xi|\xi\rangle} e^{it\langle y|\xi\rangle} e^{-ir_t(\xi)} \\ & . [a_{LR}(ty, \xi) - a_{LR}(2t\xi, \xi)] \hat{u}(\xi) d\xi + (2\pi)^{-3} \int (1 - g(\xi)) \\ & . e^{it\langle\xi|\xi\rangle} e^{it\langle y|\xi\rangle} e^{-ir_t(\xi)} [a_{LR}(ty, \xi) - a_{LR}(2t\xi, \xi)] \hat{u}(\xi) d\xi \end{aligned}$$

The domain of integration of the first integral $H_{LR}^{(1)}$ is contained in $B\left(\frac{y}{2}, 2\gamma\right)$ and that of the second integral in supp $\hat{u} \cap \complement B\left(\frac{y}{2}, \gamma\right)$. The latter domain does not contain the stationary point provided that $t > T_5$.

— Majorization of $H_{LR}^{(2)}(ty, t)$.

This function can be estimated by using the techniques of paragraph 3.1.1. The inequality $\| (\vec{\nabla}_\xi \phi)(x, t, \xi) \| > d$ is replaced by the formula (19), and if we apply three times equation (10) we find

$$\| H_{LR}^{(2)}(ty, t) \| = \mathcal{O}(t^{-3}),$$

uniformly in y such that $|y| \in [r, R]$.

— Majorization of $H_{LR}^{(1)}(ty, t)$.

We make the change of variables $\xi = \xi_0 + w$ in order to distinguish the stationary point. Let us write the function $H_{LR}^{(1)}$ in the new variable w

$$H_{LR}^{(1)}(ty, t(= (2\pi)^{-3} \int e^{-it\langle\xi_0|\xi_0\rangle} e^{it\langle y|\xi_0\rangle} e^{-ir_t(\xi_0)} e^{-it\langle w|w\rangle}$$
$$. e^{it\langle y|w\rangle} e^{-i(r_t(\xi_0+w)-r_t(\xi_0))} e^{-2it\langle\xi_0|w\rangle} \hat{u}(\xi_0 + w)$$
$$[a_{LR}(ty, \xi_0 + w) - a_{LR}(2t(\xi_0 + w), \xi_0 + w)] g(\xi_0 + w) dw .$$

The phase of this integral is proportional to

$$\psi(w, y, t) = \langle w | w \rangle - \langle y | w \rangle + t^{-1}(r_t(\xi_0 + w) - r_t(\xi_0)) + 2 \langle \xi_0 | w \rangle .$$

Furthermore one has

$$\begin{cases} \psi(0, y, t) = 0 \\ (\vec{\nabla}_w \psi)(0, y, t) = 0 . \end{cases}$$

With this new phase one can prove the Morse lemma

LEMME 3 (J. W. Morse [*10*]). — *There exists a number* $0 < T_6 < +\infty$ *such that, if* $t > T_6$, *there exists a change of variable* $w \to v(w)$ *in the ball* $B(0, 2\gamma)$ *satisfying*

i) $v(0) = 0$

ii) $\psi(w(v), y, t) = \langle v | Q | v \rangle$ *where* Q *is a real invertible hermitian* 3×3 *matrix depending only on* t *and on* y,

iii) $Q = 2Id + A$ *with* $\| A \| < 1/2$ *uniformly in* t *and* y *for* $t > T_6$ *and* $|y| \in [r, R]$.

Proof. — By Taylor's formula to the second order we get

$$\psi(w) = \langle w | B(w) | w \rangle \quad \text{where} \quad B(w) = 2 \int_0^1 (D^2\psi)(sw, y, t)(1 - s) ds .$$

From the expression for ψ we have

$$(D^2\psi)(w, y, t) = 2Id + t^{-1}(D_\xi^2 r_t)(\xi_0 + w) .$$

By condition (C_2) there exists a positive real number T′ such that

$$\| t^{-1} D^2 r_t(\xi_0) \| \langle 1/2, \quad \text{if } t \rangle T' .$$

We define:

$$\begin{cases} Q = 2Id + t^{-1}(D^2 r_t(\xi_0)) \\ A = t^{-1}(D^2 r_t)(\xi_0) . \end{cases}$$

Let $B \to R(B)$ be the mapping of the set of all invertible 3×3 matrices into itself which associates to B the solution R(B) of the equation

$$R(B)^{-1} B R(B) = Q , \qquad \text{with} \qquad R(Q) = Id$$

This application is defined in a neighbourhood of Q, that is to say there exists a positive real number ε, such that if $\| B - Q \| < \varepsilon$, R(B) is well defined. By condition (C_2), there exists a real positive number $T_6 > T'$ such that if $t > T_6$ we have

$$\| (D^2\psi)(w, y, t) - Q \| < \varepsilon \qquad \text{if} \qquad w \in B(\xi_0, 2\gamma).$$

The change of variables $v(w) = R[(D^2\psi)(w, y, t)]$ verifies all the conditions of the lemma.

Remark. — Since the application $B \to R(B)$ is C^∞ in a neighbourhood of Q, we can now suppose that the Jacobian $\frac{Dv}{Dw}$ is in $C^3(\mathbb{R}^3)$.

We can now write the function $H^{(1)}_{LR}(y, t)$ in the following form

$$H^{(1)}_{LR}(ty, t) = \int_{\mathbb{R}^3} g(\xi_0 + w(v))e^{-it\langle v|Q|v\rangle}[a_{LR}(ty, \xi_0 + w(v)) - a_{LR}(2t\xi_0 + 2tw(v), \xi_0 + w(v))] \frac{Dw}{Dv} \hat{u}(\xi_0 + w(v))dv.$$

In order to control this integral, we use the lemma of the stationary phase.

LEMMA 4. — *Let f be a function in $C^3_0(\mathbb{R}^3)$, and Q a positive invertible 3×3 matrix. For every real ε with $\varepsilon \in [0, 1/2[$ there exists a constant $c(\varepsilon)$ such that*

$$(20) \qquad \left| \int_{\mathbb{R}^3} e^{-it\langle v|Q|v\rangle} f(v)dv - \left(\operatorname{Det}\left(\frac{Q}{2\pi}\right)\right)^{1/2} t^{-3/2} f(0) \right| \leqslant t^{-3/2-\varepsilon} c(\varepsilon) \left\| \sum_{k=1}^{k=3} \partial_k (1 + \partial_i \partial_j Q^{ij}) f \right\|_{L^2}$$

where in the last formula we have used the Einstein convention.

Proof. — The integral $\int_{\mathbb{R}^3} e^{-it\langle v|Q|v\rangle} f(v)dv$ becomes after a Fourier transformation

$$(21) \qquad \left(\operatorname{Det}\left(\frac{Q}{2\pi}\right)\right)^{-1/2} t^{-3/2} \int_{\mathbb{R}^3} e^{\frac{i\langle v|Q|v\rangle}{4t}} \hat{f}(v)dv.$$

Now for every real number ε such that $0 \leqslant \varepsilon < 1/2$, there exists a finite constant $c_1(\varepsilon)$ such that

$$(22) \qquad | e^{i\frac{\alpha}{4t}} - 1 | \leqslant c_1(\varepsilon) \frac{(1 + \alpha)^\varepsilon}{t^\varepsilon} \qquad \text{for every } \alpha \text{ and } t \text{ in } R^+$$

The integral (21) is equal to

$$(23) \qquad \left(\operatorname{Det}\left(\frac{Q}{2\pi}\right)\right)^{-1/2} t^{-3/2} \left\{ f(0) + \int_{\mathbb{R}^3} (e^{i\frac{\langle v|Q|v\rangle}{4t}} - 1)\hat{f}(v)dv \right\}.$$

By formula (22) we have

$$(24)\quad \left| \int_{\mathbb{R}^3} (e^{i\frac{\langle v|Q|v\rangle}{4t}} - 1)\hat{f}(v)dv \right| \leqslant \frac{c_1(\varepsilon)}{t^\varepsilon} \int_{\mathbb{R}^3} (1 + \langle v \,|\, Q \,|\, v \rangle)^\varepsilon \,|\, \hat{f}(v) \,|\, dv .$$

From this one obtains by using the Schwarz inequality

$$\left| \int_{\mathbb{R}^3} (e^{i\frac{\langle v|Q|v\rangle}{4t}} - 1)\hat{f}(v)dv \right| \leqslant \frac{c_1(\varepsilon)}{t^\varepsilon} \int_{\mathbb{R}^3} \frac{\langle v \,|\, v \rangle^{1/2}(1+\langle v \,|\, Q \,|\, v \rangle)}{\langle v \,|\, v \rangle^{1/2}(1+\langle v \,|\, Q \,|\, v \rangle)^{1-\varepsilon}} \,|\, \hat{f}(v) \,|\, dv$$
$$\leqslant \frac{c(\varepsilon)}{t^\varepsilon} \left(\int_{\mathbb{R}^3} \langle v \,|\, v \rangle (1 + \langle v \,|\, Q \,|\, v \rangle)^2 \,|\, \hat{f}(v) \,|^2 dv \right)^{1/2}$$

Now (20) follows from the formula $\langle v \,|\, v \rangle = \sum_{k=1}^{3} v_k^2$ and Minkowski's inequality.

We may remark that the constant $c(\varepsilon)$ is given by

$$c(\varepsilon) = c_1(\varepsilon) \left(\int_{\mathbb{R}^3} \frac{dv}{\langle v \,|\, v \rangle (1 + \langle v \,|\, Q \,|\, v \rangle)^{2-2\varepsilon}} \right)^{1/2}$$

By the Morse lemma we have

$$\frac{1}{1 + \langle v \,|\, Q \,|\, v \rangle} \leqslant \frac{1}{1 + \langle v \,|\, v \rangle} \qquad \forall v \in \mathbb{R}^3$$

Thus we may suppose that $c(\varepsilon)$ does not depend on t and y if $t > T_6$ and $|\, y \,| \in [r, R]$. We now apply the lemma of the stationary phase to the integral occuring in $H_{LR}^{(1)}$. The function f is given by

$$f(v) = \{ a_{LR}(ty, \xi_0 + w(v)) - a_{LR}(2t\xi_0 + 2tw(v), \xi_0 + w(v)) \} \left(\frac{Dw}{Dv}\right)(v)\hat{u}(\xi_0 + w(v)).g(\xi_0 + w(v)) .$$

It remains to estimate $f(0)$ and $\| \partial_k(1 + Q^{ij}\partial_i\partial_j)f \|_{L^2}$.

— Majorization of $f(0)$.

From $w(0) = 0$ we deduce

$$f(0) = [a_{LR}(ty, \xi_0) - a_{LR}(2t\xi_0, \xi_0)] \left(\frac{Dw}{Dv}\right)(0)g(\xi_0) .$$

$\left(\frac{Dw}{Dv}\right)(0)g(\xi_0)$ is uniformly bounded, and the difference

$$a_{LR}(ty, \xi_0) - a_{LR}(2t\xi_0, \xi_0)$$

is $\mathcal{O}(t^{-2\eta})$ by condition (C_2) and $\left\| \frac{y}{2} - \xi_0 \right\| = \mathcal{O}(t^{-\eta})$.

We can conclude that

$$f(0)t^{-3/2} = \mathcal{O}(t^{-3/2-2\eta}) .$$

— Majorization of $\| \partial_k(1 + Q^{ij}\partial_i\partial_j)f \|_{L^2}$.

From condition (C_2) and the Morse lemma we deduce that the derivatives of the function f up to the third order are $\mathcal{O}(t^{-\eta})$ uniformly in y. As the domain of integration is contained in the ball $B\left(\frac{y}{2}, 2\gamma\right)$ we have

$$\| \partial_k(1 + Q^{ij}\partial_i\partial_j)f \|_{L^2} = \mathcal{O}(t^{-\eta})$$

It follows that

$$H^{(1)}_{LR}(y, t) = \mathcal{O}(t^{-3/2-\eta-\varepsilon}).$$

By using the change of variables $y = xt^{-1}$ we can write

$$H(t) = t^{3/2}\left(\int_{|y|\in[r,R]} |(T_{LR}u_t)(ty)|^2 dy \right)^{1/2}$$

From the preceding estimate on $H_{LR}(y, t)$ we deduce

$$H(t) = \mathcal{O}(t^{-\eta-\varepsilon}).$$

Now if $\eta > 1/2$ we can find ε with $0 < \varepsilon < 1/2$ such that $\eta + \varepsilon > 1$.

With such an ε, $H(t)$ belongs to $L^1(T_3, +\infty)$ if $T_3 > T_6$, which completes the proof of Theorem 2.

3.2 Majorization of the integral $I_{SR} = \int_{T_1}^{+\infty} \| T_{SR}u_t \| \, dt.$

We split I_{SR} into a sum of two integrals as in (7). Since $a_{SR}(x, \xi)$ has compact support in the variable x and r_u is strictly positive, the second term is zero provided that T_1 is sufficiently large. To estimate the first term, one applies the same method as in § 3.1.1. It suffices to apply twice the general formula of integration by parts, equation (10). The derivatives of the symbol arising from the phase $\phi(x, t, \xi)$ are again uniformly bounded in x and t for $\xi \in \operatorname{supp} \hat{u}$ by condition (C_2). Those arising from the function $f(x, t, \xi)$, which is here given by $f(x, t, \xi) = a_{SR}(x, \xi)\hat{u}(\xi)$, are estimated by using condition (C_3, *a*).

Remark. — The convergence proof of the modified wave operators has been given under the condition that the interaction decreases sufficiently rapidly at infinity (condition (C_2)). For the case of a simple potential this condition corresponds to the requirement that $|V(\vec{x})| < \text{const.}\, |\vec{x}|^{-\eta}$ with $\eta > 1/2$. If $\eta \leqslant 1/2$ the proofs become longer [*1*] [*9*] but the techniques are not essentially different.

In our previous paper [*4*] the existence of the wave operators has been established for interactions the symbol of which is decreasing at infinity more rapidly than const. $|\vec{x}|^{-\eta}$ with $\eta > 1$. The conditions imposed in the present article differ as follows from those in [*4*]:

— In [*4*] we imposed an integrability condition on $\partial_\xi^\alpha a(\pm tx, \xi)$ over the variables x and t, whereas here this condition is replaced by (C_2) which prescribes some differentiability of the symbol $a(x, \xi)$ with respect to x and a sufficiently rapid decrease of $a(x, \xi)$ and its derivatives for large values of x.

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