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by

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ABSTRACT. — Formulas that express the multiple normal-threshold discontinuities of scattering functions for 2-to-4 processes as sums of products of scattering amplitudes for related processes are derived from a combination of field-theoretic and S-matrix principles.

RÉSUMÉ. — A partir des principes généraux de la théorie des champs et des postulats classiques de la théorie de la matrice S, on démontre certaines formules de discontinuité pour la fonction analytique de diffusion relative au processus comportant 2 particules initiales et 4 particules finales. Ces formules expriment les discontinuités multiples de la fonction précédente à travers les coupures associées aux seuils normaux comme des sommes de produits d'amplitudes de diffusion relatives à d'autres processus.

I. INTRODUCTION

Multiparticle dispersion relations have been used recently in the development of Regge theory [1, 2]. The principal contributions to these dispersion

relations are multiple Cauchy integrals of the multiple discontinuities of the scattering function across certain sets of normal-threshold cuts. Formulas that express these multiple discontinuities as sums of products of scattering amplitudes have been derived for the 3-to-3 scattering process in references [3] and [4]. In the present work analogous formulas are derived for 2-to-4 and 4-to-2 processes. These formulas together with the earlier 3-to-3 results determine the strengths of the principal contributions to the dispersion relations for the six-particle scattering function.

It is sufficient to treat explicitly only the 2-to-4 case because the 4-to-2 results may be obtained from the 2-to-4 results by reflection.

II. THE RESULTS

The notation that will be used in what follows is that of references [3] and [4]. For 2-to-4 processes the complete set E of normal-threshold cuts consists of the one total-energy cut t, the four final 3-particle subenergy cuts f, where $f = 1, 2, 3$, or 4 labels the final particle that is not grouped with the other three final particles, and the six final 2-particle subenergy cuts $(f'f'')$, where f' and f'' label the two final particles that are not grouped with the two initial particles. The letter G will stand for a subset of the set E and the symbol M^G will be used for the scattering function evaluated below the set of cuts G and above the set of cuts $\bar{G} = E - G$. The set G can be the entire set E or the empty set $\varnothing$.

The function M_H^G defined by

$$M_H^G = \sum_{H' \subset H} (-1)^{n(H')} M^{GH'} \tag{2.1}$$

is the multiple discontinuity across the set of cuts $H \subset E$ evaluated below the set of cuts G and above the set of cuts $E - GH$. The set GH' is $G \cup H'$ and $n(H')$ is the number of cuts in the set H'. The sum in (2.1) and in all similar sums includes the terms where $H' = \varnothing$ and where $H' = H$. Equation (2.1) implies that

$$M_H^G = \sum_{G' \subset G} (-1)^{n(G')} M_{HG'} . \tag{2.2}$$

This formula allows all the multiple discontinuities M_H^G to be expressed in terms of the multiple discontinuities M_H which are evaluated above all cuts not in H. Our results will be stated in terms of these basic discontinuities M_H.

The single discontinuities M_h are given by the following formulas:

$$M_t = \quad (2.3\ a)$$

$$M_f = \quad (2.3\ b)$$

and

$$M_{(f'f'')} = $$

$$= \quad (2.3\ c)$$

The non-zero double discontinuities are

$$M_{tf} = $$

$$= $$

$$= - \quad (2.4\ a)$$

$$M_{t(f'f'')} = $$

$$= \quad (2.4\ b)$$

$$M_{f(f'f'')} = $$

$$= - \quad (2.4\ c)$$

and

$$M_{(f'f'')(ff''')} = $$

$$= \quad (2.4\ d)$$

In these equations, as in references [3] and [4],

$$\equiv \quad (2.5\ a)$$

and

$$(2.5\ b)$$

The expression $(2.5\ a)$ represents all the terms in the cluster expansion of the $\pm$ box in which line f does not go straight through (i. e. particle f interacts with some other final particle). The expression $(2.5\ b)$ represents the sum of all terms of the cluster expansion of the $\pm$ box in which each of the two lower lines touches some bubble that touches some line in the upper set of lines.

The non-zero triple discontinuities are

$$M_{tf(f'f'')} =$$

$$(2.6\ a)$$

and

$$M_{t(f'f'')(ff''')} =$$

$$(2.6\ b)$$

where

$$(2.7)$$

is the set of terms in the cluster decomposition of the $\pm$ box in which some bubble touches both a line in F′ and a line in F″.

All quadruple and higher multiple discontinuities vanish.

III. PROPERTIES OF THE M^G

It will be shown in what follows that the functions M^G satisfy the following conditions:

1) The M^G obey the generalized Steinmann relations.

2) The M^G coincide with the corresponding cell functions of field theory where the latter are defined.

3) The M^G satisfy generalized hermitian analyticity

$$M^G = - (M^{E-G})^\dagger. \tag{3.1}$$

4) The M^G_H have the appropriate support property: $M^G_H(p)$ vanishes if for any $h \in H$ the corresponding channel invariant $s_h(p)$ is less than the leading normalthreshold for the channel J_h. The letters g and h are used to label both cuts and their corresponding channels, and the channel invariant $s_h(p)$ is

$$s_h(p) = \left(\sum_{j \in J_h} p_j \right)^2. \tag{3.2}$$

5) The M^G continue in the appropriate way around all normal-threshold singularities: M^G can be expressed in a form that has no singularity corresponding to any positive-α normal-threshold diagram D_g^+ for any $g \in G$; and M^G can be expressed in a form that has no singularity corresponding to any negative-α normal-threshold diagram D_g^- for any g in $E - G$. Thus if only normal-threshold diagram singularities are considered, then the function M^G continues into itself by passing into the lower-half s_g plane near each normal-threshold singularity surface $s_g = $ (a sum of masses)^2 for each g in G, and similarly into the upper-half s_g plane for each g in $E - G$.

6) The $M^{G'}s$ can be classified as good $M^{G'}s$ and bad $M^{G'}s$. The bad $M^{G'}s$ are those such that for some pair $(f'f'')$ of final particles either

$$f' \in G, \ f'' \in G, \ t \in E - G, \quad \text{and} \quad (ff''') \in E - G \tag{3.3 a}$$

or

$$f' \in E - G, \ f'' \in E - G, \ t \in G, \quad \text{and} \quad (ff''') \in G, \tag{3.3 b}$$

where f, f', f'' are the four final particles. The remaining $M^{G'}s$ are the good $M^{G'}s$. Analytic continuation of a good M^G is never blocked by the canonical mechanism.

The possibility that the continuation of some good $M^{G'}s$ might be blocked by a non-canonical mechanism is not ruled out. However, blockage by a non-canonical mechanism is in some sense accidental; and we believe, on the basis of the analysis of reference [1], that those $M^{G'}s$ whose continuations are not blocked by the canonical mechanism are boundary values of the physical-sheet scattering function, in the sense required for multiparticle dispersion relations. This point is discussed at the end of section IX.

Proofs of the six properties described above are given in the following six sections.

IV. GENERALIZED STEINMANN RELATIONS

A scattering function M is said to satisfy the generalized Steinmann relations [1, 3-6] if the multiple discontinuity M_H^G vanishes identically

$$M_H^G = 0 \tag{4.1}$$

whenever the set H contains two overlapping channels. [A channel J_h is a non-empty proper subset of the whole set $X = \{1, 2, 3, 4, 5, 6\}$ of six particle labels. Two channels J_g and J_h are said to be overlapping if none of the four sets $J_h \cap J_g$, $J_h \cap (X - J_g)$, $(X - J_h) \cap J_g$, and $(X - J_h) \cap (X - J_g)$ is empty.]

Now it is evident from an inspection of the formulas (2.3-2.7) for the fundamental non-zero discontinuities M_H that the set H consists in each case of channels that are non-overlapping. Thus for the two-to-four case, the generalized Steinmann relations (4.1) are a consequence of Eq. (2.2) which expresses an arbitrary M_H^G in terms of these M_H.

V. AGREEMENT WITH THE CELL FUNCTIONS

It has been shown in references [5] and [6] that the cell function $\tilde{r}^G$ on the real mass shell possesses the representation

$$\tilde{r}^G = \sum_{H \subset G} (-1)^{n(H)} N_H \tag{5.1}$$

where the functions N_H are the same as the functions M_H

$$N_H = M_H \tag{5.2}$$

with the following exceptions:

$$N_{(ff')(f''f''')} = N_{t(ff')(f''f''')} = 0, \tag{5.3}$$

$$N_{t(ff')} = M_{t(ff')} + \text{(diagram)}$$

$$= M_{t(ff')} + M_{(ff')(f''f''')} - M_{t(ff')(f''f''')}, \tag{5.4}$$

and

$$N_t = \text{(diagram)} + \sum_{(f,f')}^{(3)} \text{(diagram)}$$

$$= M_t + \sum_{(ff')}^{(3)} [M_{(ff')(f''f''')} - M_{t(ff')(f''f''')}] \tag{5.5}$$

in which the sum is over the three distinct possible summands. In obtaining these formulas use has been made of the identity

$$(5.6)$$

In order to prove that $M^G = \tilde{r}^G$ for all cells G, it will be necessary to observe two properties of cells:

If (ff') and $(f''f''')$ are both in G, then t must also be in G.   (5.7)

If t is in G, then either (ff') or $(f''f''')$ or both must be in G.   (5.8)

In the statement of these two properties of cells, f, f', f'', and f''' represent the four different particle labels in the set $t = \{1, 2, 3, 4\}$.

Let δ_t^G be one if t is in G and zero otherwise. Then it follows from eqs. (5.1-5) and from property (5.7) that the cell function $\tilde{r}^G$ may be written in the form

$$\tilde{r}^G = \sum_{H \subset G} (-1)^{n(H)} N_H$$

$$= \sum_{\substack{H \subset G \\ H \neq t, t(ff'), \\ (ff')(f''f'''), \\ t(ff')(f''f''')}} (-1)^{n(H)} M_H$$

$$- \delta_t^G \left\{ M_t + \sum_{(ff')}^{(3)} [M_{(ff')(f''f''')} - M_{t(ff')(f''f''')}] \right.$$

$$\left. - \sum_{(ff') \in G} [M_{t(ff')} + M_{t(ff')(f''f''')} - M_{t(ff')(f''f''')}] \right\}$$

$$= \sum_{\substack{H \subset G \\ (ff')(f''f''') \notin H}} (-1)^{n(H)} M_H + \delta_t^G \left\{ \sum_{(ff') \in G} [M_{(ff')(f''f''')} - M_{t(ff')(f''f''')}] \right.$$

$$\left. - \sum_{(ff')}^{(3)} [M_{(ff')(f''f''')} - M_{t(ff')(f''f''')}] \right\}. \quad (5.9)$$

Now by combining the formula (2.2) for M^G with the proposition (5.7) one may express M^G for any cell G as

$$M^G = \sum_{\substack{H \subset G \\ (ff')(f''f''') \notin H}} (-1)^{n(H)} M_H$$

$$+ \delta_t^G \sum_{(ff')(f''f''') \in G} [M_{(ff')(f''f''')} - M_{t(ff')(f''f''')}]. \quad (5.10)$$

It may now be seen that these two formulas, (5.9) for $\bar{r}^G$ and (5.10) for M^G, differ only in the manner in which the terms in volving two of the (ff')-type cuts appear. There are three distinct such pairs of cuts. For each pair either (1) (ff') and $(f''f''')$ are both in G, or (2) (ff') is in G but $(f''f''')$ is not in G, or (3) (ff') is not in G but $(f''f''')$ is in G, or (4) neither (ff') nor $(f''f''')$ is in G. In case (1) the term

$$\delta_t^G [M_{(ff')(f''f''')} - M_{t(ff')(f''f''')}] \quad (5.11)$$

occurs in the formula (5.9) twice with a positive sign and once with a negative sign, and it occurs in the formula (5.10) once with a positive sign. In case (2) the term (5.11) occurs in the formula (5.9) once with a positive sign and once with a negative sign, and it does not occur in the formula (5.10). Case (3) is the same as case (2). In case (4) the coefficient δ_t^G vanishes in both formulas (5.9) and (5.10) because of the property (5.8). Thus for each cell G the function M^G defined by the formula (2.2) is the same as the cell function $\bar{r}^G$ defined by the formulas (5.1-5).

VI. GENERALIZED HERMITIAN ANALYTICITY

In S-matrix theory, the operation of hermitian conjugation, which is represented by an dagger, changes a bubble diagram F into the diagram $F^\dagger$ obtained by reversing the sign inside each bubble, box, and modified box and by multiplying by the factor $(-1)^{N_b}$ where N_b is the number of bubbles in the diagram,

$$F^\dagger = (-1)^{N_b} F(\pm \rightarrow \mp). \quad (6.1)$$

The operation of hermitian conjugation acts linearly on any linear combination of diagrams. The purpose of this section is to show that under hermitian conjugation each function M^G is turned into $-M^{E-G}$:

$$(M^G)^\dagger = -M^{E-G} \quad (6.2\,a)$$

or equivalently

$$M^G = -(M^{E-G})^\dagger. \quad (6.2\,b)$$

Since the number of boundary values M^G is very much larger than the

number of non-zero multiple discontinuities M_H, it will be easier to prove the corresponding property

$$(M_H)^\dagger = -(-1)^{n(H)} \sum_{G \subseteq E-H} (-1)^{n(G)} M_{HG}.$$ (6.3)

That this proposition (6.3) does in fact imply the desired result (6.2) follows from the identity

$$\sum_{\substack{H \subseteq G \\ K \subseteq E-H \\ HK=F}} (-1)^{n(K)} M_{HK} = \begin{cases} (-1)^{n(F)} M_F & \text{if } F \subset E-G & (6.4\,a) \\ 0 & \text{if } F \not\subset E-G & (6.4\,b) \end{cases}$$

which is itself an elementary consequence of the rule

$$\delta_L^\phi = \sum_{H \subset L} (-1)^{n(H)}$$ (6.5)

where δ_L^ϕ is one if $L = \varnothing$ and zero otherwise. For by eqs. (2.2), (6.3) and (6.4)

$$(M^G)^\dagger = \sum_{H \subset G} (-1)^{n(H)} M_H^\dagger$$

$$= -\sum_{H \subset G} \sum_{K \subset E-H} (-1)^{n(K)} M_{HK}$$

$$= -\sum_F \sum_{\substack{H \subset G \\ K \subset E-H \\ HK=F}} (-1)^{n(K)} M_{HK}$$

$$= -\sum_{F \subset E-G} (-1)^{n(F)} M_F = -M^{E-G}$$ (6.6)

Thus the problem of proving the relation (6.2) reduces to that of proving the property (6.3) for the ten different multiple discontinuities M_H.

This task was carried out by one of us (K. C.); but since the details of the calculation are tedious, we shall limit the present demonstration to showing how the computation procedes only for the two triple discontinuities, for three of the four double discontinuities, and for the null discontinuity M_ϕ.

The application of hermitian conjugation (6.1) to the triple discontinuity $M_{tf(f'f'')}$ turns the second line of equation (2.6 a) into the third line of that equation. Thus

$$[M_{tf(f'f'')}]^\dagger = M_{tf(f'f'')}$$ (6.7)

which verifies the property (6.3) for this triple discontinuity.

Similarly, by using the definition (2.7) to express the triple discontinuity $M_{t(ff')(f''f''')}$ (2.6 *b*) in the form

$$M_{t(ff')(f''f''')} = \qquad\qquad\qquad\qquad\qquad \tag{6.8}$$

one finds that

$$[M_{t(ff')(f''f''')}]^\dagger = M_{t(ff')(f''f''')} \tag{6.9}$$

which verifies the property (6.3) for this case.

Now from eq. (2.4 *d*) one has

$$[M_{(ff')(f''f''')}]^\dagger = \qquad - \qquad\qquad\qquad \tag{6.10}$$

which, together with eq. (6.8), gives

$$[M_{(ff')(f''f''')}]^\dagger = M_{t(ff')(f''f''')} - M_{(ff')(f''f''')} \tag{6.11}$$

thereby establishing the desired result (6.3) for this double discontinuity.

By eqs. (2.4 *c*) and (2.5 *a*) one may find

$$[M_{f(f'f'')}]^\dagger = \quad - \qquad\qquad\qquad\qquad$$

$$= \qquad\qquad\qquad\qquad\qquad$$

$$= M_{tf(f'f'')} - M_{f(f'f'')} \tag{6.12}$$

which verifies the property (6.3) in this case.

Now from eq. (2.4 *a*) and from the *n*-to-3-particle unitarity equation, one gets the relations

$$(M_{tf})^\dagger = \quad - \qquad\qquad\qquad\qquad$$

$$= - M_{tf} \quad - \sum_{f' \neq f}\left[\qquad\qquad + \qquad\qquad \right]$$

$$= - M_{tf} + \sum_{f' \neq f}^{(3)} M_{tf(f''f''')} \tag{6.13}$$

which is the desired result (6.3) for this double discontinuity.

The calculations required for the cases of the remaining double discontinuity and for the three single discontinuities involve too many terms to be worth reproducing here. Finally, in the case of the null discontinuity $M_\phi = M^\phi$ one has

$$(M_\phi)^\dagger = (M^\phi)^\dagger = - \; \exists\bigcirc\exists \; = \tilde{r}^\dagger \qquad (6.14\,a)$$

$$= - \; \tilde{r}^E \qquad (6.14\,b)$$

$$= - \; M^E \qquad (6.14\,c)$$

$$= - \sum_{G \subset E} (-1)^{n(G)} M_G. \qquad (6.14\,d)$$

The first line is the definition of the minus-bubble diagram, the second line follows from the hermitian-analyticity property proved in ref. [4], the third line follows from a special case of the equality $\tilde{r}^G = - M^G$ proved in section V, and the fourth line follows from a special case of eq. (2.2).

VII. SUPPORT PROPERTIES OF THE M_H^G

The functions M_H^G have the support property

$$M_H^G(p) = 0 \quad \text{if} \quad s_h(p) < s_h^0 \quad \text{for some} \quad h \in H \qquad (7.1)$$

where s_h^0 is the lowest normalthreshold of the channel J_h. This property holds also if s_h^0 is the lowest multiparticle threshold, provided $s_h(p)$ is not equal to the square of the mass of any single particle that can communicate with the sets of particles that define channel h.

Inspection of the formulas given in section II for the discontinuities M_H shows that eq. (7.1) holds in the case $G = \emptyset$. For if h belongs to H then there is in every bubble diagram contributing to $M_H(p)$ a set of particles that carries the invariant energy $s_h(p)$. The same result holds if H is replaced by any larger set $HG' \subset E$. Thus eq. (7.1) holds by virtue of eq. (2.2).

VIII. CONTINUATION AROUND NORMAL-THRESHOLD SINGULARITY SURFACES

The proofs of analyticity properties given in this section are based on the formal method described in references [1], [3] and [7]. This method involves algebraic manipulations of infinite series without regard to questions of convergence. In the 3-to-3 case the results obtained by this formal method were derived also by rigorous methods. We believe that this could be done here also but have not attempted to do so.

In the formal method the S matrix is identified with its infinite series

expansion in terms of the minus-bubble-diagram functions F^{B^-}. This expansion is

$$S = \sum_{B^- \in \mathscr{B}^-} F^{B^-} \tag{8.1}$$

where $\mathscr{B}^-$ is the set of all bubble diagrams having only minus bubbles. By virtue of the structure theorem, the function F^{B^-} can have a singularity associated with a diagram that contracts to a positive-α Landau diagram D^+ only if the Landau diagram $D(B^-)$ obtained by contracting the bubbles of B^- to points can be contracted to D^+. [In this contraction the originally unsigned lines of $D(B^-)$ are assigned plus signs.]

As in the heuristic development given in reference [3], the functions T_H are defined by

$$T_H = \sum_{B^- \in \mathscr{B}_{\overline{H}}^-} F^{B^-} \tag{8.2}$$

where $\mathscr{B}_{\overline{H}}^-$ is the subset of $\mathscr{B}^-$ consisting of all connected B^- such that for each $h \in H$ the diagram $D(B^-)$ contracts to D_h, where D_h is the generic (arbitrary number of internal lines) h-channel normal-threshold diagram. Equivalently, $\mathscr{B}_{\overline{H}}^-$ is the set of $B^- \in \mathscr{B}^-$ that have for each $h \in H$ an h cut-set, which is a set of explicit lines of B the cutting of which separates B^- into two connected parts in the manner associated with channel h. (See ref. [3], p. 1297, for a more detailed discussion. There an h cut-set is called an explicit h-channel cut-set.) If H is the empty set $\varnothing$, then there are no conditions on the elements of $\mathscr{B}_{\overline{H}}^-$ except connectedness: $\mathscr{B}_{\phi}^- = \mathscr{B}_c^-$. Thus

$$T \equiv T_\phi = M \equiv S_c.$$

The function T^G defined by

$$T^G \equiv \sum_{H \subset G} (-1)^{n(H)} T_H \tag{8.3}$$

has no singularity corresponding to any D that can be contracted to any D_h^+ with $h \in G$. This is because the term $T_\phi = M$ is the sum of the functions F^{B^-} over all $B^- \in \mathscr{B}_c^-$ while the terms $-T_g$ in (8.3) subtract all the F^{B^-} that have singularities corresponding to any D that contracts to any D_h^+ with $h \in G$. The remaining terms in (8.3) correct for the fact that a term F^{B^-} might have been subtracted more than once, as was discussed in reference [3].

The functions M^G are given by a formula similar to eq. (8.3) but with T_H replaced by M_H. It will be shown presently that

$$M_H = T_H \qquad \text{for} \qquad H \in \mathscr{H} \tag{8.4}$$

where $\mathscr{H}$ is the set of H such that no pair of channels in H are overlapping. The functions M_H satisfy

$$M_H = 0 \qquad \text{for} \quad H \notin \mathscr{H} . \tag{8.5}$$

However, the T_H do not obey this rule. Thus we put

$$M^G = T^G - D^G \tag{8.6}$$

where

$$D^G \equiv \sum_{\substack{H \subset G \\ H \notin \mathscr{H}}} (- 1)^{n(H)} T_H . \tag{8.7}$$

Our first task is to confirm (8.4). We shall use $S^+(\alpha ; \beta)$ to denote the S matrix associated with an initial set of lines α and a final set of lines β. Its adjoint $S^\dagger$ will be written S^-. A set of n lines, with n fixed, will generally be identified by the number n. The present arguments will be similar to the ones given in reference [3] and hence will be merely outlined.

Consider first the function T_t. This function is the sum of all F^{B^-} with $B^- \in \mathscr{B}_t^-$ where $\mathscr{B}_t^-$ is the set of $B^- \in \mathscr{B}^-$ having a t cut-set. Every 2-to-4 diagram B^- must begin with a minus bubble $b^-(2 ; \alpha)$ on the far left. The set $\mathscr{B}_t^-$ consists precisely of those 2-to-4 B^- that remain connected when this initial minus bubble is removed.

Consider the product $F \equiv \Sigma_\alpha [- S_c^-(2 ; \alpha)] S_c^+(\alpha ; 4)$. Let Σ_t be the sum of bubble diagrams obtained by replacing $S_c^+(\alpha ; 4)$ in F by its expansion in terms of minus bubbles and by replacing $[- S_c^-(2 ; \alpha)]$ by the corresponding minus bubble. Then every B in the sum Σ_t is a $B_t \in \mathscr{B}_t^-$, and every $B^- \in \mathscr{B}_t^-$ occurs at least once in Σ_t. In fact each $B^- \in \mathscr{B}_t^-$ appears precisely once in Σ_t. One can see this by considering the left-most t cut-set. It will lie just to the right of $b^-(2 ; \alpha)$. To construct $\mathscr{B}_t^-$ one must place on the right of this cut α every possible connected $B^-(\alpha ; 4)$. But the set of these $B^-(\alpha ; 4)$ is precisely the set $\mathscr{B}^-(\alpha ; 4)$, and the corresponding sum of terms F^{B^-} is just $S_c^+(\alpha ; 4)$. Thus one obtains the second form of eq. (2.3 a).

Consider next the function $T_{(f'f'')}$. It is the sum of the F^{B^-} over B^- in $\mathscr{B}_{(f'f'')}^-$, where $\mathscr{B}_{(f'f'')}^-$ is the set of B^- that contain an $(f'f'')$ cut-set. Consider the right-most of these cut-sets. Then arguments almost identical to those of the preceding paragraph give the second form of eq. (2.3 c).

Consider next the function T_f. It is the sum of the F^{B^-} over all B^- in $\mathscr{B}_f^-$. For any B^- in B_f^- consider the right-most f cut-set α. The part of B^- standing to the right of this cut-set α is either a minus bubble $b^-(\alpha ; 3)$ or a term in the sum represented by the first member of
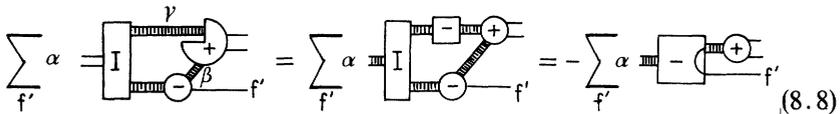
$$\tag{8.8}$$

The pie-shaped figure in the first member of this equation represents the sum of all connected diagrams $B^-(\beta + \gamma; 2)$ in $\mathscr{B}^-$ such that every bubble lies on a path that starts at one of the β lines and moves always to the right. The identity

$$\tag{8.9}$$

which is analogous to eq. (3.3) of ref. [7] or to eq. (V.4.5) of ref. [1], is used together with unitarity to obtain the second member of eq. (8.8). The minus sign in the third member comes from our convention that the connected part of S^- is represented by *minus* the minus bubble.

The part of any B^- in $\mathscr{B}_f^-$ standing to the left of this right-most f cut-set α is a term of the minus-bubble expansion of $S_c^+(2; \alpha + f)$. To obtain each B^- in $\mathscr{B}_h^-$ precisely once, one must multiply independently each term of this expansion of $S_c^+(2; \alpha + f)$ with each term of the sum of eq. (8.8) and the single minus bubble $b^-(\alpha; 3)$. This latter sum, by virtue of unitarity, is $\Sigma_\delta S^-(\alpha; \delta)S_c(\delta; 3)$. By combining this with $S^+(2; \alpha + f)$, one obtains eq. (2.3 b). (This result could also be obtained by considering, alternatively, the left-most f cut-set.)

These results obtained above by the formal (infinite-series) method are contained in eq. (6.4) of reference [8], which was obtained by finite methods.

Consider next $T_{t(f'f'')}$. It is the sum of the F^{B^-} for B^- in $\mathscr{B}_{t(f'f'')}^-$ where $\mathscr{B}_{t(f'f'')}^-$ is the set of all B^- that contain both a t cut-set and an $(f'f'')$ cut-set. By pushing the t cut-set as far left as possible and the $(f'f'')$ cut-set as far right as possible, one isolates the left-most minus bubble $b^-(z; \alpha)$ and the right-most minus bubble $b^-(\beta; 2)$. In between one must put all minus bubble diagrams that connect the remaining two final lines, f and f''', to β. (This connection is demanded by the t-cut requirement.) These diagrams are obtained by expanding all terms of $S^+(\alpha; \beta + f + f'')$ in which both f and f'' are connected to β. Thus one obtains the second form of eq. (2.4 b).

Consider next T_{tf}. By pushing the t-cut-set as far left as possible and the f cut-set as far right as possible and by using the arguments used for cases T_f and $T_{t(f'f'')}$, one obtains the second form of eq. (2.4 a).

Consider next $T_{(f'f'')(ff''')}$. Pushing the $(f'f'')$ and (ff''') cut-sets as far right as possible, one isolates the two right-most minus bubbles, $b^-(\alpha; 2)$ and $b^-(\beta; 2)$. On the left of these two cuts, α and β, one must put every connected $B^-(2; \alpha + \beta)$ in $\mathscr{B}^-$. Thus one obtains the second form of eq. (2.4 d).

Consider next $T_{f(f'f'')}$. Push the f cut to the right. Without the condition that the $(f'f'')$ cut be present, one would obtain for the part standing to the right of the f cut α the sum of the minus bubble $b^-(\alpha; 3)$ and the expansion shown in eq. (8.8). One must now take the subset having also the $(f'f'')$ cut. This condition eliminates the $b^-(\alpha; 3)$ term and two of the three terms

in eq. (8.8). By combining the expansion of the remaining term of (8.8) (the one with f''' in place of f') with the expansion of $S_c^+(2; \alpha + f)$, one obtains the second form of eq. (2.4 c). [By pushing the f cut to the left and using arguments similar to the ones used for T_f, one can obtain the first form of eq. (2.4 c) directly.]

Consider next $T_{t(f'f'')(ff''')}$. By pushing the t cut-set to the left and the $(f'f'')$ and (ff''') cut-sets to the right, one isolates the left-most minus bubble $b^-(2; \alpha)$ and the two right-most minus bubbles $b^-(\beta; 2)$ and $b^-(\gamma; 2)$. In between one must put all $B^-(\alpha; \beta + \gamma)$ in $\mathscr{B}^-$ such that β is connected to γ in $B^-(\alpha; \beta + \gamma)$. [If $B^-(\alpha; \beta + \gamma)$ has this connectedness property, then the t, $(f'f'')$, and (ff''') cut-sets are all present in $B^-(2; 4)$; otherwise they are not.] This set of $B^-(\alpha; \beta + \gamma)$ is generated by expanding all plus bubbles of those terms of $S^+(\alpha; \beta + \gamma)$ in which some bubble connects a line of β to a line of γ. Thus one obtains the second form of eq. (2.6 b).

Consider finally $T_{tf(f'f'')}$. Shifting the t cut-set to the left and the $(f'f'')$ cut-set to the right, one isolates the left-most minus bubble $b^-(2; \beta)$ and the right-most minus bubble $b^-(\delta; 2)$. Shifting the f cut-set α to the right, one must put in all possible $B^-(\beta; f + \alpha)$ such that f is connected to α. This is obtained by expanding all terms of $S^+(\beta; f + \alpha)$ in which f is connected to α. To the right of α one must put all the terms of the expansion of eq. (8.8) in which f' and f'' come into the same minus bubble $b^-(\delta; 2)$. Thus one obtains the second form of eq. (2.6 a). This completes the verification of eq. (8.4).

The functions T_H for $H \notin \mathscr{H}$ are evaluated by using skeleton diagrams. These diagrams were introduced in reference [3] in the following way: for each diagram D its skeleton diagram D_s contains the external lines and external vertices of D and certain paths connecting them. A path runs always from left to right. The diagram D_s contains a path running through a certain set of external vertices of D_s if and only if (1) there is a path in D that contains these vertices (in the same order) and (2) there is no path in D that contains all these external vertices and at least one other external vertex. For the 3-to-3 case considered in ref. [3], there were 76 different skeleton diagrams, of which 67 were tree diagrams and 9 were box diagrams. The set of cut-sets in a diagram D is identical to the set of cut-sets in the corresponding D_s, and each skeleton graph D_s has a different set of cut-sets. Thus the skeleton of a graph $D(B^-)$ determines which of the sets $\mathscr{B}_H^-$ contain B^-:

B^- belongs to $\mathscr{B}_H^-$ if and only if for every $h \in H$ the skeleton diagram $D_s(B^-)$ has cut-set h.

Thus all of the T_H, and hence all of the T^G, are constructed out of the 76 functions

$$F(D_s) = \sum_{B^- \in \mathscr{B}^-(D_s)} F^{B^-} \qquad (8.10)$$

where $\mathscr{B}^-(D_s)$ is the set of minus-bubble diagrams B^- such that $D(B^-)$ has skeleton D_s.

For the 2-to-4 case there is one modification: skeleton diagrams of the form shown in figure (8.1) must also be considered.
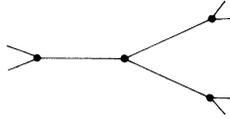

FIG. 8.1. — A skeleton diagram for the 2-to-4 case.

This skeleton diagram has an *internal* vertex. A diagram D has the skeleton shown in figure (8.1) if and only if there are two paths that start at the initial vertex, that coincide over a non-zero segment, and that then separate and go to the two final vertices shown. For all D not having skeletons of the type shown in figure (8.1) the rules of reference [3] apply.

Any bubble diagram B^- having a tree-type skeleton $D_s(B^-)$ contributes only to the T_H with $H \in \mathscr{H}$. Thus the only B^- that contribute to the T_H with $H \notin \mathscr{H}$ are those for which $D_s(B^-)$ is of box form. Hence the T_H with $H \notin \mathscr{H}$ are linear combinations of the functions $F(D_s)$.

For the 2-to-4 case there are six box-diagram skeletons. If D_s is the box diagram shown in figure (8.2 a) and $B(D_s)$ is the bubble diagram shown in figure (8.2 b), then

$$F(D_s) = F^{B(D_s)} . \tag{8.11}$$
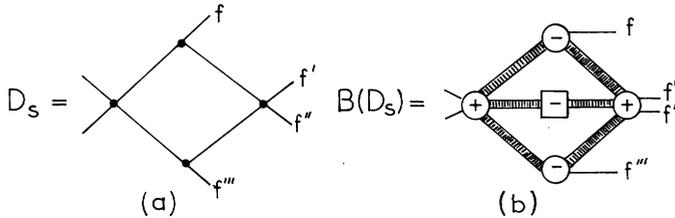
FIG. 8.2. — A box-diagram skeleton D_s
and the corresponding bubble diagram $B(D_s)$.

This result is proved by pushing the t cut-set to the right and by pushing the $(f'f'')$ cut-set to the left. The two external minus bubbles are then isolated, and the internal minus box compensates for double counting in the region lying to the left of the right-most t cut and to the right of the left-most $(f'f'')$ cut. That is, the minus box cancels, by virtue of unitarity, the part of the expansion of the left-hand (or right-hand) plus bubble that is generated already by the expansion of the right-hand (or left-hand) plus bubble.

The definition of T_H now gives

$$T_{ff'''} = T_{ff'''(f'f'')} = T_{tff'''} = T_{tff'''(f'f'')}$$
$$= F^{B(D_s)}. \tag{8.12}$$

Substitution of this result into eq. (8.7) then gives

$$D^G = F^{B(D_s)} \tag{8.13}$$

provided G contains both f and f''' but neither t nor $(f'f'')$. If this condition is not satisfied for some way of labelling the four final lines, then $D^G = 0$.

The good G are defined in eq. (3.3). For these G, $D_G = D_{E-G} = 0$. Hence

$$M^G = T^G \tag{8.14 a}$$

and

$$M^{E-G} = T^{E-G}. \tag{8.14 b}$$

Thus the generalized hermitian-analyticity relation

$$M^G = -(M^{E-G})^\dagger \tag{8.15}$$

gives for the good $G's$

$$M^G = T^G \tag{8.16 a}$$

and

$$M^G = -(T^{E-G})^\dagger. \tag{8.16 b}$$

The first of these equations expresses M^G as a function that has no singularities corresponding to any D that contracts to a D_h^+ with h in G. The second equation expresses M^G as a function that has no singularities corresponding to any D that contracts to a D_h^- with h in $E - G$.

Thus the good $M^{G's}$, by virtue of the structure theorem, must continue in the correct way around all singularity surfaces that correspond to normal-threshold diagrams. The bad $M^{G's}$ enjoy the same properties apart from singularities coming from the box-diagram contributions D^G and D^{E-G}. Hence if one considers these singularities coming from D^G and D^{E-G} to be singularities associated with box diagrams, and not with normal-threshold diagrams, then all the M^G continue in the appropriate way around all normal-threshold-diagram singularities.

IX. ANALYTICITY PROPERTIES OF THE GOOD M^G

The arguments of reference [3] show in the 3-to-3 case that for each good G and each skeleton D_s there is an open (off-mass-shell) cone $\Gamma(G, D_s)$ such that, if p is any real point that lies on the closure of $L(D)$ only if the diagram D has skeleton D_s, then in some real neighborhood of p the function M^G is the boundary value of a function analytic near p in the set Im $p \in \Gamma(G, D_s)$. Moreover, each good M^G continues into itself staying in the mass shell past every codimension-one portion of the union L^G

of its singularity surfaces, except for certain exceptional surfaces described in ref. [3].

The results just described carry over intact to the 2-to-4 case, except for singularities associated with diagrams having skeletons of the form shown in figure (8.1). In this last case the methods of reference [3] fail. However, a more limited result continues to hold: analytic continuation of the good $M^{G}s$ is not blocked by the canonical mechanism.

The canonical mechanism for blocking the analytic continuation of a sum of bubble-diagram functions occurs when the sum has singularities corresponding to two diagrams D_1 and D_2 that differ only by a reversal of all the signs α_i of all the internal lines of the diagram.

The Landau surfaces $L(D_1)$ and $L(D_2)$ associated with any such pair D_1 and D_2 coincide, but the associated $i\varepsilon$ rules of continuation are opposite. This clash of the $i\varepsilon$ rules means that no path of continuation past the surface $L(D_1) = L(D_2)$ is guaranteed by the structure theorem, and the parts of the function on the two sides of such a surface are, in general, not parts of a single analytic function.

The argument that yields this conclusion is as follows: if the skeleton of a diagram D has a form other than that shown in figure (8.1), then, by virtue of arguments almost identical to those given in reference [3], there is for each good M^{G} a pair of external vertices v_r and v_s of D, depending only on the skeleton D_s of D and on G, such that the corresponding vector $v_s - v_r$ in every space-time representation of any D corresponding to a singularity of M^{G} [or of $(M^{E-G})^{\dagger}$] is strictly time-like with the sign of its time component $v_s^0 - v_r^0$ determined solely by D_s and G. This restriction on the sign of $v_s^0 - v_r^0$ precludes the existence of another representation in which the signs α_i of all internal lines of D are reversed.

The result just described holds also for diagrams with skeletons of the form shown in figure (8.1), except that now not all the vertices v_r and v_s need be external. In particular, theorem (6.1) of reference [3] entails that if D has a skeleton D_s of the form shown in figure (8.1), then, for each of the three external vertices v_r of D and for each internal vertex v_s of D that can be identified with the internal vertex of D_s (i. e., for each internal vertex v_s of D that is connected by a correctly oriented path in D to each of the three external vertices v_r of D), the vector $v_s - v_r$ in every representation of D corresponding to a singularity of M^{G} [or of $(M^{E-G})^{\dagger}$] must be time-like with its time component $v_s^0 - v_r^0$ having a well-defined sign determined solely by D_s and G. This sign restriction rules out the possibility of a second representation corresponding to a diagram generated from D by a reversal of all signs α_i of all internal lines. Consequently the continuation of the good $M^{G}s$ is not blocked by the canonical mechanism.

In reference [3] the possibility that the continuation of good $M^{G}s$ could be blocked by non-canonical mechanisms was examined for the 3-to-3 case, and it was found that certain exceptional surfaces could indeed block

these continuations. However, the analysis of reference [1] suggests that in the context of dispersion relations one should nevertheless regard a function M^G as a single analytic function unless its continuation is blocked by the canonical mechanism. For in the cases examined the blocking by non-canonical mechanisms was due to cuts that arise in the construction of the physical sheet of the scattering function. These cuts emerge from the normal-threshold cuts and then loop back and divide the real region into separate parts. Consequently the physical-sheet scattering function has in these separated real regions boundary values represented by different analytic functions. These different analytic functions are, however, the boundary values in these separated regions of the physical-sheet function that is represented by the multiparticle dispersion relation.

X. CONCLUSIONS

Discontinuity formulas very similar to those previously obtained for 3-to-3 processes have been obtained here for 2-to-4 processes. The multiple discontinuities M_H, from which all the functions M_H^G are constructed, have the same form as for the 3-to-3 case. Each non-zero M_H corresponds to a tree diagram τ_H. A tree diagram τ_H is a simply connected set of lines and vertices. The lines run from left to right. The external lines correspond to the final and initial particles; the internal lines correspond to the cuts h in H. The multiple discontinuity M_H is constructed from the tree diagram τ_H by the following steps:

1) replace each vertex by a plus vertex-box, which will be defined below,
2) replace each internal line by a minus box,
3) join each minus box to each of the two adjacent plus vertex-boxes by a complete set of intermediate lines,
4) replace each external line by the corresponding external particle-line that runs into or out of the corresponding plus vertex-box.

A plus-vertex-box is the sum of all terms in the cluster decomposition of the plus box that have the following connectedness properties: (1) if all the minus boxes immediately to the left of a given plus vertex-box were replaced by minus-bubbles, then these bubbles together with the plus vertex-box in question would form a connected structure and (2) the same as (1) but with left replaced by right.

The functions M_H^G constructed from such M_H automatically satisfy the generalized Steinmann relations and have the required support properties. They have been shown here for the 2-to-4 case to agree with the cell functions of field theory where the latter are defined, to respect generalized hermitian analyticity, and to exhibit suitable analyticity. The analyticity properties derived here are not quite as good as those obtained in reference [3] for the 3-to-3 case. Diagrams with skeletons of the form shown in figure (8.1)

cause a breakdown of the argument that showed, in the 3-to-3 case, that the good $M^{G'}s$ could be continued past every real singularity surface by a small off-mass-shell detour. However, the continuation of the good $M^{G'}s$ is not blocked by the canonical mechanism, and this probably means that these good $M^{G'}s$ are boundary values of the physical-sheet scattering function in the sense required for multiparticle dispersion relations.

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(Manuscrit reçu le 26 janvier 1976).