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Models for Infrared Dynamics
I. Classical Currents

by

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ABSTRACT. — Scattering of charged particles is accompanied by the emission of infinitely many soft photons, which necessitates the use of non-Fock representations for the free asymptotic photon fields. This problem is investigated here for a model in which the charged particles are described as classical currents. Non-Fock representations are constructed explicitly, which permit a rigorous solution of the model with the following properties. (*i*) The interacting photon field is an operator-valued distribution satisfying canonical commutation relations. (*ii*) The time evolution of the interacting field is unitarily implementable. (*iii*) Free asymptotic photon fields may be introduced as LSZ limits of the interacting field. (*iv*) Total energy, momentum and angular momentum of the free asymptotic fields exist as self-adjoint operators. (*v*) A unitary S matrix transforms the outgoing into the incoming free photon field. (*vi*) If restricted to photons with energies above some (arbitrarily given) threshold ω_0, the representations for the free asymptotic fields are unitarily equivalent to the Fock representation. The requirements (*i*) to (*vi*) do not fix the representation but are fulfilled for infinitely many unitarily inequivalent representations. The particular representations studied here are infinite direct products of Fock representations, and are obtained from the Fock representation by a very simple canonical transformation.

CONTENTS

1. INTRODUCTION

From a heuristic point of view, the infrared problem may be considered as solved since the classical papers of Bloch and Nordsieck [*1*]. It is known since then that almost all accelerated charges, in particular charged particles during collisions, emit infinitely many soft photons. Other quantities as, e. g., the total energy loss of the charges due to radiation, remain finite. Moreover, any experimental set-up will detect a single photon only when its energy exceeds some finite threshold, and the number of such photons remains also finite. Infrared divergences thus appear in unobservable quantities only. One might then be satisfied with any recipe which yields finite results for the observable quantities, and such recipes actually exist [*2*].

Nevertheless, the infrared problem has attracted the attention of theorists again and again, since some more subtle questions remain unanswered by such more or less heuristic calculation methods. In particular, the following important problem arises. Contrary to what holds true for the scattering of massive particles, for photons the Fock representation of the asymptotic fields is inappropriate since the photon numbers before and after collision differ by an infinite amount. Thus if one takes the Fock representation for the incoming field, the outgoing field turns out to be unitarily inequivalent to it. This is very unsatisfactory for many reasons, and one would prefer to have a common irreducible representation for both the incoming and outgoing field so that, e. g., a unitary photon S matrix exists. One of us has recently constructed such a representation and a unitary S matrix for a model with classical currents [3]. The present paper is mainly an extension of this idea and contains, in addition, a discussion of the field dynamics at finite and at very large times, of the asymptotic conditions, and of observables like energy, momentum, and angular momentum for the incoming and outgoing radiation field.

Contrary to a wide-spread belief [4] [5], the representations appropriate for such models are not generalized coherent state representations. Any irreducible coherent state representation shares with the Fock representation (which is just one of them) the defect that incoming and outgoing field are unitarily inequivalent. This may be remedied formally [4] by going over to a suitable infinite direct sum of irreducible coherent state representations. But even if one accepts the use of such a huge state space, this will only save the existence of an S matrix whereas, e. g., the existence of observables like energy, momentum or angular momentum for the free asymptotic fields has not yet been thoroughly investigated in this framework. The work of Roepstorff [6] seems to indicate, however, that physically meaningful angular momentum operators do not exists in coherent state representations.

Our model may be generalized to the case where the photon source is a charged quantum mechanical particle scattered by some short range potential. But some approximations are needed in this case, whereas the model with classical currents may be treated rigorously. We postpone this to a separate paper, in which our approach will also be compared with those of Faddeev and Kulish [7] and Blanchard [8]. One main difference again is that we will not use coherent state representations. In fact, the representation which is used in this paper is appropriate also for the quantum mechanical model.

For notational convenience, we first treat a model with scalar « photons » coupled to a classical scalar « current ». The changes necessary for an analogous treatment of true photons are almost trivial, and are discussed afterwards.

2. SOLUTION OF THE FIELD EQUATION

We consider the inhomogeneous wave equation

$$\Box \mathrm{A}(x) = j(x) \tag{2.1}$$

for a quantized scalar field $\mathrm{A}(x)$ coupled to a classical « current » of the form

$$j(x) = \sum_{\nu=1}^{\mathrm{N}} c_\nu \gamma^\nu(t)\rho(\underset{\sim}{x} - \underset{\sim}{\mathrm{X}}^\nu(t)), \qquad \gamma^\nu(t) \underset{\mathrm{df.}}{=} (1 - |\dot{\underset{\sim}{\mathrm{X}}}^\nu(t)|^2)^{1/2} \tag{2.2}$$

with trajectories $\underset{\sim}{X}^\nu(t)$, $\nu = 1, \ldots, N$, of N « charges » c_ν and a smooth cutoff function ρ depending on $|\underset{\sim}{x}|$ only, with $\int d^3\underset{\sim}{x}\rho(\underset{\sim}{x}) = 1$.

$\left(\text{Notation: } x = (\underset{\sim}{x}, t),\ x^2 = t^2 - |\underset{\sim}{x}|^2,\ \square = \frac{\partial^2}{\partial t^2} - \underset{\sim}{\nabla}^2.\right)$

The explicit form of the current is chosen such that in the no cutoff limit $\rho(\underset{\sim}{x}) \to \delta(\underset{\sim}{x})$ it becomes a scalar with respect to Lorentz transformations. The trajectories are assumed to be twice continuously differentiable with respect to time and to satisfy the asymptotic conditions

$$\underset{\sim}{X}^\nu(t) \to \underset{\sim}{s}^\nu_{\substack{\text{in}\\\text{out}}} + \underset{\sim}{v}^\nu_{\substack{\text{in}\\\text{out}}} t \qquad \text{for} \qquad t \to \mp\infty . \tag{2.3}$$

More precisely, the estimates

$$|\overset{(\cdot)}{\underset{\sim}{d}}{}^\nu_{\substack{\text{in}\\\text{out}}}(t)| \leq \frac{C}{(1 + |t|)^{1+\varepsilon}} \qquad \text{for} \qquad t \lesseqgtr 0 \tag{2.4}$$

with suitable $C > 0$, $\varepsilon > 0$ are assumed to hold true for

$$\underset{\sim}{d}^\nu_{\substack{\text{in}\\\text{out}}}(t) = \underset{\sim}{X}^\nu(t) - \underset{\sim}{s}^\nu_{\substack{\text{in}\\\text{out}}} - \underset{\sim}{v}^\nu_{\substack{\text{in}\\\text{out}}} t , \qquad \dot{\underset{\sim}{d}}{}^\nu_{\substack{\text{in}\\\text{out}}}(t) = \dot{\underset{\sim}{X}}{}^\nu(t) - \underset{\sim}{v}^\nu_{\substack{\text{in}\\\text{out}}} . \tag{2.5}$$

(This corresponds, e. g., to classical particles scattered by some potential of sufficiently short range.) The absolute values of all velocities $\dot{\underset{\sim}{X}}{}^\nu(t)$ and $\underset{\sim}{v}^\nu_{\substack{\text{in}\\\text{out}}}$ shall be smaller than the velocity of light which in our units is 1. Mainly for convenience of notation, we will explicitly discuss the case $N = 1$, $c_1 = (2\pi)^{3/2}$ only, thus dropping the index ν. All essential conclusions remain valid for $N > 1$. From a physical point of view, however, a model with $N > 1$ is not very realistic, since one has to expect long-range interactions between the « charges » which invalidate estimates like (2.4).

It is convenient to decompose the current as

$$j(x) = j_{\substack{\text{in}\\\text{out}}}(x) + j_{\mp}(x) \tag{2.6}$$

with the asymptotic currents

$$j_{\substack{\text{in}\\\text{out}}}(x) = (2\pi)^{3/2}\gamma_{\substack{\text{in}\\\text{out}}}\,\rho(\underset{\sim}{x} - \underset{\sim}{s}_{\substack{\text{in}\\\text{out}}} - \underset{\sim}{v}_{\substack{\text{in}\\\text{out}}} t), \qquad \gamma_{\substack{\text{in}\\\text{out}}} \underset{\text{df.}}{=} (1 - |\underset{\sim}{v}_{\substack{\text{in}\\\text{out}}}|^2)^{1/2} . \tag{2.7}$$

The solution of (2.1) decomposes, accordingly, as

$$A(x) = A_{\substack{\text{in}\\\text{out}}}(x) + C_{\substack{\text{ret}\\\text{adv}}}(x) \tag{2.8}$$

with asymptotic fields $A_{\substack{\text{in}\\\text{out}}}(x)$ satisfying

$$\square A_{\substack{\text{in}\\\text{out}}}(x) = j_{\substack{\text{in}\\\text{out}}}(x) \tag{2.9}$$

and the retarded resp. advanced c-number solutions $C_{\substack{\text{ret}\\\text{adv}}}(x)$ of

$$\square C_{\substack{\text{ret}\\\text{adv}}}(x) = j_{\mp}(x) . \tag{2.10}$$

The latter exist since $j_{\mp}(x) \to 0$ sufficiently fast for $t \to \mp \infty$. Furthermore, we decompose the asymptotic fields as

$$A_{\substack{\text{in}\\\text{out}}}(x) = A^0_{\substack{\text{in}\\\text{out}}}(x) + B_{\substack{\text{in}\\\text{out}}}(x), \tag{2.11}$$

where $A^0_{\substack{\text{in}\\\text{out}}}(x)$ are free q-number fields satisfying canonical commutation relations, and $B_{\substack{\text{in}\\\text{out}}}(x)$ are suitable c-number fields with

$$\Box B_{\substack{\text{in}\\\text{out}}}(x) = j_{\substack{\text{in}\\\text{out}}}(x). \tag{2.12}$$

The solutions may be written down explicitly in terms of Fourier transforms of $j(x)$, $j_{\substack{\text{in}\\\text{out}}}(x)$ and $j_{\mp}(x)$,

$$\tilde{j}(\underset{\sim}{k}, t) = (2\pi)^{-3/2} \int d^3\underset{\sim}{x} e^{-i\underset{\sim}{k}\underset{\sim}{x}} j(\underset{\sim}{x}, t), \qquad \text{etc.} \tag{2.13}$$

We have

$$\tilde{j}(\underset{\sim}{k}, t) = \tilde{\rho}(\omega)\gamma(t)e^{-i\underset{\sim}{k}\underset{\sim}{X}(t)}, \tag{2.14}$$

$$\tilde{j}_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t) = \tilde{\rho}(\omega)\gamma_{\substack{\text{in}\\\text{out}}} e^{-i\underset{\sim}{k}(\underset{\sim}{x}_{\substack{\text{in}\\\text{out}}} + \underset{\sim}{v}_{\substack{\text{in}\\\text{out}}} t)}, \tag{2.15}$$

$$\tilde{j}_{\mp}(\underset{\sim}{k}, t) = \tilde{\rho}(\omega)e^{-i\underset{\sim}{k}\underset{\sim}{X}(t)}(\gamma(t) - \gamma_{\substack{\text{in}\\\text{out}}} e^{i\underset{\sim}{k}\underset{\sim}{d}_{\substack{\text{in}\\\text{out}}}(t)}) \tag{2.16}$$

with

$$\tilde{\rho}(\underset{\sim}{k}) = \int d^3\underset{\sim}{x} e^{-i\underset{\sim}{k}\underset{\sim}{x}} \rho(\underset{\sim}{x})$$

which is real, satisfies $\tilde{\rho}(0) = 1$ and depends on $|\underset{\sim}{k}| = \omega$ only. Then, with $d\mu(\underset{\sim}{k}) = \dfrac{d^3\underset{\sim}{k}}{2\omega}$ and $k = (\underset{\sim}{k}, \omega)$, the solutions of (2.12) and (2.10) are

$$\left.\begin{aligned} B_{\substack{\text{in}\\\text{out}}}(x) &= (2\pi)^{-3/2} \int d\mu(\underset{\sim}{k}) \{ e^{-ikx} b_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t) + \text{complex conjugate} \}, \\ b_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t) &= \frac{\gamma_{\substack{\text{in}\\\text{out}}} \tilde{\rho}(\omega)}{\omega - \underset{\sim}{k}\underset{\sim}{v}_{\substack{\text{in}\\\text{out}}}} e^{i(\omega t - \underset{\sim}{k}\underset{\sim}{x}_{\substack{\text{in}\\\text{out}}} - \underset{\sim}{k}\underset{\sim}{v}_{\substack{\text{in}\\\text{out}}} t)} \end{aligned}\right\} \tag{2.17}$$

and

$$\left.\begin{aligned} C_{\substack{\text{ret}\\\text{adv}}}(x) &= (2\pi)^{-3/2} \int d\mu(\underset{\sim}{k}) \{ e^{-ikx} c_{\substack{\text{ret}\\\text{adv}}}(\underset{\sim}{k}, t) + \text{c. c.} \}, \\ c_{\text{ret}}(\underset{\sim}{k}, t) &= i \int_{-\infty}^{t} d\tau e^{i\omega\tau} \tilde{j}_{-}(\underset{\sim}{k}, \tau), \\ c_{\text{adv}}(\underset{\sim}{k}, t) &= -i \int_{t}^{\infty} d\tau e^{i\omega\tau} \tilde{j}_{+}(\underset{\sim}{k}, \tau), \end{aligned}\right\} \tag{2.18}$$

as easily checked. The solution (2.17) is the scalar analog of the Liénard-Wiechert potentials of a uniformly moving extended charge. The integrals in (2.18) converge since, by (2.16) and (2.4),

$$
\begin{aligned}
|\tilde{j}_{\mp}(\underset{\sim}{k}, \tau)| &\leq |\tilde{\rho}(\omega)| \{ |\gamma(\tau) - \gamma_{\substack{\text{in}\\\text{out}}}| + \gamma_{\substack{\text{in}\\\text{out}}} |1 - e^{i\underset{\sim}{k}\underset{\sim}{d}_{\substack{\text{in}\\\text{out}}}(\tau)}| \} \\
&\leq \frac{C(M + \omega)|\tilde{\rho}(\omega)|}{(1 + |\tau|)^{1+\varepsilon}} \qquad \text{for} \qquad \tau \lesseqgtr 0. \qquad (2.19)
\end{aligned}
$$

(This follows from the elementary estimates

$$
|1 - e^{i\underset{\sim}{k}\underset{\sim}{d}_{\substack{\text{in}\\\text{out}}}(\tau)}| \leq |\underset{\sim}{k}\underset{\sim}{d}_{\substack{\text{in}\\\text{out}}}(\tau)| \leq \omega |\underset{\sim}{d}_{\substack{\text{in}\\\text{out}}}(\tau)|
$$

and

$$
\begin{aligned}
|\gamma(\tau) - \gamma_{\substack{\text{in}\\\text{out}}}| &= |(1 - |\dot{\underset{\sim}{X}}(\tau)|^2)^{1/2} - (1 - |\underset{\sim}{v}_{\substack{\text{in}\\\text{out}}}|^2)^{1/2}| \\
&\leq M \, | \, |\dot{\underset{\sim}{X}}(\tau)| - |v_{\substack{\text{in}\\\text{out}}}| \, | \leq M |\dot{\underset{\sim}{d}}_{\substack{\text{in}\\\text{out}}}(\tau)| \, .
\end{aligned}
$$

The latter is valid since $|\underset{\sim}{\dot{X}}(t)| \leq N < 1$ for all t and

$$
\left| \frac{d}{dz}(1 - z^2)^{1/2} \right| \leq M = M(N) \qquad \text{for} \qquad |z| \leq N.)
$$

For $A^0_{\substack{\text{in}\\\text{out}}}(x)$, we take as usual

$$
A^0_{\substack{\text{in}\\\text{out}}}(x) = (2\pi)^{-3/2} \int d\mu(\underset{\sim}{k}) \{ e^{-ikx} a_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}) + \text{hermitian conjugate} \} \qquad (2.20)
$$

with creation and annihilation operators $a^*_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k})$, $a_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k})$ satisfying

$$
[a_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}), a_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}')] = 0 \text{ and h. c.,} \qquad [a_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}), a^*_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}')] = 2\omega\delta(\underset{\sim}{k} - \underset{\sim}{k}') \, . \qquad (2.21)
$$

Eqs. (2.8), (2.11), (2.17), (2.18) and (2.20) imply

$$
A(x) = (2\pi)^{-3/2} \int d\mu(\underset{\sim}{k}) \{ e^{-ikx}(a_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}) - g_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t)) + \text{h. c.} \} \qquad (2.22)
$$

with

$$
g_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t) = - b_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t) - c_{\substack{\text{ret}\\\text{adv}}}(\underset{\sim}{k}, t) \qquad (2.23)
$$

and

$$
\dot{A}(x) = (2\pi)^{-3/2} \int d\mu(\underset{\sim}{k}) \{ - i\omega e^{-ikx}(a_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}) - g_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t)) + \text{h. c.} \} \, . \qquad (2.24)
$$

The latter follows since, by (2.17), (2.18) and (2.15),

$$
\dot{g}_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t) = - ie^{i\omega t}\tilde{j}(\underset{\sim}{k}, t) \, , \qquad (2.25)
$$

and the contributions from $\dot{g}_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t)$ are thus easily shown to cancel in

(2.24) because $j(x)$ is real which implies $\tilde{j}(-\underline{k}, t) = \overline{\tilde{j}(\underline{k}, t)}$ (a bar denoting the complex conjugate). Eqs. (2.22) and (2.24) together imply

$$a_{\text{out}}(\underline{k}) - a_{\text{in}}(\underline{k}) = g_{\text{out}}(\underline{k}, t) - g_{\text{in}}(\underline{k}, t). \tag{2.26}$$

By (2.25), the r. h. s. is indeed independent of t, and for $t = 0$ we get

$$a_{\text{out}}(\underline{k}) - a_{\text{in}}(\underline{k}) = b_{\text{in}}(\underline{k}, 0) + c_{\text{ret}}(\underline{k}, 0) - b_{\text{out}}(\underline{k}, 0) - c_{\text{adv}}(\underline{k}, 0). \tag{2.27}$$

A formal calculation yields

$$\begin{aligned} i\int_{-\infty}^{\infty} dt e^{i\omega t}\tilde{j}(\underline{k}, t) &= i\int_{-\infty}^{0} dt e^{i\omega t}(\tilde{j}_{\text{in}}(\underline{k}, t) + \tilde{j}_{-}(\underline{k}, t)) + i\int_{0}^{\infty} dt e^{i\omega t}(\tilde{j}_{\text{out}}(\underline{k}, t) + \tilde{j}_{+}(\underline{k}, t)) \\ &= b_{\text{in}}(\underline{k}, 0) + c_{\text{ret}}(\underline{k}, 0) - b_{\text{out}}(\underline{k}, 0) - c_{\text{adv}}(\underline{k}, 0) \end{aligned}$$

if oscillating terms at $t = \mp\infty$ are dropped since, by (2.15),

$$\begin{aligned} i\int_{-\infty}^{0} dt e^{i\omega t}\tilde{j}_{\text{in}}(\underline{k}, t) &= \frac{\gamma_{\text{in}}\tilde{\rho}(\omega)}{\omega - \underline{k}\underline{v}_{\text{in}}} e^{i(\omega t - \underline{k}\underline{s}_{\text{in}} - \underline{k}\underline{v}_{\text{in}} t)}\Big|_{-\infty}^{0} \\ &= b_{\text{in}}(\underline{k}, 0) + \text{oscillating term at } t = -\infty, \end{aligned}$$

and similarly for $j_{\text{out}}(\underline{k}, t)$. In this sense, (2.27) may be rewritten as

$$a_{\text{out}}(\underline{k}) - a_{\text{in}}(\underline{k}) = i\int_{-\infty}^{\infty} dt e^{i\omega t}\tilde{j}(\underline{k}, t) = i(2\pi)^{-3/2}\int d^4x e^{ikx} j(x),$$

the r. h. s. being proportional to the four-dimensional Fourier transform of the current on the mass shell. Usually this formula is used without comment, but as shown here it need not be literally true. By (2.17),

$$\begin{aligned} b_{\text{in}}(\underline{k}, 0) - b_{\text{out}}(\underline{k}, 0) &= \tilde{\rho}(\omega)\left(\frac{\gamma_{\text{in}} e^{-i\underline{k}\underline{s}_{\text{in}}}}{\omega - \underline{k}\underline{v}_{\text{in}}} - \frac{\gamma_{\text{out}} e^{-i\underline{k}\underline{s}_{\text{out}}}}{\omega - \underline{k}\underline{v}_{\text{out}}}\right) \\ &= \frac{\tilde{\rho}(\omega)}{\omega}\left(\frac{\gamma_{\text{in}}}{1 - \underline{n}\underline{v}_{\text{in}}} - \frac{\gamma_{\text{out}}}{1 - \underline{n}\underline{v}_{\text{out}}} + \gamma_{\text{in}}\frac{e^{-i\underline{k}\underline{s}_{\text{in}}} - 1}{1 - \underline{n}\underline{v}_{\text{in}}} - \gamma_{\text{out}}\frac{e^{-i\underline{k}\underline{s}_{\text{out}}} - 1}{1 - \underline{n}\underline{v}_{\text{out}}}\right) \end{aligned}$$

with $\underline{n} = \frac{\underline{k}}{\omega}$. Since $1 - \underline{n}\underline{v}_{\substack{\text{in}\\\text{out}}} \geq 1 - |\underline{v}_{\substack{\text{in}\\\text{out}}}| > 0$, the first two terms yield a contribution to $a_{\text{out}}(\underline{k}) - a_{\text{in}}(\underline{k})$ which is singular like $1/\omega$ for small ω (unless $\underline{v}_{\text{in}} = \underline{v}_{\text{out}}$, a case which will not be considered here), and is thus not square-integrable with respect to $d\mu(\underline{k})$. On the other hand, by

$$|e^{-i\underline{k}\underline{s}_{\substack{\text{in}\\\text{out}}}} - 1| \leq |\underline{k}\underline{s}_{\substack{\text{in}\\\text{out}}}| \leq \omega|\underline{s}_{\substack{\text{in}\\\text{out}}}|,$$

the third and fourth term remain finite for $\omega \to 0$. Likewise, by (2.18) and (2.19),

$$|c_{\rm ret}(\underset{\sim}{k}, t)| \leq \int_{-\infty}^{t} d\tau\, |\tilde{j}_-(\underset{\sim}{k}, \tau)| \leq \frac{C}{\varepsilon}(M + \omega)\,|\tilde{\rho}(\omega)|\,(1 + |t|)^{-\varepsilon} \quad \text{for} \quad t \leq 0$$

and similarly for $c_{\rm adv}(\underset{\sim}{k}, t)$, i. e.,

$$|c_{\substack{\rm ret \\ \rm adv}}(\underset{\sim}{k}, t)| \leq \frac{C}{\varepsilon}(M + \omega)\,|\tilde{\rho}(\omega)|\,(1 + |t|)^{-\varepsilon} \qquad \text{for} \qquad t \lesseqgtr 0\,. \quad (2.28)$$

Therefore

$$|c_{\rm ret}(\underset{\sim}{k}, 0) - c_{\rm adv}(\underset{\sim}{k}, 0)| \leq \frac{2C}{\varepsilon}(M + \omega)\,|\tilde{\rho}(\omega)|\,,$$

i. e., the contribution of $c_{\substack{\rm ret \\ \rm adv}}(\underset{\sim}{k}, 0)$ is also finite for small ω, and cannot compensate the $1/\omega$ singularity from $b_{\substack{\rm in \\ \rm out}}(\underset{\sim}{k}, 0)$. A unitary S matrix satisfying

$$Sa_{\rm out}(\underset{\sim}{k})S^* = a_{\rm in}(\underset{\sim}{k}) \qquad \text{and h. c.} \quad (2.29)$$

thus cannot exist in any irreducible coherent state representation (and, in particular, not in the Fock representation) since this requires square-integrability of $a_{\rm out}(\underset{\sim}{k}) - a_{\rm in}(\underset{\sim}{k})$ [4]. A more appropriate representation [3] will be constructed in the following Sections. Before doing so, however, we first recall some general properties of canonical field operators. This will also serve to fix our notation.

3. CANONICAL COMMUTATION RELATIONS

We start from (2.21), dropping the suffices $\substack{\rm in \\ \rm out}$, and define

$$p(\underset{\sim}{k}) = \frac{i}{\sqrt{2}}(a^*(\underset{\sim}{k}) - a(\underset{\sim}{k}))\,, \qquad q(\underset{\sim}{k}) = \frac{1}{\sqrt{2}}(a^*(\underset{\sim}{k}) + a(\underset{\sim}{k})) \quad (3.1)$$

with

$$\left.\begin{aligned} [p(\underset{\sim}{k}), p(\underset{\sim}{k}')] &= 0 = [q(\underset{\sim}{k}), q(\underset{\sim}{k}')]\,, \\ i[p(\underset{\sim}{k}), q(\underset{\sim}{k}')] &= 2\omega\delta(\underset{\sim}{k} - \underset{\sim}{k}')\,. \end{aligned}\right\} \quad (3.2)$$

A rigorous formulation of this is the following. There shall exist two real linear spaces $\mathscr{L}_1$ and $\mathscr{L}_2$ of test functions on $\underset{\sim}{k}$-space and, for all $f_1 \in \mathscr{L}_1$ and $f_2 \in \mathscr{L}_2$, unitary operators $V(f_1)$, $U(f_2)$ on the representation space $\mathscr{H}$ which satisfy the Weyl relations

$$\left.\begin{aligned} V(f_1)V(g_1) = V(f_1 + g_1), \qquad U(f_2)U(g_2) &= U(f_2 + g_2) \\ V(f_1)U(f_2) = e^{i\langle f_1, f_2\rangle}U(f_2)V(f_1)& \end{aligned}\right\} \quad (3.3)$$

with the real bilinear form

$$\langle f_1, f_2 \rangle = \int d\mu(\underset{\sim}{k}) f_1(\underset{\sim}{k}) f_2(\underset{\sim}{k}) \quad (3.4)$$

on $\mathscr{L}_1 \times \mathscr{L}_2$. $V(\lambda f_1)$ and $U(\lambda f_2)$ are assumed to be strongly continuous in the real parameter λ. Their self-adjoint generators are identified with

$$\langle p, f_1 \rangle = \int d\mu(\underline{k}) p(\underline{k}) f_1(\underline{k}), \qquad \langle q, f_2 \rangle = \int d\mu(\underline{k}) q(\underline{k}) f_2(\underline{k}), \quad (3.5)$$

such that

$$V(f_1) = e^{i\langle p, f_1 \rangle}, \qquad U(f_2) = e^{i\langle q, f_2 \rangle}. \quad (3.6)$$

There shall exist a dense domain D in $\mathscr{H}$ which is invariant under all $V(f_1)$ and $U(f_2)$, and on which second-order polynomials of the operators (3.5) are defined. Then, by differentiation, (3.3) leads to

$$\left.\begin{array}{c} [\langle p, f_1 \rangle, \langle p, g_1 \rangle] = 0 = [\langle q, f_2 \rangle, \langle q, g_2 \rangle], \\ i[\langle p, f_1 \rangle, \langle q, f_2 \rangle] = \langle f_1, f_2 \rangle \end{array}\right\} \quad (3.7)$$

as valid on D, which is a rigorous version of (3.2).

For complex test functions $f(\underline{k}) = \frac{1}{\sqrt{2}}(f_1(\underline{k}) + i f_2(\underline{k}))$ with $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$, we define unitary Weyl operators

$$W(f) = e^{\frac{i}{2}\langle f_1, f_2 \rangle} V(-f_1) U(f_2). \quad (3.8)$$

By (3.3) they satisfy

$$W(f)W(g) = e^{-i\{f,g\}} W(f+g) = e^{-2i\{f,g\}} W(g)W(f) \quad (3.9)$$

with the real bilinear antisymmetric form

$$\{ f, g \} = \frac{1}{2}(\langle f_1, g_2 \rangle - \langle g_1, f_2 \rangle) = \int d\mu(\underline{k}) \, \mathrm{Im} \, (\overline{f(\underline{k})} g(\underline{k})). \quad (3.10)$$

In particular, $W(\lambda f)$ for real λ is a unitary one-parameter group which is also strongly continuous in λ. We denote its self-adjoint generator by $2\{ a, f \}$, such that

$$W(f) = e^{2i\{a,f\}}. \quad (3.11)$$

Differentiation of (3.8) leads to

$$\{ a, f \} = \frac{1}{2}(\langle q, f_2 \rangle - \langle p, f_1 \rangle) \quad (3.12)$$

as valid on D. Therefore, second-order polynomials of $\{ a, f \}$'s are also defined on D, and differentiation of (3.9) yields on D the commutation relations

$$[\{ a, f \}, \{ a, g \}] = \frac{i}{2}\{ f, g \}. \quad (3.13)$$

Finally, with (3.1) and (3.5), the creation and annihilation operators

$a^*(\underset{\sim}{k})$ and $a(\underset{\sim}{k})$ make sense as operator-valued distributions with domain D if we define

$$\left.\begin{aligned}(a, f) &= \int d\mu(\underset{\sim}{k}) a^*(\underset{\sim}{k}) f(\underset{\sim}{k}) \\ &= \frac{1}{2}(\langle q, f_1 \rangle + \langle p, f_2 \rangle - i(\langle p, f_1 \rangle - \langle q, f_2 \rangle)) \\ &= \{ a, if \} + i \{ a, f \}, \\ (f, a) &= \int d\mu(\underset{\sim}{k}) \overline{f(\underset{\sim}{k})} a(\underset{\sim}{k}) \\ &= \frac{1}{2}(\langle q, f_1 \rangle + \langle p, f_2 \rangle + i(\langle p, f_1 \rangle - \langle q, f_2 \rangle)) \\ &= \{ a, if \} - i \{ a, f \} \\ \text{for complex test functions } f &= \frac{1}{\sqrt{2}}(f_1 + if_2) \text{ with } f_{\substack{1\\2}} \in \mathscr{L}_1 \cap \mathscr{L}_2. \end{aligned}\right\} \quad (3.14)$$

(For such f we have

$$\{ a, f \} = \frac{1}{2i}((a, f) - (f, a)) = \int d\mu(\underset{\sim}{k}) \frac{1}{2i}(a^*(\underset{\sim}{k}) f(\underset{\sim}{k}) - \overline{f(\underset{\sim}{k})} a(\underset{\sim}{k}))$$

which, if compared with (3.10), justifies our notation.) With (3.7) or (3.13) we obtain on D, as a rigorous version of (2.21), the commutation relations

$$\left.\begin{aligned}[(f, a), (g, a)] &= 0 = [(a, f), (a, g)], \\ [(f, a), (a, g)] &= \int d\mu(\underset{\sim}{k}) \overline{f(\underset{\sim}{k})} g(\underset{\sim}{k}) = (f, g). \end{aligned}\right\} \quad (3.15)$$

Moreover, also on D, we have

$$(a, f) = (f, a)^*. \quad (3.16)$$

When investigating canonical commutation relations one usually starts from (3.3) or (3.9) and tries to derive rigorously as many as possible of the relations listed above. Our considerations here should not be misunderstood to be another such attempt. It was solely desired to collect relations which will be used later on. A simple example where all these relations hold true is the Fock representation. In this case $\mathscr{L}_1 = \mathscr{L}_2 = L^2$, the space of real functions which are square-integrable with respect to $d\mu(\underset{\sim}{k})$. For D we may take the set of finite linear combinations of coherent states, i. e., of vectors $W(f)\Omega$ with the vacuum vector Ω. The non-Fock representations appropriate for our model are further examples for the validity of all relations listed above, since they will be constructed explicitly in terms of Fock operators (Section 5).

This explicit construction will also lead to an immediate generalization of the following fact. In the Fock representation the test function spaces

$\mathscr{L}_1 = \mathscr{L}_2 = L^2$ are Hilbert (and thus normed) spaces, and the operator functions

$$\left.\begin{array}{l}V(f_1) \text{ and } U(f_2) \text{ are strongly continuous with} \\ \text{respect to the norm topologies of } \mathscr{L}_1 \text{ and } \mathscr{L}_2 .\end{array}\right\} \quad (3.17)$$

4. TEST FUNCTION SPACES

We denote by $\mathscr{L}$ the space of real functions $f(\underset{\sim}{k})$ which are square-integrable (with respect to $d\mu(\underset{\sim}{k})$) when restricted to volumes $V : \omega = |\underset{\sim}{k}| \geq \omega_0$, with arbitrary $\omega_0 > 0$. Functions which coincide for almost all $\underset{\sim}{k}$ will be identified in $\mathscr{L}$. For pairs of functions $f, g \in \mathscr{L}$ we define

$$\langle f, g \rangle = \int d\mu(\underset{\sim}{k}) f(\underset{\sim}{k}) g(\underset{\sim}{k}) \tag{4.1}$$

whenever this integral exists (which evidently does not for all $f, g \in \mathscr{L}$). Besides $\mathscr{L}$ we will need the space L^2 of all real $f(\underset{\sim}{k})$ which are square-integrable on all of $\underset{\sim}{k}$-space, and the space $\mathscr{M}$ of all $f \in \mathscr{L}$ with $f(\underset{\sim}{k}) \equiv 0$ for $\omega \leq \omega_0$ with some $\omega_0 > 0$. Obviously $\mathscr{M} \subset L^2 \subset \mathscr{L}$, both inclusions being proper.

We introduce:

i) A sequence of radii ω_i, $i = 2, 3 \ldots$, with $\omega_i > \omega_{i+1}$ for all i and $\lim_{i \to \infty} \omega_i = 0$, and the corresponding decomposition of $\underset{\sim}{k}$-space into volumes V_i defined as $V_i : \omega_{i+1} \leq \omega \leq \omega_i$ for $i = 2, 3 \ldots$ (spherical shells) and $V_1 : \omega \geq \omega_2$.

ii) An infinite sequence of functions $g^\nu(\underset{\sim}{k}) \in L^2$, $\nu = 1, 2 \ldots$, which are orthonormal,

$$\langle g^\nu, g^\mu \rangle = \delta_{\nu\mu}, \tag{4.2}$$

and such that

$$\text{Supp } g^\nu \subseteq V_i \qquad \text{for} \qquad \nu = N_{i-1} + 1 \ldots N_i \tag{4.3}$$

(with $N_0 = 0$, $\lim_{i \to \infty} N_i = \infty$, and Supp f denoting the support of a function $f(\underset{\sim}{k})$). Since the number $n_i = N_i - N_{i-1}$ of g^ν's with support in V_i is finite for each i, the orthonormal set of all g^ν cannot be complete in L^2.

iii) A sequence of real numbers b_ν, $\nu = 1, 2 \ldots$, with $0 < b_\nu < 1$ and $\lim_{\nu \to \infty} b_\nu = 0$.

With *i*) to *iii*), we define operators

$$T_1 : f(\underset{\sim}{k}) \to f(\underset{\sim}{k}) + \sum_{\nu=1}^{\infty} (b_\nu - 1) \langle g^\nu, f \rangle g^\nu(\underset{\sim}{k}), \tag{4.4}$$

$$T_2 : f(\underset{\sim}{k}) \to f(\underset{\sim}{k}) + \sum_{\nu=1}^{\infty} \left(\frac{1}{b_\nu} - 1\right) \langle g^\nu, f \rangle g^\nu(\underset{\sim}{k}), \tag{4.5}$$

which map $\mathscr{L}$ into itself in virtue of (4.3). In fact, $\langle g^\nu, f \rangle$ clearly exists for all $f \in \mathscr{L}$, and $\sum_\nu c_\nu g^\nu(\underset{\sim}{k})$ defines a function in $\mathscr{L}$ for arbitrary real c_ν, since in each volume $V : \omega \geq \omega_0$ the infinite ν sum reduces to a finite sum.

The subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ of $\mathscr{L}$ which will serve as test function spaces for our representation are defined by

$$f \in \mathscr{L}_{\frac{1}{2}} \quad \text{iff} \quad T_{\frac{1}{2}} f \in L^2 . \tag{4.6}$$

For $f \in \mathscr{M}$ the ν sums in (4.4), (4.5) are finite by (4.3), which implies

$$\mathscr{M} \subseteq \mathscr{L}_{\frac{1}{2}} . \tag{4.7}$$

By (4.6), $\langle T_1 f_1, T_2 f_2 \rangle$ exists for arbitrary $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$. We will prove below that

$$\langle T_1 f_1, T_2 f_2 \rangle = \sum_i \int_{V_i} d\mu(\underset{\sim}{k}) f_1(\underset{\sim}{k}) f_2(\underset{\sim}{k}) , \tag{4.8}$$

the i sum being absolutely convergent. Eq. (4.8) may be rewritten as

$$\langle T_1 f_1, T_2 f_2 \rangle = \int d\mu(\underset{\sim}{k}) f_1(\underset{\sim}{k}) f_2(\underset{\sim}{k}) \equiv \langle f_1, f_2 \rangle \tag{4.9}$$

if either $f_1 \in \mathscr{M}$ or $f_2 \in \mathscr{M}$ since the i sum then terminates, or if $f_1 \in L^2$ since $f_2 \in L^2$ anyway (see below, Lemma 1). One may construct examples for which (4.9) is not true for all $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$. Nevertheless, (4.8) and (4.9) suggest the definition

$$\langle f_1, f_2 \rangle \underset{\text{df.}}{=} \langle T_1 f_1, T_2 f_2 \rangle \tag{4.10}$$

for all $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$ as a natural extension of (4.1).

In order to prove (4.8), we write

$$\langle T_1 f_1, T_2 f_2 \rangle = \sum_i \int_{V_i} d\mu(\underset{\sim}{k})(T_1 f_1)(\underset{\sim}{k})(T_2 f_2)(\underset{\sim}{k}) .$$

This sum is absolutely convergent since $T_{\frac{1}{2}} f_{\frac{1}{2}} \in L^2$. By (4.3), (4.4) and (4.5), only g^ν's with $\nu = N_{i-1} + 1 \ldots N_i$ contribute to $T_1 f_1$ and $T_2 f_2$ on V_i. With the infinite sums in (4.4) and (4.5) thus replaced by finite sums, an elementary calculation yields

$$\int_{V_i} d\mu(\underset{\sim}{k})(T_1 f_1)(\underset{\sim}{k})(T_2 f_2)(\underset{\sim}{k}) = \int_{V_i} d\mu(\underset{\sim}{k}) f_1(\underset{\sim}{k}) f_2(\underset{\sim}{k}) .$$

Using (4.9), the mappings $T_{\substack{1\\2}} : \mathscr{L}_{\substack{1\\2}} \to L^2$ are easily shown to be separating:

$$T_{\substack{1\\2}} f_{\substack{1\\2}} = T_{\substack{1\\2}} g_{\substack{1\\2}} \text{ in } L^2 \qquad \text{implies} \qquad f_{\substack{1\\2}} = g_{\substack{1\\2}} \text{ in } \mathscr{L} \tag{4.11}$$

(i. e., $f_{\substack{1\\2}}(\underset{\sim}{k}) = g_{\substack{1\\2}}(\underset{\sim}{k})$ almost everywhere). Namely, let $T_1 f_1 = T_1 g_1$ in L^2. Then (4.9) yields $\langle f_1, f_2 \rangle = \langle g_1, f_2 \rangle$ for all $f_2 \in \mathscr{M}$, and thus $f_1(\underset{\sim}{k}) = g_1(\underset{\sim}{k})$ almost everywhere. An analogous argument applies if $T_2 f_2 = T_2 g_2$. This also shows that the bilinear form $\langle f_1, f_2 \rangle$ on $\mathscr{L}_1 \times \mathscr{L}_2$ defined by (4.10) is separating, i. e.,

$$\left.\begin{aligned} \langle f_1, f_2 \rangle = \langle g_1, f_2 \rangle \qquad & \text{for all } f_2 \text{ implies } f_1 = g_1 \,, \\ \langle f_1, f_2 \rangle = \langle f_1, g_2 \rangle \qquad & \text{for all } f_1 \text{ implies } f_2 = g_2 \,. \end{aligned}\right\} \tag{4.12}$$

Moreover, the mappings $T_{\substack{1\\2}} : \mathscr{L}_{\substack{1\\2}} \to L^2$ are onto. In fact, for an arbitrary $f \in L^2$, an elementary calculation with (4.4), (4.5) and (4.2) yields

$$T_{\substack{1\\2}} T_{\substack{2\\1}} f = f. \tag{4.13}$$

Together with (4.11) this implies

$$T_{\substack{1\\2}} \mathscr{L}_{\substack{1\\2}} = L^2 \,, \qquad T_{\substack{2\\1}} L^2 = \mathscr{L}_{\substack{1\\2}} \,. \tag{4.14}$$

If we define on $\mathscr{L}_1$ and $\mathscr{L}_2$ inner products

$$\langle f_{\substack{1\\2}}, g_{\substack{1\\2}} \rangle_{\substack{1\\2}} \underset{\text{df.}}{=} \langle T_{\substack{1\\2}} f_{\substack{1\\2}}, T_{\substack{1\\2}} g_{\substack{1\\2}} \rangle \tag{4.15}$$

and the corresponding norms

$$\| f_{\substack{1\\2}} \|_{\substack{1\\2}} = (\langle f_{\substack{1\\2}}, f_{\substack{1\\2}} \rangle_{\substack{1\\2}})^{1/2} = \| T_{\substack{1\\2}} f_{\substack{1\\2}} \| \tag{4.16}$$

(with $\| f \| = (\langle f, f \rangle)^{1/2}$ for $f \in L^2$), $\mathscr{L}_1$ and $\mathscr{L}_2$ become Hilbert spaces which, by (4.14), are isomorphic to L^2, the isomorphisms being given by T_1 and T_2. By its definition (4.10), the bilinear form $\langle f_1, f_2 \rangle$ on $\mathscr{L}_1 \times \mathscr{L}_2$ is continuous in f_1 and f_2 with respect to the norm topologies on $\mathscr{L}_1$ and $\mathscr{L}_2$.

We will finally prove

LEMMA 1. — *a)* $\mathscr{M} \subset \mathscr{L}_2 \subset L^2 \subset \mathscr{L}_1 \subset \mathscr{L}$, each inclusion being proper.

b) $\| f \|_1 \leq \| f \|$ on L^2, but the norms $\| \cdot \|$ and $\| \cdot \|_1$ are inequivalent on L^2, i. e., there is no C_1 such that $\| f \| \leq C_1 \| f \|_1$ on L^2. Similarly, $\| f_2 \| \leq \| f_2 \|_2$ on $\mathscr{L}_2$, but there is no C_2 with $\| f_2 \|_2 \leq C_2 \| f_2 \|$ on $\mathscr{L}_2$.

c) $\mathscr{M}$ is dense in $\mathscr{L}_2$, L^2, and $\mathscr{L}_1$ with respect to the norm topologies of these spaces. (Thus $\mathscr{L}_2$ is also norm dense in L^2 and $\mathscr{L}_1$, and L^2 is norm dense in $\mathscr{L}_1$.)

Proof. — $\mathcal{M} \subseteq \mathcal{L}_2$, $\mathcal{L}_1 \subseteq \mathcal{L}$ is already proved. For $f \in L^2$,

$$\sum_\nu \langle g^\nu, f \rangle^2 \leq \| f \|^2 < \infty, \quad \text{and thus} \quad \sum_\nu (b_\nu - 1)^2 \langle g^\nu, f \rangle^2 < \infty .$$

Therefore, by (4.4), $T_1 f \in L^2$, with

$$\| T_1 f \|^2 = \| f \|^2 + \sum_\nu \{ 2(b_\nu - 1) + (b_\nu - 1)^2 \} \langle g^\nu, f \rangle^2$$
$$= \| f \|^2 - \sum_\nu (1 - b_\nu^2) \langle g^\nu, f \rangle^2 \leq \| f \|^2 .$$

This proves $L^2 \subseteq \mathcal{L}_1$ and $\| f \|_1 = \| T_1 f \| \leq \| f \|$ on L^2. Moreover, by (4.14), any $f_2 \in \mathcal{L}_2$ is of the form $f_2 = T_1 f$ with $f \in L^2$, and $T_2 f_2 = f$ by (4.13). The foregoing argument then implies $f_2 \in L^2$, thus $\mathcal{L}_2 \subseteq L^2$, and $\| f_2 \| = \| T_1 f \| \leq \| f \| = \| T_2 f_2 \| = \| f_2 \|_2$ on $\mathcal{L}_2$.

Consider functions f of the particular form $f = \sum_\nu c_\nu g^\nu$. One easily proves that $f \in \mathcal{M}$, $\mathcal{L}_2$, L^2, $\mathcal{L}_1$, and $\mathcal{L}$, respectively, iff $c_\nu = 0$ for almost all ν, $\sum_\nu \left(\frac{c_\nu}{b_\nu} \right)^2 < \infty$, $\sum_\nu c_\nu^2 < \infty$, $\sum_\nu (c_\nu b_\nu)^2 < \infty$, and c_ν arbitrary, respectively. Therefore all inclusions in *a*) are proper.

$\mathcal{M}$ is trivially dense in L^2. For an arbitrary $f \in \mathcal{M}$, $f_{\frac{1}{2}} = T_{\frac{2}{1}} f \in \mathcal{M}$ by (4.3), (4.4) and (4.5), and $T_{\frac{1}{2}} f_{\frac{1}{2}} = f$ by (4.13). Thus we have mappings $T_{\frac{1}{2}} : \mathcal{M} \to \mathcal{M}$ which are onto. If $f_1 \in \mathcal{L}_1$ with

$$\langle f_1, f \rangle_1 = \langle T_1 f_1, T_1 f \rangle = 0$$

for all $f \in \mathcal{M}$, this implies $\langle T_1 f_1, g \rangle = 0$ for all $g \in \mathcal{M}$, and thus $T_1 f_1 = 0$ and $f_1 = T_2 T_1 f_1 = 0$. Therefore $\mathcal{M}$ is dense in $\mathcal{L}_1$. Similarly we prove that $\mathcal{M}$ is dense in $\mathcal{L}_2$.

Finally, consider the sequence g^ν $(\nu = 1, 2 \ldots)$ in $\mathcal{M} \subset L^2$. Since $\| g^\nu \| = 1$ for all ν whereas $\| g^\nu \|_1 = \| T_1 g^\nu \| = b_\nu \xrightarrow[\nu \to \infty]{} 0$, the norms $\| \cdot \|$ and $\| \cdot \|_1$ are inequivalent on L^2. Similarly we use the sequence $b_\nu g^\nu$ in $\mathcal{M} \subset \mathcal{L}_2$ for which $\| b_\nu g^\nu \|_2 = \| g^\nu \| = 1$ and $\| b_\nu g^\nu \| = b_\nu$. ■

5. REPRESENTATION OF THE CANONICAL COMMUTATION RELATIONS

We start from the Fock representation of the canonical commutation relations, denoting the representation space by $\mathcal{H}$, the Fock vacuum by Ω, and all operators belonging to the Fock representation by a subscript F. A new representation is defined on $\mathscr{H}$ by

$$V(f_1) = V_F(T_1 f_1), \qquad U(f_2) = U_F(T_2 f_2) \qquad \text{for } f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}, \qquad (5.1)$$

which makes sense because of (4.6). With the original Fock representation, this new one is also irreducible. The Weyl relations (3.3) are satisfied if the bilinear form $\langle f_1, f_2 \rangle$ is defined by (4.10). With the norms (4.16) on $\mathscr{L}_{\frac{1}{2}}$, the continuity property (3.17) of Fock operators carries over to the new representation.

The slight generalization of the bilinear form (3.4) indicated by Eqs. (4.8) and (4.10) is very natural here. We could, e. g., start with the definition (5.1) for $f_{\frac{1}{2}} \in \mathscr{M}$, thus obtaining a representation with restricted test function spaces for which the Weyl relations (3.3) with (3.4) are literally true. By Lemma 1 (property *c*) and (3.17), this representation may then be extended by continuity to the complete test function spaces $\mathscr{L}_{\frac{1}{2}}$. Thereby the Weyl relations remain valid if the bilinear form $\langle f_1, f_2 \rangle$ is also extended by continuity, which leads back to (4.10) (See also [9]).

The representation defined by (5.1) is not unitarily equivalent to the Fock representation. This follows from general results of [9], and is true even if we consider the operators $V(f_1)$ only and restrict them to $f_1 \in \mathscr{M}$. For completeness, an elementary proof is given here. Assume there is a unitary $\mathscr{F}$ such that

$$\mathscr{F} V(f_1) \mathscr{F}^* = V_F(f_1) \qquad \text{for all } f_1 \in \mathscr{M}.$$

As already noted, the functions $g^\nu \in \mathscr{M}$ satisfy $\| g^\nu \| = 1$ and $\| g^\nu \|_1 = b_\nu \xrightarrow[\nu \to \infty]{} 0$. The above assumption implies

$$(\mathscr{F}^*\Omega, V(g^\nu)\mathscr{F}^*\Omega) = (\Omega, V_F(g^\nu)\Omega).$$

However, for $\nu \to \infty$ the l. h. s. converges to 1 by (3.17), whereas the r. h. s. is given explicitly by $e^{-\frac{1}{4}\|g^\nu\|^2} = e^{-\frac{1}{4}}$ for all ν.

Using (5.1) and the formulae of Section 3, we obtain for $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$, with

$$f = \frac{1}{\sqrt{2}}(f_1 + if_2), \qquad Tf = \frac{1}{\sqrt{2}}(T_1 f_1 + iT_2 f_2):$$

$$\langle p, f_1 \rangle = \langle p_F, T_1 f_1 \rangle, \qquad \langle q, f_2 \rangle = \langle q_F, T_2 f_2 \rangle, \qquad (5.2)$$

$$\{ a, f \} = \{ a_F, Tf \}, \qquad (5.3)$$

$$W(f) = W_F(Tf). \qquad (5.4)$$

For $f_{\frac{1}{2}} \in \mathscr{L}_1 \cap \mathscr{L}_2 = \mathscr{L}_2$ we get

$$\left.\begin{aligned}(a, f) &= \frac{1}{2}(\langle q_F, T_2 f_1 \rangle + \langle p_F, T_1 f_2 \rangle - i(\langle p_F, T_1 f_1 \rangle - \langle q_F, T_2 f_2 \rangle)) \\ &= (a_F, Tf) + \{ a_F, f' \}, \\ (f, a) &= \frac{1}{2}(\langle q_F, T_2 f_1 \rangle + \langle p_F, T_1 f_2 \rangle + i(\langle p_F, T_1 f_1 \rangle - \langle q_F, T_2 f_2 \rangle)) \\ &= (Tf, a_F) + \{ a_F, f' \}\end{aligned}\right\} \quad (5.5)$$

with

$$f' = \frac{1}{\sqrt{2}}((T_2 - T_1)f_2 + i(T_2 - T_1)f_1).$$

The last equation implies, in particular, that Ω is not annihilated by all of the new annihilation operators (f, a).

By (5.4) the dense domain D consisting of finite linear combinations of « coherent states » $W(f)\Omega$ with $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$ coincides with the domain D_F spanned by the states $W_F(f)\Omega$ with $f_{\frac{1}{2}} \in L^2$. Together with Eqs. (5.2), (5.3) and (5.5) this implies that $D = D_F$ may be taken as the domain mentioned in Section 3. The commutation relations (3.7), (3.13) and (3.15) are easily shown to be satisfied on D, if $\langle f_1, f_2 \rangle$ is always understood in the sense of Eqs. (4.8) and (4.10), and the expressions $\{ f, g \}$ and (f, g) are interpreted accordingly: for $f = \frac{1}{\sqrt{2}}(f_1 + if_2)$, $g = \frac{1}{\sqrt{2}}(g_1 + ig_2)$,

$$\{ f, g \} = \frac{1}{2}(\langle f_1, g_2 \rangle - \langle g_1, f_2 \rangle) \qquad \text{if } f_{\frac{1}{2}}, g_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}, \tag{5.6}$$

and

$$\left.\begin{aligned}(f, g) = \frac{1}{2}(\langle f_1, g_1 \rangle + \langle f_2, g_2 \rangle + i(\langle f_1, g_2 \rangle - \langle g_1, f_2 \rangle))\end{aligned}\right\} \quad (5.7)$$
$$\text{if } f_{\frac{1}{2}}, g_{\frac{1}{2}} \in \mathscr{L}_1 \cap \mathscr{L}_2 = \mathscr{L}_2 .$$

Eq. (4.8) then yields

$$\begin{aligned}\{ f, g \} &= \sum_i \int_{V_i} d\mu(\underset{\sim}{k}) \,\mathrm{Im}\, (\overline{f(\underset{\sim}{k})}g(\underset{\sim}{k})), \\ (f, g) &= \sum_i \int_{V_i} d\mu(\underset{\sim}{k})\overline{f(\underset{\sim}{k})}g(\underset{\sim}{k}),\end{aligned} \tag{5.8}$$

which slightly modifies Eqs. (3.10) and (3.15).

Another way of obtaining our representation is the following. Enlarge the system of functions $g^\nu(\underset{\sim}{k})$, $\nu = 1, 2 \ldots$ to a complete orthonormal system in L^2 by adding suitable functions $h^\mu(\underset{\sim}{k}) \in \mathscr{M}$, $\mu = 1, 2 \ldots$ From (4.4), (4.5) and (5.2) we obtain

$$\left.\begin{aligned} \langle p, g^\nu \rangle &= b_\nu \langle p_F, g^\nu \rangle, & \langle q, g^\nu \rangle &= \frac{1}{b_\nu} \langle q_F, g^\nu \rangle, \\ \langle p, h^\mu \rangle &= \langle p_F, h^\mu \rangle, & \langle q, h^\mu \rangle &= \langle q_F, h^\mu \rangle. \end{aligned}\right\} \tag{5.9}$$

The canonical field and momentum operators associated with this particular set of functions are thus obtained from the corresponding Fock operators by Eq. (5.9) which describes a canonical transformation. Instead of (5.1), we could as well take (5.9) as defining the new representation. By exponentiation and multiplication we immediately recover from (5.9) the representation (5.1) with test functions $f_{\frac{1}{2}}$ restricted to $\mathscr{M}_0$, the space of finite linear combinations of the functions g^ν and h^μ. Since $\mathscr{M}_0$ is norm dense in $\mathscr{L}_1$ and $\mathscr{L}_2$ (proof as for property c of Lemma 1), this restricted representation is again extendable by continuity to all $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$. The non-implementability of most canonical transformations of the type (5.9) has also been shown by Segal [*10*].

The very simple form of the canonical transformation (5.9) should be noted here. There is only one type of canonical transformations which looks even simpler, namely, the addition of c-numbers to the canonical fields and momenta. This, as well-known, leads to coherent state representations, which are not yet appropriate for our model. It seems natural then to proceed by investigating transformations of the slightly more complicated form (5.9); and these indeed lead to representations in which our model is exactly soluble.

Consider any fixed volume $V : \omega \geq \omega_0$, with an arbitrary but fixed infrared cutoff $\omega_0 > 0$, and denote by $\mathscr{M}_V$ the space of functions f with Supp $f \subseteq V$. Clearly $\mathscr{M}_V \subset \mathscr{M}$ and $\mathscr{M} = \bigcup_{\text{all } V} \mathscr{M}_V$. If restricted to test functions $f_{\frac{1}{2}} \in \mathscr{M}_V$, our representation becomes unitarily equivalent to the Fock representation. This fact is very important for the physical interpretation of the model, as discussed in Ref. [*3*]. It may be proved as follows. Choose i large enough such that $\omega_0 \geq \omega_{i+1}$. All functions g^ν with $\nu > N_i$ then have support outside V, and thus do not contribute to Eqs. (4.4) and (4.5) for $f \in \mathscr{M}_V$. Our representation for $f_{\frac{1}{2}} \in \mathscr{M}_V$ is therefore unchanged if one replaces in (5.9) all b_ν with $\nu > N_i$ by 1. The canonical transformation obtained this way involves a finite number of degrees of freedom only (namely, those corresponding to $g^1 \ldots g^{N_i}$). Therefore it is unitarily implementable [*10*], i. e., there is a unitary $\mathscr{F}_V$ on $\mathscr{H}$ with

$$\mathscr{F}_V \langle p, g^\nu \rangle \mathscr{F}_V^* = \langle p_F, g^\nu \rangle, \quad \mathscr{F}_V \langle q, g^\nu \rangle \mathscr{F}_V^* = \langle q_F, g^\nu \rangle \quad \text{for } \nu \leq N_i, \tag{5.10}$$

and which leaves all $\langle p, g^\nu \rangle$, $\langle q, g^\nu \rangle$ with $\nu > N_i$ and all $\langle p, h^\mu \rangle$, $\langle q, h^\mu \rangle$ unchanged. From (5.10) we immediately get

$$\mathscr{F}_V V(f_1)\mathscr{F}_V^* = V_F(f_1), \qquad \mathscr{F}_V U(f_2)\mathscr{F}_V^* = U_F(f_2) \tag{5.11}$$

for all $f_{\frac{1}{2}}$ which are finite linear combinations of g^ν's with $\nu \leq N_i$ and arbitrary h^μ's. Moreover, the norms $\|\cdot\|$, $\|\cdot\|_1$ and $\|\cdot\|_2$ are easily shown to be equivalent on $\mathscr{M}_V$. Since all $f_{\frac{1}{2}} \in \mathscr{M}_V$ are limits in the L^2 sense of finite linear combinations of the above type, Eq. (5.11) thus extends to all $f_{\frac{1}{2}} \in \mathscr{M}_V$ by continuity.

6. FREE HAMILTONIAN, MOMENTUM AND ANGULAR MOMENTUM OPERATORS

We shall now investigate, for a non-Fock representation as described in the previous Section, whether the free time evolution of creation-annihilation operators, given formally by

$$a(\underset{\sim}{k}) \rightarrow a(\underset{\sim}{k})e^{-i\omega t} \quad \text{and h. c.,} \tag{6.1}$$

is implementable. More precisely, we ask for a continuous unitary one-parameter group $U^0(t)$ on $\mathscr{H}$ satisfying

$$U^0(t)W(f)U^{0*}(t) = W(e^{i\omega t}f), \tag{6.2}$$

which follows from (6.1) by formal calculation. For the self-adjoint generator H^0 of $U^0(t)$, we shall also show that the usual formula

$$H^0 = \int d\mu(\underset{\sim}{k})a^*(\underset{\sim}{k})\omega a(\underset{\sim}{k}) \tag{6.3}$$

holds true in a sense to be specified more precisely later on.

If (6.2) with a unitary $U^0(t)$ is assumed to hold true for test functions $f = \frac{1}{\sqrt{2}}(f_1 + if_2)$ with arbitrary $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$, this means in particular that $W(e^{i\omega t}f)$ must be well-defined for all such f and all t. In other words, with $e^{i\omega t}f = \frac{1}{\sqrt{2}}(f_1^t + if_2^t)$, (6.2) implicitly requires

$$f^t_{\frac{1}{2}} = \cos \omega t \; f_{\frac{1}{2}} \mp \sin \omega t \; f_{\frac{2}{1}} \in \mathscr{L}_1 \tag{6.4}$$

for all t. These relations are trivially satisfied if $f_{\frac{1}{2}} \in \mathscr{M}$, but they are by no means obvious for more general $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$. However, it suffices to assume that (6.2) holds true for $f_{\frac{1}{2}} \in \mathcal{M}$, since the validity of (6.4) and (6.2) follows then by continuity. In order to show this, we consider an arbitrary $f = \frac{1}{\sqrt{2}}(f_1 + if_2)$ with $f_{\frac{1}{2}} \in \mathcal{L}_{\frac{1}{2}}$, choose two sequences $f_{\frac{1}{2}}^{\nu}$ in $\mathcal{M}$ ($\nu = 1, 2 \ldots$) such that $f_{\frac{1}{2}}^{\nu} \underset{\nu}{\rightarrow} f_{\frac{1}{2}}$ in $\mathcal{L}_{\frac{1}{2}}$ norm, and introduce the notations $f^{\nu} = \frac{1}{\sqrt{2}}(f_1^{\nu} + if_2^{\nu})$, $e^{i\omega t}f^{\nu} = \frac{1}{\sqrt{2}}(f_1^{\nu t} + if_2^{\nu t})$. By assumption we have

$$\mathrm{U}^0(t)\mathrm{W}(f^{\nu})\mathrm{U}^{0*}(t) = \mathrm{W}(e^{i\omega t}f^{\nu})$$

for all ν. For $\nu \rightarrow \infty$ the l. h. s. converges strongly to $\mathrm{U}^0(t)\mathrm{W}(f)\mathrm{U}^{0*}(t)$. Therefore the r. h. s. is also strongly convergent and, in particular, $\mathrm{W}(e^{i\omega t}f^{\nu})\Omega$ is a Cauchy sequence in $\mathcal{H}$. Explicit calculation yields

$$\begin{aligned} \| \mathrm{W}(e^{i\omega t}f^{\nu})\Omega - \mathrm{W}(e^{i\omega t}f^{\mu})\Omega \|^2 &= 2(1 - \mathrm{Re}\,(\Omega, \mathrm{W}(-e^{i\omega t}f^{\nu})\mathrm{W}(e^{i\omega t}f^{\mu})\Omega)) \\ &= 2\left(1 - \cos\{ f^{\nu}, f^{\mu} \}\, e^{-\frac{1}{4}(\|f_1^{\nu t} - f_1^{\mu t}\|_1^2 + \|f_2^{\nu t} - f_2^{\mu t}\|_2^2)}\right) \\ &\geq 2\left(1 - e^{-\frac{1}{4}(\|f_1^{\nu t} - f_1^{\mu t}\|_1^2 + \|f_2^{\nu t} - f_2^{\mu t}\|_2^2)}\right). \end{aligned}$$

Therefore $f_{\frac{1}{2}}^{\nu t}$ are Cauchy sequences in $\mathcal{L}_{\frac{1}{2}}$ which have limits $f_{\frac{1}{2}}' \in \mathcal{L}_{\frac{1}{2}}$ since $\mathcal{L}_{\frac{1}{2}}$ are complete, and with $f' = \frac{1}{\sqrt{2}}(f_1' + if_2')$ we get

$$\mathrm{U}^0(t)\mathrm{W}(f)\mathrm{U}^{0*}(t) = \mathrm{W}(f').$$

It remains to show that, with $f_{\frac{1}{2}}^t$ from (6.4), $f_{\frac{1}{2}}' = f_{\frac{1}{2}}^t$ for almost all $\underset{\sim}{k}$ or, equivalently,

$$\int d\mu(\underset{\sim}{k}) g_{\frac{2}{1}}(\underset{\sim}{k}) f_{\frac{2}{1}}'(k) = \int d\mu(\underset{\sim}{k}) g_{\frac{2}{1}}(\underset{\sim}{k}) f_{\frac{2}{1}}^t(\underset{\sim}{k}) \qquad \text{for all } g_{\frac{1}{2}} \in \mathcal{M}.$$

Using (4.9), the continuity of the bilinear form $\langle f_1, f_2 \rangle$ and the fact that $\cos \omega t\, g_1 \in \mathcal{M} \subset \mathcal{L}_1$ and $\sin \omega t\, g_1 \in \mathcal{M} \subset \mathcal{L}_2$, we easily obtain

$$\begin{aligned} \int d\mu(\underset{\sim}{k}) g_1(k) f_2'(\underset{\sim}{k}) &= \langle g_1, f_2' \rangle = \lim_{\nu \to \infty} \langle g_1, f_2^{\nu t} \rangle \\ &= \lim_{\nu \to \infty} (\langle f_1^{\nu}, \sin \omega t\, g_1 \rangle + \langle \cos \omega t\, g_1, f_2^{\nu} \rangle) \\ &= \langle f_1, \sin \omega t\, g_1 \rangle + \langle \cos \omega t\, g_1, f_2 \rangle = \int d\mu(\underset{\sim}{k}) g_1(\underset{\sim}{k}) f_2^t(\underset{\sim}{k}). \end{aligned}$$

The equation $f_1' = f_1^t$ follows in the same way. (By the way, the existence of $\mathrm{U}^0(t)$ also implies that the representation $f_1 \oplus f_2 \rightarrow f_1^t \oplus f_2^t$ of time translations on the Hilbert space $\mathcal{L}_1 \oplus \mathcal{L}_2$ is strongly continuous. This follows from a slight modification of the preceding argument. The same reasoning applies to space translations which are discussed below.)

From (6.2) and the strong differentiability of $W(\lambda f)$ on D we obtain that

$$U^0(t)\{ a, f \} U^{0*}(t) = \{ a, e^{i\omega t} f \}$$

holds true on D. With (3.14), finally, this leads to a rigorous version of (6.1),

$$U^0(t)(a, f)U^{0*}(t) = (a, e^{i\omega t} f) \qquad \text{and h. c.,} \tag{6.5}$$

which is also valid on D.

A sufficient condition for the existence of $U^0(t)$ is given by

LEMMA 2. — $U^0(t)$ exists if

$$\sum_{i=2}^{\infty} \omega_i \sum_{\nu=N_{i-1}+1}^{N_i} \left(b_\nu - \frac{1}{b_\nu}\right)^2 < \infty .$$

We will prove this in several steps. The proof is based on the following well-known direct product decomposition of Fock space. Denote by $\mathcal{M}_i$ the space of real square-integrable test functions $f(\underline{k})$ with support in V_i. Let $\mathcal{H}_i$ be the Fock space, with vacuum Ω_i, for an irreducible Fock representation V_{iF}, U_{iF} with test functions from $\mathcal{M}_i$. Then $\mathcal{H}$, the Fock space of the representation V_F, U_F considered before, is the incomplete direct product [11] of the spaces $\mathcal{H}_i$, spanned by product vectors strongly equivalent to $\bigotimes_i \Omega_i$. The latter, often called the reference vector, is the Fock vacuum in $\mathcal{H}$. Notation:

$$\mathcal{H} = \bigotimes_i (\mathcal{H}_i, \Omega_i), \qquad \bigotimes_i \Omega_i = \Omega . \tag{6.6}$$

Any $f \in \mathcal{M}$ is a finite sum

$$f = \sum_i f^i, \qquad f^i \in \mathcal{M}_i , \tag{6.7}$$

and for such f we have

$$V_F(f) = \bigotimes_i V_{iF}(f^i), \qquad U_F(f) = \bigotimes_i U_{iF}(f^i) . \tag{6.8}$$

(Here $V_{iF}(f^i) = U_{iF}(f^i) = \mathbb{1}_i$, the unit operator on $\mathcal{H}_i$, except for the finitely many i's for which $f^i \neq 0$ in (6.7).)

As easily shown, the operators $T_{\frac{1}{2}}$ map each $\mathcal{M}_i$ onto itself. Therefore (6.8) leads to a corresponding decomposition of the representation defined by (5.1) for $f \in \mathcal{M}$:

$$\left.\begin{aligned} V(f) = V_F(T_1 f) = \bigotimes_i V_{iF}(T_1 f^i), \\ U(f) = U_F(T_2 f) = \bigotimes_i U_{iF}(T_2 f^i). \end{aligned}\right\} \tag{6.9}$$

For $f = \frac{1}{\sqrt{2}}(f_2 + if_2)$ with $f_{\frac{1}{2}} \in \mathcal{M}$, Eqs. (3.8), (4.8) and (6.9) imply

$$W_F(f) = \bigotimes_i W_{iF}(f^i) \tag{6.10}$$

and

$$W(f) = \bigotimes_i W_{iF}(Tf^i), \tag{6.11}$$

with the decomposition

$$f = \sum_i f^i, \qquad f^i = \frac{1}{\sqrt{2}}(f_1^i + if_2^i), \qquad f_{\frac{1}{2}}^i \in \mathcal{M}_i$$

similar to (6.7), and

$$Tf^i = \frac{1}{\sqrt{2}}(T_1 f_1^i + iT_2 f_2^i).$$

(W_{iF}, of course, denotes the Weyl operators of the Fock representation V_{iF}, U_{iF} on $\mathcal{H}_i$.)

By the same method which leads to (5.11), we prove the existence of unitary operators $\mathcal{F}_i$ on $\mathcal{H}_i$ such that

$$\mathcal{F}_i V_{iF}(T_1 f_1^i)\mathcal{F}_i^* = V_{iF}(f_1^i), \qquad \mathcal{F}_i U_{iF}(T_2 f_2^i)\mathcal{F}_i^* = U_{iF}(f_2^i), \tag{6.12}$$

and thus

$$\mathcal{F}_i W_{iF}(Tf^i)\mathcal{F}_i^* = W_{iF}(f^i). \tag{6.13}$$

Since $\mathcal{H}_i$ are Fock spaces, there exist on $\mathcal{H}_i$ free Hamiltonians

$$H_{iF}^0 = \int_{V_i} d\mu(\underline{k}) a_{iF}^*(\underline{k}) \omega a_{iF}(\underline{k}) \tag{6.14}$$

and free time evolutions $U_{iF}^0(t) = e^{iH_{iF}^0 t}$ with

$$U_{iF}^0(t) W_{iF}(f^i) U_{iF}^{0*}(t) = W_{iF}(e^{i\omega t} f^i). \tag{6.15}$$

Here a_{iF}^* and a_{iF} denote the Fock creation and annihilation operators on $\mathcal{H}_i$. We define on $\mathcal{H}_i$

$$H_i^0 = \mathcal{F}_i^* H_{iF}^0 \mathcal{F}_i, \qquad U_i^0(t) = e^{iH_i^0 t} = \mathcal{F}_i^* U_{iF}^0(t) \mathcal{F}_i, \tag{6.16}$$

and denote by $\hat{H}_i^0$, $\hat{U}_i^0(t)$ the usual extension of these operators to $\mathcal{H}$:

$$\hat{H}_i^0 = H_i^0 \otimes \mathbb{1}, \qquad \hat{U}_i^0(t) = U_i^0(t) \otimes \mathbb{1}, \tag{6.17}$$

where $\mathbb{1}$ means $\bigotimes_{j \neq i} \mathbb{1}_j$. We will show below that, under the conditions of Lemma 2,

$$\left.\begin{array}{l} \mathrm{U}^0(t) = \bigotimes_i \mathrm{U}_i^0(t) = \operatorname*{s-lim}_{n\to\infty} \prod_{i=1}^{n} \hat{\mathrm{U}}_i^0(t) \\ \text{exists as a strongly continuous unitary} \\ \text{one-parameter group on } \mathscr{H}. \end{array}\right\} \tag{6.18}$$

($\mathrm{U}^0(t) = \bigotimes_i \mathrm{U}_i^0(t)$ means that it acts like $\mathrm{U}^0(t)\left(\bigotimes_i \phi_i\right) = \bigotimes_i \mathrm{U}_i^0(t)\phi_i$ on product vectors.)

Then $\mathrm{U}^0(t)$ satisfies (6.2), and is thus the desired free time evolution on $\mathscr{H}$. As shown before, it suffices to prove (6.2) for $f = \frac{1}{\sqrt{2}}(f_1 + if_2)$ with $f_{\frac{1}{2}} \in \mathscr{M}$. For such f, (6.11) and (6.18) imply

$$\begin{aligned} \mathrm{U}^0(t)\mathrm{W}(f)\mathrm{U}^{0*}(t) &= \operatorname*{s-lim}_{n\to\infty} \left(\bigotimes_{i\leq n} \mathrm{U}_i^0(t)\mathrm{W}_{i\mathrm{F}}(\mathrm{T}f^i)\mathrm{U}_i^{0*}(t)\right)\left(\bigotimes_{i>n} \mathrm{W}_{i\mathrm{F}}(\mathrm{T}f^i)\right) \\ &= \bigotimes_i \mathrm{U}_i^0(t)\mathrm{W}_{i\mathrm{F}}(\mathrm{T}f^i)\mathrm{U}_i^{0*}(t) \end{aligned}$$

since the number of $\mathrm{W}_{i\mathrm{F}}(\mathrm{T}f^i) \neq \mathbb{1}_i$ in (6.11) is finite. By (6.13), (6.15) and (6.16),

$$\mathrm{U}_i^0(t)\mathrm{W}_{i\mathrm{F}}(\mathrm{T}f^i)\mathrm{U}_i^{0*}(t) = \mathrm{W}_{i\mathrm{F}}(\mathrm{T}e^{i\omega t}f^i),$$

and thus

$$\begin{aligned} \mathrm{U}^0(t)\mathrm{W}(f)\mathrm{U}^{0*}(t) &= \bigotimes_i \mathrm{W}_{i\mathrm{F}}(\mathrm{T}e^{i\omega t}f^i) \\ &= \mathrm{W}_\mathrm{F}(\mathrm{T}e^{i\omega t}f) = \mathrm{W}(e^{i\omega t}f) \end{aligned}$$

by (6.11).

The self-adjoint generator of $\prod_{i=1}^{n} \hat{\mathrm{U}}_i^0(t)$ is

$$\mathrm{H}_n = \sum_{i=1}^{n} \hat{\mathrm{H}}_i^0,$$

and $\hat{\mathrm{H}}_i^0$ may be rewritten as

$$\hat{\mathrm{H}}_i^0 = \int_{\mathrm{V}_i} d\mu(\underset{\sim}{k})a^*(\underset{\sim}{k})\omega a(\underset{\sim}{k})$$

because, at least formally,

$$(\mathscr{F}_i^* a_{i\mathrm{F}}(\underset{\sim}{k})\mathscr{F}_i) \otimes \mathbb{1} = a(\underset{\sim}{k}) \qquad \text{for } \underset{\sim}{k} \in \mathrm{V}_i.$$

Since (6.18) implies [*12*]

$$\mathrm{H}^0 = \lim_{n\to\infty} \mathrm{H}_n$$

in a suitable sense, we thus obtain

$$H^0 = \sum_i \int_{V_i} d\mu(\underset{\sim}{k}) a^*(\underset{\sim}{k}) \omega a(\underset{\sim}{k}) . \tag{6.19}$$

If understood in this sense, Eq. (6.19) represents a somewhat more rigorous version of Eq. (6.3).

In order to complete the proof of Lemma 2, we finally have to verify (6.18). We use

LEMMA 3. — $U^0(t)$ exists (in the sense of (6.18)) iff

$$\sum_i |1 - (\Omega_i, U_i^0(t)\Omega_i)| < \infty \qquad \text{for all } t .$$

This is proved in [*13*]. A simple corollary is

LEMMA 4. — $U^0(t)$ exists if $\Omega_i \in D_{H_i^0}$ for $i \geq n$ with a suitable n, and

$$\sum_{i \geq n} (\Omega_i, |H_i^0| \Omega_i) < \infty .$$

(Here D_A denotes the domain of an operator A. Since $H_i^0 \geq 0$, the absolute value signs could be dropped here. However, Lemma 4 will also be applied to momentum operators, in which case the absolute value is essential.)

Proof. — $|1 - (\Omega_i, U_i^0(t)\Omega_i)| \leq 2$ for $i < n$ and all t. For $i \geq n$, use the spectral representation

$$H_i^0 = \int \lambda dE_i(\lambda)$$

which yields

$$\begin{aligned} |1 - (\Omega_i, U_i^0(t)\Omega_i)| &= \left| \int (1 - e^{i\lambda t}) d(\Omega_i, E_i(\lambda)\Omega_i) \right| \\ &\leq \int |1 - e^{i\lambda t}| d(\Omega_i, E_i(\lambda)\Omega_i) \\ &\leq |t| \int |\lambda| d(\Omega_i, E_i(\lambda)\Omega_i) = |t| (\Omega_i, |H_i^0| \Omega_i), \end{aligned}$$

and apply the if part of Lemma 3. ■

We will now show that

$$\Omega_i \in D_{H_i^0}, \qquad (\Omega_i, H_i^0\Omega_i) \leq \frac{\omega_i}{4} \sum_{\nu = N_{i-1}+1}^{N_i} \left(b_\nu - \frac{1}{b_\nu} \right)^2 \qquad \text{for } i \geq 2 . \tag{6.20}$$

With this, Lemma 4 (with $n = 2$) directly yields Lemma 2.

Assume $i \geq 2$. Since $\omega_{i+1} \leq \omega \leq \omega_i$ on V_i, we have

$$D_{H^0_{iF}} = D_{N_{iF}} \quad \text{and} \quad 0 \leq H^0_{iF} \leq \omega_i N_{iF} \tag{6.21}$$

with the number operator

$$N_{iF} = \int_{V_i} d\mu(\underset{\sim}{k}) a^*_{iF}(\underset{\sim}{k}) a_{iF}(\underset{\sim}{k})$$

on $\mathscr{H}_i$. Moreover, it is well-known that

$$N_{iF} \supset K_i \underset{\text{df.}}{=} \sum_{\nu=N_{i-1}+1}^{N_i} (a_{iF}, g^\nu)(g^\nu, a_{iF}) + \sum_\mu (a_{iF}, h^\mu)(h^\mu, a_{iF}), \tag{6.22}$$

if the functions h^μ ($\mu = 1, 2 \ldots$) together with the g^ν's form a complete orthonormal system in $\mathscr{M}_i$, and $\supset$ means operator extension. By (3.14), (4.4), (4.5) and (6.12),

$$\begin{aligned} \mathscr{F}_i^*(g^\nu, a_{iF})\mathscr{F}_i &= \frac{1}{\sqrt{2}} \mathscr{F}_i^*(\langle q_{iF}, g^\nu \rangle + i \langle p_{iF}, g^\nu \rangle)\mathscr{F}_i \\ &= \frac{1}{\sqrt{2}}(\langle q_{iF}, T_2 g^\nu \rangle + i \langle p_{iF}, T_1 g^\nu \rangle) \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{b_\nu}\langle q_{iF}, g^\nu \rangle + i b_\nu \langle p_{iF}, g^\nu \rangle\right) \\ &= \frac{1}{2}\left(b_\nu + \frac{1}{b_\nu}\right)(g^\nu, a_{iF}) + \frac{1}{2}\left(\frac{1}{b_\nu} - b_\nu\right)(a_{iF}, g^\nu) \end{aligned} \tag{6.23}$$

whereas, obviously,

$$\mathscr{F}_i^*(h^\mu, a_{iF})\mathscr{F}_i = (h^\mu, a_{iF}).$$

By this and $(h^\mu, a_{iF})\Omega_i = 0$, $\mathscr{F}_i^* K_i \mathscr{F}_i$ if applied to Ω_i reduces to the finite ν sum. The latter, by (6.23) and h. c., is a second-order polynomial in (g^ν, a_{iF}) and (a_{iF}, g^ν), which is well-defined on Ω_i. Thus

$$\Omega_i \in D_{\mathscr{F}_i^* K_i \mathscr{F}_i} \subseteq D_{H_i^0}$$

by (6.16), (6.21) and (6.22), and

$$(\Omega_i, H_i^0 \Omega_i) \leq \omega_i(\Omega_i, \mathscr{F}_i^* K_i \mathscr{F}_i \Omega_i).$$

Finally, by (6.22) and (6.23) one easily calculates

$$(\Omega_i, \mathscr{F}_i^* K_i \mathscr{F}_i \Omega_i) = \frac{1}{4} \sum_{\nu=N_{i-1}+1}^{N_i} \left(b_\nu - \frac{1}{b_\nu}\right)^2.$$

This completes the proof of (6.20) and Lemma 2.

For later use we note another elementary estimate. Since H^0_{iF} and N_{iF}

commute, they have a common spectral representation. Together with (6.21) this immediately implies

$$\| H^0_{iF}\Phi \|^2 \leq \omega_i^2 \| N_{iF}\Phi \|^2 \qquad \text{for} \qquad \Phi \in D_{N_{iF}} = D_{H^0_{iF}} .$$

Thus

$$\| H^0_i \Omega_i \|^2 = \| H^0_{iF} \mathscr{F}_i \Omega_i \|^2 \leq \omega_i^2 \| N_{iF} \mathscr{F}_i \Omega_i \|^2 = \omega_i^2 \| \mathscr{F}_i^* K_i \mathscr{F}_i \Omega_i \|^2$$
$$= \frac{\omega_i^2}{16} \sum_{\nu = N_{i-1}+1}^{N_i} \left(\left(b_\nu - \frac{1}{b_\nu} \right)^4 + 2 \left(b_\nu^2 - \frac{1}{b_\nu^2} \right)^2 \right), \qquad (6.24)$$

the last expression following again by an elementary calculation with (6.22) and (6.23).

The condition of Lemma 2, if satisfied, also implies the existence of translation operators $U(\underset{\sim}{x})$ with

$$U(\underset{\sim}{x}) W(f) U^*(\underset{\sim}{x}) = W(e^{-i\underset{\sim}{k}\underset{\sim}{x}} f),$$

and of the corresponding momentum operators $\underset{\sim}{P}$ with

$$U(\underset{\sim}{x}) = e^{-i\underset{\sim}{P}\underset{\sim}{x}} .$$

$U(\underset{\sim}{x})$ may be constructed in the same way as $U^0(t)$, i. e.,

$$U(\underset{\sim}{x}) = \text{s-}\lim_{n\to\infty} \left(\bigotimes_{i=1}^{n} U_i(\underset{\sim}{x}) \right) \otimes \mathbb{1} \qquad (6.25)$$

with

$$U_i(\underset{\sim}{x}) = \mathscr{F}_i^* U_{iF}(\underset{\sim}{x}) \mathscr{F}_i = e^{-i\underset{\sim}{P}_i\underset{\sim}{x}}$$

and the Fock translation operators

$$U_{iF}(\underset{\sim}{x}) = e^{-i\underset{\sim}{P}_{iF}\underset{\sim}{x}}$$

on $\mathscr{H}_i$. The spectrum of the Fock energy-momentum operators $(H^0_{iF}, \underset{\sim}{P}_{iF})$ on $\mathscr{H}_i$ is contained in the closed forward light cone (spectrum condition), which implies

$$| P^j_{iF} | \leq H^0_{iF} ,$$

and thus

$$| P^j_i | \leq H^0_i$$

for all components P^j_{iF}, P^j_i of $\underset{\sim}{P}_{iF}$ and $\underset{\sim}{P}_i$, respectively. The condition of Lemma 2 and (6.20) thus also imply $\Omega_i \in D_{P^j_i}$ for $i \geq 2$ and

$$\sum_{i\geq 2} (\Omega_i, | P^j_i | \Omega_i) < \infty ,$$

and the limit (6.25) then exists by Lemma 4.

Since $U^0_i(t)$ and $U_i(\underset{\sim}{x})$ commute for all i, the same follows from (6.18) and (6.25) for $U^0(t)$ and $U(\underset{\sim}{x})$. Moreover, the energy-momentum operator $(H^0, \underset{\sim}{P})$ also satisfies the spectrum condition, i. e., all self-adjoint generators $a_0H^0 - \underset{\sim}{a}\underset{\sim}{P}$ of translations $U(a_0)U(\underset{\sim}{a})$ into timelike or lightlike directions ($a_0^2 - \underset{\sim}{a}^2 \geq 0$) are non-negative. This follows directly [12] from

$$U(a_0)U(\underset{\sim}{a}) = \text{s-}\lim_{n\to\infty}\left(\bigotimes_{i=1}^{n} U_i(a_0)U_i(\underset{\sim}{a})\right) \otimes \mathbb{1}$$

and the spectrum conditions on $\mathcal{H}_i$. In particular, H^0 itself is non-negative. We mention without proof that, as expected, its spectrum is purely continuous and covers the non-negative real axis [14]. Another result, also not proved here [14], will turn out to be useful later on:

LEMMA 5. — If $\Omega_i \in D_{H_i^0}$ for all i,

$$\sum_i (\Omega_i, | H_i^0 | \Omega_i) < \infty \qquad \text{and} \qquad \sum_i \| H_i^0\Omega_i \|^2 < \infty\,,$$

then $\Omega = \bigotimes_i \Omega_i \in D_{H^0}$, with expectation value and mean square deviation of H^0 in state Ω given by

$$(\Omega, H^0\Omega) = \sum_i (\Omega_i, H_i^0\Omega_i)$$

and

$$\| H^0\Omega \|^2 - (\Omega, H^0\Omega)^2 = \sum_i (\| H_i^0\Omega_i \|^2 - (\Omega_i, H_i^0\Omega_i)^2)\,,$$

respectively.

By (6.20) and (6.24), the conditions of Lemma 5 are satisfied if $N_1 = 0$,

$$\sum_{i=2}^{\infty} \omega_i \sum_{N_{i-1}+1}^{N_i} \left(b_\nu - \frac{1}{b_\nu}\right)^2 < \infty$$

and

$$\sum_{i=2}^{\infty} \omega_i^2 \sum_{N_{i-1}+1}^{N_i} \left(\left(b_\nu - \frac{1}{b_\nu}\right)^4 + 2\left(b_\nu^2 - \frac{1}{b_\nu^2}\right)^2\right) < \infty\,.$$

As above, the conditions of Lemma 5 for the energy H^0 and the estimate

$$\| P_i^j\Omega_i \|^2 \leq \|H_i^0\Omega_i \|^2$$

(which follows from the spectrum condition on $\mathcal{H}_i$) imply analogous conditions for each component P^j of the momentum $\underset{\sim}{P}$.

The discussion of rotations and angular momentum operators is somewhat simpler. Spatial rotations R induce the transformations (for simplicity also denoted by R)

$$R : f(\underset{\sim}{k}) \to (Rf)(\underset{\sim}{k}) = f(R^{-1}\underset{\sim}{k}) \qquad (6.26)$$

of the function space $\mathscr{L}$. We now assume that the operators T_1 and T_2 commute with this representation (6.26) of the rotation group:

$$RT_{\frac{1}{2}}f = T_{\frac{1}{2}}Rf \qquad \text{for all R and all } f \in \mathscr{L}. \tag{6.27}$$

Then, by (4.6), the test function spaces $\mathscr{L}_{\frac{1}{2}} \subset \mathscr{L}$ are invariant under rotations R since L^2 is invariant. More precisely, the mappings $R : \mathscr{L}_{\frac{1}{2}} \to \mathscr{L}_{\frac{1}{2}}$ are onto, and are continuous unitary representations of the rotation group on the Hilbert spaces $\mathscr{L}_{\frac{1}{2}}$. (The representation property is obvious, and implies that the mappings are onto. With (6.27) and f, $g \in \mathscr{L}_{\frac{1}{2}}$, $\langle Rf, Rg \rangle_{\frac{1}{2}} = \langle T_{\frac{1}{2}}Rf, T_{\frac{1}{2}}Rg \rangle = \langle RT_{\frac{1}{2}}f, RT_{\frac{1}{2}}g \rangle = \langle T_{\frac{1}{2}}f, T_{\frac{1}{2}}g \rangle = \langle f, g \rangle_{\frac{1}{2}}$ since R is unitary on L^2. A similar argument yields the continuity of the representations.)

In the Fock representation, there is a continuous unitary representation U(R) of the rotation group on $\mathscr{H}$ with

$$U(R)V_F(f)U^*(R) = V_F(Rf), \qquad U(R)U_F(f)U^*(R) = U_F(Rf)$$

for all $f \in L^2$, and

$$U(R)\Omega = \Omega. \tag{6.28}$$

By (5.1) and (6.27) we have

$$U(R)V(f_1)U^*(R) = V(Rf_1), \qquad U(R)U(f_2)U^*(R) = U(Rf_2) \tag{6.29}$$

for all $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$, so that U(R) may serve as rotation operators also for the non-Fock representation (5.1). By (6.28), Ω is an eigenstate with eigenvalue zero of the corresponding angular momentum operators. Eqs. (4.10), (6.27) and the unitarity of R on L^2 imply

$$\langle Rf_1, Rf_2 \rangle = \langle f_1, f_2 \rangle \qquad \text{for } f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}},$$

thus (3.8) and (6.29) also yield

$$U(R)W(f)U^*(R) = W(Rf) \tag{6.30}$$

for complex test functions $f = \frac{1}{\sqrt{2}}(f_1 + if_2)$, with $Rf = \frac{1}{\sqrt{2}}(Rf_1 + iRf_2)$.

We will finally prove that, if $U^0(t)$ and $U(\underset{\sim}{x})$ also exist for such a rotationally invariant representation, the usual relations

$$U(R)U^0(t) = U^0(t)U(R) \tag{6.31}$$

and

$$U(R)U(\underset{\sim}{x}) = U(R\underset{\sim}{x})U(R) \tag{6.32}$$

are satisfied. By (6.32), U(R) and $U(\underset{\sim}{x})$ generate a representation of the Euclidean group.

In the sense of (6.18), the Fock rotation operators $U(R)$ on $\mathscr{H}$ also factorize in the form

$$U(R) = \bigotimes_i U_i(R), \tag{6.33}$$

$U_i(R)$ being Fock rotation operators on $\mathscr{H}_i$. (The existence of $\bigotimes_i U_i(R)$ follows from $U_i(R)\Omega_i = \Omega_i$ and Lemma 3, since each R belongs to some one-parameter subgroup of the rotation group. Eq. (6.33) follows because $U(R)$ and $\bigotimes_i U_i(R)$ leave the vacuum Ω invariant and induce the same transformations of the Weyl operators $W_F(f)$ with $f \in \mathscr{M}$.)

Consider $\hat{\mathscr{F}}_i = \mathscr{F}_i \otimes \mathbb{1}$ $\left(\text{where } \mathbb{1} \text{ means } \bigotimes_{j \neq i} \mathbb{1}_j\right)$ and particular Weyl operators $W(f)$ with $f \in \mathscr{M}_i$ which, by (6.11), are also of the form $W(f) = W_{iF}(Tf) \otimes \mathbb{1}$. Eqs. (6.13), (6.27) and (6.30) imply that, on $\mathscr{H}$, $U(R)\hat{\mathscr{F}}_i^* U^*(R)\hat{\mathscr{F}}_i$ commutes with $W(f)$. By (6.33) and the product form of $\hat{\mathscr{F}}_i$ and $W(f)$, this means that $U_i(R)\mathscr{F}_i^* U_i^*(R)\mathscr{F}_i$ commutes with $W_{iF}(Tf)$ on $\mathscr{H}_i$. Since the $W_{iF}(Tf)$ with $f_{\frac{1}{2}} \in \mathscr{M}_i$ are irreducible on $\mathscr{H}_i$, we obtain

$$U_i(R)\mathscr{F}_i^* = \phi_i(R)\mathscr{F}_i^* U_i(R)$$

with phase factors $\phi_i(R)$ which depend continuously on R. The representation property of the $U_i(R)$ implies that the $\phi_i(R)$ are also a representation of rotations, and thus

$$\phi_i(R) \equiv 1 .$$

The Fock operators on $\mathscr{H}_i$ are known to satisfy

$$U_i(R)U_{iF}^0(t)U_i^*(R) = U_{iF}^0(t)$$

$$U_i(R)U_{iF}(\underset{\sim}{x})U_i^*(R) = U_{iF}(R\underset{\sim}{x}) .$$

Thus, with (6.18) and (6.33)

$$\begin{aligned} U(R)U^0(t)U^*(R) &= \operatorname*{s-lim}_{n\to\infty} U(R)\left[\left(\bigotimes_{i=1}^n \mathscr{F}_i^* U_{iF}^0(t)\mathscr{F}_i\right) \otimes \left(\bigotimes_{i>n} \mathbb{1}_i\right)\right]U^*(R) \\ &= \operatorname*{s-lim}_{n\to\infty} \left[\left(\bigotimes_{i=1}^n U_i(R)\mathscr{F}_i^* U_{iF}^0(t)\mathscr{F}_i U_i^*(R)\right) \otimes \left(\bigotimes_{i>n} \mathbb{1}_i\right)\right] \\ &= \operatorname*{s-lim}_{n\to\infty} \left[\left(\bigotimes_{i=1}^n U_i^0(t)\right) \otimes \left(\bigotimes_{i>n} \mathbb{1}_i\right)\right] = U^0(t) \end{aligned}$$

and, similarly,

$$U(R)U(\underset{\sim}{x})U^*(R) = U(R\underset{\sim}{x}),$$

which proves (6.31) and (6.32).

7. PARTICULAR REPRESENTATION ADAPTED TO THE MODEL

We will now construct explicitly a representation in which the external current model is exactly soluble.

We take $\omega_i = e^{-i}$, and denote by $\chi_i(\underset{\sim}{k}) = \chi_i(\omega)$ the characteristic functions of the volumes $V_i : \omega_{i+1} \leq \omega \leq \omega_i$, $i = 2, 3 \ldots$. Introducing spherical coordinates ω, θ and ϕ in $\underset{\sim}{k}$-space, we denote by $Y_{lm}(\theta, \phi)$, $l = 0, 1 \ldots$, $m = -l \ldots +l$, real spherical harmonics, normalized according to

$$\int_0^{2\pi}\int_0^{\pi} Y_{lm}(\theta, \phi)Y_{l'm'}(\theta, \phi) \sin\theta \, d\theta \, d\phi = \delta_{ll'}\delta_{mm'} . \tag{7.1}$$

(The ϕ dependence of Y_{lm} is $\cos m\phi$ for $m \leq 0$ and $\sin m\phi$ for $m > 0$, instead of $e^{im\phi}$ as for the more usual complex spherical harmonics). For the functions $g^\nu(\underset{\sim}{k})$ defining the representation we choose

$$\left.\begin{aligned} &g^{ilm}(\underset{\sim}{k}) = \frac{\sqrt{2}\chi_i(\omega)}{\omega} Y_{lm}(\theta, \phi), \\ &i = 2, 3 \ldots; \; l = 0, 1 \ldots i-1 \; ; \; m = -l \ldots +l . \end{aligned}\right\} \tag{7.2}$$

They are orthonormal by (7.1), and the number of g^ν's with Supp $g^\nu \subseteq V_i$ is

$$n_i = \sum_{l=0}^{i-1} (2l+1) = i^2 \tag{7.3}$$

for $i \geq 2$ and $n_1 = 0$. The b_ν are taken as

$$b_{ilm} = b_i = \frac{1}{i}, \tag{7.4}$$

independent of l and m.

With these choices, the condition of Lemma 2 is satisfied:

$$\begin{aligned} \sum_{i=2}^{\infty} \omega_i \sum_{\nu=N_{i-1}+1}^{N_i} \left(b_\nu - \frac{1}{b_\nu}\right)^2 &= \sum_{i=2}^{\infty} e^{-i} n_i \left(b_i - \frac{1}{b_i}\right)^2 \\ &= \sum_{i=2}^{\infty} e^{-i} i^2 \left(\frac{1}{i} - i\right)^2 < \infty . \end{aligned}$$

Thus a free time evolution $U^0(t)$ and space translation operators $U(\underset{\sim}{x})$ exist. Moreover, as shown similarly from (6.24), the conditions of Lemma 5 are also satisfied, and thus

$$\Omega \in D_{H^0}\,, \tag{7.5}$$

as well as $\Omega \in D_{P^j}$ for each component P^j of the momentum $\underset{\sim}{P}$.

The representation is also rotationally invariant in the sense of Eq. (6.27). This follows because, for fixed l, the $2l + 1$ spherical harmonics transform under rotations R according to an orthogonal (= real unitary) representation $D^{(l)}(R)$ of the rotation group, and thus

$$(Rg^{ilm})(\underset{\sim}{k}) = \sum_{m'=-l}^{l} D^{(l)}_{mm'}(R^{-1})\, g^{ilm'}(\underset{\sim}{k})\,. \tag{7.6}$$

For arbitrary $f \in \mathscr{L}$ this implies

$$\langle g^{ilm}, Rf \rangle = \langle R^{-1} g^{ilm}, f \rangle = \sum_{m'=-l}^{l} D^{(l)}_{m'm}(R^{-1}) \langle g^{ilm'}, f \rangle \tag{7.7}$$

since the inner products $\langle\, , \rangle$ may be interpreted here as inner products in $\mathscr{M}_i$ on which R is unitary, and since $D^{(l)}_{mm'}(R) = D^{(l)}_{m'm}(R^{-1})$. With (7.6) and (7.7),

$$\begin{aligned} T_1 Rf &= Rf + \sum_{i\geq 2}(b_i - 1)\sum_{l<i}\sum_{m} \langle g^{ilm}, Rf \rangle g^{ilm} \\ &= Rf + \sum_{i\geq 2}(b_i - 1)\sum_{l<i}\sum_{m} \langle g^{ilm}, f \rangle Rg^{ilm} = RT_1 f, \end{aligned}$$

and similarly for T_2.

We next want to show that, with $b^t_{\substack{\text{in}\\\text{out}}}(k)$ and $g^t_{\substack{\text{in}\\\text{out}}}(k)$ given by (2.17), (2.18) and (2.23) (*), the Weyl operators $W(g^t_{\substack{\text{in}\\\text{out}}})$ and $W(-b^t_{\substack{\text{in}\\\text{out}}})$ exist in our representation, and are strongly continuous in t. For this it suffices, by Eq. (3.8), to prove that the real and imaginary parts of $b^t_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k})$ and $c^t_{\substack{\text{ret}\\\text{adv}}}(\underset{\sim}{k})$ belong to $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively, and are continuous in t in the $\mathscr{L}_1$ resp. $\mathscr{L}_2$ norm topology. Actually, having in mind later applications, we will even prove strong differentiability with respect to t, which is defined as follows: A one-parameter family f^t of vectors in a Hilbert space $\mathscr{H}$ is

(*) It is more convenient now to indicate the time dependence of test functions by a superfix t.

called (strongly) differentiable in $\mathscr{H}$, if there exists another one-parameter family $\dot{f}^t$ in $\mathscr{H}$ such that

$$\lim_{\substack{\tau \to 0 \\ \tau \neq 0}} \left\| \frac{f^{t+\tau} - f^t}{\tau} - \dot{f}^t \right\| = 0$$

for all t. A differentiable f^t is obviously continuous; $\dot{f}^t$ is called the (strong) derivative of f^t.

The following three lemmas are useful in this connection.

LEMMA 6. — For all $\underset{\sim}{v}$ with $|\underset{\sim}{v}| < 1$, the test functions

$$f_{\underset{\sim}{v}}(\underset{\sim}{k}) = \sqrt{2}\sum_{i=2}^{\infty} \frac{\chi_i(\omega)}{\omega - \underset{\sim}{k}\underset{\sim}{v}} = \sqrt{2}\sum_{i=2}^{\infty} \frac{\chi_i(\omega)}{\omega} \frac{1}{1 - \underset{\sim}{n}\underset{\sim}{v}}$$

$\left(\text{with } \underset{\sim}{n} = \dfrac{\underset{\sim}{k}}{\omega}\right)$ belong to $\mathscr{L}_1$.

Proof. — The function $\eta_{\underset{\sim}{v}}(\theta, \phi) = \dfrac{1}{1 - \underset{\sim}{n}\underset{\sim}{v}}$ is independent of ω and square-integrable on the unit sphere (since $|\underset{\sim}{n}\underset{\sim}{v}| \leq |\underset{\sim}{v}| < 1$), and may thus be expanded into spherical harmonics Y_{lm} with expansion coefficients $c_{lm}(\underset{\sim}{v})$. This implies

$$f_{\underset{\sim}{v}}(\underset{\sim}{k}) = \sum_{i \geq 2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm}(\underset{\sim}{v}) g^{ilm}(\underset{\sim}{k}), \tag{7.8}$$

if we extend the definition (7.2) of $g^{ilm}(\underset{\sim}{k})$ to arbitrary l. For

$$C_l(\underset{\sim}{v}) \underset{\text{df.}}{=} \sum_{m=-l}^{l} (c_{lm}(\underset{\sim}{v}))^2, \qquad l = 0, 1 \ldots \tag{7.9}$$

we will prove below the estimate

$$C_l(\underset{\sim}{v}) l^4 \leq C(\underset{\sim}{v}) \tag{7.10}$$

with some constant $C(\underset{\sim}{v})$. From (7.8) and $c_{lm}(\underset{\sim}{v}) = \langle g^{ilm}, f_{\underset{\sim}{v}} \rangle$ we obtain

$$\begin{aligned} (T_1 f_{\underset{\sim}{v}})(\underset{\sim}{k}) &= f_{\underset{\sim}{v}}(\underset{\sim}{k}) + \sum_{i \geq 2} \left(\frac{1}{i} - 1\right) \sum_{l<i} \sum_{m} \langle g^{ilm}, f_{\underset{\sim}{v}} \rangle g^{ilm}(\underset{\sim}{k}) \\ &= \sum_{i \geq 2} \left\{ \frac{1}{i} \sum_{l<i} \sum_{m} c_{lm}(\underset{\sim}{v}) g^{ilm}(\underset{\sim}{k}) + \sum_{l \geq i} \sum_{m} c_{lm}(\underset{\sim}{v}) g^{ilm}(\underset{\sim}{k}) \right\}. \end{aligned} \tag{7.11}$$

We have to show that $T_1 f_{\underset{\sim}{v}} \in L^2$ which, by (7.11) and (7.9), means

$$\sum_{i \geq 2} i^{-2} \sum_{l<i} C_l(\underset{\sim}{v}) + \sum_{i \geq 2} \sum_{l \geq i} C_l(\underset{\sim}{v}) < \infty \tag{7.12}$$

since the g^{ilm} are orthonormal. The first sum converges due to the presence of i^{-2} and because

$$\sum_{l<i} C_l(\underset{\sim}{v}) \le \sum_{l=0}^{\infty} C_l(\underset{\sim}{v}) = \sum_l \sum_m (c_{lm}(\underset{\sim}{v}))^2,$$

the last expression being finite due to the square-integrability of $\eta_{\underset{\sim}{v}}$ on the unit sphere. Since (7.10) implies

$$\sum_{l \ge i} C_l(\underset{\sim}{v}) \le C(\underset{\sim}{v}) \sum_{l \ge i} l^{-4} < \frac{C(\underset{\sim}{v})}{3}(i-1)^{-3},$$

the second sum in (7.12) is also convergent.

In order to prove (7.10), we consider a function $g \in L^2$ of the form

$$g(\underset{\sim}{k}) = \psi(\omega)\eta_{\underset{\sim}{v}}(\theta, \phi),$$

and apply to it the square I^2 of the angular momentum operator $\underset{\sim}{I} = \frac{1}{i}(\underset{\sim}{k} \times \underset{\sim}{\nabla}_k)$. Since $\underset{\sim}{I}$ acts on angles θ, ψ only, we have

$$I^2 g = \psi I^2 \eta_{\underset{\sim}{v}}.$$

Moreover, since I^2 is a real second-order differential operator in $\underset{\sim}{k}$ and $\eta_{\underset{\sim}{v}}(\underset{\sim}{k})$ is at least twice continuously differentiable, $I^2\eta_{\underset{\sim}{v}}$ is continuous, and thus square-integrable on the unit sphere, which implies $I^2 g \in L^2$. Using

$$g(\underset{\sim}{k}) = \psi(\omega) \sum_l \sum_m c_{lm}(\underset{\sim}{v}) Y_{lm}(\theta, \phi)$$

and

$$I^2 Y_{lm} = l(l+1) Y_{lm},$$

$I^2 g \in L^2$ implies

$$\| I^2 g \|^2 = \int_0^\infty d\omega \frac{\omega}{2} \psi^2(\omega) \sum_l (l(l+1))^2 \sum_m (c_{lm}(\underset{\sim}{v}))^2 < \infty,$$

from which (7.10) follows immediately. ■

LEMMA 7. — Functions $f \in \mathscr{L}$ which are bounded for small $|\underset{\sim}{k}| = \omega$ belong to $\mathscr{L}_2$.

Proof. — We have to show that $T_2 f \in L^2$, or $T_2 f - f \in L^2$ since, obviously, $f \in L^2$. By assumption, there exist M and n such that $|f(\underset{\sim}{k})| \le M$ on all V_i with $i \ge n$. Since

$$(T_2 f)(\underset{\sim}{k}) - f(\underset{\sim}{k}) = \sum_{i \ge 2} (i-1) \sum_{l<i} \sum_m c_{ilm} g^{ilm}(\underset{\sim}{k})$$

$$= \text{finite sum} + \sum_{i \ge n} (i-1) \sum_{l<i} \sum_m c_{ilm} g^{ilm}(\underset{\sim}{k}),$$

(with $c_{ilm} = \langle g^{ilm}, f \rangle$), $T_2 f - f \in L^2$ means

$$\sum_{i \geq n} (i-1)^2 \sum_{l<i} \sum_m (c_{ilm})^2 < \infty .$$

The latter is satisfied, because

$$\sum_{l<i} \sum_m (c_{ilm})^2 \leq \int_{V_i} d\mu(\underset{\sim}{k})(f(\underset{\sim}{k}))^2 \leq M^2 \int_{V_i} d\mu(\underset{\sim}{k}) = M^2 \pi(1-e^{-2})e^{-2i} . \quad \blacksquare$$

LEMMA 8. — Consider a one-parameter family f^τ in $\mathscr{L}$, defined in a neighbourhood I of τ_0 except possibly at $\tau = \tau_0$. If $| f^\tau(\underset{\sim}{k}) | \leq M$ for $|\underset{\sim}{k}| = \omega \leq \omega_0$ and all $\tau \neq \tau_0$ in I, then $\lim_{\tau\to\tau_0} \| f^\tau \| = 0$ implies $\lim_{\tau\to\tau_0} \| f^\tau \|_2 = 0$.

Proof. — $f^\tau \in \mathscr{L}_2 \subset L^2$ by Lemma 7, thus $\| f^\tau \|_2$ and $\| f^\tau \|$ actually exist. It suffices to prove that $\lim_{\tau\to\tau_0} \| f^\tau \| = 0$ implies $\lim_{\tau\to\tau_0} \| T_2 f^\tau - f^\tau \| = 0$. As in the preceding proof we get, with $c^\tau_{ilm} = \langle g^{ilm}, f^\tau \rangle$, the estimate

$$\| T_2 f^\tau - f^\tau \|^2 \leq \sum_{i<n} (i-1)^2 \sum_{l<i} \sum_m (c^\tau_{ilm})^2 + M^2\pi(1-e^{-2}) \sum_{i \geq n} (i-1)^2 e^{-2i}$$

which holds true for all $n > -\log \omega_0$ (so that $| f^\tau(\underset{\sim}{k}) | \leq M$ on all V_i with $i \geq n$). The second term on the r. h. s. goes to zero for $n \to \infty$, and is thus smaller than $\frac{\varepsilon}{2}$ for a suitable $n = n(\varepsilon)$. Using this $n(\varepsilon)$, we estimate the remaining term as

$$\sum_{i<n(\varepsilon)} (i-1)^2 \sum_{l<i} \sum_m (c^\tau_{ilm})^2 \leq n^2(\varepsilon) \sum_{ilm} (c^\tau_{ilm})^2 \leq n^2(\varepsilon) \| f^\tau \|^2 ,$$

which shows that this term also becomes smaller than $\frac{\varepsilon}{2}$ for sufficiently small $| \tau - \tau_0 |$. $\blacksquare$

As announced before, we shall now prove the following statements:

i) Re $b^t_{\substack{\text{in}\\ \text{out}}}$, Re $c^t_{\substack{\text{ret}\\ \text{adv}}}$ and their (partial) time derivatives Re $(ie^{i\omega t}\tilde{j}^t_{\substack{\text{in}\\ \text{out}}})$, Re $(ie^{i\omega t}\tilde{j}^t_{\mp})$ (cf. (2.17), (2.18) and (2.15)) belong to $\mathscr{L}_1$ for all t.

i)' The analogous statement for the imaginary parts, with $\mathscr{L}_2$ instead of $\mathscr{L}_1$.

ii) Re $b^t_{\substack{\text{in}\\ \text{out}}}$ and Re $c^t_{\substack{\text{ret}\\ \text{adv}}}$ are differentiable in $\mathscr{L}_1$, and have Re $(ie^{i\omega t}\tilde{j}^t_{\substack{\text{in}\\ \text{out}}})$ and Re $(ie^{i\omega t}\tilde{j}^t_{\mp})$ as (strong) derivatives.

ii)' The same with Im and $\mathscr{L}_2$, instead of Re and $\mathscr{L}_1$.

It suffices to prove these statements for b^t_{in} and c^t_{ret}; the proofs for b^t_{out} and c^t_{adv} are similar.

Proof of i). — We have

$$\operatorname{Re} b_{\text{in}}(\underset{\sim}{k}, 0) = \frac{\gamma_{\text{in}}\tilde{\rho}(\omega)}{\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}}} \cos \underset{\sim}{k}\underset{\sim}{s}_{\text{in}}$$
$$= \frac{\gamma_{\text{in}}}{\sqrt{2}} f_{\underset{\sim}{v}_{\text{in}}}(\underset{\sim}{k}) + \gamma_{\text{in}} \left\{ \frac{\tilde{\rho}(\omega)}{\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}}} \cos \underset{\sim}{k}\underset{\sim}{s}_{\text{in}} - \frac{1}{\sqrt{2}} f_{\underset{\sim}{v}_{\text{in}}}(\underset{\sim}{k}) \right\},$$

and $f_{\underset{\sim}{v}_{\text{in}}} \in \mathscr{L}_1$ by Lemma 6. The term in curly brackets, by definition of $f_{\underset{\sim}{v}_{\text{in}}}(\underset{\sim}{k})$, is given by

$$\frac{\tilde{\rho}(\omega)\cos \underset{\sim}{k}\underset{\sim}{s}_{\text{in}} - 1}{\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}}} = \left(\frac{\tilde{\rho}(\omega) - 1}{\omega} - \tilde{\rho}(\omega)\frac{1 - \cos \underset{\sim}{k}\underset{\sim}{s}_{\text{in}}}{\omega} \right) \frac{1}{1 - \underset{\sim}{n}\underset{\sim}{v}_{\text{in}}}$$

for $\omega \leq \omega_2 = e^{-2}$, and thus remains bounded for $\omega \to 0$ since $\tilde{\rho}(0) = 1$, $\tilde{\rho}'(0) = 0$ and $1 - \cos \underset{\sim}{k}\underset{\sim}{s}_{\text{in}} \leq \omega \,|\, \underset{\sim}{s}_{\text{in}} \,|$. Therefore this term is in $L^2 \subset \mathscr{L}_1$, and we get $\operatorname{Re} b_{\text{in}}^0 \in \mathscr{L}_1$.

Because of

$$|\, b_{\text{in}}(\underset{\sim}{k}, t) - b_{\text{in}}(\underset{\sim}{k}, 0) \,| = \frac{\gamma_{\text{in}} \,|\, \tilde{\rho}(\omega) \,|}{\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}}} \,|\, 1 - e^{i(\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}})t} \,| \leq |\, t \,|\, |\, \tilde{\rho}(\omega) \,|, \quad (7.13)$$

the real part of $b_{\text{in}}(\underset{\sim}{k}, t) - b_{\text{in}}(\underset{\sim}{k}, 0)$ is also square-integrable, and with $\operatorname{Re} b_{\text{in}}^0 \in \mathscr{L}_1$ this proves that $\operatorname{Re} b_{\text{in}}^t \in \mathscr{L}_1$ for all t.

The estimate (2.28) implies $\operatorname{Re} c_{\text{ret}}^t \in L^2 \subset \mathscr{L}_1$ for $t \leq 0$. For $t > 0$ we obtain

$$|\, c_{\text{ret}}(\underset{\sim}{k}, t) - c_{\text{ret}}(\underset{\sim}{k}, 0) \,| \leq \int_0^t |\, \tilde{j}_-(\underset{\sim}{k}, \tau) \,|\, d\tau \leq 2 \,|\, \tilde{\rho}(\omega) \,|\, t \quad (7.14)$$

from $|\, \tilde{j}_-(\underset{\sim}{k}, t) \,| \leq 2 \,|\, \tilde{\rho}(\omega) \,|$. Therefore $\operatorname{Re} c_{\text{ret}}^t - \operatorname{Re} c_{\text{ret}}^0 \in L^2 \subset \mathscr{L}_1$ also for $t > 0$. From

$$|\, ie^{i\omega t}\tilde{j}_{\text{in}}(\underset{\sim}{k}, t) \,| \leq |\, \tilde{\rho}(\omega) \,| \quad (7.15)$$

and

$$|\, ie^{i\omega t}\tilde{j}_-(\underset{\sim}{k}, t) \,| \leq 2 \,|\, \tilde{\rho}(\omega) \,| \quad (7.16)$$

we conclude that $\operatorname{Re}\left(ie^{i\omega t}\tilde{j}_{\text{in}}^t\right)$ and $\operatorname{Re}\left(ie^{i\omega t}\tilde{j}_-^t\right)$ belong to $L^2 \subset \mathscr{L}_1$ for all t.

Proof of i)'. — From

$$|\, \operatorname{Im} b_{\text{in}}(\underset{\sim}{k}, 0) \,| = \gamma_{\text{in}} \,|\, \tilde{\rho}(\omega) \,| \left| \frac{\sin \underset{\sim}{k}\underset{\sim}{s}_{\text{in}}}{\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}}} \right| \leq |\, \tilde{\rho}(\omega) \,| \frac{|\, \underset{\sim}{s}_{\text{in}} \,|}{1 - \underset{\sim}{n}\underset{\sim}{v}_{\text{in}}}$$

and Lemma 7 we see that $\operatorname{Im} b_{\text{in}}^0 \in \mathscr{L}_2$, so that (7.13) and Lemma 7 then imply $\operatorname{Im} b_{\text{in}}^t \in \mathscr{L}_2$ for all t. Similarly, Lemma 7 and Eqs. (2.28), (7.14), (7.15) and (7.16) imply the remaining statements in $i)'$.

Proof of ii). — With $\alpha = \omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}}$ and $\beta = -\underset{\sim}{k}\underset{\sim}{s}_{\text{in}}$ we have

$$b_{\text{in}}(\underset{\sim}{k}, t) = \gamma_{\text{in}} \frac{\tilde{\rho}(\omega)}{\alpha} e^{i(\alpha t + \beta)}, \quad ie^{i\omega t}\tilde{j}_{\text{in}}(\underset{\sim}{k}, t) = i\gamma_{\text{in}}\tilde{\rho}(\omega)e^{i(\alpha t + \beta)}$$

and thus

$$\left| \frac{b_{\text{in}}(\underset{\sim}{k}, t + \tau) - b_{\text{in}}(\underset{\sim}{k}, t)}{\tau} - ie^{i\omega t}\tilde{j}_{\text{in}}(\underset{\sim}{k}, t) \right| \leq |\tilde{\rho}(\omega)| \left| \frac{e^{i\alpha\tau} - 1}{\alpha\tau} - i \right| .$$

The Taylor expansion

$$e^{i\alpha\tau} = 1 + i\alpha\tau + \frac{(i\alpha\tau)^2}{2} e^{i\alpha\theta} \qquad \text{(with suitable } \theta = \theta(\alpha, \tau))$$

implies

$$\left| \frac{e^{i\alpha\tau} - 1}{\alpha\tau} - i \right| \leq \frac{|\alpha\tau|}{2} ;$$

therefore

$$\left| \frac{b_{\text{in}}(\underset{\sim}{k}, t + \tau) - b_{\text{in}}(\underset{\sim}{k}, t)}{\tau} - ie^{i\omega t}\tilde{j}_{\text{in}}(\underset{\sim}{k}, t) \right| \leq \frac{|\tau|}{2}(\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}}) |\tilde{\rho}(\omega)| . \quad (7.17)$$

A corresponding estimate for c^t_{ret} is

$$\left| \frac{c_{\text{ret}}(\underset{\sim}{k}, t + \tau) - c_{\text{ret}}(\underset{\sim}{k}, t)}{\tau} - ie^{i\omega t}\tilde{j}_{-}(\underset{\sim}{k}, t) \right| < |\tau| (3\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}} + D(t, T)) |\tilde{\rho}(\omega)| , \quad (7.18)$$

which holds true for arbitrary fixed t and T and all τ with $|\tau| \leq T$, and is obtained as follows. The l. h. s. of (7.18) may be written as

$$\left| \frac{1}{\tau} \int_t^{t+\tau} ds e^{i\omega s}\tilde{j}_{-}(\underset{\sim}{k}, s) - e^{i\omega t}\tilde{j}_{-}(\underset{\sim}{k}, t) \right| = | e^{i\omega\theta}\tilde{j}_{-}(\underset{\sim}{k}, \theta) - e^{i\omega t}\tilde{j}_{-}(\underset{\sim}{k}, t) | ,$$

where $\theta = \theta(\underset{\sim}{k}, t, \tau)$ lies between t and $t + \tau$. With α and β as before, we rewrite and estimate the last expression as

$$\begin{aligned} &|\tilde{\rho}(\omega)| \, | (\gamma(\theta) - \gamma(t))e^{i(\omega\theta - \underset{\sim}{k}\underset{\sim}{X}(\theta))} \\ &\qquad + \gamma(t)(e^{i(\omega\theta - \underset{\sim}{k}\underset{\sim}{X}(\theta))} - e^{i(\omega t - \underset{\sim}{k}\underset{\sim}{X}(t))}) - \gamma_{\text{in}}(e^{i(\alpha\theta + \beta)} - e^{i(\alpha t + \beta)}) | \\ &\leq |\tilde{\rho}(\omega)| (|\gamma(\theta) - \gamma(t)| + \gamma(t) | 1 - e^{i\{\omega(t-\theta) - \underset{\sim}{k}(\underset{\sim}{X}(t) - \underset{\sim}{X}(\theta))\}} | + \gamma_{\text{in}} | 1 - e^{i\alpha(t-\theta)} |) \\ &< |\theta - t| (3\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}} + D(t, T)) |\tilde{\rho}(\omega)| < |\tau| (3\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}} + D(t, T)) |\tilde{\rho}(\omega)| . \end{aligned}$$

Here we have used

$$| 1 - e^{i\alpha(t-\theta)} | \leq | \alpha(t - \theta) | = (\omega - \underset{\sim}{k}\underset{\sim}{v}_{\text{in}}) | \theta - t | ,$$

$$| 1 - e^{i\{\omega(t-\theta) - \underset{\sim}{k}(\underset{\sim}{X}(t) - \underset{\sim}{X}(\theta))\}} | \leq \omega | \theta - t | + | \underset{\sim}{k}(\underset{\sim}{X}(\theta) - \underset{\sim}{X}(t)) | < 2\omega | \theta - t |$$

(note that $| \underset{\sim}{X}(\theta) - \underset{\sim}{X}(t) | < | \theta - t |$ since $| \dot{\underset{\sim}{X}} | < 1$), and

$$| \gamma(\theta) - \gamma(t) | \leq D(t, T) | \theta - t | \qquad \text{for } | \theta - t | \leq T, \qquad t \text{ and T fixed.}$$

The last estimate follows since $| \ddot{\underset{\sim}{X}}(\theta) |$ is continuous, and thus bounded by some $N(t, T)$ for $| \theta - t | \leq T$. With the constant M explained after Eq. (2.19) we thus get

$$| \gamma(\theta) - \gamma(t) | \leq M \, | \, | \dot{\underset{\sim}{X}}(\theta) | - | \dot{\underset{\sim}{X}}(t) | \, | \leq M | \dot{\underset{\sim}{X}}(\theta) - \dot{\underset{\sim}{X}}(t) | \leq MN(t, T) | \theta - t | .$$

Denoting by L_c^2 the space of complex square-integrable functions $f(\underset{\sim}{k})$ and by $\|\cdot\|_c$ the L_c^2 norm, the relation $\|\mathrm{Re}\, f\| \leq \|f\|_c$, Lemma 1 *b*) and (7.17) imply

$$\left\| \frac{\mathrm{Re}\, b_{\mathrm{in}}^{t+\tau} - \mathrm{Re}\, b_{\mathrm{in}}^{t}}{\tau} - \mathrm{Re}\left(ie^{i\omega t}\tilde{j}_{\mathrm{in}}^{t}\right)\right\|_1$$
$$\leq \left\| \frac{b_{\mathrm{in}}^{t+\tau} - b_{\mathrm{in}}^{t}}{\tau} - ie^{i\omega t}\tilde{j}_{\mathrm{in}}^{t}\right\|_c \leq \frac{|\tau|}{2}\|(\omega - \underset{\sim}{k}\underset{\sim}{v}_{\mathrm{in}})\tilde{\rho}(\omega)\| \underset{\tau\to 0}{\longrightarrow} 0$$

since $(\omega - \underset{\sim}{k}\underset{\sim}{v}_{\mathrm{in}})\tilde{\rho}(\omega) \in L^2$. Thus $\mathrm{Re}\, b_{\mathrm{in}}^t$ is differentiable in $\mathscr{L}_1$, with derivative $\mathrm{Re}\left(ie^{i\omega t}\tilde{j}_{\mathrm{in}}^t\right)$. Similarly, (7.18) implies that $\mathrm{Re}\, c_{\mathrm{ret}}^t$ is differentiable in $\mathscr{L}_1$, with derivative $\mathrm{Re}\left(ie^{i\omega t}\tilde{j}_{-}^t\right)$.

Proof of ii)'. — The estimates (7.17) and (7.18) remain true for the imaginary parts of the left hand sides. The right hand sides are, in any finite τ interval, uniformly bounded for all $\underset{\sim}{k}$, and tend to zero in L^2 norm for $\tau \to 0$. Therefore Lemma 8 immediately yields the desired conclusions.

The following additional statements will also be used later on:

iii) $\mathrm{Re}\,(\dot{b}^t_{\substack{\mathrm{in}\\ \mathrm{out}}})$ and $\mathrm{Re}\,(\dot{g}^t_{\substack{\mathrm{in}\\ \mathrm{out}}})$ are continuous in t with respect to the $\mathscr{L}_1$ norm.

iii)' The same with Im and $\mathscr{L}_2$, instead of Re and $\mathscr{L}_1$.

Proof. — We use $\dot{b}^t_{\substack{\mathrm{in}\\ \mathrm{out}}} = ie^{i\omega t}\tilde{j}^t_{\substack{\mathrm{in}\\ \mathrm{out}}}$, $\dot{g}^t_{\substack{\mathrm{in}\\ \mathrm{out}}} = -ie^{i\omega t}\tilde{j}^t$ and the elementary estimates

$$|e^{i\omega(t+\tau)}\tilde{j}_{\substack{\mathrm{in}\\ \mathrm{out}}}(\underset{\sim}{k}, t+\tau) - e^{i\omega t}\tilde{j}_{\substack{\mathrm{in}\\ \mathrm{out}}}(\underset{\sim}{k}, t)| \leq |\tau|\,(\omega - \underset{\sim}{k}\underset{\sim}{v}_{\substack{\mathrm{in}\\ \mathrm{out}}})\,|\tilde{\rho}(\omega)|,$$

$$|e^{i\omega(t+\tau)}\tilde{j}(\underset{\sim}{k}, t+\tau) - e^{i\omega t}\tilde{j}(\underset{\sim}{k}, t)| < |\tau|\,(2\omega + \mathrm{D}(t, \mathrm{T}))\,|\tilde{\rho}(\omega)|.$$

(The latter is true for $|\tau| \leq \mathrm{T}$, and is proved exactly as (7.18).) These estimates lead to *iii*) and *iii*)' in the same way, as (7.17) and (7.18) have led to *ii*) and *ii*)'.

In Section 9 we will use that

$$\text{s-}\lim_{t\to\mp\infty} \mathrm{W}(-e^{-i\omega t}c^t_{\substack{\mathrm{ret}\\ \mathrm{adv}}}) = \mathbb{1}. \tag{7.19}$$

This follows immediately if

$$\lim_{t\to\mp\infty} \|\mathrm{Re}\,(e^{-i\omega t}c^t_{\substack{\mathrm{ret}\\ \mathrm{adv}}})\|_1 = 0 \tag{7.20}$$

and

$$\lim_{t\to\mp\infty} \|\mathrm{Im}\,(e^{-i\omega t}c^t_{\substack{\mathrm{ret}\\ \mathrm{adv}}})\|_2 = 0. \tag{7.21}$$

Since $(\mathrm{M} + \omega)\tilde{\rho}(\omega) \in L^2$ and thus, by (2.28),

$$\|\mathrm{Re}\,(e^{-i\omega t}c^t_{\substack{\mathrm{ret}\\ \mathrm{adv}}})\|_1 \leq \|\mathrm{Re}\,(e^{-i\omega t}c^t_{\substack{\mathrm{ret}\\ \mathrm{adv}}})\| \leq \|c^t_{\substack{\mathrm{ret}\\ \mathrm{adv}}}\|_c \leq \frac{\mathrm{C}}{\varepsilon(1+|t|)^{\varepsilon}}\|(\mathrm{M}+\omega)\tilde{\rho}(\omega)\|$$

for $t \lesseqgtr 0$, we immediately get (7.20). Moreover, again by (2.28),

$$\| \operatorname{Im} (e^{-i\omega t} c^t_{\substack{\text{ret}\\\text{adv}}}) \| \leq \| c^t_{\substack{\text{ret}\\\text{adv}}} \|_c \leq \frac{C}{\varepsilon(1 + |t|)^\varepsilon} \| (M + \omega)\tilde{\rho}(\omega) \| \xrightarrow[t \to \mp\infty]{} 0$$

and

$$| \operatorname{Im} (e^{-i\omega t} c_{\substack{\text{ret}\\\text{adv}}}(\underset{\sim}{k}, t)) | \leq \frac{C}{\varepsilon} (M + \omega) | \tilde{\rho}(\omega) | \qquad \text{for} \qquad t \lesseqgtr 0 .$$

Therefore Lemma 8 is applicable (with t instead of τ, $t_0 = \mp \infty$ and $\mathrm{I} : t \lesseqgtr 0\big)$ to $\operatorname{Im} (e^{-i\omega t} c^t_{\substack{\text{ret}\\\text{adv}}})$, and leads to (7.21).

8. RIGOROUS SOLUTION OF THE MODEL

We now assume that the creation and annihilation operators $a^*_{\text{out}}(\underset{\sim}{k})$ and $a_{\text{out}}(\underset{\sim}{k})$ are represented according to the previous Section, and denote the corresponding Weyl operators, the free time evolution, translation and rotation operators by $W_{\text{out}}(f)$, $U^0_{\text{out}}(t)$, $U_{\text{out}}(\underset{\sim}{x})$ and $U_{\text{out}}(R)$, respectively.

We want to interpret the field operators $A(x)$ and $\dot{A}(x)$ as operator-valued distributions in the representation space $\mathscr{H}$, in the sense that their space averages

$$A(f, t) = \int d^3\underset{\sim}{x} A(\underset{\sim}{x}, t) f(\underset{\sim}{x}) \tag{8.1}$$

and

$$\dot{A}(f, t) = \int d^3\underset{\sim}{x} \dot{A}(\underset{\sim}{x}, t) f(\underset{\sim}{x}) \tag{8.2}$$

exist as self-adjoint operators for all t and all real functions $f(\underset{\sim}{x})$ from a suitable test function space. They shall have a common dense domain D on which they satisfy the canonical commutation relations

$$\left.\begin{aligned} [A(f, t), A(g, t)] &= 0 = [\dot{A}(f, t), \dot{A}(g, t)] , \\ [A(f, t), \dot{A}(g, t)] &= i \int d^3\underset{\sim}{x} f(\underset{\sim}{x}) g(\underset{\sim}{x}) . \end{aligned}\right\} \tag{8.3}$$

With the Fourier transform of $f(\underset{\sim}{x})$,

$$\tilde{f}(\underset{\sim}{k}) = (2\pi)^{-3/2} \int d^3\underset{\sim}{x} e^{-i\underset{\sim}{k}\underset{\sim}{x}} f(\underset{\sim}{x}) ,$$

we obtain from the formulae of Sections 2 and 3 by formal calculation:

$$A(f, t) = 2 \{ a_{\text{out}}, ie^{i\omega t}\tilde{f} \} - 2 \{ g^t_{\text{out}}, ie^{i\omega t}\tilde{f} \} \tag{8.4}$$

and

$$\dot{A}(f, t) = - 2 \{ a_{\text{out}}, \omega e^{i\omega t}\tilde{f} \} + 2 \{ g^t_{\text{out}}, \omega e^{i\omega t}\tilde{f} \} . \tag{8.5}$$

Thus field averages are well-defined and self-adjoint if, for all t, the real and imaginary parts of $ie^{i\omega t}\tilde{f}$ and $\omega e^{i\omega t}\tilde{f}$ belong to $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively. This is satisfied most easily if we require

$$\omega \operatorname{Re} \tilde{f} \in \mathscr{M}, \qquad \omega \operatorname{Im} \tilde{f} \in \mathscr{M}, \tag{8.6}$$

which of course implies

$$\operatorname{Re} \tilde{f} \in \mathscr{M}, \qquad \operatorname{Im} \tilde{f} \in \mathscr{M}. \tag{8.7}$$

Together with the reality condition for $f(\underset{\sim}{x})$,

$$\tilde{f}(\underset{\sim}{k}) = \overline{\tilde{f}(-\underset{\sim}{k})}, \tag{8.8}$$

(8.6) thus defines a space $\mathscr{N}$ of real test functions $f(\underset{\sim}{x})$ for which (8.1) and (8.2) make sense. Since $\mathscr{M} \subset L^2$, (8.7) implies that any $f(\underset{\sim}{x}) \in \mathscr{N}$ is square-integrable. (We could as well use different test function spaces for (8.1) and (8.2), namely, $\mathscr{N}_1$ defined by (8.7) and (8.8) for (8.1), and $\mathscr{N}_2 = \mathscr{N} \subset \mathscr{N}_1$ for (8.2). This is sometimes useful, see Eq. (9.4) below. We could also enlarge the test function space $\mathscr{N}$ by requiring, instead of (8.6), $\operatorname{Im} \tilde{f}, \omega \operatorname{Re} \tilde{f} \in \mathscr{L}_1$ and $\operatorname{Re} \tilde{f}, \omega \operatorname{Im} \tilde{f} \in \mathscr{L}_2$. Eqs. (8.4) and (8.5) still make sense for such $\tilde{f}$ since the factors $e^{i\omega t}$ do not affect the admissibility of the test functions $i\tilde{f}$ and $\omega\tilde{f}$; cf. the discussion of Eq. (6.2).)

By (8.4) and (8.5), the domain D described in Section 5 is also a suitable domain for the field averages (8.1) and (8.2). The commutation relations (8.3) on D follow from (8.4), (8.5) and (3.13) by straightforward calculation. Note that all $\underset{\sim}{k}$-space test functions involved in this calculation belong to $\mathscr{M}$, so that one may use the « naive » definitions (3.4) and (3.10) of $\langle f_1, f_2 \rangle$ and $\{ f, g \}$. Thus Eq. (8.3) actually contains $\int d^3\underset{\sim}{x} f(\underset{\sim}{x}) g(\underset{\sim}{x})$, instead of some subtle generalization of this integral. Eqs. (8.4) and (8.5) also imply that, for each fixed time t, the set of operators $e^{iA(f,t)}$ and $e^{i\dot{A}(g,t)}$ with $f, g \in \mathscr{N}$ is irreducible on $\mathscr{H}$.

It is easily shown now that the time evolution of the field averages (8.1) and (8.2) is implementable. There exists a continuous unitary one-parameter family $U_+(t)$ such that, on D,

$$U_+(t)\overset{(\cdot)}{A}(f, 0)U_+^*(t) = \overset{(\cdot)}{A}(f, t). \tag{8.9}$$

In fact,

$$U_+(t) = e^{i\alpha_{\text{out}}(t)} W_{\text{out}}(g_{\text{out}}^t) U_{\text{out}}^0(t) W_{\text{out}}(-g_{\text{out}}^0), \tag{8.10}$$

with an arbitrary continuous phase $\alpha_{\text{out}}(t)$, has the required property. Continuity is obvious since $\alpha_{\text{out}}(t)$, $W_{\text{out}}(g_{\text{out}}^t)$ and $U_{\text{out}}^0(t)$ are continuous. By (8.4),

$$e^{iA(f,t)} = e^{-2i\{g_{\text{out}}^t, ie^{i\omega t}\tilde{f}\}} W_{\text{out}}(ie^{i\omega t}\tilde{f}).$$

Using this, a simple calculation with (8.10), (3.9) and (6.2) yields

$$U_+(t)e^{iA(f,0)}U_+^*(t) = e^{iA(f,t)},$$

from which (8.9) for $A(f, t)$ follows. Similarly, (8.5) yields (8.9) for $\dot{A}(f, t)$.

The representation chosen here also yields a unitary S matrix, given by

$$S = W_{out}(g^0_{out} - g^0_{in}) = W_{out}(g^t_{out} - g^t_{in}), \qquad (8.11)$$

the last equality following from (2.25). For arbitrary $f = \frac{1}{\sqrt{2}}(f_1 + if_2)$ with $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$, (8.11) and (3.9) lead to

$$\begin{aligned} SW_{out}(f)S^* &= e^{-2i\{(g^0_{out} - g^0_{in}), f\}}W_{out}(f) \\ &= e^{2i\{(a_{out} - g^0_{out} + g^0_{in}), f\}}. \end{aligned}$$

By (2.27), the r. h. s. is just $W_{in}(f)$. Since S is one of the Weyl operators which leave invariant the domain D used before for the « out » operators, we may use this same domain D also for the « in » operators. Then rigorous versions of (2.29) like

$$S\{a_{out}, f\}S^* = \{a_{in}, f\} = \{a_{out}, f\} - \{(g^t_{out} - g^t_{in}), f\} \qquad (8.12)$$

hold true on D. Free time evolution, translation and rotation operators for the « in » operators are most simply defined by

$$U^0_{in}(t) = SU^0_{out}(t)S^*, \qquad \text{etc.}$$

Using the formulae of Section 2, field averages may now also be expressed in the form

$$A(f, t) = 2\{a_{in}, ie^{i\omega t}\tilde{f}\} - 2\{g^t_{in}, ie^{i\omega t}\tilde{f}\}, \qquad (8.13)$$

$$\dot{A}(f, t) = -2\{a_{in}, \omega e^{i\omega t}\tilde{f}\} + 2\{g^t_{in}, \omega e^{i\omega t}\tilde{f}\}. \qquad (8.14)$$

The equivalence of this with (8.4) and (8.5) is obvious from (8.12). This leads to another description of time evolution, namely

$$U_-(t)\overset{(\cdot)}{A}(f, 0)U^*_-(t) = \overset{(\cdot)}{A}(f, t) \qquad (8.15)$$

with

$$U_-(t) = e^{i\alpha_{in}(t)}W_{in}(g^t_{in})U^0_{in}(t)W_{in}(-g^0_{in}), \qquad (8.16)$$

where $\alpha_{in}(t)$ is another arbitrary continuous phase. By irreducibility, $U_+(t)$ and $U_-(t)$ may differ from each other at most by a phase factor, which becomes 1 for suitably chosen $\alpha_{in}(t)$ and $\alpha_{out}(t)$. Indeed, by (8.16),

$$U_-(t) = e^{i\alpha_{in}(t)}SW_{out}(g^t_{in})U^0_{out}(t)W_{out}(-g^0_{in})S^*,$$

and inserting S from (8.11) we easily find

$$U_-(t)U^*_+(t) = e^{i(\alpha_{in}(t) - \alpha_{out}(t) + \{g^t_{in}, g^t_{out}\} - \{g^0_{in}, g^0_{out}\})}. \qquad (8.17)$$

We choose phases $\alpha_{\substack{in\\out}}(t)$ which satisfy the equations

$$\dot{\alpha}_{\substack{in\\out}}(t) = \{\dot{g}^t_{\substack{in\\out}}, g^t_{\substack{in\\out}}\}. \qquad (8.18)$$

(This is possible since the statements proved in Section 7 imply that the right hand sides of (8.18) exist and are continuous in t.) Then

$$\dot{\alpha}_{in}(t) - \dot{\alpha}_{out}(t) = \{\dot{g}^t_{in}, g^t_{in}\} - \{\dot{g}^t_{out}, g^t_{out}\} = \{\dot{g}^t_{in}, (g^0_{in} - g^0_{out})\}$$

since, by (2.25),

$$\dot{g}^t_{out} = \dot{g}^t_{in} \quad \text{and} \quad g^t_{in} - g^t_{out} = g^0_{in} - g^0_{out} .$$

Thus

$$\begin{aligned} \alpha_{in}(t) - \alpha_{out}(t) &= \{ g^t_{in}, (g^0_{in} - g^0_{out}) \} + \text{const.} \\ &= \{ g^t_{in}, (g^t_{in} - g^t_{out}) \} + \text{const.} \\ &= - \{ g^t_{in}, g^t_{out} \} + \text{const.}, \end{aligned}$$

and from this and (8.17) we see that

$$U_-(t) = U_+(t) = U(t) \tag{8.19}$$

if the integration constants in (8.18) are suitably chosen. The deeper reason for postulating (8.18) (and (8.25), see below) will become clear in Section 10.

Analogous to (8.10) and (8.16) we may also construct unitary operators which, at a fixed time t, implement the spatial rotations and translations of field averages. For each t these operators form a representation of the Euclidean group. For instance, for space rotations we have

$$\overset{(\cdot)}{A}(Rf, t) = U_{\mp}(R, t)\overset{(\cdot)}{A}(f, t)U^*_{\mp}(R, t) \tag{8.20}$$

with

$$U_{\mp}(R, t) = W_{\substack{in\\out}}(g^t_{\substack{in\\out}})U_{\substack{in\\out}}(R)W_{\substack{in\\out}}(- g^t_{\substack{in\\out}}), \tag{8.21}$$

as follows from (8.4), (8.5), (8.13), (8.14) and (6.30). (Here $(Rf)(\underset{\sim}{x}) = f(R^{-1}\underset{\sim}{x})$.) Moreover, Eqs. (8.11) and (3.9) imply $U_-(R, t) = U_+(R, t)$.

The asymptotic fields $A_{\substack{in\\out}}(x)$, defined formally in Section 2, also exist as operator-valued distributions on $\mathscr{H}$, with the same domain D and the same test function space $\mathscr{N}$ as above, since one easily obtains by formal calculation the expressions

$$\left.\begin{aligned} A_{\substack{in\\out}}(f, t) &= 2\{ a_{\substack{in\\out}}, ie^{i\omega t}\tilde{f} \} + 2\{ b^t_{\substack{in\\out}}, ie^{i\omega t}\tilde{f} \}, \\ \dot{A}_{\substack{in\\out}}(f, t) &= - 2\{ a_{\substack{in\\out}}, \omega e^{i\omega t}\tilde{f} \} - 2\{ b^t_{\substack{in\\out}}, \omega e^{i\omega t}\tilde{f} \} \end{aligned}\right\} \tag{8.22}$$

for the field averages. They also satisfy canonical commutation relations of the form (8.3). The time evolutions of the field averages (8.22) are implemented by the continuous unitary one-parameter families

$$U_{\substack{in\\out}}(t) = e^{i\beta_{\substack{in\\out}}(t)}W_{\substack{in\\out}}(- b^t_{\substack{in\\out}})U^0_{\substack{in\\out}}(t)W_{\substack{in\\out}}(b^0_{\substack{in\\out}}), \tag{8.23}$$

i. e.,

$$U_{\substack{in\\out}}(t)\overset{(\cdot)}{A}_{\substack{in\\out}}(f, 0)U^*_{\substack{in\\out}}(t) = \overset{(\cdot)}{A}_{\substack{in\\out}}(f, t). \tag{8.24}$$

The phases $\beta_{\substack{in\\out}}(t)$ are chosen here to satisfy

$$\dot{\beta}_{\substack{in\\out}}(t) = \{ \dot{b}^t_{\substack{in\\out}}, b^t_{\substack{in\\out}} \}. \tag{8.25}$$

Space translations and rotations of $A_{\substack{in\\out}}(\underset{\sim}{x}, t)$ are, of course, also implementable.

Clearly, the free fields $A^0_{\substack{in\\out}}(\underset{\sim}{x})$ are also operator-valued distributions of the type considered here, with averages

$$A^0_{\substack{in\\out}}(f, t) = 2\{ a_{\substack{in\\out}}, ie^{i\omega t}\tilde{f} \},$$

$$\dot{A}^0_{\substack{in\\out}}(f, t) = -2\{ a_{\substack{in\\out}}, \omega e^{i\omega t}\tilde{f} \},$$

and with space-time transformations implemented by $U^0_{\substack{in\\out}}(t)$, $U_{\substack{in\\out}}(\underset{\sim}{x})$ and $U_{\substack{in\\out}}(R)$.

For each fixed time t, the fields $A(x)$, $A_{\substack{in\\out}}(x)$ and $A^0_{\substack{in\\out}}(x)$—together with their time derivatives—are unitarily equivalent, and can be transformed into each other by means of suitable Weyl operators. For instance,

$$\overset{(\cdot)}{A}_{\substack{in\\out}}(f, t) = W_{\substack{in\\out}}(c^t_{\substack{ret\\adv}})\overset{(\cdot)}{A}(f, t)W^*_{\substack{in\\out}}(c^t_{\substack{ret\\adv}}),$$

as easily seen from the explicit form of the field operators and Eq. (3.9).

9. ASYMPTOTIC CONDITIONS

The formulae of the previous Section yield

$$A(f, t) - A_{\substack{in\\out}}(f, t) = 2\{ c^t_{\substack{ret\\adv}}, ie^{i\omega t}\tilde{f} \} = 2\{ e^{-i\omega t}c^t_{\substack{ret\\adv}}, i\tilde{f} \} \tag{9.1}$$

and a similar expression for $\dot{A}(f, t) - \dot{A}_{\substack{in\\out}}(f, t)$. More precisely, (9.1) is true not only on D but on the whole domain of definition $D_{A(f,t)}$ of $A(f, t)$ which coincides with $D_{A_{\substack{in\\out}}(f,t)}$. (This follows since $e^{iA(f,t)}$ and $e^{iA_{\substack{in\\out}}(f,t)}$ are, up to phase factors, the same Weyl operator.) The r. h. s. of (9.1) (and of the corresponding equation for $\dot{A}(f, t) - \dot{A}_{\substack{in\\out}}(f, t)$) converges to zero for $t \to \mp\infty$ by (7.20) and (7.21), and therefore

$$\lim_{t\to\mp\infty}(\overset{(\cdot)}{A}(f, t) - \overset{(\cdot)}{A}_{\substack{in\\out}}(f, t)) = 0. \tag{9.2}$$

One may interpret (9.2) in the sense of strong convergence on D. However, the statement is somewhat stronger since (9.2) may also be written in the form

$$\lim_{t\to\mp\infty} \| \overset{(\cdot)}{A}(f, t) - \overset{(\cdot)}{A}_{\substack{in\\out}}(f, t) \| = 0,$$

with $\|\cdot\|$ denoting the operator norm. The fields $A_{in}(x)$ and $A_{out}(x)$ are therefore the asymptotic fields associated with the current $j(x)$.

Although in a certain sense (see below) the field $A(x)$ also converges to the free fields $A^0_{\substack{\text{in}\\\text{out}}}(x)$ for $t \to \mp\infty$, a relation like (9.2) with $A_{\substack{\text{in}\\\text{out}}}(x)$ replaced by $A^0_{\substack{\text{in}\\\text{out}}}(x)$ is not generally true. The formula corresponding to (9.1) is

$$A(f, t) - A^0_{\substack{\text{in}\\\text{out}}}(f, t) = 2\{(c^t_{\substack{\text{ret}\\\text{adv}}} + b^t_{\substack{\text{in}\\\text{out}}}), ie^{i\omega t}\tilde{f}\}, \tag{9.3}$$

but whereas the contributions from $c^t_{\substack{\text{ret}\\\text{adv}}}$ vanish asymptotically in each case, the contributions from b^t_{in} or b^t_{out} are easily shown to be independent of t (and $\neq 0$ for suitable $\tilde{f}$) if $\underset{\sim}{v}_{\text{in}} = 0$ or $\underset{\sim}{v}_{\text{out}} = 0$. More generally, we could consider translations of field averages in time-like directions:

$$A(f, 0) \to A(f_{\underset{\sim}{v}t}, t)$$

with

$$f_{\underset{\sim}{v}t}(\underset{\sim}{x}) = f(\underset{\sim}{x} - \underset{\sim}{v}t), \qquad |\underset{\sim}{v}| < 1.$$

We can still prove that

$$\lim_{t\to\mp\infty} (\overset{(\cdot)}{A}(f_{\underset{\sim}{v}t}, t) - \overset{(\cdot)}{A}_{\substack{\text{in}\\\text{out}}}(f_{\underset{\sim}{v}t}, t)) = 0$$

for all $\underset{\sim}{v}$, whereas for $\overset{(\cdot)}{A}(f_{\underset{\sim}{v}t}, t) - \overset{(\cdot)}{A}{}^0_{\substack{\text{in}\\\text{out}}}(f_{\underset{\sim}{v}t}, t)$ we get t-independent contributions from B_{in} or B_{out} if $\underset{\sim}{v} = \underset{\sim}{v}_{\text{in}}$ or $\underset{\sim}{v} = \underset{\sim}{v}_{\text{out}}$ since, by (2.17),

$$B_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{x} + \underset{\sim}{v}_{\substack{\text{in}\\\text{out}}} t, t) = B_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{x}, 0).$$

However, the free fields $A^0_{\substack{\text{in}\\\text{out}}}(x)$ are limits of $A(x)$ in the LSZ sense. Consider positive-frequency solutions of the wave equation,

$$\psi(x) = (2\pi)^{-3/2}\int d\mu(\underset{\sim}{k})e^{-ikx}\tilde{\psi}(\underset{\sim}{k}), \qquad \text{Re } \tilde{\psi} \text{ and Im } \tilde{\psi} \in \mathcal{M}. \tag{9.4}$$

The corresponding LSZ averages

$$A[\psi, t] = i\int d^3\underset{\sim}{x}(A(x)\dot{\psi}(x) - \dot{A}(x)\psi(x)) \tag{9.5}$$

exist since, for each fixed t, by (9.4) the real and imaginary parts of $\dot{\psi}(x)$ and $\psi(x)$ belong to the test function spaces $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, as mentioned in Section 8. A straightforward calculation leads to

$$A[\psi, t] = (a_{\substack{\text{in}\\\text{out}}}, \tilde{\psi}) + (b^t_{\substack{\text{in}\\\text{out}}}, \tilde{\psi}) + (c^t_{\substack{\text{ret}\\\text{adv}}}, \tilde{\psi}). \tag{9.6}$$

Eq. (2.28) implies $c^t_{\substack{\text{ret}\\\text{adv}}} \in L^2_c$ and

$$\| c^t_{\substack{\text{ret}\\\text{adv}}} \|_c \leq D(1 + |t|)^{-\varepsilon} \qquad \text{for} \qquad t \lessgtr 0 \tag{9.7}$$

with a suitable constant D, and therefore

$$|(c^t_{\substack{\text{ret}\\\text{adv}}}, \tilde{\psi})| \leq \|\tilde{\psi}\|_c \| c^t_{\substack{\text{ret}\\\text{adv}}} \|_c \underset{t\to\mp\infty}{\longrightarrow} 0.$$

With the explicit form (2.17) of $b^t_{\substack{\text{in}\\ \text{out}}}$, an elementary application of the Riemann-Lebesgue lemma also yields

$$\lim_{t\to\mp\infty} (b^t_{\substack{\text{in}\\ \text{out}}}, \tilde{\psi}) = 0 .$$

Since

$$(a_{\substack{\text{in}\\ \text{out}}}, \tilde{\psi}) = \mathrm{A}^0_{\substack{\text{in}\\ \text{out}}}[\psi, t] \equiv \mathrm{A}^0_{\substack{\text{in}\\ \text{out}}}[\psi] \qquad \text{(independent of } t),$$

we therefore obtain from (9.6) that

$$\lim_{t\to\mp\infty} (\mathrm{A}[\psi, t] - \mathrm{A}^0_{\substack{\text{in}\\ \text{out}}}[\psi]) = 0 \tag{9.8}$$

in the sense of, e. g., strong convergence on D. This is slightly more than the usual (weak) LSZ asymptotic convergence.

As the last result of this Section, we shall prove that

$$\text{s-}\lim_{t\to\mp\infty} (\mathrm{U}^*_{\substack{\text{in}\\ \text{out}}}(t)\mathrm{U}(t) - e^{i\gamma_\mp(t)}\mathrm{W}_{\substack{\text{in}\\ \text{out}}}(c^0_{\substack{\text{ret}\\ \text{adv}}})) = 0 \tag{9.9}$$

with suitable phases $\gamma_\mp(t)$ which satisfy

$$\lim_{t\to\mp\infty} \dot{\gamma}_\mp(t) = 0 . \tag{9.10}$$

In some cases we may even show that

$$\lim_{t\to\mp\infty} \gamma_\mp(t) = \gamma_\mp = \text{const.}, \tag{9.11}$$

but this need not always be true. In any case Eq. (9.9) means that, up to slowly varying and physically inessential phase factors, the asymptotic time evolutions $\mathrm{U}_{\substack{\text{in}\\ \text{out}}}(t)$ approximate $\mathrm{U}(t)$ for $t \to \mp\infty$ in the sense of ordinary scattering theory. As expected, the corresponding « S matrix »

$$\tilde{\mathrm{S}} = \mathrm{W}_{\text{in}}(c^0_{\text{ret}})\mathrm{W}_{\text{out}}(-\, c^0_{\text{adv}})$$

connects the asymptotic fields at $t = 0$, i. e.,

$$\tilde{\mathrm{S}}\overset{(\cdot)}{\mathrm{A}}_{\text{out}}(f, 0)\tilde{\mathrm{S}}^* = \overset{(\cdot)}{\mathrm{A}}_{\text{in}}(f, 0) ,$$

as easily proved.

In order to prove (9.9) we start from

$$\mathrm{U}^*_{\substack{\text{in}\\ \text{out}}}(t)\mathrm{U}(t) = e^{i(\alpha_{\substack{\text{in}\\ \text{out}}}(t) - \beta_{\substack{\text{in}\\ \text{out}}}(t))}\mathrm{W}_{\substack{\text{in}\\ \text{out}}}(-\, b^0_{\substack{\text{in}\\ \text{out}}})\mathrm{U}^{0*}_{\substack{\text{in}\\ \text{out}}}(t)\mathrm{W}_{\substack{\text{in}\\ \text{out}}}(b^t_{\substack{\text{in}\\ \text{out}}}) \times \mathrm{W}_{\substack{\text{in}\\ \text{out}}}(g^t_{\substack{\text{in}\\ \text{out}}})\mathrm{U}^0_{\substack{\text{in}\\ \text{out}}}(t)\mathrm{W}_{\substack{\text{in}\\ \text{out}}}(-\, g^0_{\substack{\text{in}\\ \text{out}}}) . \tag{9.12}$$

The four t-dependent operators on the r. h. s. together yield the factor

$$e^{-i\{h^t_{\substack{\text{in}\\ \text{out}}}, g^t_{\substack{\text{in}\\ \text{out}}}\}}\mathrm{W}_{\substack{\text{in}\\ \text{out}}}(-\, e^{-i\omega t}c^t_{\substack{\text{ret}\\ \text{adv}}})$$

in virtue of (2.23), (3.9) and (6.2). By (7.19), only the phase factors survive in the last expression for $t \to \mp \infty$. Using this and Eqs. (9.12), (2.23) and (3.9), we finally arrive at (9.9) with

$$\gamma_{\mp}(t) = \alpha_{\substack{\text{in}\\\text{out}}}(t) - \beta_{\substack{\text{in}\\\text{out}}}(t) - \{ b^t_{\substack{\text{in}\\\text{out}}}, g^t_{\substack{\text{in}\\\text{out}}} \} - \{ b^0_{\substack{\text{in}\\\text{out}}}, g^0_{\substack{\text{in}\\\text{out}}} \} .$$

Since

$$\frac{d}{dt}\{ b^t_{\substack{\text{in}\\\text{out}}}, g^t_{\substack{\text{in}\\\text{out}}} \} = \{ \dot{b}^t_{\substack{\text{in}\\\text{out}}}, g^t_{\substack{\text{in}\\\text{out}}} \} + \{ b^t_{\substack{\text{in}\\\text{out}}}, \dot{g}^t_{\substack{\text{in}\\\text{out}}} \}$$

(which follows rigorously from the differentiability properties of $b^t_{\substack{\text{in}\\\text{out}}}$ and $g^t_{\substack{\text{in}\\\text{out}}}$ proved in Section 7), we find

$$\begin{aligned}\dot{\gamma}_{\mp}(t) &= \dot{\alpha}_{\substack{\text{in}\\\text{out}}}(t) - \dot{\beta}_{\substack{\text{in}\\\text{out}}}(t) + \{ \dot{b}^t_{\substack{\text{in}\\\text{out}}}, (b^t_{\substack{\text{in}\\\text{out}}} + c^t_{\substack{\text{ret}\\\text{adv}}}) \} - \{ \dot{g}^t_{\substack{\text{in}\\\text{out}}}, (g^t_{\substack{\text{in}\\\text{out}}} + c^t_{\substack{\text{ret}\\\text{adv}}}) \} \\ &= \{ \dot{b}^t_{\substack{\text{in}\\\text{out}}}, c^t_{\substack{\text{ret}\\\text{adv}}} \} - \{ \dot{g}^t_{\substack{\text{in}\\\text{out}}}, c^t_{\substack{\text{ret}\\\text{adv}}} \}\end{aligned}$$

by (8.18) and (8.25). Since $\dot{b}^t_{\substack{\text{in}\\\text{out}}}$ and $c^t_{\substack{\text{ret}\\\text{adv}}}$ belong to L^2_c, we may use (9.7) to obtain

$$|\{ \dot{b}^t_{\substack{\text{in}\\\text{out}}}, c^t_{\substack{\text{ret}\\\text{adv}}} \}| = |\operatorname{Im}(\dot{b}^t_{\substack{\text{in}\\\text{out}}}, c^t_{\substack{\text{ret}\\\text{adv}}})| \leq D \| \dot{b}^t_{\substack{\text{in}\\\text{out}}} \|_c (1 + |t|)^{-\varepsilon}$$

for $t \lessgtr 0$, and $\| \dot{b}^t_{\substack{\text{in}\\\text{out}}} \|_c$ is easily seen to be bounded uniformly in t. The same arguments apply to $\{ \dot{g}^t_{\substack{\text{in}\\\text{out}}}, c^t_{\substack{\text{ret}\\\text{adv}}} \}$, which proves (9.10). Since these considerations also yield the estimates

$$|\dot{\gamma}_{\mp}(t)| \leq E(1 + |t|)^{-\varepsilon} \qquad \text{for} \qquad t \lessgtr 0,$$

we find that (9.11) is true for all models with $\varepsilon > 1$.

Relations of the form (9.9) with $U_{\substack{\text{in}\\\text{out}}}(t)$ replaced by $U^0_{\substack{\text{in}\\\text{out}}}(t)$ are not expected to hold true. As already discussed for another model by Faddeev and Kulish [7], this is due to the asymptotically persistent interaction of the field with $j_{\substack{\text{in}\\\text{out}}}(x)$.

10. HAMILTONIANS

Time evolution operators $U(t)$ usually are obtained as solutions of the differential equation

$$\frac{d}{dt} U(t) = iH(t)U(t) \tag{10.1}$$

with a Hamiltonian $H(t)$ determined by correspondence arguments. For our model this Hamiltonian is given formally by

$$H(t) = \int d\mu(\underset{\sim}{k})(a^*_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}) - \overline{g_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t)})\omega(a_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}) - g_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}, t)) - \int d^3\underset{\sim}{x} A(\underset{\sim}{x}, t) j(\underset{\sim}{x}, t) .$$

The first term is the free energy $H^0(t)$ of the field $A(x)$ at the time t. Using

$$W(f)a(\underset{\sim}{k})W^*(f) = a(\underset{\sim}{k}) - f(\underset{\sim}{k}) \qquad \text{and h. c.}$$

(a rigourous form of which follows from (3.9)) and Eq. (6.3) for $H^0_{\substack{in\\out}}$, we obtain

$$H^0(t) = W_{\substack{in\\out}}(g^t_{\substack{in\\out}})H^0_{\substack{in\\out}}W^*_{\substack{in\\out}}(g^t_{\substack{in\\out}}).$$

Since, by (8.11),

$$\begin{aligned} W_{in}(g^t_{in})H^0_{in}W^*_{in}(g^t_{in}) &= SW_{out}(g^t_{in})H^0_{out}W^*_{out}(g^t_{in})S^* \\ &= W_{out}(g^t_{out})H^0_{out}W^*_{out}(g^t_{out}), \end{aligned}$$

$H^0(t)$ is indeed independent of whether we express it in terms of « in » or « out » operators. This is obvious for the remaining term in $H(t)$ which is the interaction energy $H^I(t)$. By (8.4), (8.13) and (2.25), $H^I(t)$ may be rewritten as

$$H^I(t) = 2\{ a_{\substack{in\\out}}, \dot{g}^t_{\substack{in\\out}} \} - 2\{ g^t_{\substack{in\\out}}, \dot{g}^t_{\substack{in\\out}} \}.$$

Therefore we use

$$H(t) = W_{\substack{in\\out}}(g^t_{\substack{in\\out}})H^0_{\substack{in\\out}}W^*_{\substack{in\\out}}(g^t_{\substack{in\\out}}) + 2\{ a_{\substack{in\\out}}, \dot{g}^t_{\substack{in\\out}} \} - 2\{ g^t_{\substack{in\\out}}, \dot{g}^t_{\substack{in\\out}} \} \qquad (10.2)$$

as a formal definition of the Hamiltonian. We want to indicate that, if interpreted properly, Eq. (10.1) is satisfied, with $H(t)$ given by (10.2), for our time evolution operator

$$U(t) = e^{i\alpha_{\substack{in\\out}}(t)}W_{\substack{in\\out}}(g^t_{\substack{in\\out}})U^0_{\substack{in\\out}}(t)W^*_{\substack{in\\out}}(g^0_{\substack{in\\out}}). \qquad (10.3)$$

For convenience of notation, we will suppress the suffices $\substack{in\\out}$ in the following.

Differentiation of (10.3) with respect to t involves differentiation of $W(g^t)$. For the latter we use

LEMMA 9. — Consider a representation of the type described in Section 5, a one parameter family of complex test functions $f^t = \frac{1}{\sqrt{2}}(f^t_1 + if^t_2)$ with $f^t_{\frac{1}{2}}$ strongly differentiable in $\mathscr{L}_{\frac{1}{2}}$, and an arbitrary vector $\Phi \in D$. Then the one-parameter family

$$\Phi^t = W(f^t)\Phi$$

is strongly differentiable in $\mathscr{H}$, with derivative

$$\dot{\Phi}^t = (2i\{ a, \dot{f}^t \} - i\{ f^t, \dot{f}^t \})W(f^t)\Phi.$$

We will only give a sketch of the method of proof [15]. One first proves a completely analogous statement for the Fock representation. This part of proof is lengthy but straightforward, since it only uses well-known properties of Fock operators and their vacuum expectation values. Then, with (5.3) and (5.4), Lemma 9 follows trivially.

We will now give a rather heuristic derivation of (10.1) from (10.3). Roughly speaking, Lemma 9 means that

$$\frac{d}{dt}\mathrm{W}(f^t) = (2i\{a, \dot{f}^t\} - i\{f^t, \dot{f}^t\})\mathrm{W}(f^t). \tag{10.4}$$

By formal differentiation of (10.3) we get

$$\frac{d}{dt}\mathrm{U}(t) = i\dot{\alpha}(t)\mathrm{U}(t) + e^{i\alpha(t)}\left(\frac{d}{dt}\mathrm{W}(g^t)\right)\mathrm{U}^0(t)\mathrm{W}^*(g^0)$$
$$+ e^{i\alpha(t)}\mathrm{W}(g^t)\left(\frac{d}{dt}\mathrm{U}^0(t)\right)\mathrm{W}^*(g^0).$$

With (10.4) and (8.18) (which now becomes essential), the first two terms on the r. h. s. give

$$(2i\{a, \dot{g}^t\} - 2i\{g^t, \dot{g}^t\})\mathrm{U}(t) = i\mathrm{H}^{\mathrm{I}}(t)\mathrm{U}(t).$$

Since $\frac{d}{dt}\mathrm{U}^0(t) = i\mathrm{H}^0\mathrm{U}^0(t)$, the remaining term becomes

$$i\mathrm{W}(g^t)\mathrm{H}^0\mathrm{W}^*(g^t)\mathrm{U}(t) = i\mathrm{H}^0(t)\mathrm{U}(t).$$

Thus, at least formally, (10.1) with (10.2) indeed follows from (10.3).

A more rigorous version of this result is the following. Consider, besides the dense domain D used before, the dense domain $\tilde{\mathrm{D}} \subset \mathrm{D}$ which consists of finite linear combinations of vectors $\mathrm{W}(f)\Omega$ with $f_{\frac{1}{2}} \in \mathscr{L}_{\frac{1}{2}}$ and $\omega f_{\frac{1}{2}} \in \mathscr{L}_{\frac{2}{1}}$. Then $\mathrm{H}(t)$ as given by (10.2) is well-defined on $\tilde{\mathrm{D}}$, and has a self-adjoint extension. (At this point it is essential that $\Omega \in \mathrm{D}_{\mathrm{H}^0}$ by (7.5).) Moreover, (10.1) is true for suitable matrix elements, namely,

$$\frac{d}{dt}(\Phi, \mathrm{U}(t)\Psi) = i(\Phi, \mathrm{H}(t)\mathrm{U}(t)\Psi) \tag{10.5}$$

if $\Phi \in \mathrm{D}$ and $\Psi \in \mathrm{U}^*(t)\tilde{\mathrm{D}}$.

The proof of these statements is rather lengthy [*15*], and we therefore omit it here. We feel, moreover, that a result like (10.5) is neither surprising nor very essential for the model discussed here.

In the same way we may treat the asymptotic time evolutions $\mathrm{U}_{\substack{\mathrm{in}\\\mathrm{out}}}(t)$ given by (8.23). For these operators we obtain

$$\frac{d}{dt}\mathrm{U}_{\substack{\mathrm{in}\\\mathrm{out}}}(t) = i\mathrm{H}_{\substack{\mathrm{in}\\\mathrm{out}}}(t)\mathrm{U}_{\substack{\mathrm{in}\\\mathrm{out}}}(t)$$

with the asymptotic Hamiltonians

$$\mathrm{H}_{\substack{\mathrm{in}\\\mathrm{out}}}(t) = \mathrm{H}^0_{\substack{\mathrm{in}\\\mathrm{out}}}(t) + \mathrm{H}^{\mathrm{I}}_{\substack{\mathrm{in}\\\mathrm{out}}}(t)$$
$$= \mathrm{W}_{\substack{\mathrm{in}\\\mathrm{out}}}(-b^t_{\substack{\mathrm{in}\\\mathrm{out}}})\mathrm{H}^0_{\substack{\mathrm{in}\\\mathrm{out}}}\mathrm{W}^*_{\substack{\mathrm{in}\\\mathrm{out}}}(-b^t_{\substack{\mathrm{in}\\\mathrm{out}}}) - \int d^3\underset{\sim}{x}\,\mathrm{A}_{\substack{\mathrm{in}\\\mathrm{out}}}(\underset{\sim}{x}, t)j_{\substack{\mathrm{in}\\\mathrm{out}}}(\underset{\sim}{x}, t)$$

which, besides the free parts, contain the persistent interactions of $A_{\substack{\text{in}\\ \text{out}}}(x)$ with the asymptotic currents $j_{\substack{\text{in}\\ \text{out}}}(x)$. Equations corresponding to (10.5) are now true for $\Phi \in D$ and $\Psi \in U^*_{\substack{\text{in}\\ \text{out}}}(t)\tilde{D}$.

11. GENERALIZATION TO THE CASE OF PHOTONS

The scalar model treated up to now was chosen mainly in order to keep the notations as simple as possible. The same treatment is also applicable to a quantized photon field interacting with classical electromagnetic currents. Such a model with photons is most easily discussed in the Coulomb gauge. The field equations for the quantized vector potential are

$$\Box \underset{\sim}{A}(x) = \underset{\sim}{j}(x), \qquad \underset{\sim}{\nabla}\underset{\sim}{A}(x) = 0 \tag{11.1}$$

with a given c-number vector field $\underset{\sim}{j}(x)$ which also satisfies $\underset{\sim}{\nabla}\underset{\sim}{j} = 0$. For a classical particle with trajectory $\underset{\sim}{X}(t)$ and electric charge e, the divergence-free part $\underset{\sim}{j}(x)$ of the current density which enters (11.1) has the Fourier transform

$$\underset{\sim}{j}(\underset{\sim}{k}, t) = (2\pi)^{-3/2} e\tilde{\rho}(\omega)\dot{\underset{\sim}{X}}^{\text{tr}}(\underset{\sim}{k}, t)e^{-i\underset{\sim}{k}\underset{\sim}{X}(t)}. \tag{11.2}$$

Here we denote by $\dot{\underset{\sim}{X}}^{\text{tr}}$ the transverse part of the velocity $\dot{\underset{\sim}{X}}(t)$, defined by

$$\dot{\underset{\sim}{X}}^{\text{tr}} = \dot{\underset{\sim}{X}} - (\underset{\sim}{n}\dot{\underset{\sim}{X}})\underset{\sim}{n}, \qquad \underset{\sim}{n} = \frac{\underset{\sim}{k}}{\omega}, \tag{11.3}$$

and we have again introduced an ultraviolet cutoff ρ.

The trajectory $\underset{\sim}{X}(t)$ is assumed to be of the type described in Section 2, and in particular we will need the estimates (2.4). It is known that estimates like (2.4) are violated for classical particles interacting with each other through Coulomb forces. Therefore models with more than one charged particle are unrealistic if (2.4) is assumed for the trajectories.

The field equations may be solved formally as in Section 2, with some obvious modifications and changes of notation. Thereby $\tilde{j}_{\substack{\text{in}\\ \text{out}}}$, $\tilde{j}_{\mp}$, $b_{\substack{\text{in}\\ \text{out}}}$, $c_{\substack{\text{ret}\\ \text{adv}}}$ and $g_{\substack{\text{in}\\ \text{out}}}$ become vector-valued functions of $\underset{\sim}{k}$ and t which are transverse in the sense of

$$\underset{\sim}{k}\tilde{\underset{\sim}{j}}_{\substack{\text{in}\\ \text{out}}} = 0, \qquad \text{etc.} \tag{11.4}$$

Free asymptotic photon fields $\underset{\sim}{A}^0_{\substack{\text{in}\\ \text{out}}}(x)$ are defined as in (2.20) in terms of creation and annihilation operators $\underset{\sim}{a}^*_{\substack{\text{in}\\ \text{out}}}(\underset{\sim}{k})$ and $\underset{\sim}{a}_{\substack{\text{in}\\ \text{out}}}(\underset{\sim}{k})$ which are also transverse vectors, and whose components $a^i_{\substack{\text{in}\\ \text{out}}}$ satisfy the commutation relations

$$\left.\begin{aligned} &[a^i_{\substack{\text{in}\\ \text{out}}}(\underset{\sim}{k}), a^j_{\substack{\text{in}\\ \text{out}}}(\underset{\sim}{k}')] = 0 \qquad \text{and h. c.,} \\ &[a^i_{\substack{\text{in}\\ \text{out}}}(\underset{\sim}{k}), a^{j*}_{\substack{\text{in}\\ \text{out}}}(\underset{\sim}{k}')] = 2\omega\left(\delta_{ij} - \frac{k_i k_j}{\omega^2}\right)\delta(\underset{\sim}{k} - \underset{\sim}{k}'). \end{aligned}\right\} \tag{11.5}$$

Sections 3, 4 and 5 may be generalized to the case of photons simply by changing the notation. All scalar operator distributions like $p(\underset{\sim}{k})$ and $q(\underset{\sim}{k})$ and test functions like $f_{\frac{1}{2}}(\underset{\sim}{k})$ and $g^{\nu}(\underset{\sim}{k})$ become transverse vectors, and this leads to obvious changes of definitions, e. g., to

$$\langle \underset{\sim}{f}_1, \underset{\sim}{f}_2 \rangle = \int d\mu(\underset{\sim}{k}) \underset{\sim}{f}_1(\underset{\sim}{k}) \underset{\sim}{f}_2(\underset{\sim}{k})$$

instead of (3.4).

The discussion of free time evolution, translation and rotation operators for photons is analogous to Section 6, except that spatial rotations R now act as

$$\mathrm{R} : \underset{\sim}{f}(\underset{\sim}{k}) \rightarrow (\mathrm{R}\underset{\sim}{f})(\underset{\sim}{k}) = \mathrm{R}\underset{\sim}{f}(\mathrm{R}^{-1}\underset{\sim}{k}) \tag{11.6}$$

on test functions $\underset{\sim}{f}$. For the construction of an explicit representation of the type described in Section 7 for photons, we only have to replace the functions $g^{ilm}(\underset{\sim}{k})$ given by (7.2) by suitable vector-valued functions. For this purpose we may take

$$\left.\begin{aligned} &\underset{\sim}{g}^{ipjm}(\underset{\sim}{k}) = \frac{\sqrt{2}\chi_i(\omega)}{\omega} \underset{\sim}{\mathrm{Y}}^p_{jm}(\theta, \phi), \\ &i = 2, 3 \ldots, j = 1, 2 \ldots i - 1, m = -j \ldots +j, p = \pm 1 \end{aligned}\right\} \tag{11.7}$$

with the transverse vector-valued spherical harmonics [*16*]

$$\left.\begin{aligned} \underset{\sim}{\mathrm{Y}}^1_{jm}(\theta, \phi) &= \frac{1}{\sqrt{j(j+1)}} \underset{\sim}{\nabla}_{\underset{\sim}{n}} \mathrm{Y}_{jm}(\theta, \phi), \\ \underset{\sim}{\mathrm{Y}}^{-1}_{jm}(\theta, \phi) &= \underset{\sim}{n} \times \underset{\sim}{\mathrm{Y}}^1_{jm}(\theta, \phi), \end{aligned}\right\} \tag{11.8}$$

where $\underset{\sim}{n} = \dfrac{\underset{\sim}{k}}{\omega}$, $\underset{\sim}{\nabla}_{\underset{\sim}{n}} = \omega \underset{\sim}{\nabla}_{\underset{\sim}{k}}$. (The superfix p indicates the parity, which is $p(-1)^j$.) The functions $\underset{\sim}{\mathrm{Y}}^p_{jm}$ transform under the rotations (11.7) according to

$$\mathrm{R}\underset{\sim}{\mathrm{Y}}^p_{jm} = \sum_{m'=-j}^{j} \mathrm{D}^{(j)}_{mm'}(\mathrm{R}^{-1}) \underset{\sim}{\mathrm{Y}}^p_{jm'}, \tag{11.9}$$

and from this we obtain rotation invariance of the representation as in Section 7. A modification is also necessary in the formulation and proof of Lemma 6. We now have to prove that

$$\underset{\sim}{f}_{\underset{\sim}{v}}(\underset{\sim}{k}) = \sqrt{2} \sum_{i=2}^{\infty} \frac{\chi_i(\omega)}{\omega} \frac{\underset{\sim}{v}^{\mathrm{tr}}(\underset{\sim}{n})}{1 - \underset{\sim}{n}\underset{\sim}{v}} \in \mathscr{L}_1, \tag{11.10}$$

with $\underset{\sim}{v}^{\mathrm{tr}} = \underset{\sim}{v} - (\underset{\sim}{n}\underset{\sim}{v})\underset{\sim}{n}$ as in (11.3). The angular momentum operator $\underset{\sim}{\mathrm{L}}$, used in the proof of Lemma 6, has to be replaced now by the total angular momentum operator $\underset{\sim}{J} = \underset{\sim}{I} + \underset{\sim}{S}$ for photons, with the spin part $\underset{\sim}{S}$ acting as a constant matrix on vector functions. Its square J^2 satisfies

$$J^2 \underset{\sim}{Y}^p_{jm} = j(j+1)\underset{\sim}{Y}^p_{jm}, \tag{11.11}$$

which can be exploited to prove (11.10) in the same way as in Section 7. The other considerations of that Section carry over to photons almost literally.

Space averages of the potential $\underset{\sim}{A}(x)$ and its time derivative could be discussed in the same way, as done in Section 8 for the scalar field. However, the observable fields are the field strengths

$$\underset{\sim}{E}(x) = -\dot{\underset{\sim}{A}}(x) - \underset{\sim}{\nabla}\Phi(x), \qquad \underset{\sim}{H}(x) = \underset{\sim}{\nabla} \times \underset{\sim}{A}(x) \tag{11.12}$$

rather than the vector potential $\underset{\sim}{A}(x)$. Eq. (11.12) contains the Coulomb potential

$$\Phi(x) = \frac{e}{4\pi}\int d^3\underset{\sim}{y}\,\frac{\rho(\underset{\sim}{y} - \underset{\sim}{X}(t))}{|\underset{\sim}{x} - \underset{\sim}{y}|} = \frac{2e}{(2\pi)^3}\int d\mu(\underset{\sim}{k})\frac{\tilde{\rho}(\omega)}{\omega}e^{i\underset{\sim}{k}(\underset{\sim}{x} - \underset{\sim}{X}(t))} \tag{11.13}$$

of the charge distribution $e\rho(\underset{\sim}{x} - \underset{\sim}{X}(t))$. We rewrite it in the form

$$\begin{aligned}\Phi(x) &= 2(2\pi)^{-3/2}\int d\mu(\underset{\sim}{k})e^{i\underset{\sim}{k}\underset{\sim}{x}}\frac{1}{\omega}\varphi^t(\underset{\sim}{k}) \\ &= \Phi_{\substack{\text{in}\\\text{out}}}(x) + \Phi_{\mp}(x)\end{aligned} \tag{11.14}$$

with asymptotic parts

$$\left.\begin{aligned}\Phi_{\substack{\text{in}\\\text{out}}}(x) &= 2(2\pi)^{-3/2}\int d\mu(\underset{\sim}{k})e^{i\underset{\sim}{k}\underset{\sim}{x}}\frac{1}{\omega}\varphi^t_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}), \\ \varphi^t_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}) &= e(2\pi)^{-3/2}\tilde{\rho}(\omega)e^{-i\underset{\sim}{k}(\underset{\sim}{s}_{\substack{\text{in}\\\text{out}}} + \underset{\sim}{v}_{\substack{\text{in}\\\text{out}}}t)}\end{aligned}\right\} \tag{11.15}$$

and

$$\left.\begin{aligned}\Phi_{\mp}(x) &= 2(2\pi)^{-3/2}\int d\mu(\underset{\sim}{k})e^{i\underset{\sim}{k}\underset{\sim}{x}}\frac{1}{\omega}\varphi^t_{\mp}(\underset{\sim}{k}), \\ \varphi^t_{\mp}(\underset{\sim}{k}) &= e(2\pi)^{-3/2}\tilde{\rho}(\omega)e^{-i\underset{\sim}{k}\underset{\sim}{X}(t)}(1 - e^{i\underset{\sim}{k}\underset{\sim}{d}_{\substack{\text{in}\\\text{out}}}(t)}),\end{aligned}\right\} \tag{11.16}$$

such that

$$\varphi^t(\underset{\sim}{k}) = \varphi^t_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{k}) + \varphi^t_{\mp}(\underset{\sim}{k}).$$

Obviously $\varphi^t_{\substack{\text{in}\\\text{out}}}$, $\varphi^t_{\mp}$ and φ^t belong to L^2_c, and (2.4) immediately leads to

$$\lim_{t\to\mp\infty} \|\varphi^t_{\mp}\|_c = 0. \tag{11.17}$$

Consider a real vector-valued test function

$$\underset{\sim}{f}(\underset{\sim}{x}) = (2\pi)^{-3/2}\int d^3\underset{\sim}{k}\,e^{i\underset{\sim}{k}\underset{\sim}{x}}\tilde{\underset{\sim}{f}}(\underset{\sim}{k}),$$

and decompose $\tilde{\underset{\sim}{f}}$ according to

$$\tilde{\underset{\sim}{f}}(\underset{\sim}{k}) = \tilde{\underset{\sim}{f}}^{\text{tr}}(\underset{\sim}{k}) + (\underset{\sim}{n}\tilde{\underset{\sim}{f}}(\underset{\sim}{k}))\underset{\sim}{n}$$

into a transverse part $\tilde{f}^{\text{tr}}$ and a longitudinal component $\underset{\sim}{n}\tilde{f}$. An elementary calculation then yields the following explicit expressions for the spatial averages of field strengths:

$$\left.\begin{aligned} \underset{\sim}{E}(\underset{\sim}{f}, t) &= \int d^3\underset{\sim}{x}\underset{\sim}{E}(\underset{\sim}{x}, t)\underset{\sim}{f}(\underset{\sim}{x}) \\ &= 2\{ \underset{\sim}{a}_{\substack{\text{in}\\\text{out}}}, \omega e^{i\omega t}\underset{\sim}{\tilde{f}}^{\text{tr}} \} - 2\{ \underset{\sim}{g}^t_{\substack{\text{in}\\\text{out}}}, \omega e^{i\omega t}\underset{\sim}{\tilde{f}}^{\text{tr}} \} - 2i(\varphi^t, \underset{\sim}{n}\tilde{f}), \\ \underset{\sim}{H}(\underset{\sim}{f}, t) &= \int d^3\underset{\sim}{x}\underset{\sim}{H}(\underset{\sim}{x}, t)\underset{\sim}{f}(\underset{\sim}{x}) \\ &= -2\{ \underset{\sim}{a}_{\substack{\text{in}\\\text{out}}}, \omega e^{i\omega t}(\underset{\sim}{n} \times \underset{\sim}{\tilde{f}}^{\text{tr}}) \} + 2\{ \underset{\sim}{g}^t_{\substack{\text{in}\\\text{out}}}, \omega e^{i\omega t}(\underset{\sim}{n} \times \underset{\sim}{\tilde{f}}^{\text{tr}}) \}. \end{aligned}\right\} \quad (11.18)$$

From this one easily concludes that $\underset{\sim}{E}(\underset{\sim}{f}, t)$ and $\underset{\sim}{H}(\underset{\sim}{f}, t)$ exist if the real and imaginary parts of $\omega\underset{\sim}{\tilde{f}}^{\text{tr}}$ belong to the (vector analog of the) space $\mathcal{M}$, and $\underset{\sim}{n}\tilde{f}$ belongs to L^2_c.

Time evolution operators U(t) and an S matrix may be defined as for the scalar model. However, whereas we still have

$$U(t)\underset{\sim}{H}(\underset{\sim}{f}, 0)U^*(t) = \underset{\sim}{H}(\underset{\sim}{f}, t)$$

for all admissible test functions $\underset{\sim}{f}(\underset{\sim}{x})$, the relation

$$U(t)\underset{\sim}{E}(\underset{\sim}{f}, 0)U^*(t) = \underset{\sim}{E}(\underset{\sim}{f}, t)$$

holds true only if $\underset{\sim}{n}\tilde{f} = 0$. In other words, the time evolution of the longitudinal (Coulomb) part $-2i(\varphi^t, \underset{\sim}{n}\tilde{f})$ of $\underset{\sim}{E}(\underset{\sim}{f}, t)$ has to be « put in by hand », since U(t) implements the time evolution of the transverse part of $\underset{\sim}{E}(\underset{\sim}{f}, t)$ only. This is also true for space translations and rotations, and is peculiar to the treatment of classical currents in the Coulomb gauge.

The discussion of asymptotic properties of the field (Section 9) has to be modified accordingly. The relations corresponding to (9.2) now read

$$\left.\begin{aligned} \lim_{t\to\mp\infty} (\underset{\sim}{E}(\underset{\sim}{f}, t) - \underset{\sim}{E}_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{f}, t)) = 0, \\ \lim_{t\to\mp\infty} (\underset{\sim}{H}(\underset{\sim}{f}, t) - \underset{\sim}{H}_{\substack{\text{in}\\\text{out}}}(\underset{\sim}{f}, t)) = 0, \end{aligned}\right\} \quad (11.19)$$

where the asymptotic fields $\underset{\sim}{E}_{\substack{\text{in}\\\text{out}}}$ and $\underset{\sim}{H}_{\substack{\text{in}\\\text{out}}}$ are given by (11.12) with the asymptotic potentials

$$\underset{\sim}{A}_{\substack{\text{in}\\\text{out}}} = \underset{\sim}{A}^0_{\substack{\text{in}\\\text{out}}} + \underset{\sim}{B}_{\substack{\text{in}\\\text{out}}} \quad \text{and} \quad \Phi_{\substack{\text{in}\\\text{out}}}$$

(cf. Eqs. (2.17) and (11.15)). These asymptotic fields contain, besides the free field operators

$$\underset{\sim}{E}^0_{\substack{\text{in}\\\text{out}}} = -\dot{\underset{\sim}{A}}^0_{\substack{\text{in}\\\text{out}}} \quad \text{and} \quad \underset{\sim}{H}^0_{\substack{\text{in}\\\text{out}}} = \underset{\sim}{\nabla} \times \underset{\sim}{A}^0_{\substack{\text{in}\\\text{out}}}, \quad (11.20)$$

the c-number fields

$$- \dot{\underset{\sim}{\mathrm{B}}}_{\substack{\mathrm{in}\\\mathrm{out}}} - \underset{\sim}{\nabla}\Phi_{\substack{\mathrm{in}\\\mathrm{out}}} \qquad \text{and} \qquad \underset{\sim}{\nabla} \times \underset{\sim}{\mathrm{B}}_{\substack{\mathrm{in}\\\mathrm{out}}} .$$

The latter are the classical Liénard-Wiechert fields of the asymptotic charge and current distributions, and are again asymptotically persistent since

$$\underset{\sim}{\mathrm{B}}_{\substack{\mathrm{in}\\\mathrm{out}}}(\underset{\sim}{x} + \underset{\sim}{v}_{\substack{\mathrm{in}\\\mathrm{out}}} t, t) = \underset{\sim}{\mathrm{B}}_{\substack{\mathrm{in}\\\mathrm{out}}}(\underset{\sim}{x}, 0), \qquad \Phi_{\substack{\mathrm{in}\\\mathrm{out}}}(\underset{\sim}{x} + \underset{\sim}{v}_{\substack{\mathrm{in}\\\mathrm{out}}} t, t) = \Phi_{\substack{\mathrm{in}\\\mathrm{out}}}(\underset{\sim}{x}, 0).$$

LSZ convergence could be discussed in terms of the vector potential $\underset{\sim}{\mathrm{A}}(x)$ (as done, e. g., in Ref. [*17*]) exactly as for the scalar model. It is preferable, however, to do this also in terms of field strengths. Consider a vector-valued positive-frequency solution of the wave equation

$$\underset{\sim}{\psi}(x) = (2\pi)^{-3/2} \int d\mu(\underset{\sim}{k}) e^{-ikx} \tilde{\underset{\sim}{\psi}}(\underset{\sim}{k})$$

with

$$\underset{\sim}{k}\tilde{\underset{\sim}{\psi}}(\underset{\sim}{k}) = 0, \qquad \omega \operatorname{Re} \tilde{\underset{\sim}{\psi}} \quad \text{and} \quad \omega \operatorname{Im} \tilde{\underset{\sim}{\psi}} \in \mathcal{M},$$

and associate with it the « field strengths »

$$\underset{\sim}{e} = -\dot{\underset{\sim}{\psi}}, \qquad \underset{\sim}{h} = \underset{\sim}{\nabla} \times \underset{\sim}{\psi}.$$

A straightforward calculation yields

$$\begin{aligned} \mathrm{F}[\underset{\sim}{\psi}, t] &\underset{\mathrm{df.}}{=} \int d^3\underset{\sim}{x}(\underset{\sim}{\mathrm{E}}(x)\underset{\sim}{e}(x) + \underset{\sim}{\mathrm{H}}(x)\underset{\sim}{h}(x)) \\ &= (\underset{\sim}{a}_{\substack{\mathrm{in}\\\mathrm{out}}}, \omega\tilde{\underset{\sim}{\psi}}) + (\underset{\sim}{b}^t_{\substack{\mathrm{in}\\\mathrm{out}}}, \omega\tilde{\underset{\sim}{\psi}}) + (\underset{\sim}{c}^t_{\substack{\mathrm{ret}\\\mathrm{adv}}}, \omega\tilde{\underset{\sim}{\psi}}). \end{aligned}$$

As for the scalar model, the last two terms may be shown to vanish for $t \to \mp \infty$. Moreover, (11.20) leads to

$$\begin{aligned} (\underset{\sim}{a}_{\substack{\mathrm{in}\\\mathrm{out}}}, \omega\tilde{\underset{\sim}{\psi}}) &= \int d^3\underset{\sim}{x}(\underset{\sim}{\mathrm{E}}^0_{\substack{\mathrm{in}\\\mathrm{out}}}(x)\underset{\sim}{e}(x) + \underset{\sim}{\mathrm{H}}^0_{\substack{\mathrm{in}\\\mathrm{out}}}(x)\underset{\sim}{h}(x)) \\ &\underset{\mathrm{df.}}{=} \mathrm{F}^0_{\substack{\mathrm{in}\\\mathrm{out}}}[\underset{\sim}{\psi}, t] \equiv \mathrm{F}^0_{\substack{\mathrm{in}\\\mathrm{out}}}[\underset{\sim}{\psi}] \qquad \text{(independent of } t\text{)}, \end{aligned}$$

so that we obtain LSZ convergence for the photon fields in the form

$$\lim_{t \to \mp\infty} (\mathrm{F}[\underset{\sim}{\psi}, t] - \mathrm{F}^0_{\substack{\mathrm{in}\\\mathrm{out}}}[\underset{\sim}{\psi}]) = 0. \tag{11.21}$$

Finally, the generalizations of Eq. (9.9) and of Section 10 to the case of photons are obvious.

12. CONCLUDING REMARKS

We have shown that the emission of infinitely many massless particles by classical currents may be described mathematically by using a suitable non-Fock representation for the free asymptotic fields. The physical interpretation of this formalism is based on the following considerations.

Any realizable experimental set-up detects single particles only if their energy exceeds some lower limit $\omega_0 > 0$. The value of ω_0, of course, depends on the apparatus, and is therefore not fixed *a priori*. With respect to a given sensitivity limit ω_0 we may classify particles as « hard » or « soft » if they have energy $\omega > \omega_0$, or $\omega \leq \omega_0$, respectively. Then the representation space $\mathscr{H}$ may be written as a direct product of the Fock space of « hard » particles and another space carrying a non-Fock representation for the « soft » part of the field. Moreover, the emission of « hard » particles is described by the familiar Fock space S matrix corresponding to the given current [3]. The operators for the total energy, momentum and angular momentum of the free incoming and outgoing fields may be decomposed accordingly. Each one of them is a sum of two terms, one representing the contribution from « hard » particles and having the usual Fock space structure, the other one containing the contribution from the infinitely many additional « soft » particles. The first term should in principle be measurable by suitable instruments, whereas in a certain sense (compare the discussion of Eq. (6.19) in Section 6) the second term goes to zero in the limit $\omega_0 \to 0$. At least in this sense of approximate measurability, therefore, the total energy, momentum and angular momentum operators may indeed be called observables. In contrast to this, the number operators for « hard » particles do not converge for $\omega_0 \to 0$.

The particular non-Fock representation which has been used here (Section 7) is by no means unique. There are in fact infinitely many representations, all unitarily inequivalent to each other, which are equally well adapted to the model. As examples we consider representations of the form described in Section 7, but with Eq. (7.4) replaced by

$$b_{ilm} = b_i = i^{-k} \tag{12.1}$$

with an arbitrary $k > \frac{1}{2}$. All the essential conclusions derived in Section 7 for the particular case $k = 1$ remain valid for general k. The unitary inequivalence of two representations I and II with different k values $k_{\mathrm{I}} < k_{\mathrm{II}}$ is easily seen as follows. Consider the sequence of test functions

$$f^i = i^{k_{\mathrm{I}}} g^{i00}, \qquad i = 2, 3 \ldots$$

in $\mathscr{M}$. Then, in a self-explanatory notation, we have

$$\mathrm{V}^{\mathrm{I}}(f^i) = \mathrm{V}_{\mathrm{F}}(\mathrm{T}_1^{\mathrm{I}} f^i) = \mathrm{V}_{\mathrm{F}}(g^{i00}), \qquad \| g^{i00} \| = 1 \qquad \text{for all } i$$

but

$$\mathrm{V}^{\mathrm{II}}(f^i) = \mathrm{V}_{\mathrm{F}}(\mathrm{T}_1^{\mathrm{II}} f^i) = \mathrm{V}_{\mathrm{F}}(i^{k_{\mathrm{I}} - k_{\mathrm{II}}} g^{i00}) \underset{i \to \infty}{\longrightarrow} \mathbb{1} .$$

This is incompatible with unitary equivalence (cf. a similar discussion in Section 5). Obviously this example is still very far from exhausting the possibilities of constructing unitarily inequivalent representations which are equally well adapted to the model.

It seems natural to ask whether among these representations there are some in which, besides space-time translations and space rotations, the Lorentz transformations

$$f(\underset{\sim}{k}) \rightarrow (\Lambda f)(\underset{\sim}{k}) = f(\Lambda^{-1}\underset{\sim}{k})$$

of test functions are also unitarily implementable. Unfortunately, the methods used in Section 6 for translations and rotations cannot be applied to Lorentz transformations. First, the construction of $U^0(t)$ and $U(\underset{\sim}{x})$ was based on the direct product decomposition (6.6) of Fock space and the invariance of the factor spaces $\mathscr{H}_i$ under space-time translations. Lorentz transformations, however, do not leave invariant the spaces $\mathscr{H}_i$. Secondly, the existence of rotation operators U(R) was achieved by choosing the functions g^ν from suitable finite-dimensional unitary representation spaces of the rotation group, whereas the Lorentz group does not admit such representations. At the moment, therefore, the above question cannot be answered.

We can prove, however, that the particular representation given in Section 7 and its generalizations by (12.1) are certainly not Lorentz invariant. The existence of unitary operators $U(\Lambda)$ with

$$U(\Lambda)W(f)U^*(\Lambda) = W(\Lambda f)$$

for sufficiently many f, e. g., for $f = \frac{1}{\sqrt{2}}(f_1 + if_2)$ with $f_{\frac{1}{2}} \in \mathscr{U}$, would in fact imply Lorentz invariance of the test function spaces $\mathscr{L}_1$ and $\mathscr{L}_2$. (Compare the discussion of Eq. (6.2).) But, for the particular representations mentioned, we may explicitly construct test functions $f_1 \in \mathscr{L}_1$ with $\Lambda f_1 \notin \mathscr{L}_1$. Such lack of Lorentz invariance is perhaps not too serious, since manifest Lorentz invariance of the model considered here is already destroyed by the ultraviolet cutoff ρ.

As mentioned in the Introduction, the methods developed here may also be applied to a model in which the source of the radiation field is the current operator of a quantum mechanical particle. In a certain approximation which preserves the infrared divergent part of the interaction (and which is less drastic than the familiar dipole approximation [8]), such a model is again soluble. The details will be discussed in a separate paper.

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