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TOSHIHIRO IWAI

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Symmetries in Relativistic Dynamics of a Charged Particle

by

Toshihiro IWAI
Department of Applied Mathematics, Gifu University,
Kagamigahara, Gifu 504, Japan

ABSTRACT. — A relativistic dynamical system is reformulated as a pseudosphere bundle, endowed with a certain geometric structure, over a pseudo-Riemannian manifold. A class of infinitesimal automorphisms (or kinematical symmetry) of the system is dealt with on this reformulation. Closer investigation is made into the dynamical system of which the base manifold is a four-dimensional pseudo-Riemannian manifold with the indefinite metric of signature $(+, -, -, -)$. When the dynamical system admits the maximal kinematical symmetry, the base manifold is locally either a product manifold or an **H**-spacetime. Finally, relations with nonrelativistic dynamical systems are discussed. It will be found that the kinematical symmetry of a relativistic dynamical system covers that of a nonrelativistic system.

1. INTRODUCTION

A number of investigations have been made into the symmetry of dynamical systems in classical as well as quantum theories. In classical theories of a charged particle, however, the symmetry has mostly been studied with some restriction imposed, for example, with the assumption that the electromagnetic field is uniform (i. e. constant in space and time) [5].

In this article a relativistic dynamical system is reformulated in terms of differential geometry. The symmetry of the system will be set up on this reformulation. To be brief, the symmetry of a relativistic dynamical system is taken as the infinitesimal automorphisms of a certain geometric structure defined on a pseudosphere bundle. In this setting, the kinematical symmetry is studied in detail. The term « kinematical » is to imply a raising from the base manifold.

Section 2 gives the definition of the pseudosphere bundle over a pseudo-Riemannian manifold, on which the motion of a charged particle is described as a flow. It is a natural formulation of the totality of the so-called four-velocities. In Section 3 a relativistic dynamical system is defined as a pseudosphere bundle endowed with a certain geometric structure which is composed of the indefinite Riemannian metric and the electromagnetic field (i. e. two-form). The infinitesimal automorphisms of the dynamical system is dealt with in Section 4. The detailed results of S. Abe and M. Ikeda on non-symmetric tensor fields are available for this article. Section 5 is concerned with relations with non-relativistic dynamical systems.

2. PSEUDOSPHERE BUNDLES

Let M be an n-dimensional pseudo-Riemannian manifold and $T(M)$ its tangent bundle. Let (x^i) be a local coordinate system of M and (x^i, u^i) the induced coordinate system of $T(M)$. Consider a connected component of the hypersurface determined by the equation in $T(M)$,

$$g_{ij}u^iu^j = 1 \text{ } (^1), \tag{2.1}$$

(g_{ij}) denoting local components of the indefinite metric of M. We call it the *pseudosphere bundle* over a pseudo-Riemannian manifold M and give the notation $H(M)$. This is because Equation (2.1) determines the pseudosphere with respect to the induced Riemannian metric in the tangent space $T_p(M)$ at each $p \in M$ to M. It is to be noted that the definition is independent of the choice of local coordinate systems and that if the metric is definite $H(M)$ becomes the sphere bundle [7].

We illustrate the pseudosphere bundle $H(M)$ over the canonical flat M of dimension four and of signature $(+, -, -, -)$. Let (x^i), $i = 0, 1, 2, 3$, be the canonical coordinate system of M and (x^i, u^i) the induced coordinate system of $T(M)$. In $T(M)$ the equation $\eta_{ij}u^iu^j = 1$, (η_{ij}) denoting the flat indefinite metric of M, determines two sheets of hypersurfaces corresponding to $u^0 \geqslant 1$ and $u^0 \leqslant -1$. We choose the one corresponding to $u^0 \geqslant 1$ on physical grounds. The pseudosphere bundle is a natural formulation of the totality of the so-called four-velocities in relativistic dynamics.

In what follows, $H(M)$ is dealt with as the submanifold of $T(M)$, and

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(1) Unless otherwise stated, Latin indices range over the values from 0 to $n - 1$, and the summation convention is adopted. Hereafter all manifolds, vector fields, functions, etc., introduced will be tacitly assumed to have a suitable order of differentiability.

the coordinate system (x^i, u^i) with the constraint (2.1) is applied to it. Moreover, a vector field, when tangent to $H(M)$, is regarded as that on $H(M)$. For example, a basis of the horizontal vector fields on $T(M)$,

$$\frac{\partial}{\partial x^i} - \Gamma^k_{ij}u^j\frac{\partial}{\partial u^k}, \tag{2.2}$$

serves as a set of vectors on $H(M)$, where Γ^k_{ij} are Christoffel symbols.

Given a vector field X with local components (ξ^i), its *lift* $\tilde{X}$ to $T(M)$ is defined in the induced coordinate system by the following [7]:

$$\tilde{X} = \xi^i\frac{\partial}{\partial x^i} + \frac{\partial \xi^i}{\partial x^k}u^k\frac{\partial}{\partial u^i}. \tag{2.3}$$

The following lemma is easily proved.

LEMMA 2.1 (Sasaki [7]). — *The lift* $\tilde{X}$ *to* T(M) *of a vector field* X *on* M *is tangent to* H(M) *and so can be regarded as the vector field on* H(M), *if and only if* X *is a Killing vector field on* M.

3. GEOMETRIC STRUCTURES ON PSEUDOSPHERE BUNDLES

We want to establish relativistic dynamics of a charged particle on a pseudosphere bundle $H(M)$. Let α be a two-form with local components (F_{ij}), representing the electromagnetic field (2) on M, and κ a one-form defined by

$$\kappa = g_{ij}u^j dx^i, \tag{3.1}$$

called the *kinetic form* in the present article. The form α is regarded as being defined on $H(M)$ as it is. We define a two-form on $H(M)$ by

$$\omega = -mcd\kappa + q\alpha, \tag{3.2}$$

where m and q denote the mass at rest and the charge of the particle, respectively, and c the speed of light.

Since $\kappa \wedge \omega^{n-1} \neq 0$, κ and ω define an *almost contact structure* on $H(M)$. A vector field Z is called the *canonical field* of the almost contact structure [*10*], if it satisfies

$$i(Z)\omega = 0, \qquad i(Z)\kappa = 1, \tag{3.3}$$

where $i(Z)$ denotes the operator of the interior product by Z.

The law of motion for a charged particle is translated into the following: *The canonical field defined by* (3.3) *describes the motion of a charged*

(2) Whether α satisfies the Maxwell's equations or not, it will be called the electromagnetic field.

particle. To be more precise, the flow defined by the canonical field on H(M) *is the trajectory lifted from that of a charged particle in* M.

This statement is verified as follows. In the induced coordinate system (x^i, u^i) of $T(M)$ (3.2) is written as

$$\omega = -mcg_{ij}Du^j \wedge dx^i + \frac{q}{2}F_{ij}dx^i \wedge dx^j \tag{3.4}$$

with the constraint

$$g_{ij}u^jDu^i = 0\,, \tag{3.5}$$

where Du^i is the covariant differential of u^i,

$$Du^i = du^i + \Gamma^i_{kj}u^k dx^j\,.$$

Assume that the canonical field is written as

$$Z = \xi^i\left(\frac{\partial}{\partial x^i} - \Gamma^k_{ij}u^j\frac{\partial}{\partial u^k}\right) + \eta^i\frac{\partial}{\partial u^i}. \tag{3.6}$$

By calculating (3.3) for (3.4), (3.5) and (3.6) to determine (ξ^i) and (η^i), we find

$$Z = u^i\left(\frac{\partial}{\partial x^i} - \Gamma^k_{ij}u^j\frac{\partial}{\partial u^k}\right) + \frac{q}{mc}F^i_ju^j\frac{\partial}{\partial u^i}, \tag{3.7}$$

where $F^i_j = F_{jk}g^{ki}$ and (g^{ij}) is the inverse of (g_{ij}). It can be easily seen that (3.7) is tangent to $H(M)$. The flow defined by (3.7) is the solution to the differential equations

$$\begin{cases} \dfrac{dx^i}{ds} = u^i\,, \\ \dfrac{du^i}{ds} = -\Gamma^i_{jk}u^ju^k + \dfrac{q}{mc}F^i_ju^j\,, \end{cases}$$

which prove to be the well-known equations of motion for a charged particle. This completes the verification. Notice that the first equation in (3.3) determines the direction of Z and that the second is merely concerned with a scalar factor of Z. Hereafter ω will be called the *generating form* (of the equations of motion).

From the above discussion, it may be said that a *relativistic dynamical system* is defined as a triple $(H(M), \kappa, \omega)$ or briefly a couple $(H(M), \omega)$. If the electromagnetic field α satisfies a part of the Maxwell's equations, $d\alpha = 0$, then the generating form ω is closed. Our main interest will center on this case.

Our setting up of a relativistic dynamical system is gauge-invariant, because no electromagnetic potential (i. e. one-form π such that $d\pi = \alpha$) appears. An analogous reformulation of relativistic dynamical systems through cotangent bundles $T^*(M)$ is to be found in [*8*].

4. INFINITESIMAL AUTOMORPHISMS

Consider a relativistic dynamical system $(H(M), \omega)$ with the closed generating form. A vector field W [3] on $H(M)$ is called an *infinitesimal automorphism* of the system $(H(M), \omega)$, if it satisfies

$$\mathscr{L}_W \omega = 0\,, \tag{4.1}$$

where $\mathscr{L}_W$ denotes the Lie derivation with respect to W. Of the infinitesimal automorphisms, we are interested in *kinematical* ones, which are, as a definition, infinitesimal automorphisms and lifted vector fields simultaneously.

THEOREM 4.1. — *A necessary and sufficient condition for a lifted field* $\tilde{X}$ *of* X *on* M *to be an infinitesimal automorphism of the relativistic dynamical system* (H(M), ω) *is that* X *is a Killing vector field that leaves the electromagnetic field invariant.*

Proof. — Let X be a vector field on M and $\tilde{X}$ its lift to $H(M)$. Then X must be a Killing vector field, as is shown in Lemma 2.1. ~~Therefore~~ the Lie derivative of κ with respect to X vanishes:

$$\mathscr{L}_{\tilde{X}} \kappa = (\mathscr{L}_X g_{ij}) u^j dx^i = 0\,. \tag{4.2}$$

On account of $\mathscr{L}_{\tilde{X}} d\kappa = d\mathscr{L}_{\tilde{X}} \kappa$ we obtain

$$\mathscr{L}_{\tilde{X}} \omega = \mathscr{L}_X \alpha = 0\,.$$

The converse is obvious

An infinitesimal automorphism can be called a *symmetry*. A symmetry gives rise to a conservation. Let W be an infinitesimal automorphism of the dynamical system $(H(M), \omega)$ with closed ω. Then one has

$$\mathscr{L}_W \omega = di(W)\omega = 0\,,$$

so that by Poincaré's lemma there is locally a function f such that

$$i(W)\omega = -\, df. \tag{4.3}$$

The function f is seen to be a conserved quantity (or a *first integral*). In fact, as the derivative of f with respect to Z vanishes:

$$-\, Zf = -\, df(Z) = (i(W)\omega)(Z) = \omega(W, Z) = 0\,, \tag{4.4}$$

f is constant along the flow defined by Z, the canonical field.

When W is kinematical, the corresponding conserved quantity is given as follows:

THEOREM 4.2. — *Let* $\tilde{X}$, *lifted field of* X, *be a kinematical infinitesimal*

[3] As the following discussion is of local nature, it is sufficient for W to be defined only locally.

automorphism of the dynamical system (H(M), ω). *Then there is a conserved quantity corresponding to* $\tilde{X}$,

$$(4.5) \qquad - mci(X)\kappa + q\phi ,$$

where ϕ *is a local function on* M *such that* $i(X)\alpha = - d\phi$.

Proof. — For a kinematical automorphism $\tilde{X}$, the left hand side of (4.3) is calculated as

$$i(\tilde{X})\omega = - mci(\tilde{X})d\kappa + qi(\tilde{X})\alpha = - mc(\mathscr{L}_{\tilde{X}}\kappa - di(\tilde{X})\kappa) + qi(X)\alpha$$
$$= mcdi(X)\kappa + qi(X)\alpha ,$$

on account of (4.2). Since X leaves the closed form α invariant (Theorem 4.1); namely $\mathscr{L}_X\alpha = di(X)\alpha = 0$, there is a local function ϕ on M such that $i(X)\alpha = - d\phi$. We obtain, therefore,

$$i(\tilde{X})\omega = mcdi(X)\kappa - qd\phi = - d(- mci(X)\kappa + q\phi) .$$

From (4.3) and (4.4) with W replaced by $\tilde{X}$, it follows that (4.5) is a conserved quantity, as is wanted.

It is easily seen that in components (4.5) is of the form

$$(4.6) \qquad - mcg_{jk}u^k\xi^j + q\phi ,$$

where (ξ^j) are local components of X.

Though this result can be derived from elsewhere, our method adopted is straightforward.

In the remainder of this section, we study the kinematical infinitesimal automorphisms. To do this, by the aid of Theorem 4.1, we need only to investigate the infinitesimal isometries (i. e. Killing vector fields) which also preserve the electromagnetic field α. These transformations, however, are of much interest from other standpoints (for example, general relativity). The detailed results due to S. Abe and M. Ikeda on non-symmetric tensor fields (4) are available for our purpose.

We assume that the dimension of M is four and that the indefinite metric is of signature $(+, -, -, -)$. By $\mathfrak{G}$ we mean the infinitesimal automorphisms of M that leave both the metric and the electromagnetic field invariant:

$$(4.7) \qquad \mathfrak{G} = \{ X \mid \mathscr{L}_X g = \mathscr{L}_X\alpha = 0 \} .$$

It is clear that $\mathfrak{G}$ is a Lie algebra. The dimension of $\mathfrak{G}$ have been determined in

PROPOSITION 4.1 (Abe-Ikeda [*1*]). — *The maximal dimension of the Lie algebra* (4.7) (5) *is equal to six.*

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(4) In their articles the symmetric part of the non-symmetric tensor is identified with the pseudo-Riemannian metric and the skew symmetric part with the electromagnetic field.

(5) Here and in the sequel, α is not assumed to be closed, but when maximal $\mathfrak{G}$ is admitted α turns out to be closed [*3*], though [*3*] does not allude to it so explicitely.

The Lie algebra composed of the kinematical infinitesimal automorphisms of the relativistic dynamical system $(H(M), \omega)$ is called the *kinematical symmetry* of the system. As is easily seen, it is isomorphic to $\mathfrak{G}$ defined by (4.7).

It is well known that an electromagnetic field $\alpha = (F_{ij}) \neq 0$ is characterized by two quantities, the norm $F_{ij}F^{ij}$ and the determinant $| F_{ij} |$, where $F^{ij} = g^{ik}g^{jl}F_{kl}$. An electromagnetic field is called of *Type II* or of *Type I* according as both the norm and the determinant vanish or not. The local structure of M admitting maximal $\mathfrak{G}$ has been specified by S. Abe for each type of the electromagnetic field [2]. On the basis of his results we have, with the assumption stated above,

THEOREM 4.3. — *Let $(H(M), \omega)$ be a relativistic dynamical system admitting the maximal kinematical symmetry. Then the base manifold* M *is an* **H**-*spacetime or locally the direct product of a Riemannian and a pseudo-Riemannian manifold, each of which is two-dimensional and of constant curvature, according as the electromagnetic field is of Type II or of Type I.*

The definition of an **H**-*spacetime* is to be found in [9], and further properties of **H**-spacetimes have been discussed in [3] and [4]. An **H**-spacetime is also known as a *plane-wave spacetime.*

5. RELATIONS WITH NON-RELATIVISTIC DYNAMICAL SYSTEMS

In this section we discuss the « *non-relativistic limit* » of the dynamical system $(H(M), \omega)$. To do this, M is assumed to be the direct product

$$M = \mathbf{R} \times N, \tag{5.1}$$

where $\mathbf{R}$ and N are, respectively, the real numbers and the $(n-1)$-dimensional Riemannian manifold endowed with the positive definite metric of local components $(v_{\lambda\mu})$ (6). The indefinite metric of M is given by

$$ds^2 = c^2(dt)^2 - v_{\lambda\mu}dx^\lambda dx^\mu , \tag{5.2}$$

ct and (x^λ) denoting the natural coordinate of $\mathbf{R}$ and a local coordinate system of N, respectively.

Let (x^λ, v^λ) denote the induced coordinate system of the tangent bundle $T(N)$. It is seen that $(ct, x^\lambda, v^\lambda)$ serves as a local coordinate system of $H(M)$. In fact

$$u^0 = \frac{c}{\sqrt{c^2 - 2T}}, \qquad u^\lambda = \frac{v^\lambda}{\sqrt{c^2 - 2T}} \tag{5.3}$$

(6) Greek indices run from *1* to $n - 1$.

satisfy Equation (2.1), where

$$T = \frac{1}{2} v_{\lambda\mu} v^{\lambda} v^{\mu}, \tag{5.4}$$

and $c^2 - 2T$ is assumed to be positive. Thus the kinetic form κ defined in Section 3 is written as

$$\kappa = \frac{1}{\sqrt{c^2 - 2T}} (c^2 dt - v_{\lambda\mu} v^{\mu} dx^{\lambda}). \tag{5.5}$$

In the non-relativistic limit where T/c^2 is sufficiently small, (5.5) is reduced to

$$\kappa_0 = cdt + \frac{1}{c} (Tdt - v_{\lambda\mu} v^{\mu} dx^{\lambda}). \tag{5.6}$$

According to (5.1), an electromagnetic field (F_{ij}) splits into a vector field $(F_{\lambda 0})$ and a tensor field $(F_{\lambda\mu})$ on N. Thus in non-relativistic limit the generating form ω is reduced to

$$\omega_0 = md(v_{\lambda\mu} v^{\mu} dx^{\lambda} - Tdt) + qF_{\lambda 0} dx^{\lambda} \wedge cdt + \frac{q}{2} F_{\lambda\mu} dx^{\lambda} \wedge dx^{\mu}. \tag{5.6}$$

This form is found to be the generating form in Newtonian (i. e. non-relativistic) dynamics. For example, see [6].

We conclude this section with the discussion of the kinematical symmetry of a non-relativistic dynamical system. Theorem 4.1 gives a necessary and sufficient condition for the kinematical symmetry. Suppose that a vector field $X = (\xi^{\lambda})$ on N is a Killing vector field which preserves the electromagnetic field, so that X preserves the vector field $(F_{\lambda 0})$ and the tensor field $(F_{\lambda\mu})$:

$$\mathscr{L}_X F_{\lambda 0} = \mathscr{L}_X F_{\lambda\mu} = 0.$$

The lift of X to the tangent bundle $T(N)$ is given by

$$\tilde{X} = \xi^{\lambda} \frac{\partial}{\partial x^{\lambda}} + \frac{\partial \xi^{\lambda}}{\partial x^{\kappa}} v^{\kappa} \frac{\partial}{\partial v^{\lambda}}.$$

It is straightforward to prove that $\tilde{X}$ preserves ω_0:

$$\mathscr{L}_{\tilde{X}} \omega_0 = 0. \tag{5.7}$$

This implies that the kinematical symmetry of a non-relativistic dynamical system is covered by that of a relativistic system.

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