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# Boundary Conditions for the $P(\phi)_2$ Euclidean Field Theory

by

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## I. INTRODUCTION

Ideas and methods from statistical mechanics have had a considerable impact on constructive field theory in the last few years. In particular, Boson models can be realized as ferromagnetic spin systems and are thus subject to the analysis available from the study of the Ising model. In this paper we continue our program [29] of applying statistical mechanical methods to the $P(\phi)_2$ Euclidean field theory by examining one of the basic questions arising in statistical mechanics, namely, the role of boundary conditions. For the most part we concentrate our attention on the « classical B. C. »: free (F), Dirichlet (D), Neumann (N), and periodic (P) boundary conditions.

Just as in statistical mechanics, we expect that the use of boundary conditions in field theory will play fundamental role in the definition of equilibrium states and in establishing the existence or nonexistence of phase transitions. Such an analysis should involve more general B. C. (i. e., analogues of « $\pm$ B. C. ») than the classical ones studied here, and we shall have only a few naive remarks to make about these important questions (III.3). With statistical mechanics as a guide, we know, however, that control of the classical B. C. will provide great flexibility in the study of the thermodynamic limit. Certain operations and assertions are trivial with one choice of B. C. and impossible or very difficult with other choices of B. C. It then becomes important to decide which objects in the theory are independent of the choice of B. C. in the thermodynamic limit. For example, the pressure should depend on B. C. only through a surface effect in finite volume and should be independent of the choice of B. C. in the infinite volume limit. This independence (properly formulated for $P(\phi)_2$ in Theorem I.2 below) is the main result of this paper.

Our approach to this specific problem has been largely influenced by Robinson's work in quantum statistical mechanics [*51*]. In his treatment of (point) Bosons interacting via a repulsive potential the use of Neumann B. C. is critical; the technical tools he deploys are quadratic forms, positive definiteness, sub-and superadditivity; his main application of the result that the pressure is independent of B. C. is to the Gibbs Variational Principle. All of these features are reflected in our work that follows.

Related results have also been obtained by Novikov [*46*] and Ginibre [*15*] who consider the statistical mechanics of Bose hard core particles interacting via an attractive potential. In their approach which employs the Feynman-Kac formula, the use of Dirichlet B. C. is critical. We mention also the results of Fisher and Ruelle on the existence of the thermodynamic limit (with Dirichlet B. C.) for particles interacting via a stable, tempered potential [*54*, § 3.5]; and the results of Fisher and Lebowitz on the independence of classical gas pressures on B. C. [*10*].

As in [*29*], we base our analysis of the $P(\phi)_2$ theory on Nelson's model of the free Euclidean boson field in 2 dimensions [*44*] [*29*]. Let $N = \mathscr{H}_{-1}(\mathbb{R}^2)$ be the real Hilbert space with inner product

$$\langle f, g \rangle_N = (f, (-\Delta + m_0^2)^{-1} g)_{L^2} = \int f(x) G_0(x - y) g(y) d^2x d^2y \quad (I.1)$$

where $G_0$ is the Green's function

$$G_0(x - y) = \frac{1}{(2\pi)^2} \int \frac{e^{ik\cdot(x-y)}}{k^2 + m_0^2} d^2k \quad (I.2)$$

and $m_0 > 0$ is the bare mass. The free Euclidean field $\phi(f)$ is the real Gaussian random field indexed by $f \in N$ with mean zero and covariance (I.1). We denote the underlying probability space by $(Q, \Sigma, \mu_0)$ so that, regarding $\phi(f)$ as a function on Q, we have

$$\int_Q \phi(f)\phi(g) d\mu_0 = \int f(x) G_0(x - y) g(y) .$$

The Gaussian measure $d\mu_0$ is called the non-interacting measure with free (F) B. C., or for short, the « free measure ». To each closed (or open) region $\Lambda \subset \mathbb{R}^2$ we associate the sub-$\sigma$-algebra $\Sigma_\Lambda$ of $\Sigma$ generated by fields $\phi(f)$ with supp $f \subset \Lambda$ (see § II.1).

Suppose for definiteness that $\Lambda$ is a rectangle. According to a well-established tradition, one tries to construct an interacting field theory on $\Lambda$ and then to pass to the thermodynamic limit $\Lambda \to \mathbb{R}^2$. In restricting the non-interacting measure to $\Sigma_\Lambda$ there is already a choice. In addition to the measure

$$d\mu_\Lambda^{\mathrm{F}} = d\mu_0 \restriction \Sigma_\Lambda$$

with free B. C., one could choose the Gaussian measure $d\mu_\Lambda^{\mathrm{X}}$ whose convariance is given by (supp $f, g \subset \Lambda$)

$$\int \phi(f)\phi(g)d\mu_\Lambda^{\mathrm{X}} = \int f(x)\mathrm{G}_\Lambda^{\mathrm{X}}(x, y)g(y)$$

where $\mathrm{G}_\Lambda^{\mathrm{X}} = \left(-\Delta_\Lambda^{\mathrm{X}} + m_0^2\right)^{-1}$ is the Green's function corresponding to any self-adjoint B. C. « X » for $\Delta$ on $\partial\Lambda$. The classical choices mentioned above would be X = D, N or P (for a more complete discussion, see § III.1), but in fact one could accept much more general B. C. than these (see § II.1 and § II.2).

In support of the basic philosophy of this paper that flexibility in the choice of B. C. affords technical advantages, let us mention the key advantages of each of the classical B. C.:

*Free (F) B. C.* This is the simplest B. C. with which to calculate since one has a simple diagonal momentum space formula (I.2) for the covariance. One disadvantage of F B. C. is that while the covariance operator is simple, the inverse differential operator is not; in particular, the corresponding B. C. on $\partial\Lambda$ is non-local (see [*29*, § V] and § II.2 below).

*Dirichlet (D) B. C.* D B. C. give the technically simplest way of introducing barriers between regions (see, in particular, the cluster expansion of [*23*]): $\mathrm{G}_\Lambda^{\mathrm{D}}$ is dominated by $\mathrm{G}_0$, leading to simple estimates. A key advantage of D B. C. may be seen from the lattice approximation where D B. C. play the role of « free boundaries » in ferromagnetic spin systems. This observation leads to monotonicity properties of the (half-) Dirichlet Schwinger functions (e. g. Nelson's Theorem [*44*] and relations with other B. C. explained in § IX.2 below).

*Periodic (P) B. C.* The theory with P B. C. is « closest » to the infinite volume theory in the sense that $\langle \phi(x) \rangle$ is a constant, as is

$$\delta\mathrm{G}_\Lambda^{\mathrm{P}} = \lim_{y\to x} \left[\mathrm{G}_\Lambda^{\mathrm{P}}(x, y) - \mathrm{G}_0(x - y)\right].$$

It is easiest to implement transformations of the measure such as $m^2 \to m'^2$ or $\phi(x) \to \phi(x) + c$ with P B. C. (see Spencer [*66*] and § VII).

*Neumann (N) B. C.* The key advantage of N B. C. is that the partition function is submultiplicative in $\Lambda$, rather than supermultiplicative (as with D B. C.). This property (of a « repulsion » between regions) leads immediately to infinite volume estimates given a finite volume estimate (see § III.2).

Central to this paper is the lattice

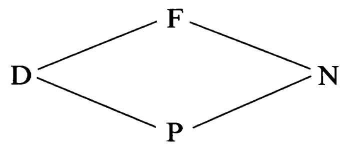


FIG. I.1.

where a B. C. at the right end-point of a line dominates the B. C. at the left endpoint. In terms of Green's functions (considered as operators on $L^2(\Lambda)$) we have (see § III.1)

$$G_\Lambda^D \leq {G_0 \atop G_\Lambda^P} \leq G_\Lambda^N . \tag{I.3 a}$$

(We prefer the notation $G_0$ to $G_\Lambda^F$ and we usually omit superscripts F.) By the theory of conditioning (see [*29*] and the review in § III.2) we immediately obtain the corresponding inequalities for the pressures

$$\alpha_\Lambda^D \leq {\alpha_\Lambda \atop \alpha_\Lambda^P} \leq \alpha_\Lambda^N . \tag{I.3 b}$$

Here the pressure $\alpha_\Lambda^X$ with X B. C. is defined in terms of the interaction polynominal P by

$$\alpha_\Lambda^X = \frac{1}{|\Lambda|} \ln \int e^{-U_\Lambda^X} d\mu_\Lambda^X \tag{I.4}$$

where $U_\Lambda^X = \int_\Lambda : P(\phi)(x) :_{X,\Lambda} d^2x$ with the subscripts X, $\Lambda$ indicating that the Wick subtractions are made with respect to the measure $d\mu_\Lambda^X$. We also consider the half-X pressures $\alpha_\Lambda^{HX}$ defined as in (I.4) except that the interaction $U_\Lambda^X$ is replaced by $U_\Lambda^F$. Our main result is:

THEOREM I.1. — For any semibounded polynomial P, the limits $\alpha_\infty^X = \lim_{\Lambda\to\infty} \alpha_\Lambda^X$ and $\alpha_\infty^{HX} = \lim_{\Lambda\to\infty} \alpha_\Lambda^{HX}$ (X = D, N, or P) all exist and equal $\alpha_\infty = \lim_{\Lambda\to\infty} \alpha_\Lambda$.

*Remarks.* — 1. Using (I.3 *b*) we prove the equality of the $\alpha_\infty^X$ by « bracketing »: first we show that $\alpha_\infty^D = \alpha_\infty$ (§ IV) and then that $\alpha_\infty^N = \alpha_\infty$ (§ V).

2. It is easy to extend the statement that the pressure is independent of B. C. to include $\pm$ B. C. (see § II.3).

3. Existence of $\alpha_\infty$ is a result of Guerra [*25*].

We also study the Schwinger functions, defined to be the moments of the interacting measures:

$$S_\Lambda^X(x_1, \ldots, x_n) \equiv \frac{\int \phi(x_1) \ldots \phi(x_n) e^{-U_\Lambda^X} d\mu_\Lambda^X}{\int e^{-U_\Lambda^X} d\mu_\Lambda^X} \tag{I.5}$$

and similarly for the half-X Schwinger functions $S_\Lambda^{HX}$ where $U_\Lambda^X$ in (I.5) is replaced by $U_\Lambda$. We are not able to draw the conclusions about the Schwinger functions suggested by the lattice of Fig. I.1, but are able only to relate D B. C. to the other B. C. For example if P is even except for a linear term,

$$0 \leq S_\Lambda^{HD} \leq S_\Lambda^{HX} \qquad X = F, P, N$$

and similarly without the H if in addition deg $P \leq 4$ (see § IX.2). In view of Nelson's result that $S_\Lambda^{HD}$ is monotone increasing in $\Lambda$ it is natural to conjecture from Fig. I.1 that $S_\Lambda^{HN}$ is monotone decreasing. We have not been able to prove this, and, indeed, such a result cannot be true for all values of the coupling constant if $P(\phi)_2$ possesses a phase transition. Our results on the independence (and existence) of $S^X = \lim_{\Lambda\to\infty} S_\Lambda^X$ on B. C. fall far short of the corresponding results for the Ising model [55] [36]; we have succeeded in showing only that $S^D = S^P$ under certain circumstances (see § IX.3).

Here is a brief guide to the organization of the paper: in § II we describe a general class of Gaussian measures associated with the operator $(-\Delta + m^2) \upharpoonright C_0^\infty(\Lambda)$ which are suitable non-interacting measures for Boson field theories. An essential regularity condition on the covariance operator G of such a measure is that $G \leq cG_0$ as operators on $L^2(\Lambda)$. In § III we explain how the theory of conditioning leads to the inequalities of (I.3) and to sub- and superadditivity properties, and we give a new proof of the « linear lower bound » using N B. C. In § III.3 we show that the method of images yields most of the bounds we require on the classical Green's functions. In § IV.1 we formulate a general statement of the vanishing influence of D B. C. as the boundary recedes to infinity and thereby show in § IV.2 that $\alpha_\infty^D = \alpha_\infty$. Among the applications we give is a proof of the Gibbs Variational Equality for the entropy. § V contains a proof that $\alpha_\infty^N = \alpha_\infty$. In § VI we give an independent proof of the convergence of $\alpha_\Lambda^P$ and we develop the machinery of periodic states. In § VII we establish covariance properties of the pressure (under translations, scaling and mass shifts) using the fact that the pressure is independent of B. C. Then we determine the dependence of the pressure on the coefficients of P (« dominant and subdominant coupling constants »). For instance we establish that if deg $P = 2n$ then as the dominant coupling constant $\lambda \to \infty$,

$$\alpha_\infty(\lambda) \sim \text{const. } \lambda (\ln \lambda)^n$$

verifying our conjecture in [28]. In § VIII we complete the proof of Theorem I.1 by showing that $\alpha_\infty^{HX} = \alpha_\infty^X$. § IX.1 is devoted to a discussion of the lattice theory and its convergence for classical B. C. The resulting correlation inequalities and some consequences are outlined in § IX.2 and § IX.3. Finally in the Appendix we extend the Checkerboard Theorem of [29] to X B. C. and we explain its significance for the question of spatial decoupling.

One interesting application of the ideas of this paper has been made in [30] where we study the $(a\phi^4 + b\phi^2 - \mu\phi)_2$ field theory. By combining (*i*) Spencer's [66] mass gap result for large $|\mu|$ with *periodic* B. C., (*ii*) the convergence of the Schwinger functions with *Dirichlet* B. C. for all $\mu$, (*iii*) the equality $\alpha_\infty^D = \alpha_\infty^P$ (Theorem I.1), (*iv*) the inequality $S^D \leq S^P$, and (*v*) the Lee-Yang Theorem [63], we show the Dirichlet $a\phi^4 + b\phi^2 - \mu\phi$ theory has a mass gap if $\mu \neq 0$. The method of proof is based on a superharmonic continuation argument of Lebowitz and Penrose [37].

We close this section by discussing two possible sources of confusion that may arise in reading this paper. Firstly, there is a discrepancy between the B. C. terminology of field theory and statistical mechanics. What we call free (F) B. C. does not correspond to « free boundaries » for the Ising model (Actually, F B. C. might be called « free at $\infty$ » since the covariance operator is $G_0$). Rather it is Dirichlet (D) B. C., often called « repulsive » B. C., which corresponds to « free boundaries » for the Ising model. The easiest way of understanding the meaning of D and N (« perfectly elastic ») B. C. is to turn to the lattice approximation. As we explain in § IX.1, the formal expression $(\phi, -\Delta\phi)$ goes over in the lattice theory to a sum over nearest neighbor spins

$$\frac{1}{2}\Sigma(q_n - q_{n'})^2 = 4\Sigma q_n^2 - \Sigma q_n q_{n'} .$$

Dirichlet data are imposed on a line L by dropping the ferromagnetic couplings $q_n q_{n'}$ across L and Neumann data by dropping the coupling terms $(q_n - q_{n'})^2$ across L.

Secondly, in this paper we shift freely between the passive and the active pictures for the free Euclidean field, and so it might be useful to review the distinction between these two pictures (see also [62]). In the active (or measure) picture we realize the free field theories corresponding to different B. C. X on $\partial\Lambda$ by choosing different Gaussian measures $d\mu_\Lambda^X$ on $\Sigma_\Lambda$ but keeping the field fixed as a coordinate function, i. e., for each $f \in C_0^\infty(\Lambda)$

$$\phi(f) : q \rightarrow \langle f, q \rangle \tag{I.6}$$

for $q \in Q$, where it is standard to realize Q as the dual $C_0^\infty(\Lambda)'$. This has been the point of view adopted throughout this section. Equivalently, we may use the passive (or field) picture in which we hold the measure $d\mu_0$

(and the space Q) fixed but « change coordinates » by realizing the field as

$$\phi_\Lambda^X(f) = \phi(p_\Lambda^X f), \qquad f \in C_0^\infty(\Lambda) \tag{I.7}$$

where the field on the right is the coordinate function (I.6) and $p_\Lambda^X$ is a suitable operator on $L^2(\Lambda)$.

These realizations of the free theory with X B. C. as $\{ \phi, d\mu_\Lambda^X \}$ or as $\{ \phi_\Lambda^X, d\mu_0 \}$ are equivalent; in particular, the covariances are equal,

$$\begin{aligned}\int \phi(f)\phi(g)d\mu_\Lambda^X &= \int \phi_\Lambda^X(f)\phi_\Lambda^X(g)d\mu_0 \\ &= \int f(x)G_\Lambda^X(x, y)g(y)\end{aligned}$$

so that $G_\Lambda^X = (p_\Lambda^X)^* G_0 p_\Lambda^X$ as operators on $L^2(\Lambda)$. We use the passive picture mainly in sections II and IV.

The passive picture lends itself to the theory of conditioning whereas the active picture is most convenient for the formulation of half-X B. C. Although we have made frequent reference to HX B. C. above, it might be helpful to make a few elementary remarks here. In the passive picture there is no ambiguity about what is meant by Wick subtractions, e. g.,

$$\begin{aligned}:\phi_\Lambda^X(f)\phi_\Lambda^X(g): &= \phi_\Lambda^X(f)\phi_\Lambda^X(g) - \int \phi_\Lambda^X(f)\phi_\Lambda^X(g)d\mu_0 \\ &= \phi_\Lambda^X(f)\phi_\Lambda^X(g) - \int f(x)G_\Lambda^X(x, y)g(y)\,.\end{aligned}$$

However in the active picture we may define the subtractions with respect to either $d\mu_\Lambda^X$ or $d\mu_0$. We denote Wick products in the former case with subscripts $\Lambda$, X and in the latter case (« free Wick ordering ») with no subscripts; e. g.,

$$\begin{aligned}:\phi(f)\phi(g):_{\Lambda,X} &= \phi(f)\phi(g) - \int \phi(f)\phi(g)d\mu_\Lambda^X \\ &= \phi(f)\phi(g) - \int f(x)G_\Lambda^X(x, y)g(y)\end{aligned}$$

and

$$\begin{aligned}:\phi(f)\phi(g): &= \phi(f)\phi(g) - \int \phi(f)\phi(g)d\mu_0 \\ &= \phi(f)\phi(g) - \int f(x)G_0(x, y)g(y)\,.\end{aligned}$$

The HX-theories are defined, in the active picture, with Gaussian measure $d\mu_\Lambda^X$ and with free Wick ordering. There is a simple relation between the different Wick powers; explicitly [*29*]

$$:\phi^r(x): = \sum_{j=0}^{[r/2]} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} [-\delta G_\Lambda^X(x)]^j :\phi^{r-2j}(x):_{X,\Lambda} \tag{I.8}$$

where $\left\{ \begin{matrix} r \\ j \end{matrix} \right\} = r!/j!(r-2j)!\,2^j$ and

$$\delta G_\Lambda^X(x) = \lim_{y \to x} \left[ G_\Lambda^X(x, y) - G_0(x - y) \right].$$

Finally we mention that we have developed a HD transfer matrix that is based on realizing D B. C. in the « time » direction by placing Q-space $\delta$-functions on the $t = \text{const.}$ boundaries. We omit a discussion of this topic from the present paper, but details may be found in [*62*]. This approach can be used to prove the convergence of the HD Wightman functions.

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## II. GAUSSIAN BOUNDARY CONDITIONS

The cutoff states we consider in this paper are of the special form $e^{-U_\Lambda}d\mu/\text{Norm}$ where the function $U_\Lambda$ in the Gibbs factor $e^{-U_\Lambda}$ differs from the free B. C. interaction only in the Wick subtractions used. Our goal in this section is to describe in detail the « unperturbed » measures $d\mu$ we will use. They will have no interaction in the sense that the Euclidean field $(\phi, d\mu)$ is still a Gaussian random field and, in a sense we make precise below, they differ from $d\mu_0$ only by a factor concentrated on the boundary of $\Lambda$.

In § II.1, we describe the allowed covariances for $d\mu$ and in § II.2 we relate these convariances to the « classical theory » of boundary conditions by showing the allowed convariances are precisely the Green's functions for a family of self-adjoint extensions of $-\Delta \upharpoonright C_0^\infty(\Lambda)$. This section is then the link with the further specialization of $d\mu$ we shall make in the remainder of the paper when we restrict to the four types of classical Green's function. In § II.3 we make some remarks about a class of B. C. which we expect will play a major role in the discussion of broken symmetry [*8*].

We use freely the language of Gaussian stochastic processes which are extensively discussed in Gelfand-Vilenkin [*13*] and Hida [*31*]; see also Segal [*58*], Dimock-Glimm [*7*] and Simon [*62*]. Throughout this section, we fix a bare mass $m_0$.

*Warning.* — With regard to a factor of $\frac{1}{2}$ in defining Gaussian processes, [*29*] and [*62*] have different conventions. We follow [*29*].

## II.1. Covariances.

We begin by recalling some notation from [29]. N denotes the Hilbert space obtained by completing $C_0^\infty(\mathbb{R}^2)$ in the norm, $G_0(f, f)^{1/2}$ where

$$G_0(f, g) = (f, (-\Delta + m_0^2)^{-1}g) \tag{II.1}$$

and $( . , . )$ is the $L^2$ inner product. Since $C_0(\mathbb{R}^2)$ is continuously imbedded in N, each element of $N^* = N$ can be viewed as a tempered distribution and, in particular, each element has a support. Given a closed set $C \subset \mathbb{R}^2$, $N_C$ is the (closed) subspace of N consisting of those elements of N with support in C. $e_C$ denotes projection (in $G_0$-inner product) onto $N_C$. If $\Lambda \subset \mathbb{R}^2$ is open, we let

$$p_\Lambda = 1 - e_{\mathbb{R}^2\backslash\Lambda} . \tag{II.2}$$

The Markov property on the one particle space (the « pre-Markov » property) implies that for $f \in C_0^\infty(\Lambda)$,

$$p_\Lambda f = f - e_{\partial\Lambda} f \tag{II.3}$$

*Remark.* — If $\Lambda$ is open, then one can show that $e_\Lambda = e_{\bar\Lambda}$ where $e_\Lambda$ is the projection onto the closure of $C_0^\infty(\Lambda)$ in N.

Finally, we recall that if $-\Delta_\Lambda$ is the Dirichlet Laplacian (Friedrichs extension of $-\Delta \upharpoonright C_0^\infty$ as an operator on $L^2(\Lambda)$) and

$$G_\Lambda^D(f, g) = (f, (-\Delta_\Lambda + m_0^2)^{-1}g) \qquad (f, g \in C_0^\infty(\Lambda)),$$

then (Corollary II.25 of [29]):

$$G_\Lambda^D(f, g) = G_0(p_\Lambda f, p_\Lambda g) \tag{II.4}$$

This extends to all $f, g \in N_{\bar\Lambda}$.

Our first theorem will motivate our choice of definition of general convariances:

THEOREM II.1. — Let

$$d\mu = F d(\mu_0 \upharpoonright \Sigma_\Lambda)$$

where $F \in L^p(Q_N, d\mu_0)$ for some $p > 1$, $F \geq 0$ and where $\int F d\mu_0 = 1$. Suppose moreover, that F is Gaussian on the boundary in the sense that

$$F = \prod_{n=1}^{\infty} [\sqrt{2\alpha_n + 1} \exp(-\alpha_n \phi(f_n)^2)]$$

where $f_n$ is some orthonormal set in $N_{\partial\Lambda}$, and for some A, $A^{-1} \leq 1 + \alpha_n \leq A$ (all $n$). For $f, g \in C_0^\infty(\Lambda)$, let

$$G(f, g) = \int \phi(f)\phi(g) d\mu . \tag{II.5}$$

Then:

(*a*) there is a constant $c$ with

$$0 \leq G(f, f) \leq cG_0(f, f) \qquad (II.6)$$

for all $f \in C_0^\infty(\Lambda)$.

(*b*) For all $f, g \in C_0^\infty(\Lambda)$:

$$G(f, g) = G_\Lambda^D(f, g) + Q(e_{\partial\Lambda}f, e_{\partial\Lambda}g) \qquad (II.7)$$

where Q is a bounded, positive definite quadratic form on $N_{\partial\Lambda}$.

*Proof.* — (*a*) Follows easily from Hölder's inequality and the fact that $\left(\int \phi(f)^p d\mu_0\right)^{1/p} \leq \sqrt{(p-1)}\left(\int \phi(f)^2 d\mu_0\right)^{1/2}$ for $p > 2$ and Gaussian random variables. To prove (*b*), we remark that (*a*) implies that G(. , .) extends to $N_{\bar\Lambda} \times N_{\bar\Lambda}$ so that, by (II.3)

$$G(f, g) = G(e_{\partial\Lambda}f, e_{\partial\Lambda}g) + G(p_\Lambda f, p_\Lambda g) + G(e_{\partial\Lambda}f, p_\Lambda g) + G(p_\Lambda f, e_{\partial\Lambda}g).$$

Now according to the breakup $N = N_{\partial\Lambda} \oplus N_{\partial\Lambda}^\perp$, the measure $d\mu_0$ factors into $d\mu_{0,\partial\Lambda} \otimes d\mu_{0,\partial\Lambda}^\perp$ and F is by hypothesis only a function of the $q_{\partial\Lambda}$ variables.

Thus

$$\begin{aligned} G(p_\Lambda f, p_\Lambda g) &= \left(\int \phi(p_\Lambda f)\phi(p_\Lambda g) d\mu_{0,\partial\Lambda}^\perp\right)\left(\int F d\mu_{0,\partial\Lambda}\right) \\ &= \int \phi(p_\Lambda f)\phi(p_\Lambda g) d\mu_0 = G_0(p_\Lambda f, p_\Lambda g) = G_\Lambda^D(f, g) \end{aligned}$$

by (II.4). Similarly $G(p_\Lambda f, e_{\partial\Lambda}g) = G(e_{\partial\Lambda}f, p_\Lambda g) = 0$. Since $Q(h, k) = G(h, k)$ for $h, k \in N_{\partial\Lambda}$ defines a bounded quadratic form by (*a*), (II.7) is proven. ■

In the above, Q is not an arbitrary bounded quadratic form since the general theory of symplectic transformations [*57*] [*60*] assures us that the operator A defined by $Q(f, g) = G_0(f, Ag)$ has $A - 1$ Hilbert-Schmidt on $N_{\partial\Lambda}$. Since this additional property does not hold e. g. for the Dirichlet B. C. theory (Q = 0), we supress it in our general definition:

DEFINITION. — Let Q be a bounded positive-definite quadratic form on $N_{\partial\Lambda}$ and let $l$ be a bounded linear functional on $N_{\partial\Lambda}$. Then, the $\{Q, l\}$-B. C. Gaussian field, $\phi$, is the Gaussian random process indexed by $C_0^\infty(\Lambda)$ with mean $l(e_{\partial\Lambda}.)$ and covariance

$$G(.\,,.) = G_\Lambda^D(.\,,.) + Q(e_{\partial\Lambda}.\,, e_{\partial\Lambda}.);$$

that is (see Remark 2),

$$\int \phi(f) d\mu = l(e_{\partial\Lambda}f) \equiv \langle \phi(f) \rangle$$

$$\int [\phi(f) - \langle \phi(f) \rangle][\phi(g) - \langle \phi(g) \rangle] d\mu = G(f, g).$$

*Remarks.* — 1. The Gaussian process $\phi$ can always be extended to a process on $N_{\bar{\Lambda}}$.

2. The process indexed by $C_0^\infty(\Lambda)$ can be realized in a standard way by a measure $d\mu$ on $C_0^\infty(\Lambda)'$ [*13*].

We shall generally realize the process in the active picture, thereby emphasizing the measure $d\mu$. In § IV, however, we shall employ the passive picture (see the Introduction and [*62*] for the distinction between the active and passive pictures) where the field $\phi$ corresponding to the covariance G can be realized as:

$$\phi(f) = \phi_{D,\Lambda}(f) \oplus \phi_{\partial\Lambda}(e_{\partial\Lambda} f) \tag{II.8}$$

In (II.8), $\phi_{D,\Lambda}$ denotes the Dirichlet field, $\oplus$ indicates direct sum (i. e. $\phi$ is realized on a product space with a product measure; see Proposition I.7 of [*62*]), and $\phi_{\partial\Lambda}(.)$ is the Gaussian process on $N_{\partial\Lambda}$ with mean $l$ and covariance Q.

EXAMPLE 1. — If $l = 0$, $Q = G_0 \upharpoonright N_{\partial\Lambda} \times N_{\partial\Lambda}$, then we obtain the free B. C. field of [*43*].

EXAMPLE 2. — If $l = 0$, $Q = 0$, then we obtain the Dirichlet B. C. field of [*29*].

*Remark.* — It is no coincidence (see § II.2) that in both cases above G is a Green's function for $-\Delta + m_0^2$ on $\Lambda \times \Lambda$.

Since we have not demanded that $Q - 1$ be Hilbert-Schmidt, we cannot hope that Theorem II.1 have a strict converse but one does have the following partial converse which makes precise the sense in which $d\mu$ « differs from $d\mu_0$ only on the boundary of $\Lambda$ »:

THEOREM II.2. — Let $\Lambda$ be a fixed open set in $\mathbb{R}^2$ and let $d\mu$ be a $\{ Q, l \}$-B. C. field. Let $\Lambda'$ be an open set with $\bar{\Lambda}'$ a compact subset of $\Lambda$ (so, in particular, $d(\partial\Lambda, \Lambda') > 0$). Then:

(*a*) $d\mu \upharpoonright \Sigma_{\Lambda'}$ is absolutely continuous with respect to $d\mu_0 \upharpoonright \Sigma_{\Lambda'}$,

(*b*) $d\mu \upharpoonright \Sigma_{\Lambda'} = F d\mu_0 \upharpoonright \Sigma_{\Lambda'}$

where F is a Gaussian measurable w. r. t. $\Sigma_{\partial\Lambda'}$.

This result generalizes Theorem II.34 of [*29*] (the case $l = Q = 0$) and as in that case depends on the fact that $e_{\Lambda'} - e_{\Lambda'} p_\Lambda$ is Hilbert-Schmidt (Lemma III.5 *b* of [*29*] given that $e_{\Lambda'} - e_{\Lambda'} p_\Lambda = e_{\Lambda'} e_{\mathbb{R}^2 \setminus \Lambda}$). We begin with a lemma that is essentially equivalent to the combined Markov property and (no interaction) DLR equations for $d\mu$!

LEMMA II.3. — Let $e_C^G$ denote the orthogonal projection onto $N_C$ in the inner product G. Then for $f \in C_0^\infty(\Lambda')$:

$$e^G_{\Lambda \setminus \Lambda'} f = e_{\partial\Lambda'} f \tag{II.9}$$

*Remark.* — We emphasize that in (II.9), $e_{\partial\Lambda'}$ is a projection w. r. t. $G_0$.

*Proof.* — Since $e_{\partial\Lambda'}f$ has support in $\Lambda\backslash\Lambda'$, we need only prove that for $g \in N_{\Lambda\backslash\Lambda'}$

$$G(f, g) = G(e_{\partial\Lambda'}f, g) \tag{II.10}$$

But

$$\begin{aligned} G_\Lambda^D(f, g) &= G_0((1 - e_{\partial\Lambda})f, g) = G_0(e_{\partial\Lambda'}(1 - e_{\partial\Lambda})f, g) \\ &= G_\Lambda^D(e_{\partial\Lambda'}f, g) \end{aligned}$$

on account of the (pre-)Markov property for $G_0$. And

$$Q(e_{\partial\Lambda}f, e_{\partial\Lambda}g) = Q(e_{\partial\Lambda}e_{\partial\Lambda'}f, e_{\partial\Lambda}g)$$

again using the (pre-)Markov property. (II.10) follows. ■

*Remark.* — The lemma immediately implies the (pre-)Markov property $e^G_{\Lambda\backslash\Lambda'}e^G_{\Lambda'} = e^G_{\partial\Lambda'}e^G_{\Lambda'}$ for $\Lambda'$ of the type considered but the proof extends for any open $\Lambda'$. We thus have as a Corollary of the lemma.

THEOREM II.4. — The field theory $(\phi, d\mu)$ has the Markov property.

*Proof of Theorem* II.2. — We consider first the case $l = 0$. To prove (*a*), we need only show that there are Hilbert-Schmidt operators $B_1$, $B_2$ on $N_{\Lambda'}$ so that for $f, g \in N_{\Lambda'}$

$$G(f, g) = G_0(f, g) - G_0(B_1 f, B_1 g) + G_0(B_2 f, B_2 g)$$

for we can then apply the theorem of Shale [*60*]. But

$$G(f, g) = G_0(f, g) - G_0(e_{\partial\Lambda}f, e_{\partial\Lambda}g) + Q(e_{\partial\Lambda}f, e_{\partial\Lambda}g).$$

Now, let C be the operator on $N_{\partial\Lambda}$ whose quadratic form is Q. Let $B_1 = e_{\partial\Lambda}e_{\Lambda'}$, $B_2 = C^{1/2}e_{\partial\Lambda}e_{\Lambda'}$. Then $B_1$, $B_2$ are Hilbert-Schmidt (by Lemma III.5 *b* of [*29*]) and G has the required form.

To prove (*b*), we note that Lemma II.3 asserts that $d\mu$ obeys the no interaction DLR equations and so by Theorem VII.2 of [*29*], $d\mu \upharpoonright \Sigma_{\Lambda'} = F d\mu_0 \upharpoonright \Sigma_{\Lambda'}$ where F is $\Sigma_{\partial\Lambda'}$-measurable. That F is a Gaussian follows from Shale's analysis.

We consider now the case $l \neq 0$. We first claim that there is an $m \in N_{\Lambda'}$ so that $l(e_{\partial\Lambda}f) = G(f, m)$ for all $f \in N_{\Lambda'}$. To assure the existence of such an $m$, we need only show that $| l(e_{\partial\Lambda}f) | \leq c G(f, f)^{1/2}$ for some fixed constant $c$ and all $f \in N_{\Lambda'}$. But by hypothesis $| l(e_{\partial\Lambda}f) | \leq c_1 G_0(f, f)^{1/2}$ for all $f$ and by the above analysis $G_0(f, f) \leq c_2 G(f, f)$ for all $f \in N_{\Lambda'}$. The $l \neq 0$ measure, $d\mu_l$, is absolutely continuous relative to the $l \neq 0$ measure, $d\mu$, by the formula

$$d\mu_l = e^{\phi(m)} d\mu / \exp\left(\frac{1}{2} G(m, m)\right). \quad ■$$

For later purposes we will need $L^p$-properties of the factor F in Theorem II.2 (*b*). We have:

THEOREM II.5. — Let A be the operator on $N_{\Lambda'}$ with quadratic form

$$\langle f, Ag \rangle_{N_{\Lambda'}} = G(f, g) - G_0(f, g)$$

(so $A = B_2^* B_2 - B_1^* B_1 = e_{\Lambda'} e_{\partial\Lambda}(C - 1) e_{\partial\Lambda} e_{\Lambda'}$ in terms of the proof above). Then:

(*a*) The factor F of Theorem II.2 lies in all $L^p$ with $p < 1 + \alpha_0^{-1}$ where $\alpha_0$ is the largest eigenvalue of A. If $\alpha_0 \leq 0$, then $F \in L^\infty$.

(*b*) For any $p > 1$, there is a constant $C_p$ so that whenever $\| A \|_{HS} \leq (2p)^{-1}$ and $A \geq 0$:

$$\left( \int | F |^p d\mu_0 \right)^{1/p} \leq (C_p \| A \|_{HS}^2) \tag{II.11 a}$$

*Remark.* — This theorem remains true, by symmetry, if (F, A, $d\mu_0$) are replaced by ($F^{-1}$, $-$ A, $d\mu$). In particular in the case where $A \geq 0$ (e. g. Neumann B. C.) so that $\alpha_0(-A) \leq 0$, $F^{-1}$ is in $L^\infty$.

*Proof.* — By the Shale theory (see e. g. § I.6 of [*62*]) F has the form:

$$F = \prod_i \lambda_i^{-1} \exp \left[ (1 - \lambda_i^{-2}) q_i^2 \right]$$

where $\lambda_i$ are related to the eigenvalues $\alpha_i$, of A, by $\lambda_i = (1 + \alpha_i)^{1/2}$ and $q_i$ are the corresponding eigencomponents. Note that the $\lambda_i$ are bounded away from zero since $A \geq - e_{\Lambda'} e_{\partial\Lambda} e_{\Lambda'} \geq - c > - 1$ by Lemma II.35 of [*29*]. $d\mu_0$ has the form:

$$d\mu_0 \restriction \Sigma_{\Lambda'} = \prod_{i=1}^{\infty} \pi^{-1/2} e^{-q_i^2} dq_i .$$

Thus if $1 > p(1 - \lambda_0^{-2}) = p\alpha_0/(1 + \alpha_0)$ then each factor is in $L^p$, and by the Shale theory we need only prove that the formal expression for $\int F^p d\mu_0 \restriction \Sigma_{\Lambda'}$ namely

$$\begin{aligned} \int F^p d\mu_0 &= \Pi \lambda_i^{-p} [1 - p(1 - \lambda_i^{-2})]^{-1/2} \\ &= \Pi [(1 + \alpha_i)^{p-1} (1 + \alpha_i - p\alpha_i)]^{-1/2} \end{aligned} \tag{II.11 b}$$

is convergent (the convergence of this formal expression is the key to justifying its use [*57*] [*60*] [*34*] [*62*]). For general A using the fact that 1 + A is invertible it is standard to prove (II.11 *b*) convergent. If $\alpha_0 \leq 0$, then $F \in L^\infty$ since $\lambda_i \leq 1$ and $\pi\lambda_i^{-1}$ is convergent because A is trace class. We concentrate on proving (II.11) in case $A \geq 0$ following a trick of Klein [*34*]. Let $f(x) = (1 + x)^{p-1}(1 + x - px)$ for $x \in [0, \alpha_0]$. Then $f(0) = 1$, $f'(0) = 0$ and for $1 \leq p < 1 + \alpha_0^{-1}$

$$f(x) \geq 1 + \alpha_0 - p\alpha_0 = (1 + \alpha_0^{-1} - p)\alpha_0 > 0$$

so $g(x) = (f(x))^{-1}$ obeys $g(0) = 1$, $g'(0) = 0$ and $\sup_{0 \leq x \leq \alpha_0} \frac{1}{2}(g''(x)) = C < \infty$. Thus for $x \in [0, \alpha_0]$:

$$g(x) \leq 1 + Cx^2 \leq e^{Cx^2}.$$

Thus the product in (II.11 *b*) is estimated by

$$\prod_i \exp\left(\frac{1}{2} C\alpha_i^2\right) = \exp\left(\frac{1}{2} C \| A \|_{HS}^2\right).$$

If $\| A \|_{HS} \leq (2p)^{-1}$, then $\alpha_0 \leq (2p)^{-1}$ so $1 + \alpha_0 - p\alpha_0 \geq 1 - p\alpha_0 \geq \frac{1}{2}$ so that C may be chosen in a universal ($p$-dependent) manner. ■

## II.2. **Connection with self-adjoint extensions.**

The covariances G considered in the last section (see (II.6) and (II.7)) arise from self-adjoint extensions of $- \Delta \restriction C_0^\infty(\Lambda)$ satisfying certain regularity properties corresponding to (II.6) and (II.7). More precisely:

THEOREM II.6. — Let G be a quadratic form on $N_\Lambda$ obeying (II.6) and (II.7). Then there is a self-adjoint extension H of $- \Delta \restriction C_0^\infty(\Lambda)$ related to G by

$$(f, (H + m_0^2)^{-1} g) = G(f, g) \tag{II.12}$$

so that:

(*i*) $H \geq - m_0^2 + \varepsilon$ for $\varepsilon > 0$.

(*ii*) $Q(H) \subset Q(H_0^\Lambda)$ where $H_0^\Lambda$ is the self-adjoint operator on $L^2(\Lambda)$ with $(f, (H_0^{\Lambda})^{-1} g) = G_0(f, g)$ for $f, g \in L^2(\Lambda)$.

Conversely given a self-adjoint extension H of $- \Delta \restriction C_0^\infty(\Lambda)$ obeying (*i*), (*ii*) above, then G defined by (II.12) obeys (II.6) and (II.7).

*Remark.* — $H_0^\Lambda$ is an extension of $- \Delta + m_0^2$ and is both $m_0$ and $\Lambda$ dependent.

*Proof of the first half of Theorem* II.6. — Let G obey (II.6, 7). Since $G(f, f) \geq G_\Lambda^D(f, f) > 0$ if $f \neq 0$, there is an unbounded operator H on $L^2(\Lambda)$ with (II.12) holding. Since, by (II.6) we have $(H + m_0^2)^{-1} \leq c(H_0^\Lambda)^{-1}$ we conclude by the theory of operator inequalities (Kato [*33*], p. 330; see also § III.1 below) that $Q(H_0^\Lambda) \supset Q(H)$ and $(H + m_0^2) \geq c^{-1} H_0^\Lambda \geq c^{-1} m_0^2$ so (*i*) holds. Thus we need only prove that H extends $- \Delta \restriction C_0^\infty(\Lambda)$. Let $g \in C_0^\infty(\Lambda)$ and let $T \in N_{\partial\Lambda}$, then:

$$\langle T, (- \Delta + m_0^2) g \rangle_N = (T, g)_{L^2} = 0$$

so $e_{\partial\Lambda}(- \Delta + m_0^2) g = 0$. It follows by (II.7) that for $f \in L^2$

$$\begin{aligned}(f, (H + m_0^2)^{-1}(- \Delta + m_0^2) g) &= G(f, (- \Delta + m_0^2) g) \\ &= G_\Lambda^D(f, (- \Delta + m_0^2) g) = (f, g)_{L_2}\end{aligned}$$

so that $(H + m_0^2)^{-1}(- \Delta + m_0^2) g = g$, i. e. $g \in D(H)$ and $Hg = - \Delta g$.

Thus H extends $-\Delta \upharpoonright C_0^\infty(\Lambda)$ and the first half of the theorem is proven. ■

For the second half of the theorem, we need:

LEMMA II.7. — Let $\mathscr{H}_{+1,\Lambda} \equiv Q(-\Delta_\Lambda) \cap \mathscr{H}_{+1}$. Let $\mathscr{H}_{-1,\Lambda}$ be the image of $\mathscr{H}_{+1,\Lambda}$ under the isometry $-\Delta + m_0^2$ from $\mathscr{H}_{+1}$ to N. Then $\mathscr{H}_{-1,\Lambda}$ is the closure in $N_{\bar{\Lambda}}$ of $(1 - e_{\partial\Lambda})[C_0^\infty(\Lambda)]$.

*Proof.* — By the theory of symmetric quadratic forms, $\mathscr{H}_{-1,\Lambda}$ is the closure of $C_0^\infty(\Lambda)$ in the norm

$$\| f \|_{-1,\Lambda}^2 = \langle f, (-\Delta_\Lambda + m_0^2)^{-1} f \rangle = \| (1 - e_{\partial\Lambda}) f \|_{N_{\bar{\Lambda}}}^2 .$$

The conclusion is thus clear. ■

*Conclusion of the proof of Theorem* II.6. — Suppose H obeys (*i*), (*ii*), that it extends $-\Delta \upharpoonright C_0^\infty(\Lambda)$ and that G is given by (II.12). By (*ii*), the Hilbert space Q(H) with norm $(f, (H + m^2)f)^{1/2}$ is imbedded in the Hilbert space $Q(H_0^\Lambda)$ with norm $(f, (H_0^\Lambda)f)^{1/2}$. By the closed graph theorem the map is continuous, so there is a C with

$$(f, (H_0^\Lambda)f) \leq C(f, (H + m_0^2)f)$$

all $f \in Q(H)$. Thus $H_0^\Lambda \leq C(H + m_0^2)$ as forms so [33, p. 330]

$$(H + m_0^2)^{-1} \leq C(H_0^\Lambda)^{-1},$$

i. e. for $f \in L^2(\Lambda)$, $G(f, f) \leq CG_0(f, f)$ proving (II.6). Now, by (II.6), G extends to the closure of $L^2(\Lambda)$ in N-norm and in particular we can write for $f \in C_0^\infty(\Lambda)$

$$G(f, f) = G(e_{\partial\Lambda} f, e_{\partial\Lambda} f) + G((1 - e_{\partial\Lambda})f, (1 - e_{\partial\Lambda})f) + \text{cross terms.} \quad \text{(II.13)}$$

Letting $Q(g, h) = G(g, h)$ for $g, h \in N_{\partial\Lambda}$ we see that the first term in (II.13) is $Q(e_{\partial\Lambda} f, e_{\partial\Lambda} f)$. Thus we need only prove that $G(p_\Lambda f, p_\Lambda f) = G_\Lambda^D(f, f)$ and that the cross terms vanish to conclude that (II.7) holds.

Now let $g = p_\Lambda f = (1 - e_{\partial\Lambda})f$. Then, by Lemma II.7, $g \in \mathscr{H}_{-1,\Lambda}$ so there exists $h_n \in C_0^\infty(\Lambda)$ with $(-\Delta + m_0^2)h_n \to g$ in N-norm and $h_n \to h$ in $\mathscr{H}_{+1,\Lambda}$ norm. Thus:

$$\begin{aligned} G(g, g) &= \lim_{n\to\infty} G(g, (-\Delta + m_0^2)h_n) \qquad \text{(by II.6)} \\ &= \lim_{n\to\infty} (g, h_n) \\ &= (g, h) = G_\Lambda^D(g, g) \end{aligned}$$

so $G(p_\Lambda f, p_\Lambda f) = G_\Lambda^D(f, f)$.

Finally we must show the cross terms vanish in (II.13). Since H extends $-\Delta \upharpoonright C_0^\infty$, the quadratic form of H is an extension of that of $-\Delta_\Lambda$ so $H \leq -\Delta_\Lambda$ as forms. Thus, $(-\Delta_\Lambda + m_0^2)^{-1} \leq (H + m_0^2)^{-1}$, i. e. $G_\Lambda^D(f, f) \leq G(f, f)$, all $f \in N_{\bar{\Lambda}}$. Now, let $f \in \text{Ran}\, p_\Lambda$; $g \in \text{Ran}\, e_{\partial\Lambda}$. Then if $h = f + \lambda g$

$$G_\Lambda^D(h, h) = G_\Lambda^D(f, f) \leq G(f, f) + 2\,\text{Re}\,(\lambda G(f, g)) + |\lambda|^2 G(g, g).$$

Since we have proven that $G(f, f) = G_\Lambda^D(f, f)$ above, we conclude that for all $\lambda$

$$|\lambda|^2 G(g, g) + 2 \operatorname{Re} (\lambda G(f, g)) \geq 0 .$$

This implies that $G(f, g) = 0$ completing the proof of (II.7). ■

There are two natural questions associated with Theorem II.6, namely:

1. For what class of B. C. and regions does G satisfy the bound (II.6)?
2. Is it possible to obtain an explicit expression for the B. C. on H in terms of the boundary form Q of (II.7)?

We do not attempt to give a complete answer to these questions in this paper, because we are not interested in all of the self-adjoint extensions of $\Delta \restriction C_0^\infty(\Lambda)$ satisfying (II.6) and (II.7) but only in the « classical » extensions (D, N, P, F B. C.). Now $G_\Lambda^D$ and $G_\Lambda^F = G_0$ clearly satisfy the bound (II.6) for all $\Lambda$; and in § III we verify that $G_\Lambda^P$ (for $\Lambda$ a rectangle) and $G_\Lambda^N$ (for a large class of regions $\Lambda$) satisfy (II.6) and hence have the various properties established here (e. g., (*a*) and (*b*) of Theorem II.5).

In general the B. C. on H will be quite complicated and *non-local*. As a specific example, we (formally) derive here the form of the F B. C. (Example 1 in the previous section). Let $\Lambda \subset \mathbb{R}^2$ be a region with suitably regular boundary $\partial\Lambda$ and let $G_E(x, y)$ be the Green's function for the exterior Dirichlet problem, i. e.,

$$\begin{aligned} (-\Delta + m_0^2)G_E(x, y) &= \delta(x - y) \qquad & x, y \in \Lambda' = \mathbb{R}\backslash\bar{\Lambda} \\ G_E(x, y) &= 0 & x \in \partial\Lambda, \quad y \in \Lambda' \end{aligned} \tag{II.14}$$

Let $\dfrac{\partial}{\partial n}$ be the outward normal derivative on $\partial\Lambda$. Then for $x, y \in \Lambda'$

$$G_E(x, y) = G_0(x, y) - \int_{\partial\Lambda} \frac{\partial G_E(x, z)}{\partial n_z} G_0(z, y) dz \tag{II.15}$$

since the right side of (II.15) clearly satisfies the defining equations (II.14) for $G_E$. Using the relation

$$\frac{\partial G_E}{\partial n_x}(x, y_e) = \delta(x - y)$$

where $f(y_e)$ denotes the limit of $f(y')$ as $y'$ in $\Lambda'$ approaches $y$ on $\partial\Lambda$, we thus obtain from (II.15)

$$\delta(x - y) = \frac{\partial G_0(x, y_e)}{\partial n_x} - \int_{\partial\Lambda} \frac{\partial^2 G_E(x, z)}{\partial n_x \partial n_z} G_0(z, y_e) dz . \tag{II.16}$$

But $G_0(x, y)$ is continuous in $y$ as $y$ crosses $\partial\Lambda$ whereas $\dfrac{\partial G_0}{\partial n_x}(x, y)$ suffers a jump:

$$\frac{\partial G_0}{\partial n_x}(x, y_e) - \frac{\partial G_0}{\partial n_x}(x, y_i) = \delta(x - y) . \tag{II.17}$$

Therefore by (II.16) and (II.17)

$$\frac{\partial G_0}{\partial n_x}(x, y_i) = \int_{\partial\Lambda} \sigma(x, z) G_0(z, y) dz .$$

where $\sigma(x, z) = \dfrac{\partial^2 G_E(x, z)}{\partial n_x \partial n_z}$. It follows that the F B. C. on a function $f$ in the range of $G_0[L^2(\Lambda)]$ is

$$\frac{\partial f}{\partial n} = \int_{\partial\Lambda} \sigma(x, z) f(z) . \qquad \text{(II.18)}$$

As an example of the nonlocal B. C. (II.18), let $\Lambda$ be the half-plane $\Lambda = \{ (x_1, x_2) \mid x_1 > 0 \}$. Then, for $x, y \in \Lambda'$,

$$G_E(x, y) = G_0(x - y) - G_0(rx - y)$$

where $rx = (-x_1, x_2)$, so that for $x \in \Lambda'$

$$\begin{aligned} \frac{\partial G_E(x, y)}{\partial n_y} &= \frac{1}{(2\pi)^2} \int \frac{ip_1 e^{ip_2(x_2 - y_2)}(e^{ip_1 x_1} - e^{-ip_1 x_1})}{p^2 + m_0^2} d^2p \\ &= \frac{1}{2\pi} \int e^{\mu_2 x_1} e^{ip_2(x_2 - y_2)} dp_2 \end{aligned}$$

where $\mu_2 = (p_2^2 + m_0^2)^{1/2}$. Therefore

$$\sigma(x_2, y_2) = -\frac{1}{2\pi} \int \mu_2 e^{ip_2(x_2 - y_2)} dp_2 ,$$

a non-local pseudo-differential operator.

## II.3. +, – Boundary Conditions.

In the previous two sections, we have examined the role of the covariance operator G. In this section, we turn to the « mean functional », $l$, and, in particular, define + and – B. C. For simplicity, we specialize to the case where $\Lambda$ is a circle, but our analysis applies to more general regions with smooth boundary.

Definition. — Let $\Lambda$ be a circle and let $ds = r d\theta$ be the usual Lebesgue measure on the boundary of $\Lambda$. Then the $\{ Q, l \}$ field with $Q = G_0$ $l(f\delta_r) = c \int f(\theta) ds$ is called the $c$ — B. C. theory.

*Remark.* — 1. We use the fact that any $g \in N_{\partial\Lambda}$ is of the form $f(\theta)\delta_r$ where $\delta_r(x) = \delta(|x| - r)$.

2. Our results below would remain true if $Q = 0$ were taken.

With respect to obtaining pure states in a situation where broken symmetry occurs, so that $\langle \phi \rangle \neq 0$ in some pure states, there are three conjectures:

(*a*) If we take $c \geq \alpha > 0$ for all $\Lambda$, we obtain a pure state in the $\Lambda \to \infty$ limit for any fixed $\alpha$.

(*b*) As in (*a*), we obtain a pure state so long as $\alpha \geq \bar{\alpha}$ where $\bar{\alpha}$ in some interaction dependent constant, e. g. $\langle \phi \rangle_{\text{pure}}$.

(*c*) We must take $c \to \infty$ as $\Lambda \to \infty$ to get a pure state.

While we lean towards the first alternative (due to preliminary results of J. Lebowitz (private communication)) we would like to make some elementary remarks about how quickly $c$ can go to infinity with a reasonable theory resulting. If $c$ goes to infinity too fast, we expect sickness similar to that in [*29*, § VII]. One criterion is clearly that when there is no interaction, we should recover the free field:

THEOREM II.8. — Let $d\mu_{c(r)}$ be the Gaussian measure for the non-interacting $\{ G_0, c(r) \}$ – B. C. theory in the circle $\Lambda_r$ of radius $r$. If $c(r) \leq Ae^{Br}$, with $B < m_0$, then as $r \to \infty$

$$\int e^{i\phi(f)}d\mu_{c(r)} \to \int e^{i\phi(f)}d\mu_0$$

for any $f \in C_0^\infty(\mathbb{R}^2)$.

*Proof.* — Let $l_r$ be the linear form on $N_{\partial\Lambda}$, $l_r(f\delta_r) = \int_0^{2\pi} f(\theta)rd\theta$. We need only prove that $c(r)l_r(e_{\partial\Lambda_r}f) \to 0$ as $r \to \infty$ for each fixed $f$. But clearly

$$| l_r(e_{\partial\Lambda}f) | \leq \| l_r \|_N \| e_{\partial\Lambda}e_R \| \; \| f \|_{-1}$$

where $R = \text{supp } f$. By Lemma III.4 of [*29*], $\| e_{\partial\Lambda}e_R \| \leq A_1 e^{-(m_0-\varepsilon)r}$ so we need only prove that $\| l_r \|_N \leq e^{\varepsilon r/2}$. An elementary computation (see Lemma II.10, below) shows that in fact $\| l_r \|_N = 0(r^{1/2})$ as $r \to \infty$ so $c(r)l_r(e_{\partial\Lambda}f) \to 0$. ■

A more restrictive criterion than that of Theorem II.8 is natural in view of the ideas of this paper concerning the independence of the pressure on B. C.:

THEOREM II.9. — If $c(r) \leq Ar^{(1/2-\varepsilon)}$, then for any $P(\phi)_2$ theory:

$$\lim_{r\to\infty} | \Lambda_r |^{-1} \ln \int \exp(-U(\Lambda_r))d\mu_{c,\Lambda_r} \to \alpha_\infty ,$$

the free B. C. pressure.

*Proof.* — $d\mu_{c(r),\Lambda(r)}$ can be explicitly written down in terms of $d\mu_0$, namely:

$$d\mu_{c(r),\Lambda(r)} = \exp(c(r)\phi(l_r))d(\mu_0 \upharpoonright \Sigma_{\Lambda_r})/Z \qquad \text{(II.19)}$$

where Z is the normalization factor

$$Z_r(c(r)) = \exp\left(\frac{1}{2}c(r)^2 \| l_r \|_N^2\right). \tag{II.20}$$

(II.19) follows easily if one notes that $\langle l_r, f \rangle = \langle l_r, e_{\partial\Lambda} f \rangle$ by the fact that $l_r \in N_{\partial\Lambda}$.

Now let

$$p(r, c(r), \beta) = (\pi r^2)^{-1} \ln \int e^{-\beta U(\Lambda_r)} d\mu_{c(r),\Lambda_r}.$$

We seek to prove that $p(r, c(r), 1) \to \alpha_\infty = \lim_{r\to\infty} p(r, 0, 1)$. By Hölder's inequality one easily finds that

$$p(r, c(r), 1) \leq (1 + \varepsilon)^{-1} p(r, 0, 1 + \varepsilon) + \varepsilon(1 + \varepsilon)^{-1} \ln Z_r[(1 + \varepsilon^{-1})c(r)](\pi r^2)^{-1}$$

and

$$p(r, c(r), 1) \geq (1 - \varepsilon)^{-1} p(r, 0, 1 - \varepsilon) - \varepsilon(1 - \varepsilon)^{-1}(\pi r^2)^{-1} \ln Z_r(\varepsilon^{-1}c(r)).$$

Thus, if we can prove that for each fixed $\gamma$,

$$\lim_{r\to\infty} r^{-2} Z_r(\gamma c(r)) = 0 \tag{II.21}$$

we have

$$(1 - \varepsilon)^{-1}\alpha_\infty(1 - \varepsilon) \leq \underline{\lim}\, p(r, c(r), 1) \leq \overline{\lim}\, p(r, c(r), 1) \leq (1 + \varepsilon)^{-1}\alpha_\infty(1 + \varepsilon)$$

so our result follows from the continuity of $\alpha_\infty(\lambda)$.

Thus we need only prove (II.21), i. e. by (II.20), that

$$\frac{1}{2}\gamma^2 r^{-2} c(r)^2 \| l_r \|_N^2 \to 0.$$

Since $\| l_r \| = O(r^{1/2})$ (Lemma II.10) and by hypothesis $c(r) = o(r^{1/2})$, (II.14) holds. ∎

In the last two theorems we have used:

LEMMA II.10. — Let $l_r \in N_{\partial\Lambda_r}$ be defined by $l_r(f\delta_r) = r \int f(\theta) d\theta$. Then $\| l_r \|_N = O(r^{1/2})$ as $r \to \infty$.

*Proof.* — We first note that

$$\langle f(\theta)\delta_r, g(\theta)\delta_r \rangle = \int K_r(\theta, \theta') f(\theta) g(\theta') d\theta d\theta' \tag{II.22 a}$$

where

$$K_r(\theta, \theta') = r^2 g_0(2r \sin[(\theta - \theta')/2]) \tag{II.22 b}$$

where $g_0$ is defined on $(0, \infty)$ by $g_0(|x|) = G_0(x)$. It follows that $l_r = \alpha_r \delta_r$ where $\alpha_r$ is defined by

$$\alpha_r \int_0^{2\pi} K_r(\theta, 0) = r.$$

It is not hard to show, using the fact that $G_0(|x|)$ is integrable at $|x| = 0$ and falls off exponentially at infinity, that

$$O(r) \leq \int_0^{2\pi} K_r(\theta, 0)d\theta \leq O(r)$$

so that $\alpha_r = O(1)$ as $r \to \infty$. Thus, by (II.22):

$$\| l_r \|_N^2 = \alpha_r^2(2\pi) \int_0^{2\pi} K_r(\theta, 0)d\theta = O(r). \quad \blacksquare$$

## III. ESTIMATES ON CLASSICAL GREEN'S FUNCTIONS

Many of the estimates required in field theory models reduce to estimates on the related Green's functions, i. e. on the non-interacting two point functions. This section is devoted to a detailed study of the classical Green's functions $G_\Lambda^X$. In § III.1 we define the operators $(-\Delta_\Lambda^X + m^2)$ with X B. C. on $\partial\Lambda$ be means of quadratic forms. It turns out that one can read off many of the pertinent properties of

$$G_\Lambda^X \equiv (-\Delta_\Lambda^X + m^2)^{-1}$$

from properties of the quadratic forms. For instance, one can immediately deduce the inequalities of (I.3 *a*).

We begin § III.2 by reviewing the theory of conditioning (see [*29*] [*62*]). Conditioning provides a powerful technique for deriving estimates in Boson models. For example, the sub-and supermultiplicativity relations of § III.2 are direct consequences of conditioning; conditioning enables us to reduce the $L^p$ estimates for X B. C. to those of F B. C. (see § III.4); our new proof of the linear lower bound using N B. C. follows from conditioning (§ III.2).

In § III.3 we outline the method of images in the case where $\Lambda$ is a rectangle and we show how this method leads easily to a number of estimates for $G_\Lambda^X$; e. g.

$$G_\Lambda^X \leq cG_0 \qquad \text{on } L^2(\Lambda). \tag{III.1}$$

Finally in § III.5 we consider non-rectangular $\Lambda$ and derive the inequality (III.1) for N B. C. for a large class of $\Lambda$.

### III.1. Quadratic form domains for $-\Delta$.

Let $\Lambda$ be a bounded open region in $\mathbb{R}^2$. We wish to define and analyze the self-adjoint extensions $-\Delta_\Lambda^D$ and $-\Delta_\Lambda^N$ of $-\Delta \restriction C_0^\infty(\Lambda)$, corresponding to Dirichlet and Neumann B. C. It is convenient to do so by means of quadratic forms. In fact, in two dimensions (unlike the case in one dimension), it is difficult to characterize the operator domains of $-\Delta_\Lambda^{\mathrm{D}}$ and $-\Delta_\Lambda^{\mathrm{N}}$ directly, but the form domains are simple to work with. For a more complete discussion of much of this material see [33] and [51].

We define the Dirichlet and Neumann forms by

$$t_\Lambda^{\mathrm{X}}(f, g) = \int_\Lambda (\nabla f \cdot \nabla g + m_0^2 fg) d^2x$$

with $\mathrm{D}(t_\Lambda^{\mathrm{D}}) = \mathrm{C}_0^1(\Lambda)$ and $\mathrm{D}(t_\Lambda^{\mathrm{N}}) = \mathrm{C}^1(\bar{\Lambda})$; here by $\mathrm{C}^1(\bar{\Lambda})$ we mean the real continously differentiable functions on the closure $\bar{\Lambda}$ of $\Lambda$ which are bounded together with their derivatives in the sup norm, and by $\mathrm{C}_0^1(\Lambda)$ we mean those functions vanishing in a neighborhood of $\partial\Lambda$. Clearly, $t_\Lambda^{\mathrm{D}}$ and $t_\Lambda^{\mathrm{N}}$ are positive and densely defined; moreover they are closeable since the vector-valued operator $\vec{\nabla}$ is closeable on $\mathrm{C}_0^1(\Lambda)$ and $\mathrm{C}^1(\Lambda)$. The closures $\bar{t}_\Lambda^{\mathrm{X}}$ thus uniquely determine positive self-adjoint operators $\mathrm{T}_\Lambda^{\mathrm{X}}$ by means of

$$\bar{t}_\Lambda^{\mathrm{X}}(f, g) = (f, \mathrm{T}_\Lambda^{\mathrm{X}} g), \qquad \text{(III.1)}$$

for all $f \in \mathrm{D}(\bar{t}_\Lambda^{\mathrm{X}})$ and $g \in \mathrm{D}(\mathrm{T}_\Lambda^{\mathrm{X}})$ with $\mathrm{D}(\mathrm{T}_\Lambda^{\mathrm{X}}) \subset \mathrm{D}(\bar{t}_\Lambda^{\mathrm{X}})$.

Formally, $\mathrm{T}_\Lambda^{\mathrm{X}} \equiv -\Delta_\Lambda^{\mathrm{X}} + m^2$ is the differential operator $-\Delta + m^2$ with B. C.

$$g = 0 \quad \text{on } \partial\Lambda \quad (\mathrm{X} = \mathrm{D}) \qquad \text{(III.2)}$$

and

$$\frac{\partial g}{\partial n} = 0 \quad \text{on } \partial\Lambda \quad (\mathrm{X} = \mathrm{N}). \qquad \text{(III.3)}$$

But this statement cannot be taken literally since vectors in $\mathrm{D}(\mathrm{T}_\Lambda^{\mathrm{X}})$ are not necessarily once differentiable in the usual sense. Thus while it happens that (III.2) may be interpreted in the ordinary sense when $\Lambda$ is sufficiently regular, the B. C. (III.3) may not (see, for example [39]). However for $f \in \mathrm{C}^1(\bar{\Lambda})$, $g \in \mathrm{C}^2(\bar{\Lambda}) \cap \mathrm{D}(\mathrm{T}_\Lambda^{\mathrm{N}})$, we see from integration by parts in (III.1) that

$$\int_{\partial\Lambda} f \frac{\partial g}{\partial n} + \int_\Lambda f(-\Delta + m_0^2) g = (f, \mathrm{T}_\Lambda^{\mathrm{N}} g)$$

so that $g$ must satisfy (III.3) and $\mathrm{T}_\Lambda^{\mathrm{N}} g = (-\Delta + m_0^2) g$ in the usual sense.

We define $\mathrm{G}_\Lambda^{\mathrm{X}}$ to be the operator $(\mathrm{T}_\Lambda^{\mathrm{X}})^{-1}$, which is obviously bounded by $m_0^{-2}$. We denote the (distribution) kernel of $\mathrm{G}_\Lambda^{\mathrm{X}}$ by $\mathrm{G}_\Lambda^{\mathrm{X}}(x, y)$, the so-called Green's function. If $\partial\Lambda$ is sufficiently smooth, it is a standard argument of elliptic partial differential equations that $\mathrm{G}_\Lambda^{\mathrm{X}}(x, y)$ is $\mathrm{C}^\infty$ in its arguments for $x \neq y$ [39].

It is also straightforward to treat the B. C.

$$\frac{\partial g}{\partial n} = \sigma g \qquad \text{(III.4)}$$

by means of quadratic forms, where $\sigma(x)$ is a given real smooth function on $\partial\Lambda$. For the form $b$

$$b_\sigma(f, g) = \int_{\partial\Lambda} f\sigma g$$

is relatively bounded with respect to $t_\Lambda^N$ with relative bound 0 [51], and so $t_\Lambda^\sigma = t_\Lambda^N + b_\sigma$ uniquely determines a self-adjoint operator

$$T_\Lambda^\sigma \equiv -\Delta_\Lambda^\sigma + m_0^2 \equiv (G_\Lambda^\sigma)^{-1},$$

corresponding to the B. C. (III.4).

Now consider two disjoint bounded open regions $\Lambda_1$ and $\Lambda_2$ with $B = \partial\Lambda_1 \cap \partial\Lambda_2$ not necessarily empty. Let $\Lambda = \text{int}(\bar{\Lambda}_1 \cup \bar{\Lambda}_2)$. In the obvious way we regard $t_{\Lambda_1}^N$ as a form on $L^2(\Lambda) = L^2(\Lambda_1) \oplus L^2(\Lambda_2)$ with $D(t_{\Lambda_1}^N) = C^1(\bar{\Lambda}_1) \oplus L^2(\Lambda_2)$, and similarly for $t_{\Lambda_2}^N$. Clearly

$$t_\Lambda^N \subset t_{\Lambda_1}^N + t_{\Lambda_2}^N \tag{III.5}$$

since the domain of the form on the right includes functions with discontinuities across B. Now $(t_{\Lambda_1}^N + t_{\Lambda_2}^N)^- = \bar{t}_{\Lambda_1}^N + \bar{t}_{\Lambda_2}^N$ determines a unique operator S, usually denoted by $S = T_{\Lambda_1}^N \dotplus T_{\Lambda_2}^N$, which, *a priori*, is a self-adjoint extension of the usual sum $T_{\Lambda_1}^N \oplus T_{\Lambda_2}^N$. However $T_{\Lambda_1}^N \oplus T_{\Lambda_2}^N$ is obviously self-adjoint so that $S = T_{\Lambda_1}^N \oplus T_{\Lambda_2}^N$. Since the forms in (III.5) are bounded from below, we may rewrite (III.5) as $t_\Lambda^N \geq t_{\Lambda_1}^N + t_{\Lambda_2}^N$ or in operator notation as $T_\Lambda^N \geq T_{\Lambda_1}^N \oplus T_{\Lambda_2}^N$ where these two inequalities *by definition* mean the inclusion (III.5). By [33, p. 330] these inequalities are equivalent to

$$G_\Lambda^N \leq G_{\Lambda_1}^N \oplus G_{\Lambda_2}^N. \tag{III.6}$$

This is the basic inequality for Neumann B. C.

On the other hand for Dirichlet B. C.

$$t_\Lambda^D \supset t_{\Lambda_1}^D + t_{\Lambda_2}^D$$

since functions in the domain of the form on the right must vanish in a neighborhood of B. Consequently,

$$G_\Lambda^D \geq G_{\Lambda_1}^D \oplus G_{\Lambda_2}^D \tag{III.7}$$

If we now suppose that $\Lambda_2 = R^2 \backslash \bar{\Lambda}_1$, then we deduce from (III.6) and (III.7) that

$$G_{\Lambda_1}^D \leq G_0 \leq G_{\Lambda_1}^N \tag{III.8}$$

since $G_{\mathbb{R}^2}^X = G_0$. There is an obvious monotonicity result for the $\sigma$ B. C. of (III.4); namely

$$G_\Lambda^{\sigma_1} \leq G_\Lambda^{\sigma_2} \tag{III.9}$$

if $\sigma_1(x) \geq \sigma_2(x)$ for almost all $x$ on $\partial\Lambda$. Clearly Neumann B. C. are given by $\sigma = 0$ and it is easy to prove that the Dirichlet form is obtained as the monotone limit $\lim_{\sigma\uparrow\infty} t_\Lambda^\sigma$ so that

$$G_\Lambda^D \leq G_\Lambda^\sigma \leq G_\Lambda^N \quad \text{if } 0 \leq \sigma(x). \tag{III.10}$$

If we specialize to the case where $\Lambda$ is a rectangle then we can introduce periodic B. C. via the basic form $t_\Lambda^X$ by taking $D(t_\Lambda^P) \cap C^1(\Lambda)$ to consist of those functions periodic in the sides of $\Lambda$. By the same considerations as above we obtain

$$G_\Lambda^D \leq G_\Lambda^P \leq G_\Lambda^N . \tag{III.11}$$

EXAMPLE. — If we restrict to 1 dimension, then we can give can explicit demonstration of the inequality (III.6) (or, indeed, of any of the other above inequalities) that shows clearly how an additional N B. C. introduces an additional degree of freedom. Thus consider an interval $\Lambda = (a, c)$, and for $a < b < c$ let $\Lambda_1 = (a, b)$ and $\Lambda_2 = (b, c)$. Then it is not hard to verify that

$$G_\Lambda^N(x, y) = \frac{1}{m_0 \sinh [m_0(c-a)]} \cosh [m_0(x-a)] \cosh [m_0(c-y)] \quad \text{if} \quad x \leq y$$

with $x$ and $y$ interchanged if $y \leq x$, and consequently that

$$\Delta G \equiv G_{\Lambda_1}^N(x, y) \oplus G_{\Lambda_2}^N(x, y) - G_\Lambda^N(x, y) = g(x)g(y)$$

where

$$g(x) = \begin{cases} \left[\dfrac{\sinh m_0(c-b)}{m_0 \sinh m_0(c-a) \sinh m_0(b-a)}\right]^{1/2} \cosh m_0(x-a) & a \leq x \leq b \\ -\left[\dfrac{\sinh m_0(b-a)}{m_0 \sinh m_0(c-a) \sinh m_0(c-b)}\right]^{1/2} \cosh m_0(c-x) & b \leq x \leq c . \end{cases}$$

Thus $\Delta G$ is a positive rank one operator, verifying (III.6).

### III.2. **Conditioning Inequalities : A new proof of the linear lower bound.**

The theory of conditioning [*29*] [*62*] leads directly to the inequalities of (I.3 *b*). For the reader's convenience we begin by briefly reviewing this theory. One can discuss conditioning either in the formalism of second quantization or in terms of integration in Q-space. We shall choose the latter setting.

Let $\phi_A$ and $\phi_B$ be two Gaussian random processes (G. r. p.) indexed by the same vector space V with zero means and covariances $S_A(f, g)$ and $S_B(f, g)$ respectively. Denote the measure space for $\phi_A$ by $Q_A$, the Gaussian measure by $d\mu_A$, and the Hilbert space completion of V in the norm $\| f \|_A \equiv S_A(f, f)^{1/2}$ by $\mathscr{H}_A$; $Q_B$, $d\mu_B$, $\mathscr{H}_B$ denote the corresponding objects for $\phi_B$. By

$$\phi = \phi_A + \phi_B \tag{III.12}$$

we mean the *independent* sum of $\phi_A$ and $\phi_B$; i. e. the G. r. p. $\langle \phi, Q, d\mu, S \rangle$ realized on $Q = Q_A \times Q_B$ with measure $d\mu = d\mu_A \times d\mu_B$, covariance

$S = S_A + S_B$, and associated Hilbert space $\mathscr{H}$, the completion of V in the norm $\| f \| \equiv S(f, f)^{1/2}$. Clearly, because of the relation

$$S_A(f, f) \leq S(f, f) \qquad \text{for} \qquad f \in V, \tag{III.13}$$

every $f \in \mathscr{H}$ can be identified with a unique vector $f_A$ in $\mathscr{H}_A$ (or similarly with an $f_B$ in $\mathscr{H}_B$); hence for $f \in \mathscr{H}$, $\phi(f)$ is well-defined as $\phi_A(f_A) + \phi_B(f_B)$.

An important feature of the independence in the sum (III.12) is the consistency with Wick ordering; for example,

$$\int : \phi(f)^n :_S d\mu_B = : \phi_A(f)^n :_{S_A} . \tag{III.14}$$

Here, and throughout this section, it is understood that the subtractions in the Wick powers of a field are made with respect to the associated covariance, so that the subscripts S and $S_A$ in (III.14) are for emphasis only. (III.14) follows from integrating the relation

$$: \phi(f)^n : = \sum_{r=0}^{n} \binom{n}{r} : \phi_A(f)^{n-r} : : \phi_B(f)^r : \tag{III.15}$$

and using the fact that $\int : \phi_B(f)^r : d\mu_B = \delta_{0,r}$. As for (III.15) we can verify it easily by means of the generating functional for Wick powers [62]:

$$\exp\left(\phi(f) - \frac{1}{2}\| f \|^2\right) = \sum_{n=0}^{\infty} : \phi(f)^n : /n! . \tag{III.16}$$

For the left side of (III.16) can be rewritten as

$$\exp\left(\phi_A(f) - \frac{1}{2}\| f \|_A^2\right) \exp\left(\phi_B(f) - \frac{1}{2}\| f \|_B^2\right)$$
$$= \sum_{r=0}^{\infty} : \phi_A(f)^r : /r! \sum_{s=0}^{\infty} : \phi_B(f)^s : /s!$$

and (III.15) follows upon equating equal powers of $f$.

Turning the above discussion around slightly, we may suppose that we begin with two G. r. p. 's $\phi$ and $\phi_A$, both indexed by V and with covariances S and $S_A$. If (III.13) is satisfied we can define $S_B = S - S_A$, construct the associated G. r. p. $\phi_B$, and thereby realize $\phi$, $\phi_A$ and $\phi_B$ simultaneously on the measure space $(Q, d\mu) \equiv (Q_A \times Q_B, d\mu_A \times d\mu_B)$ where $(Q_A, d\mu_A)$ and $(Q_B, d\mu_B)$ are measure spaces for $\phi_A$ and $\phi_B$. Because of the relation (III.14) we say that $\phi_A$ *is obtained from* $\phi$ *by conditioning if the inequality* (III.13) *holds*. It is easy to see that (III.13) is also necessary in order for there to be a relation like (III.14) [29, Prop. II.22].

The power of the theory of conditioning is best seen from the following estimates. If P is a semibounded polynomial and $g \in L^{1+\varepsilon}(\mathbb{R}^2)$, it is a standard argument to make sense of interactions like $U = \int : P(\phi(x)) : g(x)dx$ and $U_A = \int : P(\phi_A(x)) : g(x)dx$ for suitable covariances S and $S_A$ (see e. g. § III.4). Assuming that this has been done, we simply wish to note here that:

THEOREM III.1 *Conditioning Comparison Theorem.* — If $\phi_A$ is obtained from $\phi$ by conditioning, then for any $p > 1$

$$\int | U_A |^p d\mu_A \leq \int | U |^p d\mu$$

and

$$\int e^{-U_A} d\mu_A \leq \int e^{-U} d\mu . \tag{III.17}$$

*Proof.* — We prove only (III.17). From (III.14) it follows that

$$U_A = \int U d\mu_B$$

so that by Jensen's inequality

$$e^{-U_A} \leq \int e^{-U} d\mu_B .$$

We then integrate with respect to $d\mu_A$. ∎

By combining the Conditioning Comparison Theorem with the inequalities (III.6)-(III.11) we immediately deduce a number of useful inequalities among the *partition functions*

$$Z_\Lambda^X = \int e^{-U_\Lambda^X} d\mu_\Lambda^X \tag{III.18}$$

where

$$U_\Lambda^X = \int_\Lambda : P(\phi(x)) :_{X,\Lambda} g(x)dx \tag{III.19}$$

is defined in terms of a semibounded polynomial P in the field and a fixed function $g(x) \geq 0$. The Wick subtractions in (III.19) are with respect to $d\mu_\Lambda^X$. In this section we simply assume that the $Z_\Lambda^X$ are finite; in § III.4 we return to this point.

Suppose we consider the G. r. p. $\langle \phi, Q, d\mu, S \rangle$ with covariance operator

$$S = G_{\Lambda_1}^N \oplus G_{\Lambda_2}^N$$

of (III.6), where $\Lambda_1 \cap \Lambda_2 = \phi$ and $\Lambda = \operatorname{int}(\bar{\Lambda}_1 \cup \bar{\Lambda}_2)$. As explained after (III.12) this G. r. p. can be realized as the independent sum

$$\phi = \phi_1 + \phi_2$$

where the G. r. p. $\phi_j$ has covariance $G_{\Lambda_j}^N$.

From (III.15) we deduce by standard arguments that

$$U \equiv \int_\Lambda : P(\phi(x)) : g(x)dx = U^N_{\Lambda_1} + U^N_{\Lambda_2} \tag{III.20}$$

Therefore

$$\int e^{-U} d\mu = Z^N_{\Lambda_1} \cdot Z^N_{\Lambda_2} . \tag{III.21}$$

Now according to (III.6), the G. r. p. $\phi$ with covariance $G^N_\Lambda$ is obtained from $\phi$ by conditioning so that by the Conditioning Comparison Theorem and (III.21),

$$Z^N_\Lambda \leq Z^N_{\Lambda_1} \cdot Z^N_{\Lambda_2} \tag{III.22}$$

This is the basic submultiplicativity relation for Neumann B. C.

By similar reasoning, the following relations are all immediate consequences of (III.7)-(III.11):

$$Z^D_\Lambda \geq Z^D_{\Lambda_1} \cdot Z^D_{\Lambda_2} \tag{III.23}$$

$$Z^D_\Lambda \leq Z_\Lambda \leq Z^N_\Lambda \tag{III.24}$$

$$Z^D_\Lambda \leq Z^{\sigma_1}_\Lambda \leq Z^{\sigma_2}_\Lambda \leq Z^N_\Lambda \quad \text{if} \quad \sigma_1(x) \geq \sigma_2(x) \geq 0 \tag{III.25}$$

$$Z^D_\Lambda \leq Z^P_\Lambda \leq Z^N_\Lambda \tag{III.26}$$

Among the class of B. C. we are considering, then, Dirichlet B. C. are « minimal » and Neumann B. C. « maximal ». The minimality property of Dirichlet B. C. actually holds for a more general class of B. C.: if $-\Delta_B$ is *any* self-adjoint extension of $-\Delta \upharpoonright C^\infty_0(\Lambda)$ such that $-\Delta_B + m_0^2 > 0$, then by a general theorem concerning the Friedrichs extension (see, e. g., [*33*, p. 331])

$$(-\Delta^D_\Lambda + m_0^2)^{-1} \leq (-\Delta_B + m_0^2)^{-1} \equiv S_B ;$$

hence

$$Z^D_\Lambda \leq \int e^{-U_B} d\mu_B .$$

An interesting consequence of the submultiplicativity property (III.22) of Neumann B. C. is a new proof of the « linear lower bound » [*19*] [*61*] [*20*] [*42*] [*28*]: various estimates proved for a finite volume lead « for free » to the corresponding infinite volume estimates. For, example, the following two estimates are statements of the linear lower bound:

1. Let $E^X_l$ be the ground state energy for the $P(\phi)_2$ Hamiltonian $H^X_l$ with B. C. X = D, P, F or N on $[-l/2, l/2]$. If $E^N_l$ is bounded from below uniformly, say for $l \in [1, 2]$, then $E^X_l/l$ is bounded from below uniformly for $l \in [1, \infty)$.

2. Consider the pressure $\alpha^X_\Lambda$ defined in (I.4) for X = D, P, F or N. If $\Lambda_1$ is a unit square and $\Lambda$ is a union of unit squares, then

$$\alpha^X_\Lambda \leq \alpha^N_{\Lambda_1} . \tag{III.27}$$

The second result follows immediately from the estimates (III.8), (III.11), (III.17) and the submultiplicativity property (III.22), after one has carved up $\Lambda$ into a union of unit squares with N B. C. on their boundaries.

The first result follows from the standard formula

$$- E_l^X = \lim_{t\to\infty} \frac{1}{t} \log Z_{lt}^X \tag{III.28}$$

where $Z_{lt}^X = \int e^{-U_{lt}^X} d\mu_l^X$ ; here $d\mu_l^X$ is the free measure with X B. C. on the strip $[-l/2, l/2] \times \mathbb{R}$ and $U_{l,t}$ is the interaction in the region $[-l/2, l/2] \times [-t/2, t/2]$ with $d\mu_l^X$-Wick subtractions. For as above

$$Z_{lt}^X \leq \prod_i Z_{l_i t}^N$$

with $l = l_1 + l_2 + \ldots$ where $l_i \in [1, 2]$; and the result follows from (III.28). The hypothesis that $E_l^N$ is bounded for $l \in [1, 2]$ may be verified as in § III.4.

Later in this paper we indicate two other approaches to the above results which are not quite as clean as the use of N B. C. The first in § VII.4 uses hypercontractivity as in Nelson [*42*], and Guerra-Rosen-Simon [*28*]. The second in the Appendix is based on the Checkerboard Theorem.

Similar considerations based on the submultiplicativity (III.23) of Dirichlet B. C. give a bound in the other direction, e. g.,

$$\alpha_\Lambda^X \geq \alpha_{\Lambda_1}^D$$

if $\Lambda$ is a union of unit squares and $\Lambda_1$ a unit square. Of course, for $P(\phi)_2$ a bound in this directions follows more directly from Jensen's inequality.

### III.3. **Method of images.**

When $\Lambda \subset \mathbb{R}^2$ is a rectangle, the method of images provides a convenient representation for the Green's functions $G_\Lambda^X$. Suppose, without loss of generality, that $\Lambda$ is the rectangle $(-l_1/2, l_1/2) \times (-l_2/2, l_2/2)$. For $n \in \mathbb{Z}^2$, define the « reflection » operators $p_n^X$ on $\mathbb{R}^2$ by

$$p_n^N y = (-1)^n(y - nl) \equiv ((-1)^{n_1}(y_1 - n_1 l_1), (-1)^{n_2}(y_2 - n_2 l_2)),$$

$p_n^D = p_n^N$, and

$$p_n^P y = (y_1 - n_1 l_1, y_2 - n_2 l_2).$$

Then, according to the method of images, for X = D, N, P,

$$G_\Lambda^X(x, y) = \sum_{n\in\mathbb{Z}^2} (S^X)^{|n|} G_0(x - p_n^X y) \tag{III.29}$$

where $S^D = -1$, $S^N = S^P = 1$, and $|n| = |n_1| + |n_2|$. It is clear that the right side of (III.29) equals $G_\Lambda^X$ : for there is only one term ($n = 0$) with a singularity in $\Lambda$ so that $(-\Delta_X + m_0^2)G_\Lambda^X(x, y) = \delta(x - y)$ and the B. C. are obviously satisfied since, for $X = N$ say, the right side, viewed as a function of $x, y \in \mathbb{R}^2$, is even with respect to reflections in the sides of $\Lambda$ and hence has a vanishing normal derivative on $\partial\Lambda$.

In Wick-reordering formulas (see (I.8)) we encounter the functions

$$\delta G_\Lambda^X(x) = \lim_{y \to x} \left[ G_\Lambda^X(x, y) - G_0(x, y) \right].$$

When $\Lambda$ is a rectangle, we see from (III.29) that $\delta G_\Lambda^X$ is well-defined and

$$\delta G_\Lambda^X(x) = \sum_{n \neq 0} (S^X)^{|n|} G_0(x - p_n^X x). \tag{III.30}$$

These image formulas lead both to pointwise bounds on $G_\Lambda^X$ and $\delta G_\Lambda^X$ (Lemmas III.2 and III.3) and to operator bounds (Theorem III.4 and Lemma III.5). Let $|x - y|_X$ denote ordinary Euclidean distance for $X = D, N, F$ and periodic distance for $X = P$, i. e., $|x - y|_P = \min_n |x - p_n^P y|$. The first lemma asserts that $G_\Lambda^X(x, y)$ has essentially the same small and large $|x - y|$ behavior as $G_0(x, y)$:

LEMMA III.2. — Assume that $l_1, l_2 \geq 1$, and $\varepsilon > 0$. Then for $X = N$, P, D there are constants $a$ and $b$ independent of $\Lambda$ such that for $x, y \in \Lambda$

$$0 < G_\Lambda^X(x, y) < a \log |x - y|_X^{-1}, \qquad |x - y|_X \leq \frac{1}{2} \tag{III.31 a}$$

$$< b \exp(-(m_0 - \varepsilon)|x - y|_X), \qquad |x - y|_X > \frac{1}{2} \tag{III.31 b}$$

*Remark.* — In the case $X = D$, the inequality (III.31) was established in [*29*] for arbitrary open $\Lambda$ by showing that

$$G_\Lambda^D(x, y) \leq G_0(|x - y|).$$

*Proof.* — ($X = N$ only) Positivity of $G_\Lambda^N$ is obvious from (III.29) since $G_0(x) > 0$. For the small distance bound (III.31 *a*) we write

$$G_\Lambda^N(x, y) = \sum_{|n_i| \leq 1} G_0(x - p_n^N y) + R(x, y) \tag{III.32}$$

Since $G_0(x)$ is monotone decreasing in $|x|$, the first term in (III.32) is bounded by $9G_0(|x - y|) = 0(\log |x - y|^{-1})$. Because of the asymptotic behavior of $G_0$

$$G_0(x) \sim \text{const.}\ |x|^{1/2} e^{-m_0|x|}, \qquad \text{as} \qquad |x| \to \infty \tag{III.33}$$

the remainder term $R(x, y)$ is clearly bounded uniformly in $x$, $y$, and $\Lambda$.

The long distance behavior (III.31 *b*) follows from (III.33) and an elementary computation. Note that the asymptotic behavior of $G_\Lambda^N$ is not quite as good as that of $G_0$ since if one side of $\Lambda$, say $l_1$, is small then a large number ($\sim |x-y|/l_1$) of images will each contribute a term of the order of $G_0(|x-y|)$ to $G_\Lambda^N(x, y)$. ■

LEMME III.3. — Assume that $l_1, l_2 \geq 1$.

(*i*) For X = N, P, $\delta G_\Lambda^X(x) > 0$.

(*ii*) Suppose X = D, N. Let $r = \text{dist}(x, \partial\Lambda)$. For $r \leq 1/2$

$$|\delta G_\Lambda^X(x)| \leq a |\log r| \qquad \text{(III.34 } a\text{)}$$

and for $r \geq 1/2$

$$|\delta G_\Lambda^X(x)| \leq b r^{-1/2} e^{-2m_0 r} \qquad \text{(III.34 } b\text{)}$$

where the constants $a$ and $b$ are independent of $\Lambda$.

(*iii*) $\delta G_\Lambda^P$ is a constant which satisfies

$$0 < \delta G_\Lambda^P \leq d \exp[-m_0 \min(l_1, l_2)]$$

for some constant $d$ independent of $\Lambda$.

*Remark.* — In [*29*] it was noted that for general $\Lambda$, $\delta G_\Lambda^D$ is negative and satisfies (III.34).

*Proof.* — (*i*) is obvious from (III.30).

(*ii*) follows from (III.30) and the known small and large $x$ behaviour of $G_0(x)$,

(*iii*) clearly $\delta G_\Lambda^P = \sum_{n \neq 0} G_0(|(n_1 l_1, n_2 l_2)|)$ is exponentially small as claimed. ■

We may view $G_\Lambda^X(x, y)$ as the kernel of an integral operator (denoted $G_\Lambda^X$) applied to functions with support in $\Lambda$, belonging to, say, $L^2(\Lambda)$ or $N = \mathscr{H}_{-1}(\mathbb{R}^2)$. Interpreting (III.29) as an operator equation, we have

$$G_\Lambda^X = G_0 \sum_n P_n^X \qquad \text{(III.35)}$$

where $P_n^X f(x) = (S^X)^{|n|} f((p_n^X)^{-1} x)$ is a unitary operator on $L^2(\mathbb{R}^2)$ or on N. With this notation

$$\delta G_\Lambda^X = G_0 \sum_{n \neq 0} P_n^X . \qquad \text{(III.36)}$$

The formula (III.35) leads at once to an operator inequality for $G_\Lambda^X$:

THEOREM III.4. — Suppose $\Lambda$ is a rectangle and X = D, N or P. Then for some constant $c = c_\Lambda^X$ we have on $L^2(\Lambda) \times L^2(\Lambda)$

$$G_\Lambda^X \leq c G_0 \qquad \text{(III.37)}$$

*Remarks.* — 1. By the inequalities of § III.1 we know that

$$G^D_\Lambda \leq \genfrac{}{}{0pt}{}{G_0}{G^P_\Lambda} \leq G^N_\Lambda$$

so that the important case of (III.37) is for $X = N$. In § III.5 we show how to extend (III.37) for $X = N$ to a class of non-rectangular regions.

2. The constant $c^N_\Lambda$ that we find diverges as $|\Lambda| \to 0$. This is true for the best constant; see Lemma III.12.

*Proof.* — Let $f \in L^2(\Lambda)$, denote the inner product on $L^2$ by $(.\,,.)$ and on N by $\langle\, .\,,.\,\rangle$ and let $e_\Lambda$ be the projection in N onto elements with support in $\Lambda$. Then by (III.35)

$$\begin{aligned}(f, G^X_\Lambda f) &= \sum_n \langle f, P^X_n f \rangle \\ &= \sum_n \langle f, e_\Lambda P^X_n e_\Lambda f \rangle \\ &= \sum_n \langle f, e_\Lambda e_{\Lambda_n} P^X_n f \rangle\end{aligned}$$

since $P^X_n e_\Lambda = e_{\Lambda_n} P^X_n$ where $\Lambda_n = \{\, p^X_n x \mid x \in \Lambda \,\}$. But by Lemma III.5 *b* of [29], $\| e_\Lambda e_{\Lambda_n} \| = O(e^{-m_0 d_n})$ where $d_n = \operatorname{dist}(\Lambda, \Lambda_n)$. Since $\sum_n e^{-m_0 d_n} < \infty$ and $P^X_n$ is unitary, we obtain

$$(f, G^X_\Lambda f) \leq c \| f \|^2_N = c(f, G_0 f). \quad \blacksquare$$

By similar reasoning applied to (III.36) we obtain:

LEMMA III.5. — Let $\Lambda'$ be a compact subset of $\Lambda$ with $r = \operatorname{dist}(\Lambda', \partial\Lambda) > 0$. Define the operator $A^X = A^{X,\Lambda}_{\Lambda'}$ on N by

$$A^X = e_{\Lambda'} \sum_{n \neq 0} P^X_n e_{\Lambda'}\,. \tag{III.38}$$

Then $A^X$ is Hilbert-Schmidt and for $r \geq 1$,

$$\| A^X \|_{HS} \leq a(1 + |\Lambda'|^{1/2}) e^{-2m_0 r} \tag{III.39}$$

where the constant $a$ does not depend on $\Lambda$ or $\Lambda'$.

*Proof.* — As in the previous theorem $(\Lambda'_n = \{\, p^X_n x \mid x \in \Lambda' \,\})$,

$$\begin{aligned}\| A^X \|_{HS} &= \sum_{n \neq 0} \| e_{\Lambda'} e_{\Lambda'_n} P^X_n \|_{HS} \\ &\leq \sum_{n \neq 0} \| e_{\Lambda'} e_{\Lambda'_n} \|_{HS}\,.\end{aligned}$$

But by Lemma III.58 of [29],

$$\| e_{\Lambda'} e_{\Lambda'_n} \|_{HS} \leq c(r)(1 + |\Lambda'|^{1/2}) \exp(-m_0 \operatorname{dist}(\Lambda', \Lambda'_n))$$
$$\leq c(r)(1 + |\Lambda'|^{1/2}) \exp(-2m_0 r - m_0 d_n)$$

where $d_n = \operatorname{dist}(\Lambda, \Lambda_n)$ and the constant $c(r)$ is independent of $r$ for $r \geq 1$. This implies (III.39). ■

*Remark.* — By the Markov property [29, Prop. II.3], $e_{\Lambda'} e_{\Lambda'_n} = e_{\Lambda'} e^2_{\partial\Lambda} e_{\Lambda'_n}$ for $n \neq 0$; hence the trace class norm

$$\| e_{\Lambda'} e_{\Lambda'_n} \|_{TR} \leq \| e_{\Lambda'} e_{\partial\Lambda} \|_{HS} \| e_{\partial\Lambda} e_{\Lambda'_n} \|_{HS}$$

and we see that $A^X$ is actually trace class with bounds on $\| A^X \|_{TR}$ similar to (III.39).

We conclude this subsection by applying the general theory of § II.1 to compare the measures $d\mu^X_\Lambda$, X = F, D, N, P. The following estimates are useful when working in the active picture (as we do, for example, in § V). In general, the measures $d\mu^X_\Lambda$ are *not* mutually absolutely continuous (see Remark 2 below). However, if the closure of $\Lambda'$ is a compact subset of $\Lambda$ then, according to Theorem III.4 and Theorem II.2

$$d\mu^X_\Lambda \upharpoonright \Sigma_{\Lambda'} = F_X d\mu_0 \upharpoonright \Sigma_{\Lambda'}$$

where $F_X \in L^1(d\mu_0)$ is a Gaussian measurable with respect to $\Sigma_{\partial\Lambda'}$. In fact, by Theorem II.5, we can assert more about $F_X$ on the basis of the operator $A^X$ of (III.38) which satisfies, for $f, g \in N_{\Lambda'}$,

$$\int \phi(f)\phi(g) d\mu^X_\Lambda = \int f(x) G^X_\Lambda(x, y) g(y) dx dy$$
$$= \langle f, (I + A^X) g \rangle .$$

In the case of N B. C., $A^N \geq 0$, so that $F_N \in L^p$ for all $p < 1 + 1/\alpha^N_0$ where $\alpha^N_0$ is the largest eigenvalue of $A^N$ and in addition $F_N^{-1} \in L^\infty$ (by the Remark following Theorem II.5). In the case of D B. C., $A^D \leq 0$, so that the situation is reversed in the sense that $F_D^{-1} \in L^p$ for all $p < -1/\alpha^D_0$ where $\alpha^D_0$ is the smallest eigenvalue of $A^D$ and in addition $F_D \in L^\infty$. This result was already obtained in [29, Theorem II.34]. In the case of P B. C., $A^P$ is neither positive nor negative so that the conclusions about $F_P$ are that $F^P \in L^p$ for all $p < 1 + 1/\alpha^P_0$ and that $F_P^{-1} \in L^p$ for all $p < -1/\alpha^P_1$ where $\alpha^P_0$ and $\alpha^P_1$ are the maximum and minimum eigenvalues of $A^P$.

Finally, by Theorem II.5 (*b*), we can write down an implicit bound on the $L^p$ norms of $F_X$. We state the results only for the case X = N:

THEOREM III.6. — Let $\Lambda'$ be a compact subset of $\Lambda$.

(*a*) $d\mu^N_\Lambda \upharpoonright \Sigma_{\Lambda'}$ and $d\mu_0 \upharpoonright \Sigma_{\Lambda'}$ are equivalent measures; explicitly,

$$d\mu^N_\Lambda \upharpoonright \Sigma_{\Lambda'} = F_N d\mu_0 \upharpoonright \Sigma_{\Lambda'}$$

where $F_N^{-1} \in L^\infty$ and $F_N \in L^p(d\mu_0 \upharpoonright \Sigma_{\partial\Lambda'})$ for all $p < 1 + 1/\alpha_0^N$ where $\alpha_0^N$ is the maximum eigenvalue of the operator $A^N$ of (III.38).

(*b*) Let $p > 1$ be given. Then $\| F_N \|_p$ is bounded independently of $\Lambda'$ and $\Lambda$ provided that

$$(1 + |\Lambda'|^{1/2}) \exp(-2m_0 \operatorname{dist}(\Lambda', \partial\Lambda)) \leq c/p \tag{III.40}$$

for some constant $c$ independent of $\Lambda'$, $\Lambda$.

*Proof.* — Part (*a*) is just Theorem II.5 (*a*) while part (*b*) is just Theorem II.5 (*b*) combined with the estimate (III.39). ■

*Remarks.* — 1. It is instructive to examine the assertion of the theorem in one dimension where everything can be explicitly computed. For simplicity take $\Lambda = (0, \infty)$, $\Lambda' = [r, \infty) \subset \Lambda$, and $m_0 = 1$. As calculated in [*29*, Theorem II.31]

$$d\mu_\Lambda^N = 2^{-1/2} e^{\frac{1}{2}q(0)^2} \, d\mu_0 \upharpoonright \Sigma_\Lambda$$

where $q(t)$ denotes the field at « time » $t$. We see that in one dimension $d\mu_\Lambda^N$ and $d\mu_0 \upharpoonright \Sigma_\Lambda$ are equivalent without the necessity of restricting to $\Lambda'$. Upon restricting to $\Lambda'$, the Radon-Nikodym derivative

$$F_N = d\mu_\Lambda^N \upharpoonright \Sigma_{\Lambda'} / d\mu_0 \upharpoonright \Sigma_{\Lambda'}$$

is given by

$$\begin{aligned} F_N &= 2^{-1/2} e^{-rH_0} e^{\frac{1}{2}q(r)^2} \\ &= (1 + e^{2r})^{-1/2} \exp[q(r)^2/(1 + e^{2r})] \end{aligned}$$

by Mehler's formula (see e. g., [*64*]). Clearly, $F_N \in L^p(Q, \Sigma_{\Lambda'}, d\mu_0)$ if $p < 1 + e^{2r}$. Finally, we can compute the operator $A^N$ and we find that $A^N \doteq e^{-2r} e_{\partial\Lambda'}$; it maximum eigenvalue $\alpha_0^N$ is $e^{-2r}$, confirming the condition $p < 1 + 1/\alpha_0^N$.

2. In contrast to the one-dimensional result, $d\mu_\Lambda^N \upharpoonright \Sigma_{\Lambda'}$ and $d\mu_0 \upharpoonright \Sigma_{\Lambda'}$ are not absolutely continuous in general if $\operatorname{dist}(\Lambda', \partial\Lambda) = 0$ in two or more dimensions. For example, let $\Lambda$ be the half-plane $\Lambda = \{ x \in \mathbb{R}^2 \mid x_1 > 0 \}$ and $\Lambda' \subset \Lambda$ the rectangle $\Lambda' = \{ x \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < 1 \}$ so that $\partial\Lambda'$ and $\partial\Lambda$ have a common segment $L = \{ x \mid x_1 = 0, 0 < x_2 < 1 \}$. By the method of images, if $x, y \in L$, then $G_\Lambda^N(x, y) = 2G_0(x, y)$. Therefore the operator A defined by $(f, G_\Lambda^N g) = \langle f, (I + A)g \rangle$ equals I if supp $f, g \in L$. Clearly A is not a Hilbert-Schmidt operator since there is a denumerable infinity of $\{ f_n \}$ with supp $f_n \subset L$ and $\langle f_n, f_m \rangle = \delta_{n,m}$. Consequently [*60*], $d\mu_\Lambda^N \upharpoonright \Sigma_{\Lambda'}$ is not absolutely continuous with respect to $d\mu_0 \upharpoonright \Sigma_{\Lambda'}$.

## III.4. $L^p$ properties of the interaction.

The estimates for the classical Green's functions of the previous two sections and the theory of conditioning lead at once to $L^p$ properties of the interaction. To begin with:

LEMMA III.7. — For any $\varepsilon > 0$, let $g \in L^{1+\varepsilon}(\Lambda)$ where $\Lambda \subset \mathbb{R}^2$ is a rectangle. There is a constant $a$ (independent of $\Lambda$, $g$ and $p$) such that for any $p < \infty$ and $X = F, D, N, P$,

$$\| : \phi^r(g) : \|_p^{\Lambda,X} \leq a p^{r/2} \| g \|_{1+\varepsilon} \qquad \text{(III.41)}$$

where $\| \cdot \|_p^{\Lambda,X}$ denotes the $L^p(Q, d\mu_\Lambda^X)$ norm of $: \phi^r(g) :$ ($d\mu_\Lambda^X$ Wick substractions).

*Remarks.* — 1. The same estimate holds if the Wick substractions are made with respect to $d\mu_\Lambda^X$, $Y \neq X$ (see § VII.4).

2. For F and D B. C. $\Lambda$ may be any open set while for N B. C. $\Lambda$ may be fairly general (as described in § III.5).

3. Results such as (III.41) or (III.45) below may of course be formulated in the passive as well as the active picture.

*Proof.* — It is sufficient to prove (III.41) for $X = F$. For by (III.37) the theory with covariance $G_\Lambda^X$ may be obtained by conditioning from the theory with covariance $cG_0$ so that by the Conditioning Comparison Theorem (Theorem III.1)

$$\| : \phi^r(g) : \|_p^{\Lambda,X} \leq \| : \phi^r(g) : \|_p^{cF}$$

where the superscript $cF$ indicates that the covariance in the measure (and in the Wick ordering) is $cG_0$. Now by hypercontractivity (see [*44*]) for $p \geq 2$

$$\begin{aligned} \| : \phi^r(g) : \|_p^{cF} &\leq (p-1)^{r/2} \| : \phi^r(g) : \|_2^{cF} \\ &= [c(p-1)]^{r/2} \| : \phi^r(g) : \|_2^{F} \end{aligned}$$

For $p < 2$, $\| : \phi^r : \|_p \leq \| : \phi^r : \|_2$ so that (III.41) reduces to showing that

$$\| : \phi^r(g) : \|_2 \leq \text{const.} \| g \|_{1+\varepsilon} \,. \qquad \text{(III.42)}$$

This estimate is standard:

$$\begin{aligned} \| : \phi^r(g) : \|_2^2 &= r! \int g(x) G_0^r(x-y) g(y) dx dy \\ &\leq r! \| g \|_{1+\varepsilon} \| G_0^r * g \|_{(1+\varepsilon)'} \qquad \text{(Hölder's Inequality)} \\ &\leq r! \| g \|_{1+\varepsilon}^2 \| G_0^r \|_q \qquad \text{(Young's Inequality)} \end{aligned}$$

where $(1+\varepsilon)'$ is the dual Hölder index to $1+\varepsilon$ and

$$q^{-1} + (1+\varepsilon)^{-1} = 1 + ((1+\varepsilon)')^{-1}$$

so that $q = (1+\varepsilon)/2\varepsilon$. Finally by the Hausdorff-Young Inequality (assuming $qr \geq 2$)

$$\| G_0^r \|_q = \| G_0 \|_{qr}^r \leq \text{const.} \| \hat{G}_0 \|_{(qr)'}^r < \infty$$

proving, (III.42). ■

As remarked in (I.8), a change in Wick ordering introduces lower order terms according to the formula

$$
\begin{aligned}
:\delta\phi^r(x):_{X,\Lambda} &\equiv :\phi^r(x): - :\phi^r(x):_{X,\Lambda} \\
&= \sum_{j=1}^{[r/2]} \begin{Bmatrix} r \\ j \end{Bmatrix} \delta G_\Lambda^X(x)^j : \phi^{r-2j}(x) :_{X,\Lambda}
\end{aligned}
\tag{III.43}
$$

Our control of $\delta G_\Lambda^X$ in Lemma III.3 leads to:

LEMMA III.8. — For any $\varepsilon > 0$, let $g \in L^{1+\varepsilon}(\Lambda)$ where $\Lambda \subset \mathbb{R}^2$ is a rectangle, and define $\delta U = \int :\delta\phi^r(x):_{X,\Lambda} g(x)dx$. There is a constant $b$ (independent of $\Lambda$, $g$ and $p$) such that for any $p < \infty$, and $X = D, N, P$

$$\|\delta U\|_p^{\Lambda,X} \leq b p^{r/2-1} e^{-2m_0 d} \|g\|_{1+\varepsilon} \tag{III.44}$$

where $d = \text{dist}(\text{supp } g, \partial\Lambda)$.

*Remark.* — By Lemma II.37 of [29] we can prove (III.44) for $X = D$ if $\Lambda$ is any open set in $\mathbb{R}^2$.

*Proof.* — Applying the previous lemma to (III.43), we get

$$\|\delta U\|_p^{\Lambda,X} \leq \text{const. } p^{r/2-1} \sum_{j=1}^{[r/2]} \|g(\delta G)^j\|_q$$

for any $q > 1$. Choosing $q < 1 + \varepsilon$ and applying Hölder's inequality, we note that

$$\|g(\delta G)^j\|_q \leq \|g\|_{1+\varepsilon} \|(\delta G)^j \chi\|_s$$

where $\chi$ is the characteristic function of supp $g$ and $s^{-1} = q^{-1} - (1+\varepsilon)^{-1}$. By (III.34), $\|(\delta G)^j\chi\|_s$ is bounded and for large $d$

$$\|(\delta G)^j \chi\|_s = o(e^{-2m_0 d}). \quad \blacksquare$$

Next let P be a semibounded polynomial and consider the interaction

$$U_\Lambda^X = \int_\Lambda : P(\phi)(x) :_{X,\Lambda} dx.$$

LEMMA III.9. — Let $\Lambda$ be a rectangle, $X = F, D, P$, or $N$ and let $p < \infty$. There is a constant $b$ independent of $\Lambda$ such that

$$\|e^{-U_\Lambda^X}\|_p^{\Lambda,X} \leq e^{b|\Lambda|} \tag{III.45}$$

*Proof.* — By the theory of conditioning the left side of (III.45) as a function of X is greatest for X = N in which case submultiplicativity immediately gives the correct volume dependence. Thus we need only show that $\| e^{-U_\Lambda^N} \|_p^{\Lambda,N} < \infty$ for $\Lambda$ a unit square. As in the proof of Lemma III.7, Theorems III.4 and III.1 imply that

$$\| e^{-U_\Lambda^N} \|_p^{\Lambda,N} \leq \| e^{-U_\Lambda^{cF}} \|_p^{cF} .$$

If we make the change of variable $\tilde{\phi} = c^{-1/2}\phi$, then we see that

$$\| e^{-U_\Lambda^{cF}} \|_p^{cF} = \| e^{-\tilde{U}_\Lambda} \|_p \tag{III.46}$$

where the right side is expressed in terms of the usual free field with $\tilde{U}_\Lambda = \int_\Lambda : \tilde{P}(\phi(x)) : dx$ where the polynomial $\tilde{P}(\xi) = P(c^{1/2}\xi)$. That (III.46) is finite is perhaps the most celebrated estimate in $P(\phi)_2$ theories. ■

*Remark.* — As we show in § VII.4, the estimate (III.45) continues to hold even if the Wick subtractions in the interaction are made with respect to $d\mu_\Lambda^Y$, $Y \neq X$.

## III.5. **Bounds on $G_\Lambda^N$ for more general regions.**

In this section, we wish to examine when the inequality

$$G_\Lambda^N \leq cG_0 \tag{III.47}$$

hold as an operator inequality on $L^2(\Lambda)$. This inequality and its companion inequality $G_0 \leq G_\Lambda^N$, which holds by the theory of conditioning (see § III.2), are basic to our discussion of general covariance operators (see § II.1, in particular (II.6)). In Theorem III.4 we have already established (III.47) for the case of rectangular $\Lambda$ by using the method of images. In our field theory applications, the case of rectangles will suffice, but since the question of (III.47) for general $\Lambda$ is raised by our considerations, we shall make a slight detour to prove it for a larger class of regions than rectangles. We do this in two ways. First we shall prove (III.47) for two additional classes of regions, namely isosceles right triangles and circles. Then we shall prove a principle which allows us to extend (III.47) from one region to another.

LEMMA III.10. — (III.47) holds for isosceles right triangles.

*Proof.* — Let $\Lambda$ be the triangle and let R be reflection in the line containing the hypothenuse of the triangle. Let $\tilde{\Lambda}$ be the square obtained by taking the closure of $\Lambda \cup R[\Lambda]$. Then

$$G_\Lambda^N(x, y) = G_{\tilde{\Lambda}}^N(x, y) + G_{\tilde{\Lambda}}^N(x, Ry)$$

from which one easily concludes that

$$G_\Lambda^N(f, f) \leq 2G_{\tilde{\Lambda}}^N(f, f)$$

for any $f \in L^2(\Lambda)$. Since (III.47) holds for $\tilde{\Lambda}$, it holds for $\Lambda$. ■

THEOREM III.11. — (III.47) holds for any circle.

*Proof.* — Without loss of generality take $m_0 = 1$ and $\Lambda = \{ x \mid |x| < R \}$. By standard calculations $G_0$ and $G_\Lambda^N$ have expansion in terms of Bessel functions and polar coordinates

$$2\pi G_0(x, x') = g_0^{(0)}(r, r') + 2\sum_{l=1}^{\infty} g_l^{(0)}(r, r') \cos [l(\theta - \theta')] \quad \text{(III.48 } a)$$

$$2\pi G_\Lambda^N(x, x') = g_0^{(N)}(r, r') + 2\sum_{l=1}^{\infty} g_l^{(N)}(r, r') \cos [l(\theta - \theta')] \quad \text{(III.48 } b)$$

where $g_l^{(0)}$ (resp $g_l^{(N)}$) are symmetric and given for $r < r'$ by:

$$g_l^{(0)}(r, r') = I_l(r)K_l(r') \quad \text{(III.49 } a)$$

$$g_l^{(N)}(r, r') = I_l(r)K_l(r') + a_l I_l(r')I_l(r) \quad \text{(III.49 } b)$$

$$a_l = - K_l'(R)/I_l'(R). \quad \text{(III.49 } c)$$

For normalization and properties of the Bessel functions of imaginary argument, see [*1*, p. 375]. Given a real-valued function $f \in C_0^\infty(\Lambda)$, write $f(r, \theta) = \sum_{l=-\infty}^{\infty} y_l(\theta) f_l(r)$ where $y_l(\theta) = 1$ if $l = 0$, $\sqrt{2} \cos (l\theta)$ if $l > 0$ and $\sqrt{2} \sin (l\theta)$ if $l < 0$. Then, letting $\int f(x)f(y)G_0(x, y)dxdy = (f, f)_0$ we see that (III.47) follows if we can prove that

$$(2\pi)^{-1}a_l \left| \int I_l(r)f_l(r)(2\pi r)dr \right|^2 \leq c(f_l y_l(\theta), f_l y_l(\theta))_0 \quad \text{(III.50)}$$

for some constant $c$ independent of $l$. Let $h_l$ be the function defined by:

$$h_l(r, \theta) = \begin{cases} I_l(r) y_l(\theta) & r \leq R. \\ K_l(r) \dfrac{I_l(R)}{K_l(R)} y_l(\theta) & r \geq R \end{cases}$$

Then, since $\left| \int h(x)f(x)d^2x \right| \leq (f, f)_0[\| \nabla h \|_2^2 + \| h \|_2^2]$ (III.50) follows from:

$$(2\pi)^{-1}a_l[\| \nabla h_l \|_2^2 + \| h_l \|_2^2] \leq c. \quad \text{(III.51)}$$

Now $h_l$ obeys $(- \Delta + 1)h_l = 0$ away from $r = R$, so an integration by parts shows that

$$\| \nabla h_l \|_2^2 + \| h_l \|_2^2 = 2\pi I_l(R)[+ I_l'(R) - K_l'(R)I_l(R)K_l(R)^{-1}]$$
$$= 2\pi R^{-1} I_l(R)/K_l(R)$$

since $I_l'K_l - K_l'I_l = R^{-1}$. Thus, (III.51) is equivalent to a bound

$$- I_l(R)K_l'(R)/K_l(R)I_l'(R) \leq cR \tag{III.52}$$

independent of $l$ (with R fixed).

Now for $l \geq 1$ [1]

$$- K_l'(R)/K_l(R) = l/R + K_{l-1}(R)/K_l(R)$$

But $K_l > 0$ and $K_{l-1} = RK_l/2(l-1) - K_{l-2}$ so that

$$l/R \leq - K_l'/K_l \leq l/R + R/2(l-1)\,.$$

Similarly

$$l/R \leq I_l'/I_l \leq l/R + R/2(l+1)$$

so that

$$- I_lK_l'/K_lI_l' \leq 1 + R^2/2l(l-1). \tag{III.53}$$

(III.53) implies (III.52) and hence the lemma. ■

The reader may have noticed from (III.52) that as the radius R of the circle shrinks to zero, our bound on the constant $c$ in (III.47) diverges. In fact, the worst behavior is on the $l = 0$ subspace for

$$- I_0(R)K'(R)/K_0(R)I'(R) \sim \frac{1}{R \ln R^{-1}}$$

so that the constant $c$ diverges as $1/R^2 \ln R^{-1}$. This is not coincidental:

LEMMA III.12. — Fix $m_0$ and $\varepsilon > 0$. Then there exists a constant $b$ independent of $\Lambda$ so that if (III.47) holds on $L^2(\Lambda)$ then the constant $c$ must be larger than $b\,|\,\Lambda\,|^{\varepsilon-1}$.

*Proof.* — Let $\chi$ be the characteristic function of $\Lambda$. Since $\chi$ is an eigenfunction for $\Delta_\Lambda^N$ with eigenvalue 0,

$$(\chi, G_\Lambda^N\chi) = m_0^{-2}(\chi, \chi) = m_0^{-2}\,|\,\Lambda\,|\,.$$

On the other hand, by the standard argument in Lemma III.7,

$$(\chi, G_0\chi) \leq c_\delta \|\,\chi\,\|_{1+\delta}^2 = c_\delta\,|\,\Lambda\,|^{2-\varepsilon}$$

where $\delta$ is defined by $1 + \delta = (1 - \varepsilon/2)^{-1}$ (assuming $\varepsilon < 2$). ■

We now turn to a general method of obtaining new regions for which (III.47) holds from old regions for which it holds.

DEFINITION. — Two open regions, $\Lambda$ and C, in $\mathbb{R}^2$ are said to be *strongly* $C^2$ *diffeomorphic* if and only if there exists a $C^2$ diffeomorphism, $F : \mathbb{R} \to \mathbb{R}^2$, so that

(1) $F[\Lambda] = C$;

(2) The norm of the Jacobian matrix $dF(x)$ and the norm of its inverse are bounded uniformly in $x$; $x \in \mathbb{R}^2$;

(3) The Jacobian $J(x) \equiv \det(dF(x))$ has a gradient which is uniformly bounded in $x$; $x \in \mathbb{R}^2$.

*Remark.* — « Strongly $C^2$ diffeomorphic » is an equivalence relation. It is symmetric since $\det(d(F^{-1})(x)) = J(x)^{-1}$ and $|J(x)|$ is bounded from below by condition 2.

THEOREM III.13. — If two open regions are strongly $C^2$ diffeomorphic, then (III.47) either holds for both or fails for both of them.

*Remarks.* — 1. The idea of the proof is to use F to relate the Laplace operators on $\Lambda$ and C. One might expect that a condition would be needed implying that $dF(x)$ takes normals to the boundary of $\Lambda$ into normals to the boundary of C if $x$ is a boundary point of $\Lambda$. Such a condition is not needed essentially because the condition (III.47) is a form statement and the description of the form associated to $-\Delta_\Lambda^N$ makes no mention of normal derivatives

2. The global conditions on F are quite natural given our proof. We expect that if $\bar{\Lambda}$ is compact, a $C^2$-diffeomorphism of a neighborhood of $\bar{\Lambda}$ into a neighborhood of C will extend to a strong $C^2$ diffeomorphism but since this section is something of an aside, we do not examine this question.

*Proof.* — Let the regions be $\Lambda$ and C and let F be a strong $C^2$ diffeomorphism taking $\Lambda$ into C. Suppose that C obeys condition (III.47). Consider the map $U : L^2(\mathbb{R}, d^2x) \to L^2(\mathbb{R}, d^2x)$ given by:

$$(Uf)(x) = J(x)^{1/2} f(F(x))$$

where $J(x) \equiv \det(dF(x))$. Then U is unitary on $L^2(\mathbb{R}^2, d^2x)$ and also maps $L^2(C, d^2x)$ onto $L^2(\Lambda, d^2x)$ unitarily. Define a quadratic form $a$ on $L^2(\mathbb{R}^2, d^2x)$ with form domain $C_0^1(\mathbb{R}^2)$ and a quadratic form $a_C^N$ on $L^2(C, d^2x)$ with form domain $C^1(C)$ by the formula:

$$a(f, g) = \int_{\mathbb{R}^2} [\vec{\nabla}(Uf)(x)] \cdot [\vec{\nabla}(Ug)(x)] d^2x \qquad \text{(III.54)}$$

and similarly for $a_C^N$ (in which case $f$ and $g$ have support in C). By passing to the closure of these forms we obtain operators A and $A_C^N$ which clearly obey:

$$A = U^{-1}(-\Delta)U; \qquad A_C^N = U^{-1}(-\Delta_\Lambda^N)U.$$

(The forms are closable since they are unitarily equivalent to closable forms.) Let $K_0$ (resp. $K_C^N$) be the integral kernel of the inverse of $(A + m_0^2)$ (resp. $A_C^N + m_0^2$). Because of the relations (the last is by hypothesis)

$$K_0(f, f) = G_0(Uf, Uf)$$
$$K_C^N(f, f) = G_C^N(Uf, Uf); \qquad f \in L^2(C)$$
$$G_C^N(f, f) \leq c G_0(f, f); \qquad f \in L^2(C)$$

it suffices to prove that:

$$K_C^N(f, f) \le c_2 G_C^N(f, f); \qquad f \in L^2(C)$$
$$G_0(f, f) \le c_3 K_0(f, f).$$

By the theory of quadratic forms [33], this in turn follows from the inequalities:

$$\int_C |\nabla f|^2 d^2x \le c_4[a_C^N(f, f) + (f, f)]; \qquad f \in C^1(C) \quad \text{(III.55 } a\text{)}$$

$$a(f, f) \le c\left[\int(|\nabla f|^2 + |f|^2)d^2x\right] \qquad \text{(III.55 } b\text{)}$$

By (III.54):

$$a_C^N(f, f) = \int_\Lambda J(x)\left[dF(x) \circ \vec{\nabla} f)(F(x)) + \frac{1}{2}J(x)^{-1}(\vec{\nabla} J)f \circ F\right]^2 d^2x$$
$$= \int_C \left|(B(y) \circ \vec{\nabla})f(y) + \frac{1}{2}\vec{D}(y)f(y)\right|^2 d^2y$$

where $B(y) = dF(F^{-1}(y))$ and $\vec{D}(y) = J(F^{-1}(y))^{-1}(\vec{\nabla}J)(F^{-1}(y))$. (III.55 *a*, *b*) now follow easily from the bounds in the definition of a strong $C^2$ diffeomorphism. ■

EXAMPLE 3. — Linear homomorphisms are strong $C^2$-diffeomorphisms so we conclude that (III.47) holds for any triangle and for any ellipse.

THEOREM III.14. — Suppose that $\Lambda$ is a star shaped region with $C^2$ boundary. Then $\Lambda$ is strongly $C^2$-diffeomorphic to a disc, and in particular, (III.47) holds for $\Lambda$.

*Proof.* — By translation, we can suppose that $\Lambda$ is star shaped relative to zero. Thus there is a $C^2$ periodic function $g$ on $[0, 2\pi]$ so that in polar coordinates:

$$\Lambda = \{(r, \theta) \mid r < g(\theta)\}.$$

Let $a = \min g(\theta)$, $b = \max g(\theta)$. Let $\chi$ be a $C^\infty$, monotone non-increasing function on $[0, \infty)$ which is 1 on $\left[0, \frac{a}{3}\right)$ and 0 on $\left(\frac{2a}{3}, \infty\right)$. Let $\psi$ be a $C^\infty$ monotone non-decreasing function which is 0 on $(0, 2b)$ and 1 on $(3b, \infty)$. Define F by:

$$F(r, \theta) = (f(r, \theta), \theta)$$
$$f(r, \theta) = \chi(r)\frac{r}{b} + (1 - \chi(r))(1 - \psi(r))\frac{r}{g(\theta)} + \psi(r)\frac{r}{a}.$$

Then F is $C^2$, invertible (since $f(., \theta)$ is strictly monotone for each fixed $\theta$) and is linear outside a compact and hence a strong $C^2$-diffeomorphism. Clearly, F takes $\Lambda$ into the disc $\{(r, \theta) \mid r < 1\}$. ■

EXAMPLE 4. — Lest the reader think that (III.47) holds for all $\Lambda$, we note that the constant in (III.47) diverges as $r \to 0$ for the disc of radius $r$. Thus, (III.47) fails for an infinite union of disjoint discs of smaller and smaller radii. It is fairly clear that (III.47) will still fail if we join the discs together by narrow corridors and thus we expect that (III.47) fails for the interior of a suitable Jordan curve. We conjecture that (III.47) holds for the interior of any sufficiently smooth Jordan curve.

## IV. THE DIRICHLET PRESSURE

In this section, we prove the equality $\alpha_\infty^D = \alpha_\infty$ and discuss several of its applications including the proof of the Gibbs variational equality (this application was sketched in [*26*]). Most of our work on Dirichlet B. C. has been in an « active » picture in which the field is the same as the free field (namely the « coordinate » functions on $C_0^\infty(\mathbb{R}^2)'$) but the measure on $C_0^\infty(\mathbb{R}^2)'$ changes. In this section, we will work mainly in the passive picture in which the measure $d\mu_0$ on $C_0^\infty(\mathbb{R}^2)'$ is fixed but a Dirichlet field $\phi_\Lambda^D(f) = \phi(p_\Lambda f)$ replaces the coordinate functions when Dirichlet B. C. are introduced. In the passive picture, the sole change in going from $\alpha_\Lambda$ to $\alpha_\Lambda^D$ is the change $\phi \to \phi_\Lambda^D$ and moreover the two interactions $\exp(-U_\Lambda)$ and $\exp(-U_\Lambda^D)$ are realized on the same space so that we can try to control their difference directly. Such control will occur in § IV.1. By comparison, we note that in the active picture in going from $\alpha_\Lambda$ to $\alpha_\Lambda^D$ the dual change of Wick ordering and of measure are involved. A proof based on controlling these changes is possible; the basic ideas occur in § V below where we do, in fact, use the active picture to prove $\alpha_\infty^N = \alpha_\infty$.

A third proof involving the realization of Dirichlet B. C. in a transfer matrix formalism is possible and our version of this proof can be found in [*62*]. The main advantage of this latter proof is that it provides a transfer matrix formalism for Half-Dirichlet Schwinger functions. The proof we present here is simpler than that in [*62*] for two reasons: the method of § IV.1 is a simpler way of removing the Dirichlet « surfaces » than that of « $\delta$-functions in Q-space » and secondly no Wick reordering is necessary.

### IV.1. The principle of not feeling the boundary.

We use a passive picture so that $\phi_\Lambda^D(f) = \phi(p_\Lambda f)$, $p_\Lambda = 1 - e_{\mathbb{R}^2\setminus\Lambda}$. Given $\Lambda \subset \tilde{\Lambda}$, we define

$$U_\Lambda^{(D,\tilde{\Lambda})} = \int_\Lambda :P(\phi_{\tilde{\Lambda}}^D): d^2x$$

We recall that $\Gamma_m(N)$ is the set of all homogeneous polynomials of degree $m$ in the fields. The basic method for « eliminating » Dirichlet B. C. will be the following « Principle of Not Feeling the Boundary ».

THEOREM IV.1. — Let « $\Lambda$ » be a sequence of compact regions whose volumes approach infinity and for each $\Lambda$, let $\tilde{\Lambda}$ be another region with $\Lambda \subset \tilde{\Lambda}$ and

$$d(\Lambda, \partial\tilde{\Lambda}) \geq \text{const.} \mid \Lambda \mid^{\alpha}; \qquad \alpha > 0 \tag{IV.1}$$

Let $n$ be fixed and for each $\Lambda$ let $B_\Lambda \in \bigoplus_{m=0}^{n} \Gamma_m(N)$ with

$$\| B_\Lambda \|_2 \leq \exp(\text{const.} \mid \Lambda \mid^{\beta}); \qquad \beta < \alpha \tag{IV.2}$$

Then:

$$R_\Lambda \equiv \frac{\int_Q (e^{-U_\Lambda} - e^{-U_\Lambda^{(D,\tilde{\Lambda})}}) B_\Lambda d\mu_0}{\int_Q e^{-U_\Lambda} d\mu_0} \tag{IV.3}$$

goes to zero as $\mid \Lambda \mid \to \infty$.

In the proof of Theorem IV.1, we need the following two Lemmas:

LEMMA IV.2 A. — $\| U_\Lambda - U_\Lambda^{(D,\tilde{\Lambda})} \|_2^2 \leq \text{const.} \mid \Lambda \mid^2 e^{-ad(\Lambda, \partial\tilde{\Lambda})}$ for suitable $a > 0$ and all $\Lambda$, $\tilde{\Lambda}$ with $\Lambda \subset \tilde{\Lambda}$, $d(\Lambda, \partial\tilde{\Lambda}) \geq 1$ and $\mid \Lambda \mid \geq 1$.

*Proof.* — Write $\phi(x)$ as a sum of independent fields

$$\phi(x) = \phi^{(D,\tilde{\Lambda})}(x) + \phi_{\partial\tilde{\Lambda}}(x)$$

according to the passive interpretation of the theory in § II.1. Then:

$$: \phi^n(x) : - : (\phi^{(D,\tilde{\Lambda})})^n(x) : = \sum_{j=1}^{n} \binom{n}{j} : \phi_{\partial\tilde{\Lambda}}^j(x) : (\phi^{(D,\tilde{\Lambda})})^{n-j}(x) :$$

so

$$\| : \phi_\Lambda^n : - : [\phi_\Lambda^{(D,\tilde{\Lambda}_n)}]^n : \|_2^2 = \sum_{j=1}^{n} c_j \int_\Lambda \int_\Lambda S_{\partial_\Lambda}^j(x, y) [S^{(D,\partial\tilde{\Lambda})}(x, y)]^{n-j} dx dy$$

where $S_{\partial\tilde{\Lambda}}$(resp. $S^{(D,\tilde{\Lambda})}$) is the two point function for $\phi_{\partial\tilde{\Lambda}}$(resp. $\phi^{(D,\tilde{\Lambda})}$). Now [*29*]:

$$S_{\partial\tilde{\Lambda}}(x, y) \leq C e^{-ad}; \qquad a > 0$$

all $x, y \in \Lambda$, $d = d(\Lambda, \partial\tilde{\Lambda}) \geq 1$. Moreover for $k \geq 0$

$$\int_\Lambda \int_\Lambda [S^{(D,\tilde{\Lambda})}(x - y)]^k dx dy \leq \int_\Lambda \left[ \int_\Lambda [G_0(x - y)]^k dx dy \right] \leq c \mid \Lambda \mid^2$$

Thus:

$$\| : \phi_\Lambda^n : - : [\phi_\Lambda^{(D,\tilde{\Lambda})}]^n : \|_2^2 \leq c \mid \Lambda \mid^2 e^{-ad}. \quad \blacksquare$$

LEMMA IV.2 B. — Let $d\mu$ be a probability measure and define $d\nu_\Lambda = f_\Lambda d\mu \Big/ \int f_\Lambda d\mu$, where for each $\Lambda \subset \mathbb{R}^2$, $f_\Lambda \geq 0$ satisfies

$$\int f_\Lambda d\mu \geq e^{-a|\Lambda|}$$

and

$$\left[ \int f_\Lambda^q d\mu \right]^{1/q} \leq e^{b|\Lambda|}$$

for some $q > 1$ and constants $a$ and $b$. Then, there is a constant $c$, independent of $\Lambda$ so that for any $g$ and any $\Lambda$ with $|\Lambda| > 1$:

$$\left| \int g d\nu_\Lambda \right| \leq c \| g \|_{|\Lambda|}$$

(where $\| \,.\, \|_{|\Lambda|}$ is the $L^p$-norm with $p = |\Lambda|$).

*Proof.* — Let $|\Lambda|'$ be the index dual to $|\Lambda|$, i. e. $|\Lambda|' = (1 - |\Lambda|^{-1})^{-1}$. Define $\theta(\Lambda)$ by:

$$(|\Lambda|')^{-1} = \theta(\Lambda) + (1 - \theta(\Lambda))q^{-1}.$$

Then, by Hölder's inequality:

$$\| f_\Lambda \|_{|\Lambda|'} \leq \| f_\Lambda \|_1^\theta \| f_\Lambda \|_q^{1-\theta} = \| f_\Lambda \|_1 (\| f_\Lambda \|_q / \| f_\Lambda \|_1)^{1-\theta}.$$

Now

$$\begin{aligned} 1 - \theta &= q[(|\Lambda|')^{-1} - \theta] \\ &= q[(1 - \theta) - |\Lambda|^{-1}] \end{aligned}$$

so that

$$(1 - \theta) = (q - 1)^{-1} |\Lambda|^{-1}.$$

Thus, by the assumed bounds on $\| f_\Lambda \|_q$ and $\| f_\Lambda \|_1$:

$$\| f_\Lambda \|_{|\Lambda|'} / \| f_\Lambda \|_1 \leq \exp [(a + b)(q - 1)^{-1}] \equiv c.$$

By Hölder's inequality once again:

$$\begin{aligned} \left| \int g d\nu_\Lambda \right| &\leq \| g \|_{|\Lambda|} \| f_\Lambda \|_{|\Lambda|'} \| f_\Lambda \|_1^{-1} \\ &\leq c \| g \|_{|\Lambda|}. \quad \blacksquare \end{aligned}$$

*Proof of Theorem* IV.1. — Using $| e^x - e^{-y} | \leq \frac{1}{2} | x - y | \,.\, (e^x + e^y)$, we find that:

$$\begin{aligned} N^{-1} \left| \int_Q (e^{-U_\Lambda} - e^{-U_\Lambda^{(D,\tilde{\Lambda})}}) B_\Lambda d\mu \right| \\ \leq \frac{1}{2} \left| \int g(f_\Lambda + f'_\Lambda) d\mu \right| \end{aligned}$$

where $f_\Lambda = e^{-U_\Lambda}/\| e^{-U_\Lambda} \|_1$, $f'_\Lambda = e^{-U_\Lambda^{(D,\tilde\Lambda)}}/\| e^{-U_\Lambda} \|_1$, $N = \int e^{-U_\Lambda} d\mu_0$, and $g = | U_\Lambda - U_\Lambda^{(D,\tilde\Lambda)} | \, | B_\Lambda |$. By the conditioning inequality,

$$\| e^{-U_\Lambda^{(D,\tilde\Lambda)}} \|_p \leq \| e^{-U_\Lambda} \|_p$$

the « linear lower bound » $\| e^{-U_\Lambda} \|_p \leq e^{B|\Lambda|}$ and Jensen's inequality, $\| e^{-U_\Lambda} \|_1 \geq e^{-A|\Lambda|}$, one sees that $f_\Lambda$ and $f'_\Lambda$ obey the hypothesis of Lemma IV.2 B. Thus:

$$N^{-1} \left| \int_Q (e^{-U_\Lambda} - e^{-U_\Lambda^{(D,\tilde\Lambda)}}) B_\Lambda d\mu \right| \leq c \| g \|_{|\Lambda|}$$
$$\leq \| U_\Lambda - U_\Lambda^{(D,\tilde\Lambda)} \|_{2|\Lambda|} \| B_\Lambda \|_{2|\Lambda|} \quad \text{(IV.4)}$$

The conclusion of the theorem that $R_\Lambda \to 0$ follows from (IV.4) and the estimates:

$$\| U_\Lambda - U_\Lambda^{(D,\tilde\Lambda)} \|_{2|\Lambda|} \leq c_1 (2|\Lambda| - 1)^{\deg P/2} e^{-c_2|\Lambda|^\alpha} \quad \text{(IV.5 } a)$$
$$\| B_\Lambda \|_{2|\Lambda|} \leq C_3 (2|\Lambda| - 1)^{n/2} e^{C_4|\Lambda|^\beta} . \quad \text{(IV.5 } b)$$

(IV.5 *a*, *b*) follow from the hypothesis, Lemma IV.2 A and Nelson's best hypercontractive estimates [*44*]. ■

For later purposes, we also need a one-dimensional version of Theorem IV.1 whose proof is identical to the one above if one passes to a « mixed » picture in which the space Dirichlet B. C. are actively realized and the time ones are passively realized:

THEOREM IV.3. — Let $V_t = U_\Lambda^{(D,\tilde\Lambda)}$ where $\Lambda = (-l/2, l/2) \times (-t/2, t/2)$ and $\tilde\Lambda = (-l/2, l/2) \times \mathbb{R}$. Let $V_t^{(D,t')} = U_\Lambda^{(D,\tilde\Lambda)}$ where $\Lambda = (-l/2, l/2) \times (-t/2, t/2)$ and $\tilde\Lambda = (-l/2, l/2) \times (-t'/2, t'/2)$. Let $t_k \to \infty$ with $t'_k \geq t_k$, $| t'_k - t_k | \geq c | t_k |^\alpha$ ($\alpha > 0$). Let $n$ be fixed with $B_k \in \bigoplus_{m=0}^{n} \Gamma_m(N)$. Suppose that

$$\| B_k \|_2 \leq \exp(\text{const.} \, | t_k |^\beta) ; \qquad \beta < \alpha .$$

Let $V_k = V_{t_k}$, $V_k^D = V_{t_k}^{(D,t'_k)}$. Then:

$$R_k \equiv \frac{\int (e^{-V_k} - e^{-V_k^D}) B_k d\mu_l}{\int e^{-V_k} d\mu_l} \to 0$$

where $d\mu_l$ is the Gaussian measure for the field with Dirichlet B. C. on $x = \pm l/2$.

*Remark.* — While we employ the Principle of Not Feeling the Boundary only to prove equality of the pressures, we expect that it can also be used to prove the equality of certain kinds of states in the infinite volume limit.

## IV.2. $\alpha_\infty = \alpha_\infty^D$.

Let $H_{0,l}^D$ be the transfer matrix for the free measure with Dirichlet B. C. on $(-l/2, l/2) \times \mathbb{R}$, and let $V_l^D = \int_{-l/2}^{l/2} : P(\phi) : dx$ with Dirichlet Wick ordering on $(-l/2, l/2) \times \mathbb{R}$ and let $H_l^D = H_{0,l}^D + V_l^D$ (see [62] for detailed definitions). We will prove that $\alpha_\infty = \alpha_\infty^D$ by proving that $\alpha_{l,t}^D \to -E_l^D/l$ as $t \to \infty$ ($E_l^D = \inf \operatorname{spec}(H_l^D)$) and then that $-E_l^D/l \to \alpha_\infty$ as $l \to \infty$.

Our proof begins by noting some consequences of conditioning. First $\lim_{t\to\infty} \alpha_{l,t}^D$ exists and $\lim_{l\to\infty}(\lim_{t\to\infty} \alpha_{l,t}^D) = \lim_{t,l\to\infty} \alpha_{l,t}^D \equiv \alpha_\infty^D$ exist by subadditivity. Let $\alpha_{l,t}^{X,Y}$ denote the pressure in region $(-l/2, l/2) \times (-t/2, t/2)$ with X B. C. in the $l$-direction and Y in the $t$-direction. By conditioning

$$\alpha_{l,t}^D \leq \alpha_{l,t}^{D,F} \leq \alpha_{l,t} \tag{IV.6}$$

while by the standard transfer matrix argument [19], [27], [29]

$$\lim_{t\to\infty} \alpha_{l,t}^{D,F} = -E_l^D/l.$$

Thus

$$\lim_{t\to\infty} \alpha_{l,t}^D \leq -E_l^D/l. \tag{IV.7}$$

Moreover subadditivity of $l\alpha_{l,t}^{D,F}$ in $l$ then implies subadditivity of $-E_l^D$ and so the existence of $\lim_{l\to\infty} -E_l^D/l$. (IV.6) then implies that

$$\lim_{l\to\infty} -E_l^D/l \leq \alpha_\infty. \tag{IV.8}$$

THEOREM IV.4. — $\lim_{t\to\infty} \alpha_{l,t}^D = -E_l^D/l$.

*Proof.* — Throughout this proof we fix Dirichlet B. C. on $[-l/2, l/2]$ by means of the active picture; $d\mu_l$ will denote the corresponding free measure with D B. C. on the strip $[-l/2, l/2] \times \mathbb{R}$. On the other hand, D B. C. in the $t$-direction will be imposed in a passive way, i. e. with a new field $\phi^{(D,t')}$ vanishing at $\pm t'/2$.

Let $t' = t + t^{1/2}$. Define the « pressures »

$$\Gamma_t = \frac{t'}{t}\alpha_{l,t'}^D = \frac{1}{lt}\ln \int e^{-U_{t'}^D} d\mu_l$$

$$\Sigma_t = \frac{1}{lt}\ln \int e^{-U_t^D} d\mu_l$$

where $U_s^D = \int_{-l/2}^{l/2}\int_{-s/2}^{s/2} : P(\phi^{(D,t')}) : (x) d^2x$ with the D B. C. *always* at $\pm t'/2$. Then by the Principle of Not Feeling the Boundary (Theorem IV.3 with $B_k = 1$)

$$\lim_{t\to\infty} \Sigma_t = -E_l^D/l \tag{IV.9}$$

and clearly

$$\lim_{t\to\infty} \Gamma_t = \lim_{t\to\infty} \alpha^{\mathrm{D}}_{l,t}\,.$$

By (IV.9) and (IV.7), $\lim \Gamma_t \leq \lim \Sigma_t$ so the theorem follows if we can prove that

$$\underline{\lim}\, (\Gamma_t - \Sigma_t) \geq 0\,. \tag{IV.10}$$

We write $\mathrm{U}^{\mathrm{D}}_{t'} = \mathrm{U}^{\mathrm{D}}_t + \mathrm{W}^{\mathrm{D}}_{t,+} + \mathrm{W}^{\mathrm{D}}_{t,-}$ where $\mathrm{W}^{\mathrm{D}}_{t,\pm}$ are the parts of the interaction between times $[t/2, t'/2]$ and $[-t'/2, -t/2]$ respectively. Then by Jensen's inequality

$$\Gamma_t - \Sigma_t \geq \frac{1}{lt}\left[\int (-\mathrm{W}^{\mathrm{D}}_{t,+} - \mathrm{W}^{\mathrm{D}}_{t,-})e^{-\mathrm{U}^{\mathrm{D}}_t}d\mu_l \Big/ \int e^{-\mathrm{U}^{\mathrm{D}}_t}d\mu_l\right].$$

Thus to prove (IV.10) we need only show that

$$\lim_{t\to\infty} \frac{1}{t} \int \mathrm{W}^{\mathrm{D}}_{t,\pm} e^{-\mathrm{U}^{\mathrm{D}}_t} d\mu_l \Big/ \int e^{-\mathrm{U}^{\mathrm{D}}_t} d\mu_l = 0\,. \tag{IV.11}$$

Let $\mathrm{W}_{t,\pm}$, $\mathrm{U}_t$ be the same objects as $\mathrm{W}^{\mathrm{D}}_{t,\pm}$, $\mathrm{U}^{\mathrm{D}}_t$ except that the field $\phi^{(\mathrm{D},t')}$ is replaced by the field $\phi$ (Wick subtractions in both cases are with respect to $d\mu_l$). We claim that in (IV.11) we can replace $\mathrm{W}^{\mathrm{D}}_{t,\pm}$ by $\mathrm{W}_{t,\pm}$ without changing the value of the integral. For $d\mu_l$ factors as $d\mu^{\mathrm{D}} \otimes d\mu$, where $d\mu^{\mathrm{D}}$ is the Gaussian measure associated with D B. C. on the sides of $[-l/2, l/2] \times [-t'/2, t'/2]$, and $\phi$ decomposes as an independent sum $\phi = \phi^{\mathrm{D}} + \phi^{\perp}$ so that

$$\int \mathrm{W}_{t,\pm} d\mu = \mathrm{W}^{\mathrm{D}}_{t,\pm}\,.$$

Hence

$$\int \mathrm{W}_{t,\pm} e^{-\mathrm{U}^{\mathrm{D}}_t} d\mu_l = \int\left(\int \mathrm{W}_{t,\pm} d\mu\right) e^{-\mathrm{U}^{\mathrm{D}}_t} d\mu^{\mathrm{D}} = \int \mathrm{W}^{\mathrm{D}}_{t,\pm} e^{-\mathrm{U}^{\mathrm{D}}_t} d\mu_l\,.$$

Having replaced $\mathrm{W}^{\mathrm{D}}_{t,\pm}$ by $\mathrm{W}_{t,\pm}$ in (IV.11), we can next replace $\mathrm{U}^{\mathrm{D}}_t$ by $\mathrm{U}_t$ without changing the value of the limit; this is an elementary consequence of the Principle of Not Feeling the Boundary (Theorem IV.3). In this way we reduce the proof of the theorem of a statement involving only free interactions, namely:

$$\lim_{t\to\infty} \frac{1}{t} \int \mathrm{W}_{t,\pm} e^{-\mathrm{U}_t} d\mu_l \Big/ \int e^{-\mathrm{U}_t} d\mu_l = 0\,. \tag{IV.12}$$

Consider (IV.12) in the case $\mathrm{W}_{t,+}$. Since $\mathrm{W}_{t,+}$ lives at times after $t$ and $e^{-\mathrm{U}_t}$ at times before $t$, we can write

$$\int \mathrm{W}_{t,+} e^{-\mathrm{U}_t} d\mu_0 = \langle \mathrm{J}^*_\pi \mathrm{W}_{t,+}, \mathrm{J}^*_\pi e^{-\mathrm{U}_t} \rangle \tag{IV.13}$$

where $\mathrm{J}_\pi$ is the embedding of the time zero field onto the plane $\pi = \{ \langle x, s \rangle \mid s = t \}$. (IV.13) is just an expression of the Markov property. Using the Schwarz inequality and reflection symmetry in the plane $\pi$ we have:

$$\left| \int W_{t,+} e^{-U_t} d\mu_0 \right| \leq \| J_\pi^* W_{t,+} \|_2 \left( \int e^{-\tilde{U}_t - U_t} d\mu_0 \right)^{1/2}$$

where $\tilde{U}_t$ is the reflection of $U_t$ in the plane $\pi$. Thus by the Feynman-Kac formula the l. h. s. of (IV.12) is bounded by ($H \equiv H_l^P$)

$$t^{-1} \| J_\pi^* W_{t,+} \|_2 \langle \Omega_0, e^{-2tH}\Omega_0 \rangle^{1/2} / \langle \Omega_0, e^{-tH}\Omega_0 \rangle .$$

Since $\langle \Omega_0, e^{-tH}\Omega_0 \rangle = \langle \Omega_0, \Omega_l \rangle e^{-tE_l^P} + 0(e^{-t(E_l^P + m_l)})$ and

$$\| J_\pi^* W_{t,+} \|_2 \leq \| W_{t,+} \|_2 \leq C(t' - t)^{1/2} = Ct^{1/4},$$

we see that this last quantity indeed goes to zero and thus (IV.12) is proven. ■

We can now prove the basic equality of the pressures:

THEOREM IV.5.

$$\alpha_\infty^D = \alpha_\infty .$$

*Proof.* — Since we already know that $\lim_{t,l \to \infty} \alpha_{l,t}^D$ exists and we have Theorem IV.4, we need only prove that

$$\lim_{l \to \infty} - E_l^D / l = \alpha_\infty .$$

Given $l$, let $l' = l + l^{1/2}$ and place Dirichlet data on $x = \pm l'/2$ by the passive method. Consider

$$\lim_{l \to \infty} \alpha_{l,t}^{D,F} .$$

By mimicking the proof of Theorem IV.4 (thinking, in the spirit of Nelson's symmetry, of $l$ as the « time » direction) one finds that:

$$\lim_{l \to \infty} \alpha_{l,t}^{D,F} \geq - E_t / t$$

where $E_t$ is the free B. C. energy. But by the Schwarz inequality:

$$\alpha_{l,t}^{D,F} = \frac{1}{lt} \ln \langle \Omega_0, e^{-tH_l^P}\Omega_0 \rangle$$
$$\leq - l^{-1} E_l^D .$$

Thus, for any $t$

$$\lim_{l \to \infty} - E_l^D / l \geq - \frac{1}{t} E_t$$

Taking $t$ to $\infty$,

$$\lim_{l \to \infty} - E_l^D / l \geq \alpha_\infty .$$

The opposite inequality is a consequence of conditioning. ■

## IV.3. Application 1 : convergence of $-E(g)/\|g\|_2^2$.

The quantity $-E_l$ in $\alpha_\infty = \lim_l -E_l/l$ is defined with sharp cutoffs. Elsewhere [28], [29], we have considered the question (raised first by Osterwalder-Schrader [47]) of showing that $-E(g)\Big/\int g(x)^2dx \to \alpha_\infty$ for a sequence of $g$ approaching 1 in a suitable sense. A bound on the lim sup is fairly easy because of the « improved linear lower bound » [28]:

$$-E(g) \leq \int \alpha_\infty(g(x))dx.$$

The opposite inequality is harder and our previous results have required the $g$'s to have compact (increasing) supports. The problem is that we had no effective mechanism for showing that making $g$ small outside some set does not decrease the binding. Dirichlet B. C. turns to be perfect for that (see (IV.14) below). We have:

THEOREM IV.7. — Let P be normalized, i. e. $P(0) = 0$. Let $g_n$ be a sequence of non-negative functions and $I_n$ a sequence of intervals so that:

(i) $|I_n| \to \infty$

(ii) $0 \leq g_n(x) \leq C$ for all $x \in I_n$ and some C independent of $n$

(iii) $|I_n|^{-1}\int_{I_n} |g_n - 1|\,dx \to 0$

(iv) $|I_n|^{-1}\int_{\mathbb{R}\setminus I_n} g_n^2 \to 0$ as $n \to \infty$.

Then $\lim_{n\to\infty} -E(g_n)/\|g_n\|_2^2 = \alpha_\infty$.

*Proof.* — Let $h_n = g_n\chi_n$ where $\chi_n$ is the characteristic function of $I_n$. From (ii) – (iv), one easily concludes that

$$\lim_{n\to\infty} |I_n|^{-1}\|g_n\|_2^2 = \lim_{n\to\infty} |I_n|^{-1}\|h_n\|_2^2 = 1.$$

Let

$$H(g) = H_0 + \int g(x) : P(\phi) : dx$$

$$H_l^D(g) = H_{0,l}^D + \int g(x) : P(\phi) :_{D,l} dx$$

Let $E(g) = \inf \operatorname{spec} H(g)$ and $E_l^D(g) = \inf \operatorname{spec} H_l^D(g)$. By translation invariance, we can suppose $I_n = (-l_n/2, l_n/2)$ and we write $H_n^D(g)$ for $H_{l_n}^D(g)$. Now, by the improved linear lower bound:

$$- E(g_n) \leq \int \alpha_\infty(g_n(x))dx$$
$$= \int_{x \notin I_n} \alpha_\infty(g_n(x))dx + \int_{x \in I_n} \alpha_\infty(g_n(x)) .$$

Now, since P is normalized, $\alpha_\infty(\lambda) \leq c\lambda^2$ so the first integral is bounded by $c \int_{x \notin I_n} g_n(x)^2 dx$ which is $o(|I_n|)$ by (*iv*). By convexity

$$| \alpha_\infty(\lambda) - \alpha_\infty(1) | \leq | \lambda - 1 | [c^{-1}(\alpha_\infty(c) - \alpha_\infty(1))]$$

so the second integral is $| I_n | \alpha_\infty(1)$ plus an error bounded by

$$\int_{x \in I_n} | g_n(x) - 1 | dx$$

which is $o(|I_n|)$ by (*iii*). Thus:

$$\lim (- E(g_n))/\| g_n \|_2^2 \leq \alpha_\infty$$

Now by conditioning,

$$- E_n^D(h_n) \leq - E(g_n) . \tag{IV.14}$$

By mimicking our convexity arguments in [*28*] and using critically $\alpha_\infty = \alpha_\infty^D$ one sees that

$$\lim (- E_n^D(h_n))/\| h_n \|_2^2 = \alpha_\infty .$$

From this we conclude that

$$\underline{\lim} (- E(g_n))/\| g_n \| \geq \alpha_\infty . \quad \blacksquare$$

## IV.4. **Application 2 : Van Hove convergence of the pressure.**

In [*29*], we proved Fisher convergence of the free pressure $\alpha_\Lambda$ to $\alpha_\infty$ as $\Lambda \to \infty$ (Fisher) and reduced the stronger Van Hove convergence as $\Lambda \to \infty$ (Van Hove) to proving that $\alpha_\infty = \alpha_\infty^D$. Since the argument is so simple, we sketch it here now that it can be completed.

DEFINITION. — For fixed $a$, $N_a^+(\Lambda)$ is the number of squares of side $a$ with vertices at $\mathbb{Z}^2 a$ that intersect $\Lambda$ and $N_a^-(\Lambda)$ is the number of such squares contained in $\Lambda$. We say that $\Lambda_n \to \infty$ (Van Hove) if $|\Lambda_n| \to \infty$ and for each $a$, $N_a^+(\Lambda_n)/N_a^-(\Lambda_n) \to 1$ as $n \to \infty$.

THEOREM IV.8. — If $\Lambda_n \to \infty$ (Van Hove), then

$$\lim_{n\to\infty} \alpha_{\Lambda_n} = \alpha_\infty .$$

*Proof.* — Without loss suppose that P is normalized. By the improved linear lower bound, $\alpha_{\Lambda_n} \leq \alpha_\infty$. On the other hand, by conditioning

$$\alpha_{\Lambda_n} \geq \alpha^{\mathrm{D}}_{\Lambda_n} \geq |\Lambda_n|^{-1} a^2 \mathrm{N}^-_a(\Lambda_n)\alpha^{\mathrm{D}}_{a\times a}$$

so that

$$\underline{\lim}\, \alpha_{\Lambda_n} \geq \alpha^{\mathrm{D}}_{a\times a}$$

since $|\Lambda_n|^{-1}a^2\mathrm{N}_a^-(\Lambda_n) \geq \mathrm{N}_a^-(\Lambda_n)/\mathrm{N}_a^+(\Lambda_n) \to 1$. Taking $a \to \infty$,

$$\underline{\lim}\, \alpha_{\Lambda_n} \geq \alpha^{\mathrm{D}}_\infty = \alpha_\infty . \quad \blacksquare$$

## IV.5. Application 3 : the Gibbs variational equality.

In [*29*], we defined the entropy of weakly tempered states and proved that (Gibbs variational inequality):

$$s(\rho) - \mathrm{U}(\rho, \mathrm{P}) \leq \alpha_\infty(\mathrm{P}) \qquad \text{(IV.15 } a)$$

but left open the Gibbs variational equality

$$\sup_\rho\, [s(\rho) - \mathrm{U}(\rho, \mathrm{P})] = \alpha_\infty(\mathrm{P}) . \qquad \text{(IV.15 } b)$$

In the above, $s(\rho)$ is the mean entropy $\lim_{|\Lambda|\to\infty} \mathrm{S}_\Lambda(\rho)/|\Lambda|$ and U is the mean interaction; see [*29*] for precise definitions. Here we will prove (IV.15 *b*). A sketch of the main ideas has appeared already in [*26*]. Below, we use notation from [*29*] freely (see especially § VI, VII of [*29*]).

THEOREM IV.9. — The Gibbs variational equality (IV.15 *b*) holds.

*Proof.* — For each $a$ we will construct a state $\rho_a$ with

$$s(\rho_a) - \mathrm{U}(\rho_a, \mathrm{P}) \geq \alpha^{\mathrm{D}}_{a\times a}(\mathrm{P}) \qquad \text{(IV.16)}$$

so that

$$\sup_\rho\, [s(\rho) - \mathrm{U}(\rho, \mathrm{P})] \geq \alpha^{\mathrm{D}}_\infty = \alpha_\infty \qquad \text{(IV.17)}$$

(IV.15 *a*) and (IV.17) imply (IV.15 *b*).

Fix $a$, and for $i \in \mathbb{Z}^2$, let $\mathrm{C}_i$ be the square with center $ia$ and side $a$. Let $\phi_i^{(\mathrm{D})}$ be the Dirichlet field in $\mathrm{C}_i$ in a passive picture and define $e_i$ by $\phi_i^{(\mathrm{D})}(f) = \phi(e_i f)$. Let $p$ be the projection (in N) onto the distributions supported on $\bigcup_i \partial\mathrm{C}_i$. Then

$$\mathrm{N} = \mathrm{Ran}\, p \oplus \left(\bigoplus_i \mathrm{Ran}\, e_i\right) . \qquad \text{(IV.18)}$$

Corresponding to the breakup (IV.18), Q space factors in the form

$$Q = Q_p \times \bigtimes_i Q_i . \tag{IV.19}$$

Consider the measure on Q which is a product measure with respect to the decomposition (IV.19)

$$d\mu = d\mu_p \otimes \bigtimes d\mu_i \tag{IV.20 a}$$

where

$$d\mu_p = d(\mu_0 \restriction \Sigma_p) \tag{IV.20 b}$$

$$d\mu_i = N^{-1} \exp(-U_i^{(D)}) d\mu_0 \restriction \Sigma_i . \tag{IV.20 c}$$

In the above, $\Sigma_p$ (resp. $\Sigma_i$) is the $\Sigma$-algebra of measurable sets on $Q_p$(resp. $Q_i$), $U_i^{(D)}$ is the *Dirichlet* interaction on $C_i$ and

$$N = \int e^{-U_i^{(D)}} d\mu_0$$

which is $i$ independent. The state $\tilde{\rho}_a$ defined by $\mu$ is easily seen to be tempered (if $A \subset \mathbb{Z}^2$ is finite and $\Lambda = \bigcup_{i \in A} \bar{C}_i$, then the corresponding density $f_\Lambda$ is given by $f_\Lambda d\mu_0 = \bigtimes_{i \in A} d\mu_i$) and periodic under space translations. If we define $\rho_a$ by:

$$\rho_a = a^{-2} \int_{|x| \leq a/2} (\tau_x \tilde{\rho}_a) d^2x$$

then $\rho$ is a translation invariant and tempered. We will verify (IV.16) for $\rho_a$. Since $-x \ln x$ is concave we have:

$$S_\Lambda(\rho_a) \geq a^{-2} \int S_\Lambda(\tau_x \tilde{\rho}_a) d^2x .$$

Thus if we can show that $\lim_{\Lambda \to \infty} S_\Lambda(\tilde{\rho}_a) \equiv S(\tilde{\rho}_a)$ exists then $S(\rho_a) \geq S(\tilde{\rho}_a)$. Since our proof of convergence of entropy easily extends to periodic states since there is a checkerboard estimate for such states (see Appendix):

$$S(\rho_a) \geq S(\tilde{\rho}_a) . \tag{IV.21}$$

(Remark: one can actually prove equality in (IV.21) and then obtain equality in (IV.16)). Now, since $\tilde{\rho}_a$ is a product measure

$$a^{-2} S_{C_0}(\tilde{\rho}_a) = S(\tilde{\rho}_a) ,$$

so

$$S(\tilde{\rho}_a) = + a^{-2} \ln N + a^{-2} N^{-1} \int U_0^{(D)} e^{-U_0^{(D)}} d\mu_0 . \tag{IV.22}$$

Since $\mu$ is periodic:

$$\begin{aligned} U(\rho_a, P) &= a^{-2}\int U_{C_0}d\mu \\ &= a^{-2}\int U_{C_0}d\mu_{C_0}\otimes d(\mu_0 \upharpoonright \Sigma_p) \\ &= a^{-2}\int\left(\int U_{C_0}d(\mu_0 \upharpoonright \Sigma_p)\right)d\mu_{C_0} \\ &= a^{-2}\int U_{C_0}^{(D)}d\mu_{C_0} \\ &= a^{-2}N^{-1}\int U_0^{(D)}e^{-U_0^{(D)}}d\mu_0\,. \end{aligned} \tag{IV.23}$$

(IV.21-23) imply (IV.16) and so complete the proof. ■

## V. THE NEUMANN PRESSURE

### V.1. $\alpha_\infty^N = \alpha_\infty$.

In this subsection we prove that $\alpha_\infty^N = \alpha_\infty$. The inequality $\alpha_\Lambda^N \geq \alpha_\Lambda$ follows from the theory of conditioning (see (II.24)) and so this section will be devoted to showing the reverse inequality

$$\alpha_\infty^N \leq \alpha_\infty\,. \tag{V.1}$$

Note that by a standard argument based on subadditivity (see (III.22)) we already know the existence of the limit $\alpha_\infty^N = \lim_{\Lambda\to\infty} \alpha_\Lambda^N$. However, in the course of establishing (V.1) we shall give an independent proof of the existence of $\lim_\Lambda \alpha_\Lambda^N$. It is also possible to give a proof of (V.1) in the passive picture that is parallel to the corresponding proof for Dirichlet B. C. in § IV.2.

Let $\Lambda$ be a rectangle which we assume goes to $\infty$ in a suitably regular

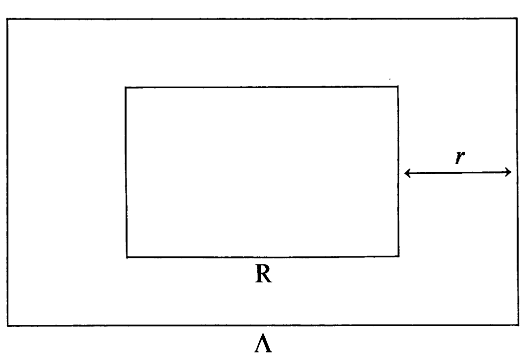


FIG. V.1.

way (say in the sense of Fisher [29, App. C]) and let $R \subset \Lambda$ be the rectangle whose sides are a distance $r = |\Lambda|^\eta$ from $\partial\Lambda$. Here $\eta \in (0, 1/2)$ is a fixed small constant. Our proof of (V.1) consists of three steps (each step is up to an error which vanishes as $\Lambda \to \infty$):

1. STRIP REMOVAL

We dominate $\alpha_\Lambda^N$ by $\alpha_R^{N,\Lambda}$, the « pressure » with interaction in R and measure with Neumann B. C. on $\partial\Lambda$, i. e.

$$\alpha_R^{N,\Lambda}(\lambda) = \frac{1}{|R|}\log \int e^{-\lambda U_R^{N,\Lambda}} d\mu_\Lambda^N \tag{V.2}$$

where

$$U_R^{N,\Lambda} = \int_R :P(\phi(x)):_{N,\Lambda} dx.$$

2. CHANGE OF WICK ORDERING

We bound $\alpha_R^{N,\Lambda}$ by $\alpha_R^{HN,\Lambda}$, defined by

$$\alpha_R^{HN,\Lambda}(\lambda) = \frac{1}{|R|}\log \int e^{-\lambda U_R} d\mu_\Lambda^N \tag{V.3}$$

where $U_R = \int_R :P(\phi(x)): dx$.

3. CHANGE OF MEASURE

Finally we dominate $\alpha_R^{HN,\Lambda}$ by $\alpha_R$.

In the following lemmas $\varepsilon_j$ denotes various constants which go to zero as $\Lambda \to \infty$ and which also depend on other fixed constants ($\lambda$, $p$ ...).

LEMMA V.1. — Let $\lambda > 1$ be given. Then

$$\alpha_\Lambda^N \leq \alpha_R^{N,\Lambda}(\lambda)/\lambda + \varepsilon_1 .$$

*Proof.* — By Hölder's inequality ($1/\lambda + 1/\lambda' = 1$)

$$\int e^{-U_\Lambda^{N,\Lambda}} d\mu_\Lambda^N \leq \left(\int e^{-\lambda U_R^{N,\Lambda}} d\mu_\Lambda^N\right)^{1/\lambda} \left(\int e^{-\lambda' U_{\Lambda\setminus R}^{N,\Lambda}} d\mu_\Lambda^N\right)^{1/\lambda'} . \tag{V.4}$$

Now by the subadditivity of Neumann B. C. (see (III.27)), we see that the last factor in (V.4) is of the order $e^{0(|\Lambda\setminus R|)}$. Hence upon taking logarithms and dividing by $|\Lambda|$,

$$\alpha_\Lambda^N \leq \left|\frac{R}{\Lambda}\right| \alpha^{N,\Lambda}(\lambda)/\lambda + 0(|\Lambda\setminus R|)/|\Lambda|$$
$$= \alpha_R^{N,\Lambda}(\lambda)/\lambda + 0(|\Lambda|^{\eta-1/2})$$

since $|\Lambda\backslash R| \sim |\Lambda|^{n+1/2}$ and since $\alpha_R^{N,\Lambda}$ is uniformly bounded in $|\Lambda|$, by subadditivity. ■

LEMMA V.2.

$$\alpha_R^{N,\Lambda} \leq \alpha_R^{HN,\Lambda} + \varepsilon_2 .$$

*Proof.* — Let $\delta U = U_R^{N,\Lambda} - U_R$ and define

$$W = U_R^{N,\Lambda} + \delta U \tag{V.5}$$

so that $U_R^{N,\Lambda} = \frac{1}{2} U_R + \frac{1}{2} W$. By the Schwarz inequality

$$\alpha_R^{N,\Lambda} \leq \frac{1}{2} \alpha_R^{HN,\Lambda} + \frac{1}{2|R|} \log \int e^{-W} d\mu_\Lambda^N . \tag{V.6}$$

Now by Jensen's inequality and (V.5)

$$\log \frac{\int e^{-U_R^{N,\Lambda}} d\mu_\Lambda^N}{\int e^{-W} d\mu_\Lambda^N} \geq \frac{\int \delta U e^{-W} d\mu_\Lambda^N}{\int e^{-W} d\mu_\Lambda^N} \equiv \langle \delta U \rangle .$$

Upon substituting into (V.6) we obtain

$$\alpha_R^{N,\Lambda} \leq \frac{1}{2} \alpha_R^{HN,\Lambda} + \frac{1}{2} \alpha_R^{N,\Lambda} - \frac{1}{2|R|} \langle \delta U \rangle$$

or

$$\alpha_R^{N,\Lambda} \leq \alpha_R^{HN,\Lambda} - \frac{1}{|R|} \langle \delta U \rangle .$$

To estimate $\langle \delta U \rangle$ we note that by submultiplicativity and the bounds (III.34) on the coefficients in $\delta U$,

$$\int e^{-2W} d\mu_\Lambda^N \leq e^{0(|R|)} ,$$

whereas by Jensen's inequality and (III.34) there is a constant C such that

$$\int e^{-W} d\mu_\Lambda^N \geq \exp\left[ - \int W d\mu_\Lambda^N \right] \geq e^{-C|R|} .$$

We may thus apply Lemma IV.2 B to deduce that

$$\begin{aligned} |\langle \delta U \rangle| &\leq \text{const.} \, \| \delta U \|_{|R|}^{\Lambda,N} \\ &= 0(|R|^{n-1} e^{-2m_0 r}) \end{aligned}$$

by Lemma III.8, where $2n$ is the degree of the interaction. This last bound clearly goes to zero as $|\Lambda| \to \infty$. ■

LEMMA V.3. — Fix $p' > 1$. Then

$$\alpha_{\mathrm{R}}^{\mathrm{HN},\Lambda} \leq \alpha_{\mathrm{R}}(p')/p' + \varepsilon_3 .$$

*Proof.* — Let $p$ be the conjugate index to $p'$, and choose $\Lambda$ sufficiently large so that (III.40) is satisfied with $\mathrm{R} = \Lambda'$. Then by Theorem III.6 $\mathrm{F_N} = d\mu_\Lambda^{\mathrm{N}} \restriction \Sigma_{\mathrm{R}}/d\mu_0 \restriction \Sigma_{\mathrm{R}}$ has $\mathrm{L}^p$ norm bounded independently of $|\Lambda|$. Therefore by Hölder's inequality

$$\begin{aligned} \alpha_{\mathrm{R}}^{\mathrm{HN},\Lambda} &= \frac{1}{|\mathrm{R}|}\log \int e^{-\mathrm{U_R}}\mathrm{F_N} d\mu_0 \\ &\leq \alpha_{\mathrm{R}}(p')/p' + \frac{1}{|\mathrm{R}|}\log \|\mathrm{F_N}\|_p . \qquad \blacksquare \end{aligned}$$

Combining the previous three lemmas, we see that for any $\mu > 1$,

$$\alpha_\Lambda^{\mathrm{N}} \leq \alpha_{\mathrm{R}}(\mu)/\mu + \varepsilon_4 .$$

Letting $|\Lambda| \to \infty$ we deduce that

$$\alpha_\infty^{\mathrm{N}} \leq \alpha_\infty(\mu)/\mu$$

and taking $\mu \to 1$ we obtain (V.1) since the convex function $\alpha_\infty(\mu)$ is continuous in $\mu$. Summarizing:

THEOREM V.4.

$$\alpha_\infty^{\mathrm{N}} = \alpha_\infty .$$

*Remark.* — Together with the result $\alpha_\infty^{\mathrm{D}} = \alpha_\infty$ of § IV, and the basic conditioning inequalities (I.3 *b*), this theorem establishes the equality $\alpha_\infty^{\mathrm{X}} = \alpha_\infty$ of Theorem I.1.

## VI. THE PERIODIC PRESSURE

In this section we shall examine the pressure and, to a lesser extent, the states associated with using a periodic Laplacian in a box

$$(-\,l/2, l/2) \times (-\,t/2, t/2) .$$

(See Hoegh-Krohn [*32*] for related results). In the first subsection, we will use the theory of conditioning and the equality $\alpha_\infty^{\mathrm{D}} = \alpha_\infty^{\mathrm{N}} = \alpha_\infty$ which we have already proven (sections IV, V) to provide quick proofs of the convergence to $\alpha_\infty$ of the periodic pressure and a pressure with mixed boundary conditions: periodic in space and free in time. By using transfer matrix techniques, we will be able to relate the latter to the convergence of a periodic Hamiltonian ground state energy per unit volume. We will then identify this energy with the quantity $\mathrm{E_V}$ used by Glimm-Jaffe [*19*]. In the second subsection, we develop a transfer matrix formalism for a situation with periodic B. C. in time. We emphasize at the outset that Nelson's symmetry for periodic B. C. has a somewhat subtle form (see (VI.11)). As an application of this transfer matrix-Nelson symmetry formalism, we provide an alternate proof of the convergence of the periodic pressure. A second application to the proof of $\phi$-bounds for periodic Hamiltonians (as proved by Glimm-Jaffe [*19*]) appears in our paper [*30*].

### VI.1. **Convergence of the periodic pressure.**

We have already defined the periodic pressure $\alpha_{l,t}^{P}$ and proved that

$$\alpha_{l,t}^{D} \leq \alpha_{l,t}^{P} \leq \alpha_{l,t}^{N}.$$

From the convergence of $\alpha_{l,t}^{D}(\alpha_{l,t}^{N})$ as $l,t \to \infty$ to the common limit $\alpha_\infty$ we immediately have:

THEOREM VI.1. — $\lim_{l,t\to\infty} \alpha_{l,t}^{P}$ exists and equals $\alpha_\infty$.

In the next subsection, we develop a transfer matrix formalism which allows us to interpret a limit of pressures in a direction with periodic B. C. in terms of a ground state energy. It is technically somewhat simpler to deal with free B. C. in the direction of transfer in the transfer matrix and so we introduce the mixed pressures:

DEFINITION. — $\alpha_{l,t}^{P,F}$ is the pressure associated with the « free » measure $d\mu_{0,l}^{P}$ whose covariance is given by the periodic Green's function in the infinite strip $(-l/2, l/2) \times \mathbb{R}$ with interaction in region $(-l/2, l/2) \times (-t/2, t/2)$ with $d\mu_{0,l}^{P}$ Wick ordering.

Since we are dealing with full pressures and not half-pressures (i. e. the interaction is $d\mu_{0,l,\infty}^{P}$ Wick ordered) we still have the conditioning inequalities:

$$\alpha_{l,t}^{D} \leq \alpha_{l,t}^{P,F} \leq \alpha_{l,t}^{N}$$

so that

THEOREM VI.2 A.

$$\lim_{l,t\to\infty} \alpha_{l,t}^{P,F} = \alpha_\infty .$$

*Remark.* — By the same argument all 16 objects $\alpha_{l,t}^{X,Y}$ have the same limit.

We next want to identify $\lim_{t\to\infty} (-l)\alpha_{l,t}^{P,F}$ as $E_l^P$, the ground state energy of a periodic B. C. Hamiltonian $H_l^P$. The situation is similar to that for the Dirichlet theory discussed in § VIII.1 of [*62*]. The key fact is that the periodic Green's function $G_l^P(x, t; y, s)$ in the strip $(-l/2, l/2) \times \mathbb{R}$ is easily seen to have an eigenfunction expansion:

$$G_l^P(x, t; y, s) = (4l)^{-1} \sum_{n=0}^{\infty} f_n^{(l)}(x) f_n^{(l)}(y) e^{-|t-s|\mu(k_n)}$$

where

$$k_n = \begin{cases} \pi n/l & n = 0, 2, 4, \dots \\ \pi(n+1)/l & n = 1, 3, 5, \dots \end{cases}$$

$\mu(k_n) = (k_n^2 + m_0^2)^{1/2}$ and where

$$f_n^{(l)}(x) = \begin{cases} (2\mu(k_n))^{-1/2} & n = 0 \\ \mu(k_n)^{-1/2} \cos(k_n x) & n = 2, 4, 6, \dots \\ \mu(k_n)^{-1/2} \sin(k_n x) & n = 1, 3, 5, \dots \end{cases}$$

Let $\mathscr{F}^{P}$ be the Fock space, $\Gamma(l_2)$, over $l_2$, let $q_n$ be the corresponding Q-space coordinates and let $h_{0,l}^{P}$ be the diagonal matrix on $l_2$ with *nth* eigenvalue $\mu(k_n)$. Let

$$H_{0,l}^{P} = d\Gamma(h_{0,l}^{P}) \tag{VI.2}$$

and

$$\phi_l^{P}(x) = (2\,l)^{-1/2} \sum_{n=0}^{\infty} f_n^{(l)}(x) q_n . \tag{VI.3}$$

Let

$$H_{I,l}^{P} = \int_{-l/2}^{l/2} :P(\phi_l^{P}(x)): dx$$

where $:\ :$ is Wick ordering with respect to the vacuum $\Omega_0$ in $\mathscr{F}^{P}$. Finally, let

$$H_l^{P} = H_{0,l}^{P} + H_{I,l}^{P} . \tag{VI.4}$$

Then

THEOREM VI.3.

$$\alpha_{l,t}^{P,F} = (lt)^{-1} \ln \langle \Omega_0, e^{-tH_l^{P}} \Omega_0 \rangle .$$

*Sketch of Proof.* — As in the Dirichlet case, one uses the Green's function expansion (VI.1) to prove that $d\mu_{0,l}^{P}$ is a path measure for $H_{0,l}^{P}$ and the theorem is then proven by developing a suitable FKN formula. ■

Let $E_l^{P} \equiv \inf \sigma(H_l^{P})$. By the standard transfer matrix arguments [17], [19] (see also [29], [62]),

$$\lim_{t\to\infty} t^{-1} \ln \langle \Omega_0, e^{-tH_l^{P}} \Omega_0 \rangle = -E_l^{P} .$$

Thus Theorem VI.2 A can be restated:

THEOREM VI.2 B.

$$\lim_{l\to\infty} -E_l^{P}/l = \alpha_\infty .$$

Finally, we wish to restate Theorem VI.2 B in terms of the periodic Hamiltonian $H_V$ introduced by Glimm and Jaffe (see e. g. [18]). Their periodic Hamiltonian which is an operator on the usual free field Fock space differs from ours but in a rather simple way. We recall their definition:

DEFINITION. — Let $\mathscr{F}$ be the Fock space $\Gamma(L^2(\mathbb{R}))$. For given V, let $\mathbb{Z}_V = (2\pi/V)\mathbb{Z}$ and for $k \in \mathbb{R}$, let $k_V$ be the lattice point closest to $k$, i. e. $k_V \in \mathbb{Z}_V$ and $-\pi/V < k - k_V \leq \pi/V$. Let

$$\begin{aligned} H_{0,V} &= \int \mu(k_V) a^*(k) a(k) dk \\ \phi_V(x) &= (4\pi)^{-1/2} \int e^{-ik_V x} \mu(k_V)^{-1/2} \{ a^*(k) + a(-k) \} dk \\ H_{I,V} &= \int_{-V/2}^{V/2} : P(\phi_V) : (x) dx \\ H_V &= H_{0,V} + H_{I,V} \\ E_V &= \inf \operatorname{spec} (H_V) \end{aligned}$$

*Remark.* — In [*18*], in the description of $\phi_V$, the factor $k_V x$ in $e^{ik_V x}$ is written as $kx$ but $k_V x$ is clearly intended (see [*17*]).

Let $\mathscr{H}$ be the subspace of $L^2(\mathbb{R})$ consisting of functions piecewise constant in such a way that $f(k) = f(k_V)$. Then since $L^2 = \mathscr{H} \oplus \mathscr{H}^\perp$ we can write

$$\mathscr{F} = \Gamma(\mathscr{H}) \otimes \Gamma(\mathscr{H}^\perp) \tag{VI.5}$$

(see e. g. § I.1 of [*62*]). Since $\mathscr{H} \cong l_2$ we have a natural isomorphism

$$\Gamma(\mathscr{H}) \simeq \Gamma(l_2) . \tag{VI.6}$$

THEOREM VI.4. — Let $V = l$. Then:

(*a*) Under the decomposition (VI.5):

$$H_V = (H_V \upharpoonright \Gamma(\mathscr{H})) \otimes I + I \otimes (H_{0,V} \upharpoonright \Gamma(\mathscr{H}^\perp)) .$$

(*b*) Under the isomorphism (VI.6)

$$H_V \upharpoonright \Gamma(\mathscr{H}) = H_l^P .$$

In particular,

$$E_V = E_l^P .$$

*Proof.* — (*a*) is Theorem 1.3.4 of [*18*]

(*b*) holds by direct computation. ■

Thus $H_V$ and $H_l^P$ are related by $H_V = H_l^P \otimes I + I \otimes d\Gamma(B)$ with an explicit $B \geq m_0$. Not only do we have $E_V = E_l^P$ but $\phi$ bounds for $H_V$ and $H_l^P$ are easily seen to be equivalent.

On account of $E_V = E_l^P$ we have:

THEOREM VI.2 C. — If $E_V$ is the energy of the Glimm-Jaffe periodic Hamiltonian [*17*], [*18*], then

$$\lim_{V \to \infty} E_V/V = \alpha_\infty .$$

## VI.2. **Traces and periodic states.**

In this subsection we shall develop a transfer matrix formalism for periodic states (related to that of Hoegh-Krohn [32]) in which the main formula reads:

$$\int \exp(-U^{P}_{l\times t})d\mu^{P}_{0,l,t} = \frac{\mathrm{Tr}(\exp(-t\tilde{H}_{l,t}))}{\mathrm{Tr}(\exp(-tH^{P}_{0,l}))} \tag{VI.7}$$

where $\tilde{H}_{l,t} = H^{P}_{0,l} + W_t$ with $W_t$ differing from $H^{P}_{I,l}$ by Wick reordering terms (which are $t$-dependent). (VI.7) is similar to formulas in statistical mechanics in the sense that it is well known that periodic B. C. in the direction of transfer lead to traces but dissimilar in the occurence of normalization factor in the denominator. This occurs because the free measure is not counting measure as in classical statistical mechanics but has periodic couplings built into it and into its normalization. In applications this normalization factor is something of a nuisance but not hard to control since we can compute it explicitly.

Our proof of (VI.7) begins with:

LEMMA VI.5. — (*a*) For any $l < \infty$, $H^{P}_{0,l}$ has compact resolvent and $\mathrm{Tr}\,(e^{-tH^{P}_{0,l}}) < \infty$ for any $t > 0$.

(*b*) For any V obeying $\frac{1}{2}H^{P}_{0,l} + V \geq$ const., $H \equiv H^{P}_{0,l} + V$ has compact resolvent and $\mathrm{Tr}\,(e^{-tH}) < \infty$ for all $t > 0$.

*Proof.* — (*a*) The eigenvalues of $H^{P}_{0,l}$ are $E_{n_1}, \ldots, {}_{n_m} = \sum_{i=1}^{m} \mu(k_{n_i})$ for each $m$ tuple $n_1, \ldots, n_m$ with $n_1 \leq n_2 \leq \ldots \leq n_m$. Since the eigenvectors are complete and only finitely many E are less than any preassigned constant C, $H^{P}_{0,l}$ has compact resolvent. To prove $\mathrm{Tr}\,(e^{-tH_{0,l}}) < \infty$ we must show $\Sigma e^{-tE_n} < \infty$ but

$$\Sigma e^{-tE_{n_1,\ldots,n_m}} = \prod_{n=0}^{\infty} (1 + e^{-\mu(k_n)t} + e^{-2\mu(k_n)t} + \ldots)$$

$$= \prod_{n=0}^{\infty} (1 - e^{-\mu(k_n)t})^{-1}$$

so since $\Sigma e^{-\mu(k_n)} < \infty$, the product converges (all the above manipulations are justified by this convergence). For later reference we note the final result of the above:

$$\mathrm{Tr}\,(e^{-tH^{P}_{0,l}}) = \prod_{n=-\infty}^{\infty} \left(1 - \exp\left[-\left(m_0^2 + \left(\frac{2\pi n}{l}\right)^2\right)^{1/2} t\right]\right)^{-1} \tag{VI.8}$$

(where we have taken the degeneracy of $\mu(k_n)$ into account by taking the product from $-\infty$ to $\infty$).

(*b*) This is a standard perturbation argument based on the minimax principle (c. f. [5]). Since $H \geq \frac{1}{2} H^P_{0,l} + C$ we have, e. g.

$$\mathrm{Tr}\,(e^{-tH}) \leq e^{-tC}\mathrm{Tr}\,(e^{-\frac{1}{2}tH^P_{0,l}}). \quad \blacksquare$$

Next we identify the periodic Green's function using traces:

LEMMA VI.6. — Let $H^P_{0,l}$, $\phi^P_l$ be given by (VI.2, 3). Let $G^P_{l,t}(x, s; y, \sigma)$ be the periodic Green's function for $(-l/2, l/2) \times (-t/2, t/2)$. Then as distributions:

$$G^P_{l,t}(x, s; y, \sigma) = \frac{\mathrm{Tr}\,(e^{-|s-\sigma|H^P_{0,l}}\phi^P_l(x)e^{-(t-|s-\sigma|)H^P_{0,l}}\phi^P_l(y))}{\mathrm{Tr}\,(e^{-tH^P_{0,l}})} \tag{VI.9}$$

*Proof.* — By the expansions (VI.1) and (VI.3) and the images formula for $G^P_{l,t}$ in terms of $G^P_l$, both sides of (VI.9) have expansions in terms of the functions $f^{(l)}_n$. We need only show that the coefficients are equal. This is the equality of a periodic Green's function in one dimension and a one dimensional trace formula. One proves such a formula by a similar method to that used for Dirichlet B. C. in one dimension (see [*29*], § II.6 or [*62*], § VII.2). $\blacksquare$

THEOREM VI.7. — Let $H^P_{I,l,t} = \int_{-l/2}^{l/2} :P(\phi(x)):_{l,t}dx$ where $:\ :_{l,t}$ is Wick ordering with respect to $G^P_{l,t}$ (as opposed to $G^P_{l,\infty}$ as in (VI.4)). And let $H^{HP}_{I,l}$ be the same object but with free ($G_0$) Wick ordering. Then:

$$\int e^{-U^P_{l,t}}d\mu^P_{0,l,t} = \mathrm{Tr}\,(\exp[-t(H^P_{0,l} + H^P_{I,l,t})])/\mathrm{Norm}$$

$$\int e^{-U^F_{l,t}}d\mu^P_{0,l,t} = \mathrm{Tr}\,(\exp[-t(H^P_{0,l} + H^{HP}_{I,l})])/\mathrm{Norm}$$

with $\mathrm{Norm} = \mathrm{Tr}\,(\exp(-tH^P_{0,l}))$.

*Proof.* — By the usual (Trotter product formula) proof of Feynman-Kac formulae [*40*], we need only prove that

$$\int f_1(\phi(x, t_1)) \ldots f_n(\phi(x, t_n))d\mu^P_{0,l,t}$$
$$= \mathrm{Tr}\,(e^{-s_0H^P_{0,l}}f_1(\phi^P(x))e^{-s_1H^P_{0,l}} \ldots f(\phi^P(x))e^{-s_nH^P_{0,l}})/\mathrm{Norm} \tag{VI.10}$$

with $s_0 = t_1 + t/2$, $s_1 = t_2 - t_1, \ldots, s_{n-1} = t_n - t_{n-1}$, $s_n = t/2 - t_n$. By Lemma VI.6, (VI.10) holds in the special case where $n = 2$ and $f_1(y) = f_2(y) = y$. Thus (VI.10) holds when the $f_i$'s are polynomials since both sides have an expansion as a sun over pairings (Wick's theorem). The case of general $f$ now follows by a standard limiting argument. $\blacksquare$

Letting $H_l^{HP} = H_{0,l}^{P} + H_{I,l}^{HP}$, we note that Nelson's symmetry does not take the form $\mathrm{Tr}\,(e^{-tH_l^{HP}}) = \mathrm{Tr}\,(e^{-lH_t^{HP}})$ but rather:

$$\mathrm{Tr}\,(e^{-lH_t^{HP}}) = \frac{\mathrm{Tr}\,(e^{-lH_{0,t}^{P}})\mathrm{Tr}\,(e^{-tH_l^{HP}})}{\mathrm{Tr}\,(e^{-tH_{0,l}^{P}})} \tag{VI.11}$$

As an illustration of the use of the transfer matrix formalism and the Nelson symmetry (VI.11) in particular, we give an alternate proof that $\alpha_{l,t}^{P}$ converges (although this method does not prove the limit is $\alpha_\infty$) and we will control the periodic surface pressure. Actually we will only prove that the half-Periodic pressure converges, but this is sufficient since making the dependence on the coefficients of $P(X) = a_{2n}X^{2n} + \ldots a_0$, explicit:

$$\alpha_{l,t}^{P}(a_{2n}, \ldots, a_0) = \alpha_{l,t}^{HP}(a_{2n}, a_{2n-1}, a_{2n-2} + c_{l,t}^{2n-2}, \ldots, a_0 + c_{l,t}^{0})$$

on account of Wick reordering ([29], § V). The coefficients $c_{l,t}^{k} \to 0$ exponentially as $l, t \to \infty$ (see § III.3). Using the convexity of $\alpha_{l,t}^{HP}$ in the $a_m$ it is easy to prove that convergence of the $\alpha_{l,t}^{HP}$ implies convergence of the $\alpha_{l,t}^{P}$ and that the difference is $0(\exp(-\min(cl, ct)))$.

Thus we shall prove the existence of the iterated limit:

$$\lim_{l\to\infty}\lim_{t\to\infty}(lt)^{-1}\ln\int e^{-U_{l,t}^{F}}d\mu_{0,l,t}^{P}. \tag{VI.12}$$

LEMMA VI.8. — If A is a self-adjoint operator with $e^{-tA}$ trace class for each $t > 0$, then $t^{-1}\ln[\mathrm{tr}\,(e^{-tA})]$ is monotone decreasing and

$$\lim_{t\to\infty} t^{-1}\ln[\mathrm{tr}\,(e^{-tA})] = -\inf \mathrm{spec}\,(A)\,.$$

*Proof.* — Since $e^{-A}$ is trace class, A has compact resolvent and eigenvalues $E_0 \le E_1 \le \ldots$ with a corresponding complete set of eigenvectors. Thus

$$\mathrm{Tr}\,(e^{-t(A-E_0)}) = \sum_{n=0}^{\infty} e^{-t(E_n-E_0)}$$

converges monotonically downwards to $m \equiv$ multiplicity of $E_0$. Thus

$$t^{-1}\ln[\mathrm{tr}\,(e^{-tA})] = -E_0 + t^{-1}\ln[\mathrm{tr}\,(e^{-t(A-E_0)})]$$

converges monotonically downwards to $-E_0$. ∎

In the first place, Lemma VI.8 and the transfer matrix formalism, Theorem VI.7, imply that

$$\lim_{t\to\infty}(lt)^{-1}\ln\alpha_{l,t}^{HP} = -l^{-1}E_l^{HP} \tag{VI.13}$$

where $E_l^{HP} = \inf\mathrm{spec}\,(H_l^{HP})$. Now we use Nelson's symmetry (VI.11) and Lemma VI.8. $(lt)^{-1}\ln$ (LHS of (VI.11)) is monotone decreasing in $l$, thus

$\lim_{t\to\infty} (lt)^{-1} \ln$ (RHS of (VI.11)) is monotone decreasing in $l$. By Lemma VI.8 and (VI.8) this quantity is

$$a_l \equiv -l^{-1}E_l^{HP} - (2\pi l)^{-1} \int_{-\infty}^{\infty} \ln(1 - e^{-\mu(k)l})dk. \qquad \text{(VI.14)}$$

Thus $\lim_{l\to\infty} a_l$ exists. But since

$$0 \leq -\ln(1 - e^{-\mu(k)l}) \leq e^{-\mu(k)l}$$

the second term on the right of (VI.14) has a limit (zero, in fact) as $l \to \infty$ so $\lim_{l\to\infty} -l^{-1}E_l^{HP}$ exists and thus the limit (VI.12) exists. ■

The above proof actually shows:

THEOREM VI.9.

$$\lim_{l\to\infty} E_l^{HP} + \alpha_\infty l \equiv -\beta_\infty^{HP} \qquad \text{exists, and} \qquad \beta_\infty^{HP} \geq 0.$$

*Remark.* — Since $|E_l^{HP} - E_l^P| = 0(e^{-cl})$, $E_l^P + \alpha_\infty l$ also converges to $-\beta_\infty^{HP}$.

*Proof.* — The proof of Lemma VI.8 actually shows that

$$\ln(\text{tr}(e^{-tA})) + t \inf \text{spec}(A)$$

is monotone decreasing in $t$ and non-negative. Thus $t^{-1}\text{Tr}(e^{-lH_t^{HP}}) + t^{-1}lE_t^{HP}$ is monotone decreasing in $l$ and non-negative. Using (VI.11) and taking $t$ to infinity, we conclude that:

$$-la_l + l\alpha_\infty = (E_l^{HP} + \alpha_\infty l) + 2\pi \int_{-\infty}^{\infty} \ln(1 - e^{-\mu(k)l})dk$$

is monotone increasing in $l$ and positive. Since the second term converges to zero, the limit $-\beta_\infty^{HP}$ exists and is non-positive. ■

This result raises two natural questions:

1. Is the approach of $E_l^{HP} + \alpha_\infty l + \beta_\infty^{HP}$ to zero exponential as it is in the small coupling free B. C. case [37]?

2. Since $\beta_\infty$ represents a « surface energy » and tori have no surfaces, is $\beta_\infty^{HP} = 0$? In any event since $\beta_\infty^{\text{Free B.C.}} < 0$, we have $\beta_\infty^P \neq \beta_\infty^{\text{Free}}$ so that (as is to be expected) the pressure theorem does not continue to surface pressures.

## VII. COUPLING CONSTANT DEPENDENCE OF THE PRESSURE

The pressure $\alpha_\infty$ is a function of a coefficients $a_{2n}, a_{2n-1}, \ldots, a_0$ of $P(X) = \sum_{j=0}^{2n} a_j X^j$ and of the « bare » mass $m_0$. We shall write $\alpha_\infty(a_{2n}, \ldots, a_0; m_0)$,

$\alpha_\infty(P; m_0)$, or $\alpha_\infty(P)$ (when $m_0$ is fixed) to indicate this dependence. There are several general properties of $\alpha_\infty$ that are easy to establish:

THEOREM VII.1 [*29*]. — (*a*) $\alpha_\infty(P + b_0) = \alpha_\infty(P) - b_0$
(*b*) $\alpha_\infty(tP + (1-t)Q) \leq t\alpha_\infty(P) + (1-t)\alpha_\infty(Q)$ for $0 \leq t \leq 1$.
(*c*) If $P(0) = 0$, then $\alpha_\infty(P) \geq 0$.

*Proof.* — (*a*) – (*c*) can be established for finite volume and then hold in the limit. (*a*) is direct, (*b*) follows by Hölder's inequality and (*c*) comes from the fact that

$$\int : P(\phi(x)) : d\mu_0 = 0$$

if $P(0) = 0$ and from Jensen's inequality:

$$\int \exp(-U_\Lambda) d\mu_0 \geq \exp\left(-\int U_\Lambda d\mu_0\right). \qquad \blacksquare$$

We begin by establishing three covariance properties of $\alpha_\infty$ under
(*i*) scaling,
(*ii*) translation of the field, and,
(*iii*) Wick reordering.
All have been used before in the study of $P(\phi)_2$ theories: (*i*) first by Glimm, Jaffe and Spencer [*24*], (*ii*) by Spencer [*66*] and (*iii*) by Baumel [*2*] (and subsequently by the present authors [*29*] and by Glimm-Jaffe [*22*]).

We shall then obtain some information about $\alpha_\infty(P; m_0)$ for fixed $m_0$ as some coefficients of P get large. In § VII.4, we obtain bounds on the behavior of $\alpha_\infty(P)$ as the subdominant couplings $a_j (j < 2n)$ goes to infinity with $a_{2n}$ held fixed and strictly positive. These bounds are useful in obtaining best constants in $: \phi^j :$ bounds [*11*, part II]. In § VII.3, we obtain a lower bound on $\alpha_\infty(\lambda P)$ as $\lambda \to \infty$ of the form

$$\alpha_\infty(\lambda P) \geq (\text{const.})\lambda(\ln \lambda)^n$$

where $n = \deg P/2$. Since an upper bound of the same form is already known [*28*], this determines the qualitative nature of the large $\lambda$ behavior.

In the discussions of this section, it is convenient to use the fact that $\alpha_\infty$ is independent of boundary conditions and also the convergence of the lattice approximation in squares with any boundary conditions. We use these results freely even though some of them are not established until § VIII and § IX.

## VII.1. Translation and scaling covariance.

In many ways, it is the polynomial $P(X) + \frac{1}{2} m_0^2 X^2$ which enters naturally in studying the pressure. This can be seen in the periodic lattice approximation where the partition function is (see § IX.1):

$$Z_{\Lambda,\delta}(P) = \Xi_{\Lambda,\delta}(P)/\Xi_{\Lambda,\delta}(0) \qquad \text{(VII.1 } a\text{)}$$

with

$$\Xi_{\Lambda,\delta}(\mathrm{P}) = \int \exp - (\mathrm{A}(\mathrm{P})) \prod_{n\delta\in\Lambda_\delta} dq_n \tag{VII.1 b}$$

$$\mathrm{A}(\mathrm{P}\,;m_0) = \delta^2\Big[\sum_{n\in\Lambda_\delta} :\mathrm{P}(q_n): + \frac{1}{2}m_0^2 :q_n^2:\Big] + \frac{1}{4}\sum_{\substack{|n-m|_{\mathrm{P}}=1\\ n,m\in\Lambda_\delta}} (q_n - q_m)^2 \tag{VII.1 c}$$

where $\Lambda$ is a square whose side is a multiple of $\delta$.

THEOREM VII.2 (Essentially in Spencer [66]). — Let $c$ be any real constant. Let P be a semibounded polynomial and define Q by

$$\mathrm{Q}(\mathrm{X}) + \frac{1}{2}m_0^2\mathrm{X}^2 = \mathrm{P}(\mathrm{X}+c) + \frac{1}{2}m_0^2(\mathrm{X}+c)^2,$$

i. e.,

$$\mathrm{Q}(\mathrm{X}) = \mathrm{P}(\mathrm{X}+c) + m_0^2c\mathrm{X} + \frac{1}{2}m_0^2c^2.$$

Then

$$\alpha_\infty(\mathrm{Q}) = \alpha_\infty(\mathrm{P}) \tag{VII.2}$$

We first note the lemma:

LEMMA VII.2 A. — Define the linear maps $\mathrm{T}_c$ and $\mathrm{W}_b$ on the space of polynomials by

$$\mathrm{T}_c\mathrm{X}^m = \sum_{j=0}^{m}\binom{m}{j}c^j\mathrm{X}^{m-j}$$

$$\mathrm{W}_b\mathrm{X}^m = \sum_{j=0}^{[m/2]}\left\{\begin{matrix} m \\ j\end{matrix}\right\}(-b^2)^j\mathrm{X}^{m-2j}$$

where $\left\{\begin{matrix} m \\ j\end{matrix}\right\} = \dfrac{m!}{2^j j!(m-2j)!}$. Then $\mathrm{T}_c\mathrm{W}_b = \mathrm{W}_b\mathrm{T}_c$.

*Remark.* — $\mathrm{T}_c$ is of course translation and $\mathrm{W}_b$ is Wick ordering or reordering. In particular, this lemma implies that $\mathrm{H}_n$ is the $n$th Hermite function)

$$\mathrm{H}_n(x+c) = \sum_{m=0}^{n}\binom{n}{m}\mathrm{H}_m(x)c^{n-m}. \tag{VII.3}$$

*Proof.* — This follows by direct calculation done most elegantly by using the generating function $f_\alpha(x) = \sum_{n=0}^{\infty}\alpha^n\mathrm{H}_n(x)/n! = \exp\left(\alpha x - \frac{1}{2}\alpha^2\right)$. ■

*Proof of Theorem* VII.2. — By (VII.3) $: P(q + c) : = : P : (q + c)$; using the fact that $[(q_n + c) - (q_m + c)]^2 = (q_n - q_m)^2$ we see that

$$A(P)(q_n + c) = A(Q)(q_n).$$

It follows from VII.1 that

$$Z_{\Lambda,\delta}(P) = Z_{\Lambda,\delta}(Q).$$

Taking $\delta \to 0$, taking logarithms, and then taking $\Lambda$ to infinity, we obtain (VII.2). ■

*Remark.* — The proof shows that for finite $\Lambda$, $\alpha_\Lambda^P(Q ; m_0) = \alpha_\Lambda^P(P ; m_0)$. It is fairly evident that the finite volume result does not hold for Dirichlet or free pressures. It does however, also hold for Neumann pressures, i. e.

$$\alpha_\Lambda^N(Q) = \alpha_\Lambda^N(P)$$

since the gradient part of the Neumann action is of the form $\Sigma(q_n - q_m)^2$ for a suitable sum (see § IX.1).

THEOREM VII.3 (Essentially in Glimm, Jaffe, Spencer [*24*]). — For any $\lambda$ positive and any semibounded polynomial P:

$$\alpha_\infty(\lambda^2 P ; \lambda m_0) = \lambda^2 \alpha_\infty(P ; m_0). \qquad \text{(VII.4)}$$

*Proof.* — Making the $m_0$ dependent of the free Euclidean field with free B. C. explicit, we have that

$$\begin{aligned} \langle \phi(\lambda^{-1}x ; \lambda m_0)\phi(\lambda^{-1}y ; \lambda m_0) \rangle_{\lambda m_0} &= (2\pi)^{-2} \int (p^2 + \lambda^2 m_0^2)^{-1} \exp(i\lambda^{-1}p \cdot (x - y)) d^2p \\ &= \langle \phi(x ; m_0)\phi(y ; m_0) \rangle_{m_0}. \end{aligned}$$

It follows that $\phi(\lambda^{-1}x ; \lambda m_0)$ and $\phi(x ; m_0)$ are « isomorphic » Gaussian processes. Thus

$$\begin{aligned} \int \exp\left[ - \int_{x\in\Lambda} : P(\phi(x ; m_0)) : d^2x \right] d\mu_{m_0} &= \exp\left[ - \int_{x\in\Lambda} : P(\phi(\lambda^{-1}x ; \lambda m_0) : d^2x \right] d\mu_{\lambda m_0} \\ &= \exp\left[ - \lambda^2 \int_{y\in\lambda^{-1}\Lambda} : P(\phi(y ; \lambda m_0)) : d^2y \right] d\mu_{\lambda m_0}. \end{aligned}$$

(VII.4) follows. ■

Theorems VII.1 and VII.3 allow us to obtain some information about the $m_0$ dependence of $\alpha_\infty(P ; m_0)$ for fixed P.

THEOREM VII.4. — Let P be fixed and normalized ($P(0) = 0$). Then $m_0^{-2}\alpha_\infty(P ; m_0)$ is monotone decreasing as a function of $m_0$ and convex as a function of $m_0^2$.

*Proof.* — By scaling covariance:

$$m_0^{-2}\alpha_\infty(\mathrm{P}\,;m_0) = \alpha_\infty(m_0^{-2}\mathrm{P}\,;1). \qquad \text{(VII.5)}$$

Let $f(\lambda) = \alpha_\infty(\lambda\mathrm{P}\,;1)$ and $g(\lambda) = f(\lambda^{-1})$. Then $f$ is convex and since P is normalized $f'(0) = 0$ (see [*28*]) so $f''(\lambda)$, $f'(\lambda) \geq 0$. It follows that $g'(\lambda) = -\lambda^{-2}f'(\lambda) \leq 0$ and that $g''(\lambda) = \lambda^{-4}f''(\lambda^{-1}) + 2\lambda^{-3}f'(\lambda^{-1}) \geq 0$. ■

We know that $f(\lambda) \sim \lambda^2$ for $\lambda$ small and we shall prove in § VII.3 that $f(\lambda) \sim \lambda(\ln\lambda)^n\left(n = \frac{1}{2}\deg \mathrm{P}\right)$. Thus for $m_0$ large, $\alpha_\infty(\mathrm{P}\,;m_0) \sim m_0^{-2}$ and for $m_0$ small $\alpha_\infty(\mathrm{P}\,;m_0) \sim (-\ln m_0)^n$. This suggests that it should be possible to improve the monotonicity of $m_0^{-2}\alpha_\infty(\mathrm{P}\,;m_0)$ to monotonicity of $\alpha_\infty(\mathrm{P}\,;m_0)$. We in fact, know this:

THEOREM VII.5. — (*a*) For fixed P, $\alpha_\infty(\mathrm{P}\,;m_0)$ is monotone decreasing in $m_0$.

(*b*) For fixed $m_0$ and P, $\lambda^{-1}\alpha_\infty(\lambda\mathrm{P}\,;m_0)$ is monotone increasing in $\lambda$.

*Proof.* — (*b*) follows from (*a*) by scaling (as in Theorem VII.10) so we need only prove (*a*). To prove (*a*) suppose that $m_0 < m_1$. Then for any $p$, $(p^2 + m_1^2)^{-1} < (p^2 + m_0^2)^{-1}$. Thus if $\mathrm{G}_0(x, y\,;m^2)$ is the covariance for the free field of mass $m^2$, $\mathrm{G}_0(f, f;\, m_1^2) \leq \mathrm{G}_0(f, f;\, m_0^2)$. The theory of conditioning thus applies (see § III.2) and implies that

$$\alpha_\Lambda(\mathrm{P}\,;m_1) \leq \alpha_\Lambda(\mathrm{P}\,;m_0). \quad ■$$

*Remarks.* — 1. That conditioning applies under change in bare mass has been noted independently by Fröhlich [*12*].

2. Alternatively, (*b*) follows from convexity in $\lambda$ and then (*a*) from (*b*) via scaling.

### VII.2. **Wick reordering.**

The imput mass $m_0$ enters in the free measure $d\mu_0$ but also in the meaning of $:\ :$ in $:\mathrm{P}(\phi(x)):$. One expects that a change in $m_0$ can therefore be compensated by changes in P that take into account both effects of $m_0$. On the level of DLR equations, we did this in [*29*] and simultaneous to our work Baumel [*2*] discussed the same question for $\alpha_\infty^{\mathrm{P}}$ (which he did not know at the time was the same as $\alpha_\infty$!). To derive Baumel's result we must as a preliminary compute $\alpha_\infty(\mathrm{P}\,;m_0)$ when $\mathrm{P} = a\mathrm{X}^2$. From a Fock space point of view, this computation is found in [*9*], [*49*], [*52*] but we give here a purely Euclidean computation exploiting the fact $\alpha_\infty$ can be computed with any boundary conditions.

THEOREM VII.6.

$$\alpha_\infty(a\mathrm{X}^2\,;m_0) = \frac{a}{4\pi}\left[\frac{m_0^2 + 2a}{2a}\ln\left(1 + \frac{2a}{m_0^2}\right) - 1\right]. \qquad \text{(VII.6)}$$

*Remark.* — Note explicity that $\alpha_\infty \sim 0(a^2)$ for a small and $0(a \ln a)$ for a large.

*Proof.* — We compute $\alpha_\Lambda^P$ with Periodic B. C. with

$$\Lambda = (-l/2, l/2) \times (-l/2, l/2).$$

For each $n \in \mathbb{Z}$, let $q_n$ be a *complex* valued Gaussian random variable with mean zero and joint covariance given by

$$\langle q_n q_m \rangle = \delta_{n;-m}$$
$$q_n^* = q_{-n}.$$

Let $\mathbb{Z}_+^2 = \{ n = \langle n_1, n_2 \rangle \mid n_1 > 0 \text{ or } n_1 = 0;\ n_2 > 0 \}$. For $n \in \mathbb{Z}_+^2$, let $x_n = \frac{1}{2}(q_n + q_{-n})$; $y_n = \frac{1}{2i}(q_n - q_{-n})$. Let $y_0 = 0$ and $x_0 = q_0/\sqrt{2}$. Thus, the $x$'s and $y$'s are independent Gaussian random variables and

$$\langle x_n^2 \rangle = \frac{1}{2};\ \langle y_n^2 \rangle = \frac{1}{2}\ (n \neq 0).$$

Since the covariance $G_\Lambda^P(x, y)$ has the expansion

$$G_\Lambda^P(x, y) = l^{-2} \sum_{n \in \mathbb{Z}^2} \mu\left(\frac{2\pi n}{l}\right)^{-2} \exp\left(2\pi i\, n \cdot (x - y)\right)$$

the field $\phi(x)$ can be represented in terms of the Gaussian process defined above via

$$\phi(x) = l^{-1} \sum_{n \in \mathbb{Z}^2} \mu\left(\frac{2\pi n}{l}\right)^{-1} \exp\left(2\pi i\, x \cdot n/l\right) q_n .$$

From this realization we see that

$$\int_\Lambda : \phi^2(x) : = \sum_{n \in \mathbb{Z}_+^2} \mu\left(\frac{2\pi n}{l}\right)^{-2} \left[2(: x_n^2 : + : y_n^2 :)\right].$$

Now, since $x_n$ and $y_n$ here variance 1/2, $: x_n^2 : = x_n^2 - 1/2$. Moreover, since the $x_n$'s and $y$'s are independent $\int d\mu_0 e^{-a\int_\Lambda :\phi^2:}$ is a product of integrals of the form

$$\pi^{-1/2} \int \exp\left(-\lambda : x^2 :\right) \exp\left(-x^2\right) dx = (1 + \lambda)^{-1/2} e^{1/2\lambda}.$$

We conclude that

$$\int \exp\left(-a \int_\Lambda : \phi^2(x) : d^2x\right)$$
$$= \sum_{n \in \mathbb{Z}^2} \left(1 + 2a\mu\left(\frac{2\pi n}{l}\right)^{-2}\right)^{1/2} \exp\left(a\mu\left(\frac{2\pi n}{l}\right)^{-2}\right)$$

so

$$\alpha_{l\times l}(aX^2\,;m_0) = l^{-2}\sum_{n\in\mathbb{Z}^2}\left[a\mu\left(\frac{2\pi n}{l}\right)^{-2} - \frac{1}{2}\ln\left(1 + 2a\mu^{-2}\right)\right].$$

As $l \to \infty$, the sum converges to an integral and so

$$\begin{aligned}\alpha_\infty(ax^2\,;m_0) &= (2\pi)^{-2}\int \frac{a}{k^2+m_0^2} - \frac{1}{2}\ln\left[1 + \frac{2a}{k^2+m_0^2}\right]d^2k \\ &= \frac{a}{4\pi}\int_{m_0^2/2a}^{\infty}\left[\frac{1}{x} - \ln\left[1+\frac{1}{x}\right]\right]dx.\end{aligned}$$

Now the antiderivative of $x^{-1} - \ln(1+x^{-1})$ is $-x\ln(1+x^{-1}) - \ln(1+x^{-1})$ so

$$\alpha_\infty(ax^2, m_0) = \frac{a}{4\pi}\left[\frac{m_0^2+2a}{2a}\ln\left(\frac{m_0^2+2a}{m_0^2}\right) - 1\right]. \quad \blacksquare$$

The main Wick reordering theorem is:

THEOREM VII.7 (essentially due to Baumel [2]). — Let $P(X) = \sum_{j=0}^{N} a_j X^j$ be a semibounded polynomial and let $m$, $\tilde{m}$ be given positive numbers. Let

$$d = -\frac{1}{4\pi}\ln\left(\frac{\tilde{m}^2}{m^2}\right) \tag{VII.7}$$

$$\tilde{P}(X) = \sum_{j=0}^{N} a_j \sum_{n=0}^{[j/2]} \frac{j!\,d^n}{2^n n!(j-2n)!}X^{j-2n} + \frac{1}{2}(m^2 - \tilde{m}^2)X^2. \tag{VII.8}$$

Then

$$\alpha_\infty(P, m) = \alpha_\infty(\tilde{P}, \tilde{m}) + f_\infty \tag{VII.9}$$

where

$$f_\infty = \frac{m^2 - \tilde{m}^2}{8\pi} + \frac{m^2}{8\pi}\ln\frac{\tilde{m}^2}{m^2}. \tag{VII.10}$$

*Remark.* — Thus, by Theorem VII.1, $\alpha_\infty(P, m) = \alpha_\infty(\hat{P}, \tilde{m})$ with $\hat{P} = P - f_\infty$. Note that when we quoted this result in [29] we made a sign orror.

*Proof.* — Let $\alpha_l^P$ be the periodic pressure in $(-l/2, l/2)\times(-l/2, l/2)$. Let

$$d_l = l^{-2}\sum_{n\in\mathbb{Z}^2}\left\{\left[\left(\frac{2\pi n}{l}\right)^2 + \tilde{m}^2\right]^{-1} - \left[\left(\frac{2\pi n}{l}\right)^2 + m^2\right]^{-1}\right\}$$

and let $\tilde{P}_l$ be given by (VII.8) with $d$ replaced by $d_l$. We shall first prove that

$$\alpha_l^P(P, m) = \alpha_l^P(\tilde{P}, \tilde{m}) + f_l \tag{VII.11}$$

for some constant $f_l$ which depends on $m$ and $\tilde{m}$ but not on P.

For pass to the lattice approximation and use (VII.1). Then

$$A(P, m) = A(\tilde{P}_{l,\delta}, \tilde{m})$$

where $\tilde{P}_{l,\delta}$ is defined by (VII.8) with $d$ replaced by $d_{l,\delta}$ and $d_{l,\delta}$ is the difference of the finite covariances of the periodic lattice fields with the masses $m$, $\tilde{m}$ (evaluated at a point). Making the $m$ dependence of $\Xi(P)$ and $Z(P)$ explicit we conclude that

$$\Xi_{\Lambda,\delta}(P, m) = \Xi_{\Lambda,\delta}(\tilde{P}_{l,\delta}, \tilde{m})$$
$$Z_{\Lambda,\delta}(P, m) = Z_{\Lambda,\delta}(\tilde{P}_{l,\delta}, \tilde{m}) \exp(|\Lambda| f_{l,\delta})$$

where $f_{l,\delta}$ is P independent. (VII.11) follows by taking $\delta$ to zero and this argument: note that $d_{l,\delta} \to d$ and that by convexity of $\ln Z_{\Lambda,\delta}$ in coupling constants the $\ln Z_{\Lambda,\delta}$ are locally Lipschitz uniformly in $\delta$ in coupling constant (see § VIII.1). Since

$$Z_{\Lambda,\delta}(P, m) \to Z_{\Lambda}(P, m) \quad \text{and} \quad Z_{\Lambda,\delta}(\tilde{P}_{l,\delta}, \tilde{m}) \to Z_{\Lambda}(\tilde{P}_l, \tilde{m}),$$

$f_{l,\delta}$ has a limit $f_l$. This establishes (VII.11).

Now take $l$ to $\infty$ and mimic the argument above. (VII.9) results where $f_\infty$ is a still to be determined constant. But taking $P = 0$ in (VII.9) we see that

$$f_\infty = -\alpha_\infty\left(\frac{1}{2}[m^2 - \tilde{m}^2]X^2, \tilde{m}\right)$$

and (VII.10) then follows from (VII.6). ■

### VII.3. Behavior of $\alpha_\infty(\lambda)$ as $\lambda \to \infty$.

Fix P and $m_0$. Let $\alpha_\infty(\lambda)$ be $\alpha_\infty(\lambda P, m_0)$. In [*28*], we showed using methods of Nelson [*42*] that for $\lambda$ large:

$$\alpha_\infty(\lambda) \leq C\lambda(\ln \lambda)^n; \qquad \deg P = 2n. \tag{VII.12}$$

We conjectured at the time that there should be a lower bound

$$\alpha_\infty(\lambda) \geq C'\lambda(\ln \lambda)^n \tag{VII.13}$$

although all we could prove was that $\alpha_\infty(\lambda)/\lambda \to \infty$ (using an argument from [*64*]).

In the spring of 1973, R. Baumel (unpublished) communicated to us an idea for proving a bound of the form (VII.13) in the case $n = 2$ (deg $P = 4$). Baumel's basic idea was to perform a variational calculation using a Gaussian « ground state vector » with mass and mean as parameters and to exploit covariance properties. In addition his method required control over an object related to the pressure. Baumel was unable to control this object, nor can we. Subsequently, Baumel [2] and we independently found a partially alternate proof of (VII.13). For the reader's convenience we now sketch this proof of Baumel's as an illustration of the use of covariance properties.

It will turn out to be easier to prove (VII.13) when $n$ is odd than when it is even. To describe the idea, let $P(X) = X^{2n}$ with $n$ odd. Change $m_0$ the bare mass to $\tilde{m}$ by Wick reordering and let $\tilde{P}(X) = X^{2n} + \ldots + \tilde{a}$ be the corresponding polynomial of Theorem VII.7. Then by Theorems VII.1 and VII.7:

$$\alpha_\infty(\lambda) = \alpha_\infty(\lambda X^{2n}; m_0) = \alpha_\infty(\lambda\tilde{P}, \tilde{m}) + f_\infty \geq -\lambda\tilde{a} + f_\infty .$$

Putting in the explicit values for $\tilde{a}$ and $f_\infty$ we see that for any $\lambda$, $\tilde{m}$

$$\alpha_\infty(\lambda) \geq -\frac{\lambda}{(4\pi)^n}\frac{(2n)!}{2^n n!}\left(-\ln\left(\frac{\tilde{m}^2}{m_0^2}\right)\right)^n + \frac{m_0^2 - \tilde{m}^2}{8\pi} + \frac{m_0^2}{8\pi}\ln\frac{\tilde{m}^2}{m_0^2}.$$

Since $-(-1)^n = 1$ ($n$ is odd!), we see upon letting $\tilde{m}^2 = \lambda m_0^2$ that

$$\alpha_\infty(\lambda) \geq C_n(\lambda(\ln\lambda)^n - \lambda)$$

for large $\lambda$ and (VII.13) follows. We thus have:

THEOREM VII.8. — If $P(X) = a_{2n}X^{2n} + \ldots + a_0$ with $a_{2n} > 0$ and $n$ odd, then:

$$\alpha_\infty(\lambda P; m_0) \geq (4\pi)^{-n}\frac{(2n)!}{2^n n!}a_{2n}\lambda(\ln\lambda)^n + 0(\lambda(\ln\lambda)^{n-1}).$$

*Proof.* — As above, one finds that:

$$\alpha_\infty(\lambda P; m_0) \geq \lambda\sum_{j=0}^{n} a_{2j}\frac{(2j)!}{2^j j!}(4\pi)^{-j}(-1)^{j+1}\ln\left(\frac{\tilde{m}^2}{m_0^2}\right)^j + f_\infty$$

and the result follows by taking $\tilde{m}^2 = \lambda m_0^2$. ■

To illustrate our method for proving (VII.13) in case $n$ is even, we consider the case $P(X) = X^4$. We first use the covariance of Theorem VII.2 to note that $\alpha_\infty(\lambda X^4; m_0) = \alpha_\infty(\lambda(X + c)^4 + m_0^2 cX + \frac{1}{2}m_0^2c^2; m_0)$. We now Wick reorder and obtain the inequality

$$\begin{aligned}\alpha_\infty(\lambda X^4; m_0) \geq & -3\lambda\left[\frac{1}{4\pi}\ln\left(\frac{\tilde{m}^2}{m_0^2}\right)\right]^2 \\ & + 6\lambda c^2\left[\frac{1}{4\pi}\ln\left(\frac{\tilde{m}^2}{m_0^2}\right)\right] - \lambda c^4 - \frac{1}{2}m_0^2c^2 + \frac{m_0^2 - \tilde{m}^2}{8\pi} + \frac{m_0^2}{8\pi}\ln\left(\frac{\tilde{m}^2}{m_0^2}\right).\end{aligned}$$

Since $-3 + 6 - 1 = 2 > 0$ if we take $c^2 = \frac{1}{4\pi}\ln\left(\frac{\tilde{m}^2}{m_0^2}\right)$ and then $\tilde{m}^2 = \lambda m_0^2$, we see that

$$\alpha_\infty(\lambda X^4; m_0^2) \geq 2(4\pi)^{-2}\lambda(\ln\lambda)^2 - 0(\lambda).$$

We systematize the following:

THEOREM VII.9 (Baumel [2]). — Let P be a polynomial of degree $2n$.

Let $\gamma_{2n} = -\inf_x H_{2n}(x)$ where $H_{2n}(x)$ is the $2n$ th Hermite polynomial. Then

$$\alpha_\infty(\lambda P ; m_0) \geq (4\pi)^{-n}\gamma_{2n}a_{2n}\lambda(\ln \lambda)^n + 0(\lambda(\ln \lambda)^{n-1}). \quad \text{(VII.14)}$$

*Remarks.* — 1. Since $\int_{-\infty}^{\infty} \gamma_{2n}(x)e^{-x^2}dx = 0$ (for $n \neq 0$), $\gamma_{2n} > 0$.

2. Even if $n$ is odd this result improves Theorem VII.8 since

$$\gamma_{2n} > -H_{2n}(0) = \frac{(2n)!}{2^n n!} \quad \text{if} \quad n > 1.$$

*Proof.* — As above translate by $c^2 = \frac{y}{4\pi}\ln\left(\frac{\tilde{m}^2}{m_0^2}\right)$ with $y$ to be determined, let $\tilde{m}^2 = \lambda m_0^2$, and identify the $\lambda(\ln \lambda)^n$ term using Lemma VII.2 A as

$$-(4\pi)^{-n}a_{2n}H_{2n}(y)[\lambda(\ln \lambda)^n].$$

Choosing $y$ so that $\gamma_{2n} = -H_{2n}(y)$, (VII.14) results. ■

*Remarks.* — 1. Basically what we see by our arguments in this section is that the $\lambda(\ln \lambda)^n$ bound on $\alpha_\infty$ is obtainable from the Gibbs variational principle with Gaussian states (with non-zero mean allowed). Such a calculation via explicit computation in the Gibbs principle has been done independently by P. Sodano [*65*]. Coleman [*4*] has used a similar idea (by using a Rayleigh-Ritz principle) in his work on the Sine-Gordon equation.

2. By hypercontractivity arguments [*28*], this $\lambda(\ln \lambda)^n$ bound on the behavior for $\alpha_\infty$ is also the large $\lambda$ behavior of $-E_l$ for any $l$ and also for

$$\ln \int \exp(-\lambda U_\Lambda)d\mu_0.$$

## VII.4. **Bounds on subdominant couplings.**

In this section we determine the dependence of the pressure on the subdominant couplings. The resulting estimates are useful in controlling the half-X pressures (see § VIII). The interaction that we consider is of the form

$$U_R^{X,\Lambda}(\vec{g}) = \int_R :\phi^{2n}(x) + g_{2n-1}(x)\phi^{2n-1}(x) + \dots + g_0(x):_{X,\Lambda} d^2x \quad \text{(VII.15)}$$

where R and $\Lambda$ are rectangles with $R \subset \Lambda$ and the subdominant couplings $\vec{g}(x) = (g_0(X), \dots, g_{2n-1}(x))$ are measurable functions on R.

To describe the behavior of $U_R^{X,\Lambda}(\vec{g})$ as the $g_j$ get large, define

$$\sigma(\vec{a}) = \sum_{j=1}^{2n} |a_{2n-j}|^{2n/j}$$

where $\vec{a} = (a_0, \ldots, a_{2n-1})$, and define the functional

$$\sigma_{\mathrm{R}}(\vec{g}) = \int_{\mathrm{R}} \sigma(\vec{g}(x))dx . \tag{VII.16}$$

Our basic estimate is then:

THEOREM VII.10. — Let $\vec{g}(x) = (g_0(x), \ldots, g_{2n-1}(x))$ be measurable functions on R where $\mathrm{R} \subset \Lambda$ are both rectangles in $\mathbb{R}^2$ and the sides of R have length greater than 1. Let X = F, D, N, P. Then there is a constant $b$ independent of R, $\Lambda$, $\vec{g}$ and X such that

$$\int e^{-\mathrm{U}_{\mathrm{R}}^{\Lambda,\mathrm{X}}(\vec{g})} d\mu_\Lambda^{\mathrm{X}} \leq e^{b(|\mathrm{R}| + \sigma_{\mathrm{R}}(\vec{g}))} . \tag{VII.17}$$

*Remarks.* — 1. The leading $g$-independent term $:\phi^{2n}:$ in $\mathrm{U}_{\mathrm{R}}^{\mathrm{X},\Lambda}$ may be replaced by any fixed semibounded polynomial of degree $2n$ and the proof of (VII.17) remains valid.

2. When X = F, D. one can remove the restriction that the sides of R be greater than 1 as well as the restriction that R and $\Lambda$ be rectangles.

There are three significant reductions in the proof of Theorem VII.10:

1. It is sufficient to prove the theorem for X = F. The reason is the same as that used in the proof of Lemma III.9: by conditioning the left side of (VII.17) is greatest for X = N; but by the inequality $\mathrm{G}_\Lambda^{\mathrm{N}} \leq c\mathrm{G}_0$ of Theorem III.4, we can in turn dominate the left side a similar expression involving F B. C. but where the powers $:\phi^j:$ of the field have been replaced by $c^{j/2} :\phi^j:$.

2. It is sufficient to consider the case where the $g_j(x)$ are constant. For the FKN formula and locality easily imply that (see, e. g., Theorem I.7 of [*28*])

$$\int e^{-\mathrm{U}_{\mathrm{R}}(\vec{g})} d\mu_0 \leq \exp\left[\int_{\mathrm{R}} \alpha_\infty(g_0(x), \ldots, g_{2n-1}(x))d^2x\right] \tag{VII.18}$$

where $\alpha_\infty(a_0, \ldots, a_{2n-1})$ is the infinite volume pressure corresponding to the polynomial $\mathrm{P}(\xi) = \xi^{2n} + a_{2n-1}\xi^{2n-1} + \ldots a_0$. Thus (VII.17) follows from (VII.18) once we show that

$$\alpha_\infty(a_0, \ldots, a_{2n-1}) \leq b(1 + \sigma(\vec{a})) \tag{VII.19}$$

3. Note that the inequalities (VII.17)-(VII.19) all display the « linear » dependence on the volume. The last reduction is that it suffices to prove a finite volume analogue of (VII.19) without care for the correct volume dependence. That is, we need only show that for $l$ finite there is a constant $b_l$ (possibly dependent on $l$) such that the pressure in the square

$$(-l/2, l/2) \times (-l/2, l/2)$$

satisfies

$$\alpha_{l,l}(\vec{a}) \leq b_l(1 + \sigma(\vec{a})) . \tag{VII.20}$$

The justification for this simplification goes back to Nelson's proof [*42*] of the linear lower bound as elaborated on in [*27*], [*28*]. For by the arguments of [*27*], [*28*] it follows that for any $\lambda > 1$ there is an $l > \infty$ such that

$$\alpha_\infty(P) \leq \alpha_{l,l}(\lambda P)/\lambda \tag{VII.21}$$

where by $\alpha_\infty(P)$ (and $\alpha_{l,l}(\lambda P)$) we denote the dependence of the pressure on the interaction polynomial.

As we have already mentioned in § III.2, we give in this paper two other methods of obtaining the correct volume dependence in estimates such as (VII.17). The first in § III.2 uses the submultiplicativity property of Neumann B. C.; the third in the Appendix is based on the Checkerboard Estimate, which may be regarded as an abstraction of the hypercontractive ideas involved in proving (VII.21).

We now begin the proof of the inequality (VII.20) which just amounts to squeezing the original NGS semiboundedness proof [*41*], [*16*], [*59*] a little harder. We introduce the ultraviolet cutoff by defining

$$\phi_\kappa(x) = \int \tilde{\chi}_\kappa(x - y)\phi(y)dy$$

and

$$G_\kappa(x) = \int \tilde{\chi}_\kappa(x - y)G_0(y)dy = \frac{1}{(2\pi)^2}\int \frac{e^{ikx}}{k^2 + m^2}\chi_\kappa(k)dk$$

where $\chi_\kappa(k)$ is the characteristic function of the set $\{ k \mid |k| \leq \kappa \}$ and $\tilde{\chi}_\kappa(x) = \frac{1}{(2\pi)^2}\int e^{-ikx}\chi_\kappa(k)dk$. Let S denote the square of side $l$ and define the ultraviolet cutoff interaction

$$U_\kappa(\vec{a}) = \int_S :P(\phi_\kappa(x)): dx$$

where $P(\xi) = \xi^{2n} + a_{2n-1}\xi^{2n-1} + \ldots + a_0$. Then:

LEMMA VII.11. — There is a constant $d_1$ independent of $\kappa$ and $\vec{a}$ such that

$$:P(\phi_\kappa(x)): \geq -d_1((\ln \kappa)^n + \sigma(\vec{a})). \tag{VII.22}$$

*Proof.* — Undoing the Wick ordering, we write

$$:P(\phi_\kappa): = \phi_\kappa^{2n} + b_{2n-1}\phi_\kappa^{2n-1} + \ldots + b_0 \tag{VII.23}$$

where $b_j(\vec{a}, \kappa)$ is linear in the $a_j$'s and a polynomial of degree $d = [n - j/2]$ in the Wick constant

$$c_\kappa = \int \phi_\kappa(x)^2 d\mu_0 = G_\kappa(0) = 0(\ln \kappa).$$

Explicitly (see, e. g., [*29*, Lemma V.27])

$$b_j = a_j - \alpha_{j,1}c_k a_{j+2} + \alpha_{j,2}c_\kappa^2 a_{j+4} \ldots + \alpha_{j,d}(-c_\kappa)^d a_{j+2d}$$

where $a_{2n} = 1$ and the $\alpha_{j,k}$ are combinatorial factors. Now a polynomial of the form (VII.23) is bounded below by

$$: P(\phi_\kappa) : \geq - \text{const.} \max_j | b_j |^{2n/(2n-j)} \qquad \text{(VII.24)}$$

since the minimum of $\xi^{2n} + b_j \xi^j$ occurs at const. $| b_j |^{\frac{1}{2n-j}}$ and has a value $-$ const. $| b_j |^{2n/(2n-j)}$.

Using Hölder's inequality in the form

$$\left( \sum_{i=1}^{n} r_i \right)^\alpha \leq n^{\alpha-1} \sum_{i=1}^{n} r_i^\alpha \quad \text{for} \quad \alpha \geq 1 \qquad \text{(VII.25)}$$

we estimate $b_j$ by

$$\begin{aligned} | b_j |^{2n/(2n-j)} &\leq \text{const.} \sum_{i=0}^{d} | a_{j+2i} c_\kappa^i |^{2n/(2n-j)} \\ &\leq \text{const.} \sum_{i=0}^{d} (| a_{j+2i} |^{2n/(2n-j-2i)} + c_\kappa^n) \end{aligned} \qquad \text{(VII.26)}$$

where we have used the arithmetic-geometric mean inequality in the form

$$| ac | \leq \frac{2n - j - 2i}{2n - j} \cdot a^{(2n-j)/(2n-j-2i)} + \frac{2i}{2n - j} c^{(2n-j)/2i} .$$

Inequalities (VII.24) and (VII.26) yield the lemma. ■

The next lemma is a standard part of the semiboundedness argument and so we omit the proof (see Lemma III.7 for a closely related proof):

LEMMA VII.12. — Assume that $g \in L^{1+\varepsilon}$ for $\varepsilon > 0$ and let

$$\delta V_\kappa^r = \int g(x)(: \phi^r(x) : - : \phi_\kappa^r(x) :) dx .$$

Then there are positive constants $d_2$ and $\alpha$ such that for any $p < \infty$

$$\| \delta V_\kappa^r \|_p \leq d_2 p^{r/2} \| g \|_{1+\varepsilon} \kappa^{-\alpha} .$$

If we set

$$\delta U_\kappa = U - U_\kappa = \int_S : P(\phi) : - \int_S : P(\phi_\kappa) :$$

then we obtain this elementary corollary:

COROLLARY VII.13. — There are constants $d_3$ and $\alpha > 0$ (independent of $\kappa$, $\vec{a}$, and $p$) such that

$$\| \delta U_\kappa \|_p \leq d_3 \kappa^{-\alpha} (p^n + \sigma(\vec{a})) \qquad \text{(VII.28)}$$

*Proof.* — By the Lemma

$$\begin{aligned} \| \delta U_\kappa \|_p &\leq \text{const. } \kappa^{-\alpha} \sum_{r=1}^{2n} p^{r/2} \mid a_r \mid \\ &\leq \text{const. } \kappa^{-\alpha}(p^n + \sigma(\vec{a})) \end{aligned}$$

since $p^{r/2} \mid a_r \mid \leq p^n + \mid a_r \mid^{2n/(2n-r)}$. ■

We are now in a position to prove Theorem VII.10. It is convenient to do so using a Duhamel expansion, following Glimm and Jaffe [20] [21]. The Duhamel or perturbation expansion for $e^{-U}$ is obtained by iterating

$$e^{-U} = e^{-U_{\kappa_1}} - \int_0^1 e^{-s_1 U} \delta U_{\kappa_1} e^{-(1-s_1)U_{\kappa_1}} ds_1$$

with a sequence of cutoffs $0 \leq \kappa_1 \leq \kappa_2 \leq \ldots$. This gives

$$e^{-U} = \sum_{m=0}^{\infty} (-1)^m \int \ldots \int ds_1 \ldots ds_m \prod_{j=1}^{m+1} e^{-\delta s_j U_{\kappa_j}} \prod_{j=1}^{m} \delta U_{\kappa_j} \qquad \text{(VII.29)}$$

where $s_0 = 1$, $s_{m+1} = 0$, $\delta s_j = s_{j-1} - s_j$, and the integration variables $s_j$ are ordered by $s_0 \geq s_1 \geq \ldots \geq s_{m+1}$. The expansion (VII.29) clearly converges pointwise in Q space, and by the majorization obtained below, it therefore converges in each $L^p(Q)$, $p < \infty$.

*Proof of* Theorem VII.10. — We choose the sequence $\kappa_j = e^{j^{1/n}}$. By (VII.22)

$$U_{\kappa_j} \geq -d_4(j + \sigma(a))$$

so that

$$\prod_{j=1}^{m+1} e^{-\delta s_j U_{\kappa_j}} \leq \prod_{j=1}^{m+1} e^{d_4(j+\sigma)\delta s_j} \leq e^{d_4(m+1+\sigma)} \qquad \text{(VII.30)}$$

On the other hand by (VII.28)

$$\begin{aligned} \int \left| \prod_{j=1}^{m} \delta U_{\kappa_j} \right| d\mu_0 &\leq \prod_{j=1} \| \delta U_{\kappa_j} \|_m \\ &\leq \prod_j d_3 \kappa_j^{-\alpha}(m^n + \sigma) \\ &\leq [d_3(m^n + \sigma)]^m e^{-d_5 m^{1+n^{-1}}} \\ &\leq (2d_3)^m (m^{mn} + \sigma^m) e^{-d_5 m^{1+n^{-1}}} \qquad \text{(VII.31)} \end{aligned}$$

by (VII.25). Inserting the estimates (VII.30-31) into (VII.29) we obtain

$$\int e^{-U} d\mu_0 \leq e^{d_4(\sigma+1)} \sum_m [(2d_3 m^n)^m + (2d_3\sigma)^m] e^{d_4 m - d_5 m^\gamma} \equiv e^{d_4(\sigma+1)}[S_1 + S_2]$$

where $\gamma = 1 + n^{-1}$. The first sum $S_1$ is clearly bounded and so to prove (VII.20) it remains to show that

$$S_2 \equiv \sum_m e^{m(\ln(2d_3\sigma)+d_4)-d_5m^\gamma} = e^{0(1+\sigma)}.$$

Let $x = \ln(2d_3\sigma) + d_4$. For $m \geq m_0 = (2x/d_5)^{\frac{1}{\gamma-1}}$ the sum over $m$ is bounded independently of $x$:

$$\sum_{m=m_0}^{\infty} \exp(mx - d_5m^\gamma) \leq \sum_{m=m_0}^{\infty} \exp\left(-\frac{1}{2}d_5m^\gamma\right).$$

For the first part of the sum we make the crude estimate

$$\begin{aligned}\sum_{m=0}^{m_0-1} \exp(mx - d_5m^\gamma) &\leq m_0e^{m_0x} \\ &\leq e^{0(x^{\gamma/(\gamma-1)})} \leq e^{0(1+\sigma)}.\end{aligned}$$

Hence $S_2$ is bounded as claimed; this implies (VII.20) and so by the reductions noted after the statement of the theorem, the proof of the theorem is complete. ■

As a special case of the above estimates, we note:

COROLLARY VII.14. — Let $\alpha_\infty(\vec{a})$ be the pressure corresponding to the polynomial $P(\xi) = \xi^{2n} + a_{2n-1}\xi^{2n-1} + \ldots + a_0$. Then

$$|\alpha_\infty(\vec{a})| \leq \text{const.}\left(1 + \sum_{j=1}^{2n} |a_{2n-j}|^{2n/j}\right). \tag{VII.32}$$

*Proof.* — The upper bound follows from (VII.20) and (VII.21). The lower bound is trivial since by Jensen's inequality

$$\alpha_\infty(\vec{a}) \geq -a_0. \quad ■$$

## VIII. THE HALF-X PRESSURES

Let $U^Y(\Lambda)$ denote the interaction $\int_\Lambda : P(\phi(x)) :_{Y,\Lambda} d^2x$ with Wick subtractions with respect to $d\mu_\Lambda^Y$. Define

$$\alpha_\Lambda^{X;Y} = \frac{1}{|\Lambda|} \ln \int d\mu_\Lambda^X \exp(-U^Y(\Lambda))$$

and so obtain 16 « pressures » as X, Y = F, D, N, P. We have thus far considered the 4 diagonal objects $\alpha_\Lambda^{X;X}$ and proven the equality of the limits as $\Lambda \to \infty$, say as a sequence of squares. In this section we shall prove the convergence of the remaining twelve to the same limit. We are primarily interested in the objects with Y = F in which we speak of the Half-X pressure but our methods handle all cases so we include all 12. The importance of Half-Dirichlet B. C. was first noted by Nelson [44] because of the convenience of Wick ordering not changing as $\Lambda$ changes. And we have used Half-Periodic states in our own work [30]. We remark that an earlier more complex proof of ours that $\lim \alpha_\Lambda^{D;F} \to \alpha_\infty$ appears in [62].

The proof of our main result on these general pressures appears in § VIII.2. The basic idea is that since $U^Y(\Lambda)$ and $U^X(\Lambda)$ only differ appreciably near the boundary, $|\alpha_\Lambda^{X;Y} - \alpha_\infty^{X;X}|$ should be of order $|\Lambda|^{-1/2}$. Our proof of a weaker fact is based on some elementary convexity arguments to be found in § VIII.1. We are indebted to Robert Isreal for bringing our attention to these bounds in a different context.

## VIII.1. **Convexity and Lipschitz bounds.**

Recall that by Hölder's inequality $\alpha_\Lambda(\lambda)$ is a convex function of the coupling constant $\lambda$.

THEOREM VIII.1. — Let $f$ be a bounded convex function on the unit ball of a Banach space, X. Let $C = \sup_{\|x\| \leq 1} |f(x)|$. Then for any $x, y \in X$ with $\|x\| \leq \frac{1}{2}$, $\|x\| \leq \frac{1}{2}$, $\|x - y\| \leq \frac{1}{2}$:

$$|f(x) - f(y)| \leq 4C \|x - y\|.$$

*Proof.* — Let $z = x + \frac{1}{2}\|x - y\|^{-1}(y - x)$. Then $\|z\| \leq \|x\| + \frac{1}{2} \leq 1$ so $z$ is in the domain of definition of $f$. Since

$$y = (1 - 2\|x - y\|)x + (2\|x - y\|)z$$

convexity assures us that:

$$f(y) \leq f(x) + 2\|x - y\|(f(z) - f(x))$$

so that

$$f(y) - f(x) \leq (2C)(2\|x - y\|).$$

By symmetry in $x$, $y$ the result follows. ■

## VIII.2. **Control of the half-X pressure.**

We begin by stating the main theorem of this paper:

THEOREM VIII.2. — For any fixed semibounded polynomial, the sixteen limits $\lim_{\Lambda \to \infty} \alpha_\Lambda^{X;Y}$ exist and are equal, where $\Lambda$ is taken to $\infty$ through a sequence of rectangles whose sides independently go to infinity and where X, Y = F, D, N, P.

*Remark.* — This theorem complements our result for the pressure with $\pm$ B. C. (Theorem II.9).

We henceforth fix $P(\xi) = a_{2n}\xi^{2n} + \ldots + a_0$, where $a_{2n} > 0$. Define the four functionals of $2n - 1$ functions $\vec{g} = (g_0, \ldots, g_{2n-2})$ on $\Lambda$:

$$\alpha_\Lambda^X(\vec{g}) = \frac{1}{|\Lambda|} \ln \int d\mu_\Lambda^X e^{-U_\Lambda^X(\vec{g})} \tag{VIII.1}$$

where

$$U_\Lambda^X(\vec{g}) = \int_\Lambda : P(\phi(x)) + g_{2n-2}(x)\phi^{2n-2}(x) + \ldots + g_0(x) : dx \tag{VIII.2}$$

where the Wick ordering in (VIII.2) is with respect to $d\mu_\Lambda^X$. Of course, the reason that the functionals $\alpha_\Lambda^X(\vec{g})$ are of interest is that there are $g_i$ depending on P, $\Lambda$, X, and Y such that

$$\alpha_\Lambda^{X;Y} = \alpha_\Lambda^X(\vec{g}).$$

By the explicit formula (I.8) for Wick reordering each $g_i(x)$ has the form

$$g_i(x) = \sum_{j=1}^{N} b_j(\delta G_\Lambda^{X;Y}(x))^j$$

where the $b_j$ are constants and $\delta G$ is a difference of Green's functions,

$$\begin{aligned} \delta G_\Lambda^{X;Y}(x) &= \lim_{y \to x} \left[ G_\Lambda^X(x, y) - G_\Lambda^Y(x, y) \right]. \\ &= \delta G_\Lambda^X(x) - \delta G_\Lambda^Y(x) \end{aligned}$$

in the notation of § III.3. Let $r = \text{dist}(x, \partial\Lambda)$ and let $l$ = length of the smallest side of $\Lambda$, which we assume satisfies $l \geq 1$. Then by Lemma III.3, each $g_i(x)$ satisfies for some N

$$|g_i(x)| \leq \begin{cases} \text{const.}\, |\log r|^N & r \leq 1/2 \\ \text{const.}\, [e^{-2m_0 r} + e^{-m_0 l}] & r \geq 1/2. \end{cases} \tag{VIII.3}$$

*Proof of* Theorem VIII.2. — Since we have already proved that $\lim \alpha_\Lambda^{X;X}$ exists independently of X (see the Remark after Theorem V.4), it is enough to show that

$$|\alpha_\Lambda^{X;Y} - \alpha_\Lambda^{X;X}| \leq C l^{-1/n} \tag{VIII.4}$$

if $l \geq l_0$, for some constants C and $l_0$. Define the norm

$$|||\vec{g}|||_\Lambda = \left\{ |\Lambda|^{-1} \sum_{i=0}^{2n-2} \int_\Lambda |g_i(x)|^n dx \right\}^{1/n}. \tag{VIII.5}$$

By Theorem VII.10, there is *a* constant $b$ such that

$$|\alpha_\Lambda^X(\vec{g})| \leq b(1 + |||\vec{g}|||_\Lambda).$$

It follows from (VIII.3) that there is an $l_0$ such that $|||\vec{g}|||_\Lambda \leq 1/2$ if $l \geq l_0$. By Hölder's inequality $\alpha_\Lambda^X(\vec{g})$ is a convex function of the $g_i$'s. According to Theorem VIII.1, if $l \geq l_0$ then

$$|\alpha_\Lambda^{X;Y} - \alpha_\Lambda^{X;X}| = |\alpha_\Lambda^X(\vec{g}) - \alpha_\Lambda^X(0)| \leq 8b\,|||\vec{g}|||_\Lambda .$$

By a simple calculation using the estimates (VIII.3) we see that $|||\vec{g}|||_\Lambda \leq \text{const. } l^{-1/n}$. This establishes (VIII.4) and the theorem. ■

## IX. CORRELATION INEQUALITIES

### IX.1. Ferromagnetism for general B. C.

Intuitively, one can understand the ferromagnetic nature of boson field theories from the formal expression for the Gaussian measure (with X B. C. in a region $\Lambda$)

$$d\mu_\Lambda^X = \text{const. exp}\left[ -\frac{1}{2}(\phi, (-\Delta_\Lambda^X + m_0^2)\phi)\right] d\phi . \qquad \text{(IX.1)}$$

For $-\Delta$ is positive « on-diagonal » and *negative* « infinitesimally off-diagonal ». Different choices of B. C. do not affect this property (even at the boundary of $\Lambda$). In this subsection we make these remarks rigorous by extending the lattice approximation for X = F, D (see [*29*]) to X = P and N. The correlation inequalities of [*29*] then extend to general B. C. as we explain in the following subsection.

Throughout this section we specialize to the case where $\Lambda$ is a rectangle, say $(-l_1/2, l_1/2) \times (-l_2/2, l_2/2)$. Our notation is as in [*29*]: $\delta > 0$ is the spacing parameter for the lattice $L_\delta = \{ n\delta \mid n = (n_1, n_2) \in \mathbb{Z}^2 \}$, $\Lambda_\delta \equiv \Lambda \cap L_\delta$ denotes the set of lattice points within $\Lambda$, and $\partial\Lambda_\delta$ denotes the points in $\Lambda_\delta$ which have nearest neighbors outside $\Lambda_\delta$. With each site $n\delta \in L_\delta$ we associate a field variable $q_n$ taking values in $\mathbb{R}$. For convenience we assume that $l_1$ *and* $l_2$ *are odd multiples of* $\delta$ so that in particular the sides of $\Lambda$ lie midway between lattice points.

We now explain how to write down the lattice measures corresponding to (IX.1) with various B. C. X. The following expressions ((IX.2) to (IX.4)) are to be regarded as formal definitions and the accompanying comments as heuristics. The actual justification of these definitions consists of the proof of convergence of the lattice approximation which will occupy the remainder of this section. If we consider F B. C. on the infinite lattice $L_\delta$, the lattice measure (see [*29*, § IV]) is (formally)

$$d\mu_\infty^F = \text{const. } e^{-\frac{1}{2}q\cdot A_\infty^F q} dq$$

where the infinite matrix $A^F_\infty$ is defined by

$$q\cdot A^F_\infty q = m_0^2\delta^2\Sigma q_n^2 + \frac{1}{2}\sum_{|n-n'|=1}(q_n - q_n')^2 \qquad \text{(IX.2 }a)$$

$$= (m_0^2\delta^2 + 4)\Sigma q_n^2 - \sum_{|n-n'|=1} q_n q_n' . \qquad \text{(IX.2 }b)$$

Here we have normed $\mathbb{Z}^2$ by $|n| = |n_1| + |n_2|$ so that the sum in (IX.2) takes place over nearest neighbors. More explicitly,

$$(A^F_\infty)_{nn'} = \begin{cases} 4 + m_0^2\delta^2 & n = n' \\ -1 & |n - n'| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A^F_\infty$ is negative off-diagonal, i. e., it is « ferromagnetic ».

Upon restricting $d\mu^F_\infty$ to the lattice $\Lambda_\delta$ and imposing B. C. X = D, N, P, F, we obtain the well-defined lattice measures $d\mu^X_{\Lambda,\delta}$ as follows. For Dirichlet B. C. the spins outside $\Lambda$ are set equal to 0; from (IX.2 $b$) we have

$$q\cdot A^D_\Lambda q = (m_0^2\delta^2 + 4)\sum_{n\in\Lambda} q_n^2 - \sum_{\substack{n,n'\in\Lambda \\ |n-n'|=1}} q_n q_n' \qquad \text{(IX.3 }a)$$

where, in a abuse of notation, we write $n \in \Lambda$ to denote $n\delta \in \Lambda_\delta$. Thus $A^D_\Lambda$ is just the restriction of $A^F_\infty$ to $\Lambda_\delta \times \Lambda_\delta$; and

$$d\mu^D_\Lambda = \text{const. } e^{-\frac{1}{2}q\cdot A^F_\infty q} dq$$

where the const. is always chosen so that $d\mu^D_\Lambda$ has measure 1 ; i. e. const. $= (2\pi)^{-N/2} | A^D_\Lambda |^{1/2}$ where N is the number of lattice points in $\Lambda_\delta$ and $| A^D_\Lambda | = \det A^D_\Lambda$. More generally, if L is a line segment (parallel to the $x_1$ or $x_2$ axis and passing midway between lattice points) we can impose Dirichlet B. C. on L by « cutting the bonds » across L, i. e. by dropping the terms $q_n q_n'$ in (IX.2 $b$) across L. For example, the infinite matrix $A^D_\Lambda \oplus A^D_{\Lambda'}$ corresponding to Dirichlet B. C. on $\partial\Lambda$ ($\Lambda' = \mathbb{R}^2\backslash\bar{\Lambda}$) would be defined by

$$q\cdot(A^D_\Lambda \oplus A^D_{\Lambda'})q = q\cdot A^F_\infty q + 2\sum_{\substack{|n-n'|=1 \\ n\in\Lambda, n'\in\Lambda'}} q_n q_n' .$$

To obtain Neumann B. C. on $\partial\Lambda$ we drop the coupling terms $(q_n - q_n')^2$ across $\partial\Lambda$ since this simulates zero normal derivative. Thus

$$q\cdot(A^N_\Lambda \oplus A^N_{\Lambda'})q = q\cdot A^F_\infty q - \sum_{\substack{|n-n'|=1 \\ n\in\Lambda, n'\in\Lambda'}} (q_n - q_n')^2 ;$$

restricting to $\Lambda_\delta$ gives

$$q \cdot A_\Lambda^N q = m_0^2 \delta^2 \sum_{n \in \Lambda} q_n^2 + \frac{1}{2} \sum_{\substack{n,n' \in \Lambda \\ |n-n'|=1}} (q_n - q_n')^2 \qquad \text{(IX.3 } b\text{)}$$

or more explicitly

$$A_\Lambda^N = A_\Lambda^D - B_\Lambda^N \qquad \text{(IX.4 } a\text{)}$$

where the diagonal matrix $B_\Lambda^N$ is concentrated on $\partial\Lambda_\delta$ with entries

$$(B_\Lambda^N)_{nn} = \begin{cases} 2 & \text{if } n\delta \in \partial\Lambda_\delta \text{ is a corner site of } \Lambda_\delta \\ 1 & \text{if } n\delta \in \partial\Lambda_\delta \text{ is not a corner site of } \Lambda_\delta \\ 0 & \text{otherwise.} \end{cases}$$

To obtain periodic B. C. on $\partial\Lambda$ we simply introduce couplings $q_n q_n'$ between boundary fields at opposite edges and this yields the matrix on $\Lambda_\delta \times \Lambda_\delta$

$$A_\Lambda^P = A_\Lambda^D - B_\Lambda^P \qquad \text{(IX.4 } b\text{)}$$

where

$$(B_\Lambda^P)_{nn'} = \begin{cases} 1 & \text{if } \; |n - n'|_P = 1 \quad \text{and} \quad |n - n'| \neq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here $|n|_P$ is the periodic distance on $\mathbb{Z}^2$

$$|n|_p = \min_{k \in \mathbb{Z}^2} (|n_1 + k_1 l_1/\delta| + |n_2 + k_2 l_2/\delta|).$$

The matrix $A_\Lambda^F$ for free B. C. is the most difficult to write down since, as explained in [*29*],

$$A_\Lambda^F = [(A^F)^{-1} \upharpoonright \Lambda_\delta \times \Lambda_\delta]^{-1}.$$

Nevertheless (see [*29*]), $A_\Lambda^F$ still has the form

$$A_\Lambda^F = A_\Lambda^D - B_\Lambda^F \qquad \text{(IX.4 } c\text{)}$$

above $B_\Lambda^F$ is a nonnegative matrix concentrated on $\partial\Lambda_\delta$. If there is no confusion we shall suppress the subscript $\Lambda$ on $A_\Lambda^X$ and $C_\Lambda^X = (A_\Lambda^X)^{-1}$.

In summary, the free (normalized) lattice measures

$$d\mu_{\Lambda,\delta}^X = \text{const. } e^{-\frac{1}{2} q \cdot A_\Lambda^X q} dq$$

are all *ferromagnetic* in the sense that $A_\Lambda^X$ has nonpositive off-diagonal entries. The interacting (normalized) lattice measures corresponding to the polynomial P are defined by

$$d\nu_{\Lambda,\delta}^X = \text{const. } e^{-V_{\Lambda,\delta}^X} d\mu_{\Lambda,\delta}^X \qquad \text{(IX.5 } a\text{)}$$

where

$$V_{\Lambda,\delta}^X = \delta^2 \sum_{n \in \Lambda} :P(q_n):_{X,\Lambda}.$$

Here the subscripts X, $\Lambda$ indicate that the Wick subtractions are made with respect to $d\mu^{X}_{\Lambda,\delta}$. The half-X (HX) interacting measures are similarly defined except that free Wick ordering is used:

$$d\nu^{HX}_{\Lambda,\delta} = \text{const. } e^{-V^{F,\delta}_{\Lambda}} d\mu^{X}_{\Lambda,\delta}. \qquad \text{(IX.5 } b\text{)}$$

The justification of the definition of $d\nu^{X}_{\Lambda,\delta}$ as the lattice measure with X B. C. consists of showing that, as $\delta \to 0$, this measure converges in an appropriate sense to the continuum measure $d\nu^{X}_{\Lambda}$. To discuss this question, we first provide different explicit expressions for the covariance matrices $C^{X}_{\Lambda} = (A^{X}_{\Lambda})^{-1}$; the first uses images and the second X-momentum space. For free B. C. on $L_\delta$ the covariance is by construction (see [*29*])

$$(C^{F}_{\infty})_{nn'} \equiv \int \phi(f_{\delta,n})\phi(f_{\delta,n'})d\mu_0 = G_{0,\delta}(n\delta - n'\delta)$$

where

$$G_{0,\delta}(x) = (2\pi)^{-2} \int_{T_\delta} e^{ik.x}\mu_\delta(k)^{-2}d^2k \qquad \text{(IX.6)}$$

$$f_{\delta,n}(x) = (2\pi)^{-2} \int_{T_\delta} e^{ik.(x-n\delta)}\mu(k)/\mu_\delta(k)d^2k \qquad \text{(IX.7)}$$

$$\mu_\delta(k)^2 = \delta^{-2}(4 - 2\cos\delta k_1 - 2\cos\delta k_2) + m_0^2 \qquad \text{(IX.8)}$$

and $T_\delta$ is the square $[-\pi/\delta, \pi/\delta]^2$.

For $k \in \mathbb{Z}^2$, let $\pi^{X}_{k}$ be the analogue of the « reflection » operator $p^{X}_{k}$ of § III.3; i. e., $\pi^{N}_{k}n = ((-1)^{k_1}(n_1 - k_1 l_1/\delta), (-1)^{k_2}(n_2 - k_2 l_2/\delta))$ and $\pi^{P}_{k}n = (n_1 - k_1 l_1/\delta, n_2 - k_2 l_2/\delta)$. Then in analogy with (III.29) we have:

LEMME IX.1. — For X = P, N,

$$(C^{X}_{\Lambda})_{mn} = \sum_{k\in\mathbb{Z}^2} G_{0,\delta}(m\delta - (\pi^{X}_{k}n)\delta). \qquad \text{(IX.9)}$$

*Remarks.* — 1. A similar formula holds for X = D provided it is arranged that $\partial\Lambda$ lies on lattice points, i. e. $l_1$ and $l_2$ are chosen to be even rather than odd multiples of $\delta$.

2. The convergence of the sum (IX.9) is obvious from Lemma IX.8 below.

*Proof.* — Denote the right side of (IX.9) by $G^{X}_{mn}$. We regard $G^{X}_{mn}$ as an infinite matrix indexed by $\mathbb{Z}^2$. Because of our assumption regarding the positioning of $\Lambda$ with respect to $L_\delta$, it is easy to see that $G^N$ is even with respect to reflections in the sides of $\Lambda$, i. e. $G^{N}_{mn} = G^{N}_{m'n}$ if $m'\delta$ is the reflection of $m\delta$ in a side of $\Lambda$. Similarly $G^P$ is invariant under translations by the sides of $\Lambda$, i. e. $G^{P}_{mn} = G^{P}_{\pi^{P}_{k}m,n}$ for any $k \in \mathbb{Z}^2$.

To verify (IX.9) for X = N we must check that for $m\delta$ and $n\delta \in \Lambda$,

$$\sum_{r\in\Lambda} A^{N}_{mr}G^{N}_{rn} = \delta_{m,n}. \qquad \text{(IX.10)}$$

Now if $m\delta \notin \partial\Lambda_\delta$, $A^N_{mr} = A^F_{mr}$ (where we have omitted the subscript $\infty$ on $(A^F_\infty)_{mr}$) so that the sum in (IX.10) equals

$$\sum_{r\in\mathbb{Z}^2} A^F_{mr} \sum_k C^F_{r,\pi^N_k n} = \sum_k \delta_{m,\pi^N_k n} = \delta_{m,n}$$

since $A^F_{mr} = 0$ if $r \notin \Lambda$ and since $\pi^N_k n \in \Lambda_\delta$ only when $k = 0$ in which case $\pi^N_k n = n$. When $m\delta \in \partial\Lambda_\delta$, the left side of (IX.10) equals

$$\sum_{r\in\Lambda} (A^D_{mr} - B^N_{mr})G^N_m = \sum_{r\in\mathbb{Z}^2} A^F_{mr}G^N_{rn} + \sum_{\substack{m'\notin\Lambda \\ |m'-m|=1}} G^N_{m'n} - \sum_r B^N_{mr}G^N_{rn}$$

by the definition of $A^D$. The last two sums on the right cancel since $G^N_{m'n} = G^N_{mn}$ for those $m'$ which are the nearest neighbors of $m$ outside $\Lambda_\delta$ and since $B^N$ is diagonal. The first sum on the right is just $\delta_{m,n}$ as above. Thus we have checked (IX.10).

The case of periodic B. C. is similar. If $m\delta \in \partial\Lambda_\delta$ we have

$$\sum_{r\in\Lambda} A^P_{mr}G^P_{rn} = \sum_{r\in\mathbb{Z}^2} A^F_{mr}G^P_{rn} + \sum_{\substack{m'\notin\Lambda \\ |m'-m|=1}} G^P_{m'n} - \sum_r B^P_{mr}G^P_{rn} = \delta_{m,n}$$

since the last two sums cancel by the periodicity of $G^P$. ■

The second formula for $C^X_\Lambda$ is based on the standard eigenfunction expansion for the Green's functions $G^X_\Lambda$. The operator $-\Delta^P + m_0^2$ has wave numbers $k \in T^P = \frac{2\pi}{l_1}\mathbb{Z} \times \frac{2\pi}{l_2}\mathbb{Z}$ with associated eigenvalues $\mu(k)^2 = k^2 + m_0^2$ and eigenfunctions $f^P_k(x) = (l_1 l_2)^{-1/2} e^{ik.x}$. The operator $(-\Delta^N_\Lambda + m_0^2)$ has wave numbers $k \in T^N = \frac{\pi}{l_1}\mathbb{Z} \times \frac{\pi}{l_2}\mathbb{Z}$ with associated eigenvalues $\mu(k)^2$ and eigenfunctions $f^N_k(x) = (l_1 l_2)^{-1/2} g^{(1)}_{k_1}(x_1) g^{(2)}_{k_2}(x_2)$ where

$$g^{(j)}_{k_j}(x_j) = \begin{cases} \cos k_j x_j & \text{if} \quad l_j k_j/\pi \quad \text{is even} \\ \sin k_j x_j & \text{if} \quad l_j k_j/\pi \quad \text{is odd}. \end{cases}$$

Note that $g^{(j)}_{k_j} = \pm\, g^{(j)}_{-k_j}$ so that there is a redundancy in our labelling but this is compensated for by the normalization $\int_{-l_j/2}^{l_j/2} (g^{(j)}_{k_j})^2 dx_j = \frac{1}{2}$ if $k_j \neq 0$. Similarly, the operator $(-\Delta^D_\Lambda + m_0^2)$ has wave numbers $k \in T^{D'} = T^N$ with eigenvalues $\mu(k)^2$ and eigenfunctions $f^D_k(x) = (l_1 l_2)^{-1/2} h^{(1)}_{k_1}(x_1) h^{(2)}_{k_2}(x_2)$ where

$$h^{(j)}_{k_j}(x_j) = \begin{cases} \sin k_j x_j & \text{if} \quad l_j k_j/\pi \quad \text{is even} \\ \cos k_j x_j & \text{if} \quad l_j k_j/\pi \quad \text{is odd}. \end{cases}$$

Note that $h_0^{(j)} = 0$ so that strictly speaking $T^D$ should not include wave numbers with zero components. With the above notation we have the standard formula for the Green's function

$$G_\Lambda^X(x, y) = \sum_{k \in T} f_k^X(x) \bar{f}_k^X(y) \mu(k)^{-2} \tag{IX.11}$$

where the eigenfunctions satisfy the completeness relation

$$\sum_{k \in T} f_k^X(x) \bar{f}_k^X(y) = \delta(x - y) . \tag{IX.12 a}$$

In order to write down the lattice version of (IX.11) we introduce lattice nomentum space defined by

$$T_\delta^X = T^X \cap T_\delta = T^X \cap [-\pi/\delta, \pi/\delta]^2 .$$

Then it is straightforward to verify that the functions $\{ \delta f_k^X(m\delta) \}_{k \in T_\delta^X}$ satisfy the completeness relation

$$\delta^2 \sum_{k \in T_\delta} f_k^X(m\delta) \bar{f}_k^X(n\delta) = \delta_{m,n} \tag{IX.12 b}$$

for $m\delta$, $n\delta \in \Lambda$. For example, in the case X = P, (IX.12 *b*) reads

$$\frac{\delta^2}{l_1 l_2} \sum_{j_1 = -\lambda_1}^{\lambda_1} \sum_{j_2 = -\lambda_2}^{\lambda_2} e^{i\frac{2\pi\delta}{l_1} j_1(m_1 - n_1) + i\frac{2\pi\delta}{l_2} j_2(m_2 - n_2)} = \delta_{m,n}$$

where $l_i = (2\lambda_i + 1)\delta$. Then:

LEMMA IX.2. — For X = P, N

$$(C_\Lambda^X)_{mn} = \sum_{k \in T^X} f_k^X(m\delta) \bar{f}_k^X(n\delta) \mu_\delta(k)^{-2} . \tag{IX.13}$$

*Remark.* — As in the case of Lemma IX.1, a similar formula holds for X = D provided we arrange that the sides of $\Lambda$ pass through lattice points so that $C_\Lambda^X = 0$ on $\partial\Lambda$.

*Proof.* — Denote the right side of (IX.13) by $G_{mn}^X$. We regard $G_{mn}^X$ as an infinite matrix indexed by $\mathbb{Z}^2$. Because of our assumption regarding the positioning of $\Lambda$ with respect to $L_\delta$ and by the definition of $f_k^X$, it is easy to see that $G_{mn}^N$ is even with respect to reflection in the sides of $\Lambda$ and $G_{mn}^P$ is invariant under translations by the sides of $\Lambda$. In particular, if $m\delta \in \partial\Lambda_\delta$ and $m'\delta$ is a neighboring site not in $\Lambda$, then

$$G_{m'n}^N = G_{mn}^N \tag{IX.14 a}$$

and

$$G^{P}_{m'n} = G^{P}_{m''n} \tag{IX.14 b}$$

where $m''$ is the site in line with $m$ and $m'$ at the opposite edge of $\partial\Lambda_\delta$.

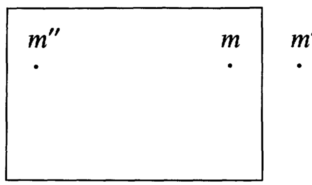


FIG. IX.1.

To verify (IX.13) we must check that for $n\delta \in \Lambda$,

$$\sum_{r\in\Lambda} A^{X}_{mr} G^{X}_{rn} = \delta_{m,n} \tag{IX.15}$$

By (IX.4), if $m\delta \notin \partial\Lambda_\delta$ then $(A^F = A^F_\infty)$

$$\begin{aligned}\sum_{r\in\Lambda} A^{X}_{mr} G^{X}_{rn} &= \sum_{r\in\mathbb{Z}^2} A^{F}_{mr} G^{X}_{rn} \\ &= (4 + m_0^2\delta^2) G^{X}_{mn} - \sum_{|r-m|=1} G^{X}_{rn}\,.\end{aligned} \tag{IX.16}$$

We claim that (IX.16) also holds if $m\delta \in \partial\Lambda_\delta$. For by (IX.4 *a*) and (IX.14 *a*)

$$\begin{aligned}\sum_{r\in\Lambda} A^{N}_{mr} G^{N}_{rn} &= \sum_{r\in\Lambda} A^{F}_{mr} G^{N}_{rn} - B^{N}_{mm} G^{N}_{mn} \\ &= \sum_{r\in\Lambda} A^{F}_{mr} G^{N}_{rn} + \sum_{m'\notin\Lambda} A^{F}_{mm'} G^{N}_{m'n} \\ &= \sum_{r\in\mathbb{Z}^2} A^{F}_{mr} G^{N}_{rn}\,.\end{aligned}$$

The argument is similar for $X = P$.

Inserting the expression (IX.13) for $G^{X}_{mn}$ in (IX.16) we must show that

$$(4 + m_0^2\delta^2) f^{X}_{k}(m\delta) - \sum_{|r-m|=1} f^{X}_{k}(r\delta) = \delta^2 \mu_\delta(k)^2 f^{X}_{k}(m\delta) \tag{IX.17}$$

for then by (IX.12 *b*) we deduce (IX.15). By definition $f^{X}_{k}$ is (up to normalization) a product $h_1(k_1x_1)h_2(k_2x_2)$ where $h_j(\theta) = e^{i\theta}$, $\sin\theta$, or $\cos\theta$. Each such function $h_j$ satisfies

$$h_j(k_j(m_j - 1)\delta) + h_j(k_j(m_j + 1)\delta) = 2\cos(k_j\delta)h_j(k_jm_j\delta) \quad \text{(IX.18)}$$

since (IX.18) is obviously true for $e^{i\theta}$ and hence for $\sin\theta$ and $\cos\theta$ by taking real and imaginary parts. (In fact, the trigonometric functions are the only $C^2$ functions satisfying (IX.18).) Consequently, the left side of (IX.17) equals

$$(4 + m_0^2\delta^2)f_k^X(m\delta) - [2\cos(k_1\delta) + 2\cos(k_2\delta]f_k^X(m\delta) = \delta^2\mu_\delta(k)^2 f_k^X(m\delta)$$

by the definition (IX.8). This concludes the proof of (IX.13). ■

The (smeared) lattice Schwinger functions $S^X_{\Lambda,\delta}$ are defined in terms of the measure (IX.5) by

$$S^X_{\Lambda,\delta}(h_1, \ldots, h_r) = \int q(h_1) \ldots q(h_r)d\nu^X_{\Lambda,\delta} \quad \text{(IX.19)}$$

where $h_j \in C_0^\infty(\Lambda)$ and

$$q(h) = \delta^2 \sum_{n\in\mathbb{Z}^2} h(n\delta)q_n .$$

Before entering into the proof of the convergence of $S^X_{\Lambda,\delta}$ as $\delta \to 0$ we should like to review briefly the case of F B. C. since the proof in [*29*] is rather sketchy. We essentially follow the discussion in [*62*], expanding on the proof of part *b*) of Theorem VIII.5 of [*62*] which is incomplete. The first step consists of rewriting (IX.19) in terms of the continuum measure $d\mu_0$. To do so we realize $q_n$ as $\phi(f_{\delta,n})$ where $f_{\delta,n}$ is defined in (IX.7); this gives.

$$S_{\Lambda,\delta}(h_1, \ldots, h_r) = \frac{\int \phi_\delta(h_1) \ldots \phi_\delta(h_r)e^{-U_{\Lambda,\delta}}d\mu_0}{\int e^{-U_{\Lambda,\delta}}d\mu_0} \quad \text{(IX.20)}$$

where

$$\phi_\delta(h) = \delta^2\Sigma h(n\delta)\phi(f_{\delta,n})$$

and

$$U_{\Lambda,\delta} = \delta^2 \sum_{n\delta\in\Lambda} :P(\phi(f_{\delta,n})): .$$

The equality of (IX.19) and (IX.20) follows from the equality of covariances $(C^F_\infty)_{nn'} = \int \phi(f_{\delta,n})\phi(f_{\delta,n'})d\mu_0$.

The key step now is to show that for suitable $g$

$$:\phi_\delta^r(g): \;\to\; :\phi^r(g): \quad \text{in any } L^p(d\mu_0), \qquad p < \infty \quad \text{(IX.21)}$$

as $\delta \to 0$. The convergence of the Schwinger functions follows from (IX.21) by a standard argument: we have

$$|e^{-U_{\Lambda,\delta}} - e^{-U_\Lambda}| \leq |U_{\Lambda,\delta} - U_\Lambda| \, . \, |e^{-U_{\Lambda,\delta}} + e^{-U_\Lambda}| \, . \qquad \text{(IX.22)}$$

By imitating the standard semiboundedness proof we can show that $e^{-U_{\Lambda,\delta}} \in L^p(Q, d\mu_0)$ uniformly in $\delta$ for any $p < \infty$. Hence (IX.21) and (IX.22) imply that $e^{-U_{\Lambda,\delta}} \to e^{-U_\Lambda}$ in any $L^p$ and so, by Hölder's inequality, the Schwinger functions (IX.20) converge.

We return to the proof of (IX.21). By hypercontractivity it is sufficient to prove (IX.21) for $p = 2$. An explicit computation gives

$$\| : \phi_\delta^r(g) : - : \phi^r(g) : \|_2^2 = r! \int \left| \frac{\tilde{g}_\delta(k_1 + \ldots + k_r)}{\mu_\delta(k_1) \ldots \mu_\delta(k_r)} X_\delta(k) - \frac{\hat{g}(k_1 + \ldots + k_r)}{\mu(k_1) \ldots \mu(k_r)} \right|^2 dk \qquad \text{(IX.23)}$$

where

$$\tilde{g}_\delta(k) = \frac{\delta^2}{2\pi} \sum_{n \in \mathbb{Z}^2} g(n\delta) e^{-ik \cdot n\delta}$$

and

$$X_\delta(k) = \chi_\delta(k_1) \ldots \chi_\delta(k_r)$$

where $\chi_\delta$ is the characteristic function of $T_\delta = [-\pi/\delta, \pi/\delta]^2$. Let $\hat{g}_\delta$ be the restriction of $\tilde{g}_\delta$ to $T_\delta$, i. e.

$$\hat{g}_\delta(k) = \chi_\delta(k) \tilde{g}_\delta(k) \, .$$

Then under fairly mild conditions on $g$, $\hat{g}_\delta(k) \to \hat{g}(k)$ in $L^2$ as $\delta \to 0$:

LEMMA IX.3. — Suppose that $g(x)$ is continuous a. e., and that for some function $h \in L^1 \cap L^2$,

$$| g([x]_\delta) | \leq h(x)$$

for all $\delta > 0$, where $[x]_\delta$ is the point in $L_\delta$ closest to $x$. Then as $\delta \to 0$, $\hat{g}_\delta(k) \to \hat{g}(k)$ in $L^2$.

*Remark.* — The case $g = \chi_\Lambda$ used in (IX.22) satisfies the conditions of the lemma, as do the $g$'s that enter for HX B. C. (see below).

*Proof.*

$$\begin{aligned} \int | \hat{g}_\delta(k) |^2 dk &= \delta^2 \Sigma | g(n\delta) |^2 \\ &= \int | g([x]_\delta) |^2 dx \to \int | g(x) |^2 dx \\ &= \int | \hat{g}(k) |^2 dk \end{aligned}$$

by the Lebesgue dominated convergence theorem. Thus $L^2$ convergence follows from weak $L^2$ convergence. Let $f \in C_0^\infty$ and denote its inverse Fourier transform by $\check{f}$. Then for sufficiently small $\delta$

$$\begin{aligned}\int \overline{f(k)}\hat{g}_\delta(k)dk &= \int \overline{f(k)}\tilde{g}_\delta(k)dk \\ &= \delta^2 \sum_n \check{f}(n\delta)g(n\delta) \\ &= \int \check{f}([x]_\delta)g([x]_\delta)dx \to \int \check{f}(x)g(x)dx \\ &= \int \overline{f(k)}\hat{g}(k)dk\end{aligned}$$

by the dominated convergence theorem. ■

We now easily deduce (IX.21):

LEMMA IX.4. — For any $g$ satisfying the hypotheses of Lemma IX.3, $:\phi_\delta^r(g): \to :\phi^r(g):$ in any $L^p(Q, d\mu_0)$, $p < \infty$, as $\delta \to 0$.

*Proof.* — It is sufficient to show that the right side of (IX.23) converges to zero. We first note that we may replace $\tilde{g}_\delta$ by $\hat{g}_\delta$. For on $T_\delta$ [*29*, Lemma IV.2]

$$\mu_\delta(k)^{-1} \leq \frac{\pi}{2}\,\mu(k)^{-1}. \tag{IX.24}$$

Therefore

$$\begin{aligned}\int \left| \frac{\tilde{g}_\delta - \hat{g}_\delta}{\mu_\delta \ldots \mu_\delta} X_\delta \right|^2 &\leq \text{const.} \int \frac{|\tilde{g}_\delta(k_1 + \ldots + k_r) - \hat{g}_\delta(k_1 + \ldots + k_r)|^2}{\mu(k_1)^2 \ldots \mu(k_r)^2} X_\delta \\ &\leq \text{const.} \int \frac{|\tilde{g}_\delta(k) - \hat{g}_\delta(k)|^2}{\mu(k)^{2-\varepsilon}} \chi_{\delta/r}(k)\end{aligned} \tag{IX.25}$$

for any $\varepsilon > 0$, where we have used the standard inequality (see, e. g., [*28*])

$$\int \frac{d^2p_1 \ldots d^2p_{r-1}}{\mu(p_1)^2\mu(p_2 - p_1)^2 \ldots \mu(k - p_{r-1})^2} \leq \frac{\text{const.}}{\mu(k)^{2-\varepsilon}}. \tag{IX.26}$$

We estimate the integral on the right side of (IX.25) by

$$\mu(\pi/\delta)^{\varepsilon-2} \int |\tilde{g}_\delta(k)|^2(1 - \chi_\delta(k))\chi_{\delta/r}(k) = (r^2 - 1)\mu(\pi/\delta)^{\varepsilon-2} \int |\hat{g}_\delta(k)|^2 \to 0$$

where we have used the periodicity with periods $\frac{2\pi}{\delta}$ of $\tilde{g}_\delta(k)$.

Having replaced $\tilde{g}_\delta$ by $\hat{g}_\delta$ in (IX.23) we dominate the resulting expression by

$$\int \frac{X_\delta |\hat{g}_\delta - \hat{g}|^2}{\mu_\delta^2 \ldots \mu_\delta^2} + \int |\hat{g}|^2 \left| \frac{X_\delta}{\mu_\delta \ldots \mu_\delta} - \frac{1}{\mu \ldots \mu} \right|^2.$$

The first term goes to zero because of (IX.24), (IX.26) and the previous Lemma; the second term goes to zero by the dominated convergence theorem together with the pointwise convergence $\mu_\delta(k) \to \mu(k)$. ■

This completes our review of the lattice convergence for free B. C. For $X = P$ or $N$ exactly the same proof goes through with the replacements

$$\mathbb{R}^2 \to \Lambda, \qquad T_\delta \to T_\delta^X, \qquad \text{and} \qquad e^{ik.x} \to f_k^X(x).$$

We now amplify somewhat on this remark:

We rewrite the lattice theory in terms of the continuum theory by realizing

$$q_n = \phi(f_{\delta,n}^X)$$

where

$$f_{\delta,n}^X(x) = \sum_{k \in T_\delta^X} f_k^X(n\delta)\bar{f}_k^X(x)\mu(k)\mu_\delta(k)^{-1}.$$

It is then easy to check from (IX.11) and (IX.13) that the covariances agree

$$\int \phi(f_{\delta,n}^X)\phi(f_{\delta,m}^X)d\mu_\Lambda^X = \int f_{\delta,n}^X(x)G_\Lambda^X(x, y)f_{\delta,n}^X(y)dxdy = (C_\Lambda^X)_{mn}.$$

Therefore

$$S_{\Lambda,\delta}^X(h_1, \ldots, h_r) = \frac{\int \phi_\delta^X(h_1) \ldots \phi_\delta^X(h_r)e^{-U_{\Lambda,\delta}}d\mu_\Lambda^X}{\int e^{-U_{\Lambda,\delta}^X}d\mu_\Lambda^X} \qquad \text{(IX.27)}$$

where

$$\phi_\delta^X(h) = \delta^2\Sigma h(n\delta)\phi(f_{\delta,n}^X)$$

etc.

As in the case $X = F$ the key step is to show that $:(\phi_\delta^X)^r(g): \to :\phi^r(g):$ in $L^2(d\mu_\Lambda^X)$. This reduces to showing the convergence to zero of

$$\sum_{k \in T^X} \left| \frac{\tilde{g}_\delta^X(k_1 + \ldots + k_r)}{\mu_\delta(k_1) \ldots \mu_\delta(k_r)} X_\delta(k) - \frac{\hat{g}^X(k_1 + \ldots + k_r)}{\mu(k_1) \ldots \mu(k_r)} \right|^2 \qquad \text{(IX.28)}$$

where for $g$ with support in $\Lambda$

$$\tilde{g}_\delta^X(k) = \delta^2 \sum_n \overline{f_k^X(n\delta)}g(n\delta)$$

$$\hat{g}^X(k) = \int_\Lambda \overline{f_k^X(x)}g(x)dx$$

and $\hat{g}_\delta^X(k) = \chi_\delta(k)\tilde{g}_\delta^X(k)$. From (IX.12 $b$) we have as usual a Parseval's identity

$$\delta^2 \sum_n |g(n\delta)|^2 = \sum_{k \in T_\delta^X} |\tilde{g}_\delta^X(k)|^2$$

and the proof that (IX.28) $\to$ 0 is virtually identical to the case X = F. In conclusion:

THEOREM IX.5. — Let X = P or N. Suppose $h_j \in C_0^\infty(\Lambda)$, $j = 1, \ldots, r$. As $\delta \to 0$, the lattice Schwinger functions $S^X_{\Lambda,\delta}(h_1, \ldots, h_r)$ of (IX.19) converge to the continuum Schwinger functions

$$S^X_\Lambda(h_1, \ldots, h_r) = \frac{\int \phi(h_1) \ldots \phi(h_r) e^{-U^X_\Lambda} d\mu^X_\Lambda}{\int e^{-U^X_\Lambda} d\mu^X_\Lambda}.$$

*Remarks.* — 1. Since we are assuming that the sides of $\Lambda$ are odd multtiples of $\delta$, we mean that $\delta \to 0$ through a sequence $\delta_j = l/(2j + 1)$, say.

2. The method used in proving Theorem IX.5 also applies to Dirichlet B. C. provided we position the lattice so that the sides of $\Lambda$ pass through lattice points (see the remark after Lemma IX.2). Thus in the case of rectangles we have a somewhat simpler proof of the convergence of $S^D_{\Lambda,\delta}$ than the proof given in [29] for more general regions.

We conclude this subsection with a proof of the convergence of the HX Schwinger functions for X = P, N. These are defined as in (IX.19) or (IX.27) except that the interaction is Wick ordered with respect to the free covariance:

$$S^{HX}_{\Lambda,\delta}(h_1, \ldots, h_r) = \frac{\int \phi^X_\delta(h_1) \ldots \phi^X_\delta(h_r) e^{-U_{\Lambda,\delta}} d\mu^X_\Lambda}{\int e^{-U_{\Lambda,\delta}} d\mu^X_\Lambda}.$$

The convergence proof amounts to a corollary of Theorem IX.5 since we can rewrite free Wick-ordered powers in terms of X-Wick ordered powers [29]: (Notational warning: in the symbol $\delta G$ for a difference of Green's functions, the $\delta$ has nothing to do with the lattice spacing $\delta$!)

$$:\phi^r(x): = \sum_{j=0}^{[r/2]} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \delta G^X_\Lambda(x)^j :\phi^{r-2j}(x):_{X,\Lambda} \qquad \text{(IX.29 } a\text{)}$$

and

$$:\phi(f^X_{\delta,n})^r: = \sum_{j=0}^{[r/2]} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \delta G^X_\delta(n\delta)^j :\phi(f^X_{\delta,n})^{r-2j}:_{X,\Lambda} \qquad \text{(IX.29 } b\text{)}$$

where

$$\left\{ \begin{matrix} r \\ j \end{matrix} \right\} = r!/j!\,(r-2j)!\,2^j$$

$$\delta G^X_\Lambda(x) = \lim_{y \to x} \left[ G^X_\Lambda(x, y) - G_0(x - y) \right] \qquad \text{(IX.30 } a\text{)}$$

and

$$\delta G_\delta^X(n\delta) = (C_\Lambda^X)_{n,n} - G_{0,\delta}(0). \qquad \text{(IX.30 } b\text{)}$$

Thus, the effect of Free Wick ordering is just to modify the spatial cutoff function $\chi_\Lambda$ by factors of $\delta G_\Lambda^X(x)^j$ (or $\delta G_\delta^X(n\delta)^j$ in the lattice case). These factors are most easily controlled by means of the image formulas (III.30) and (IX.9). As we have already noted in § III the singularities involved are only logarithmic in the « distance » of $x \in \Lambda$ to the boundary $\partial\Lambda$,

$$r = \max_{j=1,2} (\min (x_j + l_j/2, l_j/2 - x_j)).$$

Explicitly, we have:

LEMMA IX.6. — For $x \in \Lambda$, $\delta G_\delta^X([x]_\delta)$ is bounded uniformly in $\delta$ for $r \geq 1$ and satisfies for $r < 1$

$$| \delta G_\delta^X([x]_\delta) | \leq a \log r^{-1} \qquad \text{(IX.31)}$$

where the constant $a$ is independent of $\delta$. Moreover, as $\delta \to 0$,

$$\delta G_\delta^X([x]_\delta) \to \delta G_\Lambda^X(x). \qquad \text{(IX.32)}$$

Postponing the proof of Lemma IX.6 we see upon examination of the proof of Theorem IX.5 that the HX Schwinger functions also converge. For by (IX.29 $b$), (IX.31) and the subdominant coupling estimates of § VIII, $e^{-U_{\Lambda,\delta}} \in L^p(d\mu_\Lambda^X)$ for all $p < \infty$, uniformly in $\delta$. The key step (IX.21) now takes the form

$$:(\phi_\delta^X)^r(h_\delta):_{X,\Lambda} \to :\phi^r(h):_{X,\Lambda} \quad \text{in} \quad L^p(d\mu_\Lambda^X), \qquad p < \infty \qquad \text{(IX.33)}$$

where $h_\delta(x) = \chi_\Lambda(x)\delta G_\delta^X([x]_\delta)^j$ and $h(x) = \chi_\Lambda(x)\delta G_\Lambda^X(x)^j$. By Lemma IX.6 and the method of Lemma IX.3 it is clear that $(h_\delta)_\delta^\wedge \to \hat{h}$ in $L^2$ and so (IX.33) holds as before. This yields:

THEOREM IX.7. — Let $h_j \in C_0^\infty(\Lambda)$ and let $X = P$ or $N$. As $\delta \to 0$ the half-X lattice Schwinger functions $S_{\Lambda,\delta}^{HX}(h_1, \ldots, h_r)$ converge to the continuum Schwinger functions $S_\Lambda^{HX}(h_1, \ldots, h_r)$.

The main step in proving Lemma IX.6 consists of showing that each of the terms in the images sum (IX.9) has the expected behavior:

LEMMA IX.8. — Let $r = \max (|n_1|\delta, |n_2|\delta)$. There are positive constants $a$, $b$ and $c$ independent of $\delta$ such that

$$| G_{0,\delta}(n\delta) | \leq a \log r^{-1} \qquad r \leq 1/2 \qquad \text{(IX.34)}$$

and

$$| G_{0,\delta}(n\delta) | \leq b e^{-cr} \qquad r \geq 1/2. \qquad \text{(IX.35)}$$

*Proof.* — Suppose without loss of generality that $n_1 \geq |n_2|$. For each $k_2 \in [-\pi/\delta, \pi/\delta]$, we regard $f(k_1) = \mu_\delta(k)^{-1}$ as an analytic function of $k_1$. From the definition (IX.8) we see that $f(k_1)$ is analytic in a strip $|\operatorname{Im} k_1| < \kappa$ where $\kappa$ is determined by

$$2 \cosh \delta\kappa = 4 - 2 \cos \delta k_2 + m_0^2\delta^2 \equiv \alpha.$$

i. e. $\kappa = \delta^{-1} \log [(\alpha + \sqrt{\alpha^2 - 4})/2]$. Now [29, Lemma IV.2]

$$\cos y \leq 1 - \frac{2}{\pi^2} y^2 \tag{IX.36}$$

if $y \in [-\pi, \pi]$ so that

$$\alpha \geq 2 + \delta^2\left(m_0^2 + \frac{4}{\pi^2} k_2^2\right) \equiv 2 + \delta^2 v(k_2)^2 .$$

It follows that

$$\begin{aligned} \kappa &\geq \delta^{-1} \log \frac{2 + \delta^2 v^2 + \sqrt{4\delta^2 v^2 + \delta^4 v^4}}{2} \\ &\geq \delta^{-1} \log (1 + \delta v) \\ &\geq c_1 v \geq c_1' \mu \end{aligned}$$

where $c_1 = \min\limits_{0<x<x_0} \dfrac{\log (1 + x)}{x}$ where $x_0 \equiv (4 + m_0^2 \delta^2)^{1/2} = \max \delta v$. Note that the constants $c_1$, $c_1'$ are independent of $\delta$ as $\delta \to 0$.

Using Cauchy's Theorem we shift the integral over $k_1$ in the definition of $G_{0,\delta}$ to the line $\operatorname{Im} k_1 = c_2 \mu$ where $c_2 < c_1'$ is a positive constant to be determined:

$$\int_{-\pi/\delta}^{\pi/\delta} e^{ik_1 r} \mu_\delta(k)^{-2} dk_1 = \int_{-\pi/\delta}^{\pi/\delta} e^{ik_1 r - c_2 \mu r} \mu_\delta(k_1 + ic_2\mu, k_2)^{-2} dk_1 .$$

The integrals along the lines $\operatorname{Re} k_1 = \pm \pi/\delta$ cancel because of the periodicity of $e^{ik_1 r}$ and $\mu_\delta(k)$.

Now

$$\begin{aligned} &| \delta^2 \mu_\delta^2(k_1 + ic_2\mu, k_2) | \\ &\quad = | 4 - 2 \cos k_1\delta \cosh c_2\mu\delta - 2i \sin k_1\delta \sinh c_2\mu\delta - 2 \cos k_2\delta + m_0^2\delta^2 | \\ &\quad \geq 4 - 2 \cos k_1\delta - 2 \cos k_2\delta + m_0^2\delta^2 - 2 (\cosh c_2\mu\delta - 1) \\ &\quad \geq \delta^2\left(m_0^2 + \frac{4}{\pi^2} k^2\right) - c_3(c_2\mu\delta)^2 \end{aligned}$$

by (IX.36) and the fact that there is some $c_3 > 0$ (independent of $\delta$) such that $2 (\cosh x - 1) \leq c_3 x^2$ for $0 \leq x \leq c_1 x_0$. We now choose $c_2$ sufficiently small so that $c_3 c_2^2 \leq 1/10$. Then

$$| \mu_\delta^2(k_1 + ic_2\mu, k_2) | \geq \text{const. } \mu(k)^2 .$$

We thus obtain the estimate

$$\begin{aligned} | G_{0,\delta}(n\delta) | &\leq \text{const.} \int_{\mathbb{R}^2} d^2k e^{-c_2\mu(k_2) r} \mu(k)^{-2} \\ &= \text{const.} \int_{\mathbb{R}} dk_2 e^{-c_2\mu(k_2) r} \mu(k_2)^{-1} \\ &= \text{const.} \int_{m_0}^{\infty} d\mu e^{-c_2\mu r} (\mu^2 - m_0^2)^{-1/2} . \end{aligned} \tag{IX.37}$$

It is obvious from (IX.37) that $G_{0,\delta}(n\delta) = O(e^{-c_2 m_0 r})$ for large $r$, whereas for small $r$ (say $r \leq m_0$) we have

$$\begin{aligned} |G_{0,\delta}(n\delta)| &\leq \text{const.} \left\{ \int_{m_0}^{r^{-1}} \frac{e^{-c_2\mu r} d\mu}{(\mu^2 - m_0^2)^{1/2}} + \int_{r^{-1}}^{\infty} \frac{e^{-c_2\mu r}}{(\mu^2 - m_0^2)^{1/2}} d\mu \right\} \\ &\leq \text{const.} \left\{ \int_{m_0}^{r^{-1}} \frac{d\mu}{(\mu^2 - m_0^2)^{1/2}} + \int_{1}^{\infty} \frac{e^{-c_2 x}}{(x^2 - m_0^2 r^2)^{1/2}} dx \right\} \\ &= 0(\ln r^{-1}) + 0(1). \end{aligned}$$

This completes the proof of the lemma. ■

*Proof of* Lemma IX.6. — By (IX.9) we exhibit $\delta G_\delta^X$ as an absolutely convergent sum

$$\delta G_\delta^X(n\delta) = \sum_{\substack{j \in \mathbb{Z}^2 \\ j \neq 0}} G_{0,\delta}(n\delta - (\pi_j^X n)\delta). \tag{IX.38}$$

By an argument we have used already in § III, we see from (IX.38) that « nearest » images of $n\delta$ contribute at most a logarithmic singularity (by (IX.31)) while the other images contribute a rapidly convergent sum by virtue of (IX.32). Consequently,

$$|\delta G_\delta^X(n\delta)| \leq \text{const.} \ln r_\delta^{-1} \tag{IX.39}$$

where

$$r_\delta = \max_j (\min (n_j\delta + l_j/2, l_j/2 - n_j\delta)).$$

Trivial geometric considerations show that $r_\delta \geq r/2$ so that (IX.39) implies (IX.31).

As for the pointwise convergence (IX.32), it is sufficient to show convergence for each term in (IX.38):

$$G_{0,\delta}([x]_\delta - p_j^X[x]_\delta) \to G_0(x - p_j^X x).$$

Setting $z = x - p_j^X x$ we may suppose that $z_1 \neq 0$ and hence that $[z_1]_\delta \neq 0$ for sufficiently small $\delta$. Integration by parts with respect to $k_1$ gives

$$G_{0,\delta}([z]_\delta) = \int_{T_\delta} \frac{e^{ik\cdot[z]}}{i[z_1]_\delta} \frac{2 \sin \delta k_1}{\delta \mu_\delta^4} \tag{IX.40}$$

and

$$G_0(z) = \int \frac{e^{ik.z}}{iz_1} \frac{2k_1}{\mu^4} \tag{IX.41}$$

The integrand in (IX.40) is bounded by an integrable function independent of $\delta$ and converges pointwise to the integrand in (IX.41) as $\delta \to 0$. Hence by the Lebesgue dominated convergence theorem $G_{0,\delta}([z]_\delta) \to G_0(z)$ as $\delta \to 0$. ■

*Remark.* — It is not hard to see that the lattice approximation also converges with mixed B. C., i. e. one of F, D, N or P on opposite edges and another of F, D, N or P on the other two edges.

## IX.2. **Inequalities on Schwinger functions.**

In the previous section we showed that

*a*) the X and HX $P(\phi)_2$ lattice theories are ferromagnetic;

*b*) as the lattice spacing $\to 0$, the lattice Schwinger functions converge to the continuum Schwinger functions. Consequently, by the methods of [*29*] [*63*] and the correlation inequalities of the Ising model (see e. g., [*14*] [*35*]) we immediately have the standard correlation inequalities for $\sigma = $ F, D, N, P, HD, HN, HP:

i) *Griffiths inequalities.* — If $P(x) = P_e(x) - \mu x$ where $P_e$ is even and $\mu \geq 0$, then

$$S_\Lambda^\sigma(x_1, \ldots, x_n) \geq 0$$

and

$$S_\Lambda^\sigma(x_1, \ldots, x_{r+s}) \geq S_\Lambda^\sigma(x_1, \ldots, x_r)S_\Lambda^\sigma(x_{r+1}, \ldots, x_{r+s}) .$$

ii) *FKG inequalities.* — If P is an arbitrary (semibounded) polynomial and if F and G are increasing functions of the fields (see [*29*]) then

$$\langle FG \rangle_\Lambda^\sigma \geq \langle G \rangle_\Lambda^\sigma \langle G \rangle_\Lambda^\sigma$$

where $\langle \, . \, \rangle_\Lambda^\sigma$ denotes expectation with respect to the $P(\phi)$ interacting measure in $\Lambda$ with $\sigma$ B. C.

iii) *GHS and Lebowitz inequalities.* — If $P(\phi) = a\phi^4 + b\phi^2 - \mu\phi$ where $a > 0$ and $\mu \geq 0$. Then the truncated three point function

$$\langle \phi(x)\phi(y)\phi(z) \rangle_{\Lambda,T}^\sigma \leq 0 .$$

If $\mu = 0$, then

$$\langle \phi(x)\phi(y)\phi(z)\phi(w) \rangle_{\Lambda,T}^\sigma \leq 0 .$$

*Remarks.* — 1. If the infinite volume limit is known to exist then the above inequalities transfert to the infinite volume theory (see [*53*] for more details).

2. For $P(\phi) = a\phi^4 + b\phi^2 - \mu\phi$, the Lee-Yang Theorem also holds for all of the above B. C. $\sigma$ (see [*62*]). For instance the ground state energies of the various Hamiltonians $E_l^\sigma(\mu)$ are real analytic in $\mu > 0$ and have analytic continuations to the region Re $\mu > 0$ (this result uses the Remark at the end of the previous subsection).

3. Any other multilinear inequality proved for F or D B. C. extends to these B. C. $\sigma$, e. g., Neuman's new inequality [*45*] and the inequality $U_6 \leq 0$ [*3*] [*48*] [*67*].

What about the lattice of Fig. I.1? It is natural to conjecture that the

Schwinger functions for different B. C. are related as in Fig. I.1, especially since in the non-interacting case such relations are true by (I.3 *a*) (the $S^X_\Lambda$ are just sums of products of the $G^X_\Lambda$!). However, we have been able to relate *only DB. C.* to the other three B. C. Explicitly, it is clear from (IX.4) that F, N and P B. C. are « more ferromagnetic » than D B. C. so that by the second Griffiths inequality $(P(x) = P_e(x) - \mu x, \mu \geq 0)$

$$S^{HD}_\Lambda \leq S^{HX}_\Lambda, \qquad X = F, N \quad \text{or} \quad P. \tag{IX.42}$$

However the relations between F and N B. C. or between P and N B. C. are not so simple. For instance, from (IX.4) we have

$$A^P_\Lambda = A^N_\Lambda - (B^P_\Lambda - B^N_\Lambda)$$

where the matrix $B^P_\Lambda - B^N_\Lambda$ has both positive and negative elements. In order to show that $S^{HN}_\Lambda$ dominates $S_\Lambda$ it is clear from the discussion around formula (IX.3 *b*) that we must prove an inequality like

$$\langle (q_n - q_{n'})^2 F(\vec{q}) \rangle \geq \langle (q_n - q_{n'})^2 \rangle \langle F(\vec{q}) \rangle \tag{IX.43}$$

where the expectations are infinite lattice expectations that interpolate between F B. C. and N B. C. across $\partial\Lambda$, $q_n$ is a boundary spin in $\Lambda$, $q_{n'}$ a nearest neighbor outside $\Lambda$, and $F(\vec{q})$ is a product of spins inside $\Lambda$. The inequality (IX.43) is plausible since the site $n$ is « closer » than $n'$ to the spin sites inside $\Lambda$; however, intuitive considerations indicate that the Wick wells in the interaction polynomial invalidate an inequality such as (IX.43).

In the special case where deg $P \leq 4$ the analogue of (IX.42) without the H holds since we can control the change in Wick ordering. Explicitly by (IX.29 *a*)

$$:\phi^4:_{\Lambda,D} = :\phi^4:_{\Lambda,X} + bc(x) :\phi^4:_{\Lambda,X} + \text{const.}$$

where

$$c(x) = \delta G^X_\Lambda(x) - \delta G^D_\Lambda(x) \geq 0$$

by (I.3 *a*). Using the $:\phi^2:$ correlation inequality of Theorem V.11 of [*29*] we deduce that

$$S^D_\Lambda \leq S^X_\Lambda, \qquad X = F, N, P.$$

We state these results as:

THEOREM IX.9. — Let $P(x) = P_e(x) - \mu x$ where $P_e$ is even and $\mu \geq 0$. Then for X = F, N or P, $S^{HD}_\Lambda \leq S^{HX}_\Lambda$. If in addition, deg $P \leq 4$, then also $S^D_\Lambda \leq S^X_\Lambda$.

As we remarked in the Introduction it is very tempting to conjecture on the basis of the lattice Fig. I.1 that, since $S^{HD}_\Lambda$ is monotone increasing in $\Lambda$ [*44*] [*29*] for $P = P_e - \mu x$, $S^{HN}_\Lambda$ is monotone decreasing in $\Lambda$. Indeed suppose one considers the non-interacting case where $S^{HX}_\Lambda = S^X_\Lambda$ is a sum of products of $G^X_\Lambda$. Then it is immediate from the images formula (III.29) that $G^N_\Lambda(x, y)$ (and hence $S^X_\Lambda$) is monotone decreasing as $\Lambda \to \infty$. (Ironically, the decrease of $G^N_\Lambda$ is more transparent from the images formula than the increase of $G_\Lambda^D$!). However, a proof of this decrease for the interacting theory involves proving an inequality like (IX.43) which seems false in general. Certainly, if $P(\phi)_2$ possesses a phase transition, then $S_\Lambda^{HN}$ could not decrease for all values of the coupling constants or else the infinite volume theory would always have a positive mass gap since the theory infinite in one direction has a transfer matrix with discrete spectrum and unique vacuum. These remarks (and Theorem IX.9) serve to support our assertion in the Introduction that a *Dirichlet* barrier is most compatible with the ferromagnetic nature of boson field theories.

### IX.3. **On the identity of certain states.**

We have seen that the pressure is independent of a wide variety of boundary conditions. In certain regions of the space of coupling constants, one expects that the infinite volume Schwinger functions will also be independent of boundary condition. For example if $P(X) = aX^4 + bx^2 - \mu X$ with $\mu \neq 0$, we have the analogue of a classical ferromagnet in non-zero external field and such statistical mechanical systems are known to have only one equilibrium state [*36*] [*55*]. We note however, that even in the lattice approximation, it is not completely clear how one should go about extending the above results since they depend on the fact that the spins take values in a *bounded* set.

We have two results to report here. The first relies on the B. C.-independence of the pressure:

THEOREM IX.10. — Let $P(X) = Q(X) - \mu X$. Fix Q an even polynomial and $m_0$. Suppose that, for some open interval $(a, b) \subset \mathbb{R}$, there is an $M > 0$ so that both the Half-Dirichlet and free B. C. transfer matrices for the $Q - \mu X$ theory $(a < \mu < b)$ in $(-l/2, l/2)$ have a mass gap $m_l^X$ with $m_l^X(\mu) \geq M$ for all $l > 1$. Suppose the free B. C. Schwinger functions have an infinite volume limit for $a < \mu < b$. Then for all such $\mu$, the Half-Dirichlet and free B. C. Schwinger functions agree.

*Proof.* — Let $S_{n,T}^X(x_1, \ldots, x_n ; \mu)$ denote the truncated $n$ point Schwinger function for the infinite volume $Q - \mu\phi$ theory with X-B. C. where X is F of HD. Let $\alpha_\infty(\mu)$ denote the pressure for the $Q - \mu X$ theory. It is a result of Dimock [*6*] that $\alpha_\infty(\mu)$ is $C^\infty$ for $a < \mu < b$ and

$$\frac{d^k \alpha_\infty(\mu)}{d\mu^k} = \int S_{k,T}^X(0, x_1, \ldots, x_{k-1} ; \mu) d^{2k-2}x$$

The necessary bounds that Dimock requires follow in this case by the $\phi$-bounds in Fröhlich's form [*11*]. As a result we have

$$\int S_{k,T}^{HD}(0, x_1, \ldots, x_{k-1} ; \mu) d^{2k-2}x = \int S_{k,T}^{F}(0, x_1, \ldots, x_{k-1} ; \mu) d^{2k-2}x \quad \text{(IX.44)}$$

We can suppose $\mu \geq 0$ without loss of generality by the $(\mu, \phi) \to (-\mu, -\phi)$ covariance. Let us prove that

$$S^F_{n,T}(0, x_1, \ldots, x_{n-1}; \mu) = S^{HD}_{n,T}(0, x_1, \ldots, x_{n-1}; \mu) \qquad (IX.45)$$

by induction on $n$. For $n = 1$, this equality is just (IX.44). By the Griffiths inequality (IX.42)

$$S^F_m \geq S^{HD}_m$$

and, so if we know that $S^F_{n,T} = S^{HD}_{n,T}$ for $n < m$ then $S^F_{m,T} \geq S^{HD}_{m,T}$. Since their integrals are equal by (IX.44), they must be equal. As a result (IX.45) for $n < m$ implies it for $n = m$. (IX.45) thus holds by induction. ■

COROLLARY IX.11. — Let $Q(X) = aX^4 + bX^2$ ; $a > 0$. Suppose

(*i*) The free B. C. transfer matrix for the $Q(\phi)_2$ theory in $(-l/2, l/2)$ has a mass gap $m^F_l \geq M > 0$ for $l > 1$;

(*ii*) The free B. C. infinite volume theory for $Q - \mu X$ exists and is translation invariant for all $\mu \in (a, b)$.

Then for $\mu \in (a, b)$ the free B. C., HD and D theories agree.

*Proof.* — Let us prove the free and HD theories agree. The argument of the D and HD theories follows by mimicking the argument in Theorem IX.10 and in this proof. By the GHS inequalities, $m^F_l(\mu) \geq m^F_l(\mu = 0)$ and $m^{HD}_l(\mu) \geq m^{HD}_l(\mu = 0)$ (see [*63*]). By the GKS inequalities

$$m^{HD}_l(\mu = 0) \geq m^F_l(\mu = 0)$$

(see [*29*]). Therefore $m^{HD}_l(\mu)$ and $m^F_l(\mu)$ are uniformly bounded away from 0 for all $\mu$. ■

*Remarks.* — 1. In the above theorem and corollary we can replace F B. C. by HP or HN B. C.

2. Under hypothesis (*i*) of the corollary we can conclude $S^D_n$ and $S^{HD}_n$ agree for all $\mu$.

Our second general result is the elementary:

THEOREM IX.12. — Suppose that the infinite volume $P(\phi)_2$ Schwinger functions $S^P$ exists for periodic B. C. with $P(X) = aX^4 + bX^2 - \mu X$ for $a$, $\mu$ fixed and $b$ in some interval $(\alpha, \beta)$. Suppose that the infinite volume Schwinger functions $S^P$ are continuous in $b$ for $b \in (\alpha, \beta)$. Then, for $b \in (\alpha, \beta)$, the half-periodic B. C. infinite volume Schwinger functions $S^{HP}$ exist and equal $S^P$.

*Proof.* — Let $S^X_{l,t}(x; b)$ denote the X-B. C. Schwinger functions for the $aX^4 + bX^2 - \mu X$ theory in $(-l/2, l/2) \times (-t/2, t/2)$. By our discussion in § VI, $S^{HP}_{l,t}(x; b) = S^P_{l,t}(x; b + c_{l,t})$ where $c_{l,t} \to 0$ as $l, t \to \infty$. Moreover, $S^P_{l,t}(x; b)$ is monotone decreasing as $b$ increases by the $:\phi:^2$ correlation inequality [29] and the convergence of the lattice approximation with P and HP B. C. Since $c_{l,t} \to 0$ as $l, t \to \infty$ we see that for any $\varepsilon$:

$$\overline{\lim}\ S_{l,t}^{HP}(x\,;b) \leq S^{P}(x\,;b - \varepsilon)$$

$$\underline{\lim}\ S_{l,t}^{HP}(x\,;b) \geq S^{P}(x\,;b + \varepsilon)\,.$$

Thus continuity of $S^P$ implies existence of the limit $S^{HP} = \lim S_{l,t}^{HP}$ and the equality $S^{HP} = S^P$. ■

*Note added in proof.* While this paper was in press, a number of results have appeared which are related to the main themes of this paper.

We mention the following two:

1. J. Glimm, A. Jaffe and T. Spencer have established the existence of phase transitions for $\phi_2^4$ and have used $\pm$ B. C. to study the detailed properties of these phases (*Commun. Math. Phys.*, t. **45**, 1975, p. 203-216, and *Ann. Phys., to appear*).
2. J. Fröhlich and B. Simon (Princeton preprint) have studied the problem of the identity of states and have improved the results of § IX.3. They also prove that equality holds in the Gibbs Variational Principle for all states constructed so far for $P(\phi)_2$.

# APPENDIX

## CHECKERBOARD ESTIMATES AND SPATIAL DECOUPLING

In [*29*] we established the Checkerboard estimate for the case of F. B. C. and used it to control the decoupling of spatially distant regions. In this Appendix we shall extend the Checkerboard estimate to the other B. C. X = D, N, P considered in this paper, and we shall indicate its usefulness in obtaining estimates with the « correct » volume dependence. In § IV.5 we have already used the Checkerboard estimate for P B. C. to ensure the convergence of the mean entropy for periodic states. The statement of the estimate is this:

THEOREM A.1 *(Checkerboard estimate)*. — Let X = F, D, N, P. Let $\Lambda = \cup \Lambda_j$ be a rectangular array of adjacent translates $\Lambda_j$ of a rectangle $\Lambda_1$. Then if $u_j$ is $(\Sigma_{\Lambda_j}, d\mu_\Lambda^X)$ measurable,

$$\| \Pi u_j \|_1^{\Lambda,X} \leq \Pi \| u_j \|_p^{\Lambda,X} \tag{A.1}$$

where $p < \infty$ is independent of the number of rectangles $\Lambda_j$.

*Remark.* — We denote the norms on $L^p(Q, d\mu_\Lambda^X)$ by $\| \, . \, \|_p^{\Lambda,X}$ and the associated one particle space by $N_\Lambda^X$ with inner product

$$\langle f, g \rangle_\Lambda^X = \int f(x) G_\Lambda^X(x, y) g(y) dx dy .$$

For free B. C. the Checkerboard estimate is an immediate consequence of the hypercontractivity of $e^{-tH_0}$ (see [*29*]), whereas for general B. C. a more involved argument is required (as in [*29*, § III.1]). Our basic estimate on the Green's function is (III.31) whose long distance part is, for $|x - y|_X \geq 1$, say,

$$0 < G_\Lambda^X(x, y) \leq b e^{-(m_0 - \varepsilon)|x-y|_X} \tag{A.2}$$

where the constant $b$ is independent of $\Lambda$ if the sides of $\Lambda$ have length greater than 1. Here $|x - y|_X$ denotes ordinary Euclidean distance for X = F, D, N and periodic distance for X = P (see Lemma III.2).

First we note that the analogue of Lemma III.4 of [*29*] holds:

LEMMA A.2. — Let $f_1$ and $f_2$ in $N_\Lambda^X$ have supports separated by an X-distance $r$. Then

$$|\langle f_1, f_2 \rangle_\Lambda^X| \leq e(r) \| f_1 \|_\Lambda^X \| f_2 \|_\Lambda^X$$

where $e(r) = a\alpha^r$ for some $a > 0$, $0 < \alpha < 1$.

*Remarks.* — 1. The method of proof shows that $\alpha$ may be taken to be $e^{-(m_0 - \varepsilon)}$ for any $\varepsilon > 0$.

2. The proof below is merely sketched since it follows that of [*29*, Lemma III.4] whose proof in turn is based on a method due to Osterwalder and Schrader [*47*].

*Proof.* — It is sufficient to prove the lemma for $r \geq 1$. Let $\zeta_j \in C^\infty(\mathbb{R}^2)$ satisfy $\zeta_j = 1$ on supp $f_j$ and $\zeta_j = 0$ just off supp $f_j$ so that the regions $\Lambda_j = \Lambda \cap \operatorname{supp} \zeta_j$ are separated by an X-distance greater than $r - 1$. Then

$$\begin{aligned} \langle f_1, f_2 \rangle_\Lambda^X &= (f_1, \zeta_1 G_\Lambda^X \zeta_2 f_2) \\ &= (f_1, \zeta_1 G_\Lambda^X \zeta_2 (-\Delta + m_0^2) G_\Lambda^X f_2) \\ &= (f_1, \zeta_1 G_\Lambda^X [(\Delta \zeta_2) + 2(\nabla \zeta_2) \cdot \nabla] G_\Lambda^X f_2) \end{aligned}$$

by « commuting » $\zeta_2$ and $-\Delta + m_0^2$. We perform a similar operation with $\zeta_1$ and insert characteristic functions of $\Lambda$:

$$\langle f_1, f_2 \rangle_\Lambda^X = (f_1, G_\Lambda^X[(\Delta\zeta_1) - 2\nabla\cdot(\nabla\zeta_1)]\chi_\Lambda \cdot G_\Lambda^X \chi_\Lambda[(\Delta\zeta_2) + 2(\nabla\zeta_2)\cdot\nabla]G_\Lambda^X f_2).$$

Letting $\eta_j = (\Delta\zeta_j)\chi_\Lambda$ we see from (A.2) and the proof of Lemma III.5 A of [*29*] that, as an operator on $L^2(\Lambda)$,

$$\| \eta_1 G_\Lambda^X \eta_2 \| \leq \text{const. } e^{-(m_0 - \varepsilon)r}.$$

A similar bound holds for the other terms such as $(\nabla\zeta_1)\chi_\Lambda G_\Lambda^X \chi_\Lambda(\Delta\zeta_2)$. Finally by a quadratic form argument it is easy to see that $\nabla(G_\Lambda^X)^{1/2}$ is a bounded operator so that

$$| \langle f_1, f_2 \rangle_\Lambda^X | \leq \text{const. } e^{-(m_0 - \varepsilon)r} \| (G_\Lambda^X)^{1/2} f \| \, . \, \| (G_\Lambda^X)^{1/2} f \|$$

and this proves the lemma. ■

By Nelson's « Best » hypercontractive Theorem [*43*], we obtain as in [*29*]:

COROLLARY A.3. — Let $\Lambda_1$ and $\Lambda_2$ be regions in $\Lambda$ separated by an X-distance $r$. If $u_j$ is $(\Sigma_{\Lambda_j}, d\mu_\Lambda^X)$ measurable and if

$$(p_1 - 1)(p_2 - 1) \geq e(r)^2, \tag{A.3}$$

then

$$\| u_1 u_2 \|_1^{\Lambda,X} \leq \| u_1 \|_{p_1}^{\Lambda,X} \cdot \| u_2 \|_{p_2}^{\Lambda,X}.$$

*Proof of Theorem* A.1. — Assume first that $\Lambda$ is an $N \times 1$ « strip » of rectangles $\Lambda_j$ of dimension $a \times b$. Decompose $\Lambda = R_1 \cup R_2$ into two subsets obtained by taking alternate $\Lambda_j$'s in $R_1$ and the other $\Lambda_j$'s in $R_2$. Similarly decompose each of $R_1$ and $R_2$ into two subsets a distance $a$ apart. If this process is repeated until the sets reduce to the basic constituents $\Lambda_j$, it is clear that at the $n$th step the two subsets of a decomposition are a distance $d_n = (2^{n-1} - 1)a$ apart.

Consider the cases X = D or N and apply Corollary A.3 at each decomposition with $p_1 = p_2$. For $n = 1$ we have

$$\| \Pi u_j \|_1^{\Lambda,X} \leq \| \Pi_{\Lambda_j \subset R_1} u_j \|_{q_1}^{\Lambda,X} \| \Pi_{\Lambda_j \subset R_2} u_j \|_{q_1}^{\Lambda,X},$$

where $q_1 = 2$. At the $n$th decomposition we obtain $L^{q_n}$ norms, where, by (A.3), $q_n/q_{n-1} = 1 + e(d_n)$. That is,

$$q_n = \frac{q_n}{q_{n-1}} \cdot \frac{q_{n-1}}{q_{n-2}} \cdots \frac{q_2}{q_1} \cdot q_1 = 2 \prod_{i=2}^{n} (1 + e(d_i)).$$

Clearly, since $\sum_{i=2}^{\infty} e(d_i) < \infty$, $q_n$ is bounded independently of $n$, and so the theorem is established in this particular case. For X = P this argument must be slightly modified since it is the periodic distance which is involved, but we reach the same conclusion.

Finally suppose that $\Lambda$ is an $N \times M$ array. First apply the above argument where the elementary constituents are taken to be the $N$ $1 \times M$ strips and then apply the same argument to each of these strips. ■

*Remark.* — For the purposes of this section it is interesting to note that an exponential decrease function $e(r)$ is not required; in fact, $e(r) = 0(r^{-\varepsilon})$ for some $\varepsilon > 0$ would be sufficient.

The usefulness of the Checkerboard estimate lies in the fact that it allows a factorization over disjoint spatial regions even though such regions are not strictly independent. As an illustration, we shall give the third proof in this paper of the « linear lower bound »; this proof is an alternative to the proof using N B. C. in § III.2 and a partial alternative to the method of § VII.4 using hypercontractivity (the Checkerboard estimate may be regarded as an abstraction of hypercontractivity). The same technique gives an alternative proof of the subdominant coupling estimates of § VII.4. For related material involving $P(\phi)_2$ estimates see also Dimock and Glimm [7].

THEOREM A.4. — If $\Lambda$ is a rectangle, $\int e^{-U_\Lambda^X} d\mu_\Lambda^X = e^{0(|\Lambda|)}$.

*Remark.* — We make our usual assumption that the sides of $\Lambda$ have length greater than 1. In fact, for simplicity of exposition we shall assume that $\Lambda = \bigcup_{i=1}^{|\Lambda|} \Lambda_i$ where the $\Lambda_i$ are unit squares.

*Proof.* — Although this linear lower bound follows at once from (A.1) and the lower bound upon setting $u_j = \exp\left(-U_{\Lambda_j}^{X,\Lambda}\right)$, we wish to give a proof from first principles. The basic idea is to make a Duhamel expansion in each $\Lambda_i$ and to use the Checkerboard estimate to « decouple » these expansions. With $X \neq$ F B. C. some care is required in introducing an ultraviolet cutoff in such a way as to ensure that $U_{\Lambda,\kappa}^X$ remains $\Sigma_\Lambda$-measurable. It is possible to do so, for example, by using a cutoff function $h_\kappa(x)$ with compact support in $x$-space (rather than the function $\tilde{\chi}_\kappa(x)$ used in § VII.4 which had compact support in momentum space) and by introducing an additional spatial cutoff in the integral over $x$. We refrain from writing out the details but shall prove the theorem only for F B. C. in which case the definitions of § VII.4 are appropriate. Thus let $U_{\Lambda_i} = \int_{\Lambda_i} : P(\phi) :$,

$$U_{\Lambda_i,\kappa} = \int_{\Lambda_i} : P(\phi_\kappa) : , \qquad \text{and} \qquad \delta U_{\Lambda_i,\kappa} = U_{\Lambda_i} - U_{\Lambda_i,\kappa} .$$

We perform a Duhamel expansion (VII.29) in each $\Lambda_i$ (see the proof of Theorem VII.10 above):

$$e^{-U_\Lambda} = \prod_i e^{-U_{\Lambda_i}} = \sum_{m=0}^{\infty} (-1)^{|m|} \int_0^1 ds \prod_{i=1}^{|\Lambda|} \prod_{j=1}^{m_i+1} e^{-\delta s_j^i U_{\Lambda_i,\kappa_j}} \prod_{j=1}^{m_i} \delta U_{\Lambda_i,\kappa_j} \tag{A.4}$$

where $m = (m_1, \ldots, m_{|\Lambda|})$; $|m| = m_1 + \ldots + m_{|\Lambda|}$; $\int ds$ represents integration with respect to the $|m|$ variables $s_1^1 \geq \ldots \geq s_{m_1}^1$; $\ldots$ ; $s_1^{|\Lambda|} \geq \ldots \geq s_{m_{|\Lambda|}}^{|\Lambda|}$; and $\delta s_j^i = s_{j-1}^i - s_j^i$.
If we choose $\kappa_j = e^{j^{1/n}}$ then as in (VII.30) we have by the semiboundedness of $U_{\Lambda_i,\kappa}$:

$$\prod_j e^{-\delta s_j^i U_{\Lambda_i,\kappa_j}} \leq \prod_j e^{c_1 j \delta s_j^i} \leq e^{c_1(m_i+1)} . \tag{A.5}$$

By the Checkerboard estimate (since the ultraviolet cutoff introduces a slight overlap, we first apply Hölder's inequality)

$$\int \prod_{i,j} |\delta U_{\Lambda_i,\kappa_j}| \, d\mu_0 \leq \prod_i \left\| \prod_j \delta U_{\Lambda_i,\kappa_j} \right\|_p \tag{A.6}$$

where *p is independent of* $|\Lambda|$. By Hölder's inequality and the smallness of the tail $\delta U_{\Lambda_i,\kappa_j}$ (see VII.28):

$$\begin{aligned} \left\| \prod_{j=1}^{m_i} \delta U_{\Lambda_i,\kappa_j} \right\|_p &\leq \prod_{j=1}^{m_i} \| \delta U_{\Lambda_i,\kappa_j} \|_{pm_i} \\ &\leq (c_2 p m_i)^{nm_i} \prod_j \kappa_j^{-\varepsilon} \\ &\leq c_3 (c_2 p m_i)^{nm_i} e^{-c_4 m_i^{1+n^{-1}}} \end{aligned} \tag{A.7}$$

where $c_3$, $c_4$ are positive constants. Applying the estimates (A.5) and (A.7) to (A.4) we obtain an expansion which *factors*:

$$\begin{aligned}\int e^{-U_\Lambda} d\mu_0 &\leq \sum_m \prod_i \frac{1}{m_i!} e^{c_1(m_i+1)} c_3 (c_2 p m_i)^{n m_i} e^{-c_4 m_i^{1+n^{-1}}} \\ &= \prod_i \sum_{m_i} \frac{c_3}{m_i!} \exp\left[c_1(m_i+1) + n m_i \ln c_2 p m_i - c_4 m_i^{1+n^{-1}}\right] \\ &= \prod_i 0(1) = e^{0(|\Lambda|)}. \quad \blacksquare\end{aligned}$$

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