ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 25, nº 2 (1976), p. 165-175
<http://www.numdam.org/item?id=AIHPA_1976__25_2_165_0>

Ann. Inst. Henri Poincaré Section A :
Vol. XXV, n° 2, 1976, p. 165-175. *Physique théorique.*

On quantizing A-bundles over Hamilton G-spaces

by

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ABSTRACT. — It is shown that a natural generalization of Kostant's results concerning prequantization yields a characterization of quantizing A-bundles over (A, λ)-quantizable symplectic manifolds. Furthermore, proof is given for the statement that any quantizing bundle over a Hamilton G-space can be considered as a G-quantizing bundle.

1. INTRODUCTION

Geometric quantization ([*2*] [*5*]) gives a procedure for the construction of representations of the Poisson algebra $\mathfrak{F}(M)$. The basic step consists of a quantizing bundle over a symplectic manifold (M, ω). Generalizing the prequantization technique of Kostant, we introduced a more general definition of a quantizing bundle [*7*] which includes (up to association) Kostant's Hermitian line bundle and Souriau's *espace fibré quantifiant*.

In the following a mathematical description of these generalized quantizing bundles over (A, λ)-quantizable symplectic manifolds is given. A theorem of Milnor [*3*] determines a bijection between the group $\pi_1^A(M, m_0)$ of group homomorphisms $\pi_1(M, m_0) \to A$ and the group $\mathbf{F}(A, M)$ of equivalence classes of flat principal bundles over M with abelian structure group A. It then follows that the set $\mathbf{Q}(A, \lambda, M, \omega)$ of equivalence classes of quantizing bundles over (A, λ)-quantizable (M, ω) is characterized by a free and transitive action

$$\mathbf{Q}(A, \lambda, M, \omega) \times \pi_1^A(M, m_0) \to \mathbf{Q}(A, \lambda, M, \omega)$$

of $\pi_1^A(M, m_0)$ on $\mathbf{Q}(A, \lambda, M, \omega)$.

The infinitesimal action of the Poisson algebra $\mathfrak{F}(M)$ on M can be lifted to the total space P of a quantizing bundle over (M, ω). This is most useful when applied to representations by complete vector fields. Especially, Theorem 3 says that any quantizing bundle over a Hamilton G-space (G/K, ω, Φ) appears (up to equivalence) as a G-quantizing bundle. Moreover, Theorem 3 establishes a natural one-one correspondence between $\mathbf{Q}(A, \lambda, G/K, \omega)$ and a set of Lie group homomorphisms $K \to A$. This is used to generalize a theorem of Kostant [2] characterizing Hermitian line bundles over Hamilton G-spaces.

2. QUANTIZING BUNDLES

Throughout (P, A, M) will denote a smooth principal bundle over a connected manifold M with abelian structure group A. π will denote the projection $P \to M$. Let α be a connection form on P and let ω be a symplectic structure on M. Given a linear map $\lambda : \mathbf{R} \to \mathfrak{a}$ from the real numbers into the Lie algebra of A, we say that

$$(P, \alpha, A, \lambda, M, \omega)$$

is an (A, λ, M, ω)-*bundle* if

$$d\alpha = \lambda \pi^* \omega .$$

It is called a *quantizing* A-*bundle* (or simply *quantizing bundle*) if λ is injective. In this case, we will say that (M, ω) is (A, λ)-*quantizable.* Otherwise, if $\lambda = 0$, then an (A, λ, M, ω)-bundle is said to be a *flat principal bundle.*

Let $\mathfrak{U} = \{ U_i ; i \in I \}$ be a simple covering of M. If we suppose $\{ f_{ij} ; i, j \in I \}$ to be the transition functions of (P, A, M) corresponding to a trivialization $\{ U_i, \varphi_i ; i \in I \}$, a formal computation shows that an (A, λ, M, ω)-bundle is characterized by a system $\{ f_{ij}, \alpha_i ; i, j \in I \}$ of (A, λ, $\mathfrak{U}$, ω)-functions ; that is, there exist $\alpha_{ij} \in \mathfrak{F}_{\mathfrak{a}}(U_i \cap U_j)$ with

$$f_{ij} = \exp \alpha_{ij} , \qquad d\alpha_{ij} = \alpha_j - \alpha_i , \qquad d\alpha_i = \lambda \omega .$$

Here exp denotes the exponential map $\mathfrak{a} \to A$.

Furthermore, two (A, λ, M, ω)-bundles (P, α, A, λ, M, ω) and (P$'$, α', A, λ, M, ω) are equivalent iff the associated systems $\{ f_{ij}, \alpha_i ; i, j \in I \}$ and $\{ f'_{ij}, \alpha'_i ; i, j \in I \}$ of (A, λ, $\mathfrak{U}$, ω)-functions are equivalent, i. e. iff there are $\beta_i \in \mathfrak{F}_{\mathfrak{a}}(U_i)$ such that

$$f'_{ij} = \exp - \beta_i f_{ij} \exp \beta_j , \alpha'_i = \alpha_i + d\beta_i .$$

For the special case where λ is injective, the proof can be found in [7] ; exactly the same proof gives the corresponding result for the general case.

Denote by $\mathbf{P}(A, \lambda, \mathfrak{U}, \omega)$ the set of equivalence classes of systems of

$(A, \lambda, \mathfrak{U}, \omega)$-functions. The equivalence class of $\{ f_{ij}, \alpha_i ; i, j \in I \}$ is denoted by $[f_{ij}, \alpha_i ; i, j \in I]$. If λ is injective we put

$$\mathbf{P}(A, \lambda, \mathfrak{U}, \omega) = \mathbf{Q}(A, \lambda, \mathfrak{U}, \omega) ;$$

otherwise

$$\mathbf{P}(A, O, \mathfrak{U}, \omega) = \mathbf{F}(A, \mathfrak{U}) .$$

It is not hard to conclude that we may define a map

$$\varphi_\lambda^{\mathfrak{U}} : \mathbf{P}(A, \lambda, \mathfrak{U}, \omega) \times \mathbf{F}(A, \mathfrak{U}) \to \mathbf{P}(A, \lambda, \mathfrak{U}, \omega)$$

by

$$([f_{ij}, \alpha_i ; i, j \in I], [f'_{ij}, \alpha'_i ; i, j \in I]) \to [f_{ij}f'_{ij}, \alpha_i + \alpha'_i ; i, j \in I] .$$

The proof of the following result is a straightforward calculation.

PROPOSITION 1. — (1) $\varphi_0^{\mathfrak{U}}$ makes $\mathbf{F}(A, \mathfrak{U})$ into an abelian group.
(2) Suppose that $\mathbf{Q}(A, \lambda, \mathfrak{U}, \omega)$ is not empty. Then

$$\varphi_\lambda^{\mathfrak{U}} : \mathbf{Q}(A, \lambda, \mathfrak{U}, \omega) \times \mathbf{F}(A, \mathfrak{U}) \to \mathbf{Q}(A, \lambda, \mathfrak{U}, \omega)$$

is a free and transitive action of $\mathbf{F}(A, \mathfrak{U})$ on $\mathbf{Q}(A, \lambda, \mathfrak{U}, \omega)$.

Now consider a refinement $\mathfrak{V} = \{ V_j ; j \in J \}$ of the open covering $\mathfrak{U} = \{ U_i ; i \in I \}$. Choose a map $\sigma : J \to I$ such that $V_j \subset U_{\sigma_j}$ for $j \in J$. This defines a map

$$r_{\mathfrak{V}}^{\mathfrak{U}} : \mathbf{P}(A, \lambda, \mathfrak{U}, \omega) \to \mathbf{P}(A, \lambda, \mathfrak{V}, \omega)$$

by the equation

$$r_{\mathfrak{V}}^{\mathfrak{U}}[f_{ij}, \alpha_i ; i, j \in I] = [f_{\sigma_k, \sigma_l}, \alpha_{\sigma_k} ; k, l \in J] .$$

Let $\tau : J \to I$ be another map with $V_j \subset U_{\tau_j}$. Suppose $\alpha_{ij} \in \mathfrak{F}_a(U_i \cap U_j)$ such that $f_{ij} = \exp \alpha_{ij}$, $d\alpha_{ij} = \alpha_j - \alpha_i$. Then the $\beta_k = \alpha_{\sigma_k, \tau_k} \in \mathfrak{F}_a(V_k)$, $k \in J$, define an equivalence between $\{ f_{\sigma_k, \sigma_l}, \alpha_{\sigma_k} ; k, l \in J \}$ and $\{ f_{\tau_k, \tau_l}, \alpha_{\tau_k} ; k, l \in J \}$. Therefore $r_{\mathfrak{V}}^{\mathfrak{U}}$ does not depend on the choice of refinement map $\sigma : J \to I$. Notice that $r_{\mathfrak{U}}^{\mathfrak{U}}$ is the identity, and if $\mathfrak{W}$ is a refinement of $\mathfrak{V}$ then $r_{\mathfrak{W}}^{\mathfrak{U}} = r_{\mathfrak{W}}^{\mathfrak{V}} r_{\mathfrak{V}}^{\mathfrak{U}}$. Hence $\{ \mathbf{P}(A, \lambda, \mathfrak{U}, \omega), r_{\mathfrak{V}}^{\mathfrak{U}} \}$ forms a direct system over the directed set of open coverings of M. We call the direct limit

$$\mathbf{P}(A, \lambda, M, \omega) .$$

The elements of $\mathbf{P}(A, \lambda, M, \omega)$ will be denoted by $[P, \alpha, A, \lambda, M, \omega]$. Since the equivalence classes of principal bundles over M with abelian structure group A are in a natural one-one correspondence with the elements of the cohomology group $H^1(M, A)$, the above discussion gives the following theorem.

THEOREM 1. — There is a natural one-one correspondence between the elements of $\mathbf{P}(A, \lambda, M, \omega)$ and the equivalence classes of (A, λ, M, ω)-bundles.

If λ is injective then we write

$$\mathbf{P}(A, \lambda, M, \omega) = \mathbf{Q}(A, \lambda, M, \omega) ;$$

otherwise

$$\mathbf{P}(A, O, M, \omega) = \mathbf{F}(A, M).$$

It is easy to check that the φ_λ^{u} define a map

$$\varphi_\lambda^{M} : \mathbf{P}(A, \lambda, M, \omega) \times \mathbf{F}(A, M) \rightarrow \mathbf{P}(A, \lambda, M, \omega)$$

in a natural way. By Proposition 1 we have

PROPOSITION 2. — (1) φ_0^{M} makes $\mathbf{F}(A, M)$ into an abelian group.
(2) Assume that (M, ω) is (A, λ)-quantizable. Then

$$\varphi_\lambda^{M} : \mathbf{Q}(A, \lambda, M, \omega) \times \mathbf{F}(A, M) \rightarrow \mathbf{Q}(A, \lambda, M, \omega)$$

is a free and transitive action of $\mathbf{F}(A, M)$ on $\mathbf{Q}(A, \lambda, M, \omega)$. In other words, the group of equivalence classes of flat principal bundles over M with structure group A acts freely and transitively on the set of equivalence classes of quantizing bundles over (A, λ)-quantizable (M, ω).

Suppose that (P, G, M) is a principal bundle with structure group G. Let $\rho : G \rightarrow A$ be a group homomorphism. Then the ρ-bundle associated with (P, G, M) is the principal bundle

$$(P \times_\rho A, A, M),$$

where $P \times_\rho A$ is the orbit space of the right G-action on $P \times A$ given by letting $g \in G$ take (p, a) to $(pg, a\rho(g))$. The equivalence class of (p, a) is denoted by $[p, a]$. Note that the structure group A acts on $P \times_\rho A$ by $[p, a]a' = [p, a'^{-1}a]$ for $a' \in A$.

Now let $\tilde{M}$ be the universal covering manifold of the connected manifold M and let $(\tilde{M}, \pi_1(M, m_0), M)$ stand for the principal bundle with structure group $\pi_1(M, m_0)$ and covering projection $p : \tilde{M} \rightarrow M$. Next, consider the trivial principal bundle $(\tilde{M} \times A, A, \tilde{M})$ with the canonical flat connection. Since p is a local diffeomorphism, the A-equivariant principal bundle homomorphism

$$(\varphi, p) : (\tilde{M} \times A, A, \tilde{M}) \rightarrow (\tilde{M} \times_\rho A, A, M)$$

given by $\varphi(\tilde{m}, a) = [\tilde{m}, a]$ induces a flat connection form α_ρ on $\tilde{M} \times_\rho A$. Here ρ is an element in the group $\pi_1^A(M, m_0)$ of group homomorphisms $\pi_1(M, m_0) \rightarrow A$.

In the case when A is abelian, the following fact can be derived from a result of Milnor [3].

PROPOSITION 3. — The association

$$\rho \rightarrow (\tilde{M} \times_\rho A, \alpha_\rho, A, M)$$

induces a group isomorphism

$$\pi_1^A(M, m_0) \rightarrow \mathbf{F}(A, M).$$

An immediate application of Propositions 2 and 3 is the generalization of theorems of Kostant ([*2*], p. 135 and 142) to (A, λ)-quantizable manifolds.

THEOREM 2. — Assume that (M, ω) is (A, λ)-quantizable. Then there is a canonical free and transitive action of $\pi_1^A(M, m_0)$ on $\mathbf{Q}(A, \lambda, M, \omega)$.

COROLLARY. — Assume that (M, ω) is simply connected and (A, λ)-quantizable. Then $\mathbf{Q}(A, \lambda, M, \omega)$ has exactly one element.

3. LIE GROUP ACTIONS

Given a symplectic manifold (M, ω), let $\{ \varphi, \psi \}$ be the Lie algebra structure on $\mathfrak{F}(M)$ defined by

$$\{ \varphi, \psi \} = \xi_\varphi \psi = \omega(\xi_\psi, \xi_\varphi).$$

Here ξ_φ is the Hamiltonian vector field corresponding to $\varphi \in \mathfrak{F}(M)$. For any quantizing bundle (P, α, A, λ, M, ω) over (M, ω) the Lie algebra homomorphism

$$\varphi \in \mathfrak{F}(M) \to \xi_\varphi \in \mathfrak{V}(M)$$

can be lifted to an injective homomorphism

$$\delta : \mathfrak{F}(M) \to \mathfrak{V}(P)$$

by setting

$$(\delta\varphi)_p = (\xi_\varphi)_p^* - (\lambda\varphi(\pi p))_p^+ ,$$

$p \in P$ ([*2*] [*7*]). Here ξ_φ^* is the horizontal lift of ξ_φ and x^+ is the vector field on P induced by $x \in \mathfrak{a}$. The map δ is called *prequantization.*

We shall need the following fact.

LEMMA 1. — α is an invariant 1-form of $\delta\varphi$ for $\varphi \in \mathfrak{F}(M)$; that is,

$$L_{\delta\varphi}\alpha = 0 .$$

Proof. — We have

$$i(\delta\varphi)\alpha_p = - \alpha_p((\lambda\varphi(\pi p))^+) = - \lambda\varphi(\pi p),$$

i. e.

$$d(i(\delta\varphi)\alpha) = - \lambda d(\varphi\pi) = - \lambda\pi^* d\varphi .$$

On the other hand

$$i(\delta\varphi)d\alpha = i(\delta\varphi)\lambda\pi^*\omega = \lambda\pi^* i(\xi_\varphi)\omega = \lambda\pi^* d\varphi .$$

Consequently

$$L_{\delta\varphi}\alpha = i(\delta\varphi)d\alpha + d(i(\delta\varphi)\alpha) = 0 .$$

Next suppose G is a connected and simply connected Lie group. Let $\Phi : \mathfrak{g} \to \mathfrak{F}(M)$ be a Lie algebra homomorphism from the algebra $\mathfrak{g}$ of left invariant vector fields on G into the Poisson algebra $\mathfrak{F}(M)$. We assume that

$$x \in \mathfrak{g} \to \xi_{\Phi(x)} \in \mathfrak{V}(M)$$

is an infinitesimal action by complete vector fields. Then it is not hard to see that each $(\delta\Phi)(x) \in \mathfrak{V}(P)$ generates a global flow

$$F_{(\delta\Phi)(x)} : P \times \mathbf{R} \to P$$

by

$$F_{(\delta\Phi)(x)}(p, t) = F_{\xi^*_{\Phi(x)}}(p, t) \exp - t\lambda(\Phi(x))(\pi p).$$

Hence, in view of a result of Palais [*4*], we have

PROPOSITION 4. — In the above situation, there exists a G-action

$$P \times G \to P,$$

written $(p, g) \to pg$, such that

(*i*) $p \operatorname{Exp} x = F_{(\delta\Phi)(x)}(p, 1)$ for $x \in \mathfrak{g}$;
(*ii*) α is G-invariant.

Here Exp means the exponential map $\mathfrak{g} \to G$.

We come now to Hamilton G-spaces. Let $(G/K, \omega)$ be a homogeneous symplectic manifold and let

$$\theta : G/K \times G \to G/K,$$

written $([g], g') \to [g]g'$, be the natural right G-action given by $[g]g' = [g'^{-1}g]$ for $[g] \in G/K$, $g' \in G$. The infinitesimal action $\mathfrak{g} \to \mathfrak{V}(G/K)$ associated to θ will be denoted by θ, too.

Given a Lie algebra homomorphism $\Phi : \mathfrak{g} \to \mathfrak{F}(G/K)$, we call

$$(G/K, \omega, \Phi)$$

a *Hamilton G-space* if

(*i*) G is connected and simply connected;
(*ii*) $\theta(x) = \xi_{\Phi(x)}$ for $x \in \mathfrak{g}$.

LEMMA 2. — With the notation above,

$$(\Phi(\operatorname{ad} g'x))[g] = (\Phi(x))[g'^{-1}g]$$

for $x \in \mathfrak{g}$, $g, g' \in G$.

Proof. — Let $\eta_{[g]} \in T_{[g]}(G/K)$, then

$$\begin{aligned}\eta_{[g]}(\Phi(\operatorname{ad} g'x)) &= \omega_{[g]}(\xi_{\Phi(\operatorname{ad} g'x)}, \eta) \\ &= \omega_{[g]}(\theta(\operatorname{ad} g'x), \eta) = \omega_{[g]}\big((R_{g'}^{-1}) * \theta_{[g'^{-1}g]}(x), \eta_{[g]}\big).\end{aligned}$$

Since ω is G-invariant, it follows that

$$\begin{aligned}\eta_{[g]}(\Phi(\operatorname{ad} g'x)) &= \omega_{[g'^{-1}g]}(\theta(x), (R_{g'}) * \eta_{[g]}) \\ &= ((R_{g'}) * \eta_{[g]})_{[g'^{-1}g]}(\Phi(x)) = \eta_{[g]}(\Phi(x)R_g).\end{aligned}$$

Thus

$$\Phi(\operatorname{ad} g'x) = \Phi(x)R_{g'}.$$

Let $\rho : K \to A$ be a Lie group homomorphism. The ρ-bundle

$(G \times_\rho A, A, G/K)$ associated with $(G, K, G/K)$ can be regarded as a right G-bundle by $[g, a]g' = [g'^{-1}g, a]$. Observe that the actions of G and A on $G \times_\rho A$ commute.

A quantizing bundle $(G \times_\rho A, \alpha, A, \lambda, G/K, \omega)$ over a Hamilton G-space $(G/K, \omega, \Phi)$ is called a *G-quantizing bundle* if

$$(\delta\Phi)(x) = x^+$$

for $x \in \mathfrak{g}$. Here x^+ denotes the vector field on $G \times_\rho A$ induced by $x \in \mathfrak{g}$. We shall prove that each quantizing bundle over a Hamilton G-space is equivalent to a G-quantizing bundle. First, we need some material concerning invariant connections [*6*].

PROPOSITION 5. — There is a one-one correspondence between the set of G-invariant connections on $(G \times_\rho A, A, G/K)$ and the set of linear maps $\Lambda : \mathfrak{g} \to \mathfrak{a}$ with

(*i*) $\Lambda(y) = \rho(y)$ for $y \in \mathfrak{k}$;

(*ii*) $\Lambda(\operatorname{ad} kx) = \Lambda(x)$ for $k \in K$, $x \in \mathfrak{g}$,

where $\mathfrak{k}$ denotes the Lie algebra of K; the correspondence is given by

$$\Lambda(x) = -\alpha_{[e,e]}(x^+)$$

for $x \in \mathfrak{g}$.

For the proof of Proposition 5 see also [*1*].

Now let K_λ^Φ be the set of Lie group homomorphisms $\rho : K \to A$ such that

$$\rho(y) = \lambda(\Phi(y))[e]$$

for $y \in \mathfrak{k}$. As a consequence of Lemma 2 and Proposition 5 we get

PROPOSITION 6. — Let $(G/K, \omega, \Phi)$ be a Hamilton G-space. Then, for any $\rho \in K_\lambda^\Phi$, there is exactly one G-invariant connection (say α^ρ) on $(G \times_\rho A, A, G/K)$ such that

$$\lambda(\Phi(x))[e] = -\alpha^\rho_{[e,e]}(x^+)$$

for $x \in \mathfrak{g}$.

The following result generalizes a theorem of Kostant ([*2*], p. 203).

THEOREM 3. — Suppose that $(G/K, \omega, \Phi)$ is a Hamilton G-space. Then, for any $\rho \in K_\lambda^\Phi$,

$$(G \times_\rho A, \alpha^\rho, A, \lambda, G/K, \omega)$$

is a G-quantizing bundle. Moreover, this association induces a natural one-one correspondence between K_λ^Φ and $\mathbf{Q}(A, \lambda, G/K, \omega)$.

Thus each element in $\mathbf{Q}(A, \lambda, G/K, \omega)$ is represented by exactly one G-quantizing bundle. Observe that $(G/K, \omega, \Phi)$ is (A, λ)-quantizable iff K_λ^Φ is not empty.

4. PROOF OF THEOREM 3

We first prove that each $\rho \in K_\lambda^\Phi$ induces a G-quantizing bundle $(G \times {}_\rho A, \alpha^\rho, A, \lambda, G/K, \omega)$. It is sufficient to show that

(*a*) $d\alpha^\rho = \lambda\pi^*\omega$;

(*b*) $(\delta\Phi)(x) = x^+$ for $x \in \mathfrak{g}$.

(*a*) Choose
$$\xi^i_{[g,a]} = (x_i)^+_{[g,a]} + (y_i)^+_{[g,a]} \in T_{[g,a]}(G \times_\rho A)$$
with $x_i \in \mathfrak{g}$, $y_i \in \mathfrak{a}$, $i = 1, 2$. Then

(1) $$d\alpha^\rho(\xi^1_{[g,a]}, \xi^2_{[g,a]}) = \xi^1_{[g,a]}\alpha^\rho(\xi^2) - \xi^2_{[g,a]}\alpha^\rho(\xi^1) - \alpha^\rho_{[g,a]}([\xi^1, \xi^2])$$

for $\xi^i = x_i^+ + y_i^+$. By the very definition of α^ρ (compare [*1*], p. 107) we have

(2) $$\xi^1_{[g,a]}\alpha^\rho(\xi^2) = (x_1)^+_{[g,a]}\alpha^\rho(x_2^+) .$$

Next, observe that

(3) $$(R_{g'}) * x^+_{[g,a]} = (\text{ad } g'^{-1}x)^+_{[g'^{-1}g,a]}$$

for $x \in \mathfrak{g}$, $g, g' \in G$. Hence, by Proposition 6,
$$\alpha^\rho_{[g,a]}(x_2^+) = -\lambda(\Phi\,(\text{ad } g^{-1}x_2))[e] .$$

Differentiation yields

(4) $$\xi^1_{[g,a]}\alpha^\rho(\xi^2) = -\lambda(\Phi\,(\text{ad } g^{-1}[x_1, x_2]))[e] .$$

Since the actions of G and A on $G \times_\rho A$ commute, we get

(5) $$\begin{aligned}\alpha^\rho_{[g,a]}([\xi^1, \xi^2]) &= \alpha^\rho_{[g,a]}\,(\text{ad } g^{-1}[x_1, x_2])^+ \\ &= -\lambda(\Phi\,(\text{ad } g^{-1}[x_1, x_2]))[e] .\end{aligned}$$

(1), (4) and (5) imply

(6) $$d\alpha^\rho(\xi^1_{[g,a]}, \xi^2_{[g,a]}) = -\lambda(\Phi\,(\text{ad } g^{-1}[x_1, x_2]))[e] .$$

On the other hand

(7) $$\pi^*\omega(\xi^1_{[g,a]}, \xi^2_{[g,a]}) = \omega_{[g]}(\theta(x_1), \theta(x_2)) = -(\Phi[x_1, x_2])[g] .$$

If we combine (6), (7) and Lemma 2, the assertion (*a*) follows easily.

(*b*) To prove (*b*) we use the G-invariance of α^ρ. Given $x \in \mathfrak{g}$, we can write (see [*1*], p. 104)
$$x^+_{[g,a]} = (\theta(x))^*_{[g,a]} + (\alpha^\rho(x^+_{[g,a]}))^+_{[g,a]} .$$

It follows from (3) and Lemma 2 that
$$\alpha^\rho(x^+_{[g,a]}) = -\lambda(\Phi(x))[g] .$$

This proves (*b*).

Thus, given $\rho \in K_\lambda^\Phi$, we have shown how to construct a G-quantizing bundle. Conversely, any quantizing bundle $(P, \alpha, A, \lambda, G/K, \omega)$ over a

Hamilton G-space (G/K, ω, Φ) generates an element $\rho \in K_\lambda^\Phi$. To prove this, observe that

$$\theta : x \in \mathfrak{g} \rightarrow \xi_{\Phi(x)} \in \mathfrak{V}(G/K)$$

defines an infinitesimal action of G on G/K by complete vector fields. Hence, by Proposition 4, there exists a right G-action of P such that

$$p \text{ Exp } tx = F_{(\delta\Phi)(x)}(p, t)$$

for $t \in \mathbf{R}$, $x \in \mathfrak{g}$. Now define $\rho : K \rightarrow A$ by

$$(*) \qquad p_0 k = p_0 \rho^{-1}(k), \qquad p_0 \in \pi^{-1}[e]$$

for $k \in K$. Then $\rho \in K_\lambda^\Phi$ since

$$p_0 \text{ Exp } ty = p_0 \exp - t\lambda(\Phi(y))[e]$$

and

$$p_0 \rho^{-1} (\text{Exp } ty) = p_0 \exp - t\rho(y)$$

for $y \in \mathfrak{k}$. Observe that for a G-quantizing bundle ($G \times_\rho A$, α^ρ, A, λ, G/K, ω) one obtains

$$[e, e]k = [e, e]\rho^{-1}(k), \qquad k \in K.$$

Thus we have reduced the proof of Theorem 3 to the following proposition.

PROPOSITION 7. — Let (P, α, A, λ, G/K, ω) be a quantizing bundle over the Hamilton G-space (G/K, ω, Φ). Define $\rho \in K_\lambda^\Phi$ by (*). Then (P, α, A, λ, G/K, ω) and ($G \times_\rho A$, α^ρ, A, λ, G/K, ω) are equivalent.

Proof. — It is easy to check that the assignment

$$[g, a] \in G \times_\rho A \rightarrow p_0 g^{-1} a^{-1} \in P$$

defines a G, A-equivariant principal bundle isomorphism

$$\varphi : (G \times_\rho A, A, G/K) \rightarrow (P, A, G/K).$$

We show that $\varphi^*\alpha = \alpha^\rho$. Clearly

$$(\varphi^*\alpha)_{[e,e]}(x^+) = \alpha_{p_0}((\delta\Phi)(x))$$

for $x \in \mathfrak{g}$. By Lemma 1 and an argument used above we obtain

$$(\varphi^*\alpha)_{[e,e]}(x^+) = - \lambda(\Phi(x))[e],$$

$x \in \mathfrak{g}$. Since $\varphi^*\alpha$ is a G-invariant connection form, Proposition 6 gives

$$\varphi^*\alpha = \alpha^\rho.$$

The result now follows.

5. CHARACTERIZATION OF K_λ^Φ

In conclusion, we compute the action

$$K_\lambda^\Phi \times \pi_1^A(G/K, [e]) \rightarrow K_\lambda^\Phi$$

given by Theorems 2 and 3 more explicitly. For this purpose, consider the principal bundle (G, K, G/K) with structure group K. Let $K_0 \subset K$ be the identity component of K. Define an action

$$G/K_0 \times K/K_0 \rightarrow G/K_0$$

of K/K_0 on G/K_0 by setting $([g]_0, [k]_0) \rightarrow [gk]_0$. Since G is simply connected, $(G/K_0, K/K_0, G/K)$ is a principal bundle with structure group $K/K_0 \cong \pi_1(G/K, [e])$. Therefore, for $\rho \in K_\lambda^\Phi$, $\sigma \in \pi_1^A(G/K, [e])$, the association

$$(\rho, \sigma) \rightarrow \rho\sigma ,$$

$(\rho\sigma)(k) = \rho(k)\sigma([k]_0)$, defines an action of $\pi_1^A(G/K, [e])$ on K_λ^Φ. The following result proves that, in view of Theorem 3, this action can be identified with the action of $\pi_1^A(G/K, [e])$ on $\mathbf{Q}(A, \lambda, G/K, \omega)$.

PROPOSITION 8. — Let $(G/K, \omega, \Phi)$ be an (A, λ)-quantizable Hamilton G-space. Then the bijection

$$\rho \in K_\lambda^\Phi \rightarrow [G \times_\rho A, \alpha^\rho, A, \lambda, G/K, \omega] \in \mathbf{Q}(A, \lambda, G/K, \omega)$$

is a $\pi_1^A(G/K, [e])$-equivariant map.

Proof. — If (G, K, G/K) is characterized by transition functions $\{ g_{ij} ; i, j \in I \}$, then the system $\{ g_{jj}^0 ; i, j \in I \}$ defined by $g_{ij}^0(x) = [g_{ij}(x)]_0$, $x \in U_i \cap U_j \subset G/K$, represents $(G/K_0, K/K_0, G/K)$. Now

$$\{ (\rho\sigma)g_{ij} ; i, j \in I \}$$

are transition functions associated with $[G \times_{\rho\sigma} A, \alpha^{\rho\sigma}, A, \lambda, G/K, \omega]$, whereas $[G \times_\rho A, \alpha^\rho, A, \lambda, G/K, \omega]\sigma$ is described by

$$\{ (\rho g_{ij})(\sigma g_{ij}^0) ; i, j \in I \} .$$

Since, for $x \in U_i \cap U_j$,

$$((\rho\sigma)g_{ij})(x) = ((\rho g_{ij})(\sigma g_{ij}^0))(x)$$

we conclude that

$$[G \times_{\rho\sigma} A, \alpha^{\rho\sigma}, A, \lambda, G/K, \omega] = [G \times_\rho A, \alpha^\rho, A, \lambda, G/K, \omega]\sigma .$$

This proves the assertion.

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(Manuscrit reçu le 5 juin 1975)