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Section A :
Physique théorique.

The inverse problem for the one-dimensional schrödinger equation with an energy-dependent potential. II (*)

by

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ABSTRACT. — In a previous paper I, the one-dimensional Schrödinger equation $y^{+\prime\prime} + [k^2 - \mathrm{V}^+(k, x)]y^+ = 0$, $x \in \mathbb{R}$, was considered when the potential $\mathrm{V}^+(k, x)$ depends on the energy k^2 in the following way:

$$\mathrm{V}^+(k, x) = \mathrm{U}(x) + 2k\mathrm{Q}(x),$$

$(\mathrm{U}(x), \mathrm{Q}(x))$ belonging to a large class $\mathscr{V}$ of pairs of real potentials admitting no bound state. In this paper, we solve the two systems of differential and integral equations introduced in I. Then, investigating the « inverse scattering problem », we find that a necessary and sufficient condition for one of the functions

$$\mathrm{S}^+(k)=\begin{pmatrix} s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & s_{22}^+(k) \end{pmatrix} (k\in\mathbb{R}) \quad \text{and} \quad \mathrm{S}_{-1}^+(k)=\begin{pmatrix} -s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & -s_{22}^+(k) \end{pmatrix} (k\in\mathbb{R}),$$

to be the « scattering matrix » associated with a pair $(\mathrm{U}(x), \mathrm{Q}(x))$ in $\mathscr{V}$ is that $\mathrm{S}^+(k)$ $(k \in \mathbb{R})$ (or equivalently $\mathrm{S}_{-1}^+(k)$ $(k \in \mathbb{R})$) belongs to the class $\mathscr{S}$ introduced in I. This pair is the only one in $\mathscr{V}$ admitting this function as its scattering matrix. Investigating the « inverse reflection problem », we find that a necessary and sufficient condition for a function $s_{21}^+(k)$ $(k \in \mathbb{R})$ to be the « reflection coefficient to the right » associated with a pair

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$(U(x), Q(x))$ in $\mathscr{V}$ is that $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ belongs to the class $\mathscr{R}$ introduced in I. This pair is the only one in $\mathscr{V}$ admitting this function as its reflection coefficient to the right. We apply our study to the solution of an inverse problem for the one-dimensional Klein-Gordon equation of zero mass with a static potential.

RÉSUMÉ. — Dans un article précédent I, nous avons étudié l'équation de Schrödinger à une dimension $y^{+\prime\prime} + [k^2 - V^{+}(k, x)]y^{+} = 0$, $x \in \mathbb{R}$, dans le cas où le potentiel $V^{+}(k, x)$ dépend de l'énergie de la façon suivante : $V^{+}(k, x) = U(x) + 2kQ(x)$, $(U(x), Q(x))$ appartenant à une vaste classe $\mathscr{V}$ de couples de potentiels réels n'admettant pas d'état lié. Dans cet article nous résolvons les deux systèmes d'équations différentielles et intégrales introduits en I, ce qui nous permet d'étudier le problème inverse de la diffusion puis celui de la réflexion. Nous montrons qu'une condition nécessaire et suffisante pour que l'une des fonctions

$$S^{+}(k)=\begin{pmatrix} s_{11}^{+}(k) & s_{21}^{+}(k) \\ s_{12}^{+}(k) & s_{22}^{+}(k) \end{pmatrix} \ (k\in\mathbb{R}) \quad \text{et} \quad S_{-1}^{+}(k)=\begin{pmatrix} -s_{11}^{+}(k) & s_{21}^{+}(k) \\ s_{12}^{+}(k) & -s_{22}^{+}(k) \end{pmatrix} \ (k\in\mathbb{R})$$

soit la « matrice de diffusion » associée à un couple $(U(x), Q(x))$ de $\mathscr{V}$ est que $S^{+}(k)$ $(k \in \mathbb{R})$ (ou de façon équivalente $S_{-1}^{+}(k)$ $(k \in \mathbb{R})$) appartienne à la classe $\mathscr{S}$ introduite en I. Ce couple est le seul dans $\mathscr{V}$ qui admette cette fonction pour matrice de diffusion. Nous montrons également qu'une condition nécessaire et suffisante pour qu'une fonction $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ soit le « coefficient de réflexion à droite » associé à un couple $(U(x), Q(x))$ de $\mathscr{V}$ est que $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ appartienne à la classe $\mathscr{R}$ introduite en I. Ce couple est le seul dans $\mathscr{V}$ qui admette cette fonction pour coefficient de réflexion à droite. Nous appliquons notre étude à la résolution d'un problème inverse pour l'équation de Klein-Gordon de masse nulle à une dimension, avec un potentiel statique.

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1. INTRODUCTION

In a previous paper [1] referred to as I in the following, we considered the scattering problem for the one-dimensional Schrödinger equation

$$y^{+\prime\prime} + [E - V^{+}(E, x)]y^{+} = 0, \qquad x \in \mathbb{R}, \tag{1.1}$$

with the energy-dependent potential

$$V^{+}(E, x) = U(x) + 2\sqrt{E}Q(x), \qquad x \in \mathbb{R}, \quad E \in \mathbb{C}, \tag{1.2}$$

$$\sqrt{E} = -|E|^{\frac{1}{2}} \exp\left(\frac{i}{2}\operatorname{Arg} E\right), \qquad 0 < \operatorname{Arg} E \leq 2\pi. \tag{1.3}$$

The principal notations and results of I can be found in section 2 of I.

We will use them throughout. We found it useful in I to consider simultaneously both equations

$$y^{\pm''} + [k^2 - V^{\pm}(k, x)]y^{\pm} = 0, \qquad x \in \mathbb{R}, \tag{1.4}$$

$$V^{\pm}(k, x) = U(x) \pm 2kQ(x), \qquad x \in \mathbb{R}, \quad k \in \mathbb{C}; \tag{1.5}$$

indeed, setting $k = \sqrt{E}$ ($E \in \mathbb{C}$) we see that for the index « + » formulas (1.4) and (1.5) for $\operatorname{Im} k < 0$ or $k > 0$ reduce to (1.1) and (1.2). Let us recall that to each pair $(U(x), Q(x))$ in the class $\mathscr{V}$ is associated a 2×2 matrix-valued function, the « scattering matrix »

$$S^{+}(k) = \begin{pmatrix} s_{11}^{+}(k) & s_{21}^{+}(k) \\ s_{12}^{+}(k) & s_{22}^{+}(k) \end{pmatrix} \qquad (k \in \mathbb{R}),$$

for which $S^{+}(k)$ ($k > 0$) represents the « physical part » in the scattering problem associated with equations (1.1) and (1.2). The complex function $s_{21}^{+}(k)$ ($k \in \mathbb{R}$) is the « reflection coefficient to the right ». It has been proved in I that the scattering matrix—resp. the reflection coefficient to the right—associated with a pair in $\mathscr{V}$ belongs to class $\mathscr{S}$—resp. to the clads $\mathscr{R}$—.

In this paper, we will solve two closely connected inverse problems, whose investigation has been prepared for in I:

1. *The « inverse scattering problem »*

For any given function $S^{+}(k)$ ($k \in \mathbb{R}$) belonging to $\mathscr{S}$, does there exist a pair $(U(x), Q(x))$ belonging to $\mathscr{V}$ which admits it as its scattering matrix, and if so, is this pair unique?

2. *The « inverse reflection problem »*

For any given function $s_{21}^{+}(k)$ ($k \in \mathbb{R}$) belonging to $\mathscr{R}$, does there exist a pair $(U(x), Q(x))$ belonging to $\mathscr{V}$ which admits it as its reflection coefficient to the right, and if so, is this pair unique?

Many proofs in this paper are only sketched. For a more detailed version of our work we refer to [*2*].

In sections 2 and 3 we consider the inverse scattering problem. Our method of investigation is based on the solution of the two systems S_1 and S_2 of integral and differential equations introduced in I, in which $S^{+}(k)$ ($k \in \mathbb{R}$) is any given function in $\mathscr{S}$. In section 2 we prove that the system S_1 has a unique solution $(F_1^{+}(x), A_1^{+}(x, t))$ and that the system S_2 has a unique solution $(F_2^{+}(x), A_2^{+}(x, t))$. From the solution $(F_1^{+}(x), A_1^{+}(x, t))$ of S_1 we define a pair $(U_1(x), Q_1(x))$ by formulas (I.2.16). Similarly, from the solution $(F_2^{+}(x), A_2^{+}(x, t))$ of S_2 we define a pair $(U_2(x), Q_2(x))$ by formulas (I.2.17). The pair $(U_1(x), Q_1(x))$ — resp. $(U_2(x), Q_2(x))$ — satisfies the conditions D_1^+ and D_2^+ — resp. D_1^- and D_2^- — introduced in I, section 3. We prove that the function $f_1^{\pm}(k, x)$ defined from $(F_1^+(x), A_1^+(x, t))$ by formula (I.2.10) { where $F_1^-(x)$ and $A_1^-(x, t)$ are respectively defined as the complex conjugates of $F_1^+(x)$ and $A_1^+(x, t)$ } is the « Jost solution at $+\infty$ » of the differential equation (1.4) associated with the pair $(U_1(x), Q_1(x))$. Similarly the function $f_2^{\pm}(k, x)$ defined from $(F_2^+(x), A_2^+(x, t))$ by formula (I.2.11) { where $F_2^-(x)$ and $A_2^-(x, t)$ are respectively defined as the complex conjugates of $F_2^+(x)$ and $A_2^+(x, t)$ } is the « Jost solution at $-\infty$ » of the differential equation (1.4) associated with the pair $(U_2(x), Q_2(x))$.

It is clear from I that at this point in our study we can assert that there is at most one pair in $\mathscr{V}$ which admits a given function in $\mathscr{S}$ as its scattering matrix. This pair is then given by the pair $(U_1(x), Q_1(x))$ constructed from solution of S_1 as well as by the pair $(U_2(x), Q_2(x))$ constructed from solution of S_2.

In section 3 we return to the investigation of the existence question for the inverse scattering problem. The proof of existence is interconnected with the identity of the two constructed pairs $(U_1(x), Q_1(x))$ and $(U_2(x), Q_2(x))$. Precisely we prove that these two pairs are identical, that this unique pair belongs to $\mathscr{V}$ and admits as its scattering matrix the input element in $\mathscr{S}$,

$$S^+(k) = \begin{pmatrix} s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & s_{22}^+(k) \end{pmatrix} \qquad (k \in \mathbb{R}),$$

or the element in $\mathscr{V}$ deduced from it by inversing the sign of the diagonal elements

$$S_{-1}^+(k) = \begin{pmatrix} -\, s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & -\, s_{22}^+(k) \end{pmatrix} \qquad (k \in \mathbb{R}).$$

This ambiguity is not surprising since, clearly, $S^+(k)$ and $S_{-1}^+(k)$ $(k \in \mathbb{R})$ lead to the same systems of equations S_1 and S_2 and so to the same pairs $(U_1(x), Q_1(x))$ and $(U_2(x), Q_2(x))$. In this part we need to use some theorems on functions analytic in the upper half plane. These theorems are the subject of an appendix.

The principal results of our study on the inverse scattering problem may be stated as follows: *A necessary and sufficient condition for one of the functions*

$$S^+(k)=\begin{pmatrix} s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & s_{22}^+(k) \end{pmatrix} \ (k\in\mathbb{R}) \quad \textit{and} \quad S_{-1}^+(k)=\begin{pmatrix} -s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & -s_{22}^+(k) \end{pmatrix} \ (k\in\mathbb{R})$$

to be the scattering matrix associated with a pair $(U(x), Q(x))$ *in* $\mathscr{V}$ *is that* $S^+(k)$ $(k \in \mathbb{R})$ *belongs to the class* $\mathscr{S}$ *(or equivalently that* $S_{-1}^+(k)$ $(k \in \mathbb{R})$ *belongs to* $\mathscr{S}$*). Note that only one of the functions* $S^+(k)$ $(k \in \mathbb{R})$ *and* $S_{-1}^+(k)$ $(k \in \mathbb{R})$ *is the scattering matrix associated with a pair in* $\mathscr{V}$*. This pair is the only one in* $\mathscr{V}$ *admitting this function as its scattering matrix. It can be* *obtained by solving either* S_1 *and using formulas* (I.2.16) *or by solving* S_2 *and using formulas* (I.2.17).

In section 4, we show that the study of the inverse scattering problem can be easily adjusted to that of the inverse reflection problem. We have no ambiguity in the existence question in this case. We find that *a necessary and sufficient condition for a function* $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ *to be the reflection coefficient to the right associated with a pair* $(U(x), Q(x))$ *in* $\mathscr{V}$ *is that* $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ *belongs to the class* $\mathscr{R}$. *This pair is the only one in* $\mathscr{V}$ *admitting this function as its reflection coefficient to the right. It can be obtained by solving* S_1 — *with* $r_1^{+}(t)$ $(t \in \mathbb{R})$ *defined in terms of the given function* $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ *through formula* (I.2.8) — *and using formulas* (I.2.16). We then see in an indirect way that *the scattering matrix* $S^{+}(k)$ $(k \in \mathbb{R})$ *associated with a pair in* $\mathscr{V}$ *is completely determined by the reflection coefficient to the right* $s_{21}^{+}(k)$ $(k \in \mathbb{R})$.

Our study can be applied to the « physical » inverse scattering — resp. reflection — problem associated with equations (1.1) and (1.2) in which only the « physical part » of the scattering « matrix », i. e. $S^{+}(k)$ $(k > 0)$ — resp. of the reflection coefficient to the right, i. e. $s_{21}^{+}(k)$ $(k > 0)$ — is given. Choosing the part $S^{+}(k)$ $(k \leq 0)$ — resp. $s_{21}^{+}(k)$ $(k \leq 0)$ — arbitrarily provided that the input function $S^{+}(k)$ $(k \in \mathbb{R})$ — resp. $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ — belongs to the class $\mathscr{S}$ — resp. $\mathscr{R}$ —, we are reduced to the inverse scattering — resp. reflection — problem invertigated here. The degree of indeterminacy of this choice is an open question.

In section 5 we consider the particular case corresponding to $Q(x) = 0$ which has been already solved by Kay [3], Kay and Moses [4] and Faddeev [5] — note that in this case the scattering matrix $S^{+}(k)$ $(k \in \mathbb{R})$ is completely determined by its « physical part » $S^{+}(k)$ $(k > 0)$ —. We find that in this case it is easy to define the class of input functions $S^{+}(k)$ $(k \in \mathbb{R})$ in such a way that there is no more ambiguity in the existence question for the inverse scattering problem.

In section 6, we apply our study to the solution of an inverse scattering problem associated with the one-dimensional Klein-Gordon equation of zero mass with a static potential. In this case the scattering matrix $S^{+}(k)$ $(k \in \mathbb{R})$ for $k > 0$ describes the scattering of a particle and for $k < 0$ describes the scattering of the correspondent antiparticle.

2. CONSTRUCTION OF TWO PAIRS $(U_1(x), Q_1(x))$ AND $(U_2(x), Q_2(x))$ FROM GIVEN $S^{+}(k)(k \in \mathbb{R})$ IN $\mathscr{S}$

In this section, we are given a function $S^{+}(k)$ $(k \in \mathbb{R})$ belonging to the class $\mathscr{S}$ and we will solve the systems S_1 and S_2 introduced in I. For more details, we refer to [2] and to analogous proofs in [6] [5] and [7].

2.1. A lemma

Let us prove the following result which is slightly stronger that the one needed in this section but which will turn out to be useful in section 3:

LEMMA. — *For any fixed real x, the equation*

$$y(t) = \int_x^{\infty} r_1^+(t+u)\overline{y(u)}du\,, \qquad t \geq x\,, \tag{2.1}$$

has the unique solution $y(t) = 0$ in the class of square integrable complex functions $y(t)$ defined a. e. (almost everywhere) for $t \geq x$ and having their Fourier-transforms $Y(k)$,

$$Y(k) = \underset{n\to\infty}{\text{l.i.m.}} \int_x^{n} y(t)\exp(-ikt)dt\,, \qquad a.\ e.\ for \quad k \in \mathbb{R}\,, \tag{2.2}$$

essentially bounded in $\mathbb{R}$ (i. e. a number K exists such that $|Y(k)| \leq K$ a. e. for $k \in \mathbb{R}$).

From (2.1) we obtain

$$\int_x^{\infty} \overline{y(t)}\,y(t)dt = \int_x^{\infty} \overline{y(t)}dt \int_x^{\infty} r_1^+(t+u)\overline{y(u)}du\,. \tag{2.3}$$

We see from theorems on Fourier transforms that (2.3) can be written in the form

$$\int_{-\infty}^{\infty} \overline{Y(k)}Y(k)dk = -\int_{-\infty}^{\infty} (\overline{Y(k)})^2 s_{21}^+(k)dk\,. \tag{2.4}$$

Adding this equation and its complex conjugate and then using the unitarity of the 2×2 matrix $S^+(k)$, we obtain the equality

$$\int_{-\infty}^{\infty} [\,|Y(k) + s_{21}^+(k)\overline{Y(k)}|^2 + |s_{11}^+(k)Y(k)|^2]dk = 0 \tag{2.5}$$

Hence $Y(k) = 0$ a. e. for $k \in \mathbb{R}$ and therefore $y(t) = 0$ a. e. for $t \geq x$.

2.2. Solution of S_1

Let M_x be the linear operator defined as

$$(M_x y)(t) = \int_x^{\infty} r_1^+(t+u)\overline{y(u)}du\,, \qquad t \geq x\,, \tag{2.6}$$

in the Banach space $L^1(x, \infty)$ of classes of complex functions $y(t)$ integrable in $[x, \infty[$, considered as a real vector space and equipped with the norm $\|y\| = \int_x^{\infty} |y(t)|dt$. Whit the help of a theorem of Frechet-Kolmogorov ([8], p. 275) one can prove that the operator M_x is compact in $L^1(x, \infty)$. It is clear from the lemma that the equation (2.1) has the unique solution $y(t) = 0$ in $L^1(x, \infty)$. From the Fredholm alternative we conclude that the operator $I - M_x$ has an inverse in $L^1(x, \infty)$ for any real x (I is the identity operator). As a consequence, for fixed $F_1^+(x)$, the equation (I.2.19) has a unique solution $A_1^+(x, t)$ in the space of functions of (x, t) $(t \geq x, x \in \mathbb{R})$ which are, for fixed x, continuous and integrable in t for $t \geq x$.

Now we seek to make the dependence of $A_1^+(x, t)$ on $F_1^+(x)$ explicit. Let $a_{1,1}^+(x, t)$ be the solution of the equation (I.2.19) corresponding to $F_1^+(x) = 1$ and $a_{1,2}^+(x, t)$ be the one corresponding to $F_1^+(x) = -i$. Let $\alpha_1^+(x, t)$ and $\beta_1^-(x, t)$ be the functions defined for $(t \geq x, x \in \mathbb{R})$ as

$$\alpha_1^+(x, t) = \frac{a_{1,1}^+(x, t) - ia_{1,2}^+(x, t)}{2}, \quad \beta_1^-(x, t) = \frac{a_{1,1}^+(x, t) + ia_{1,2}^+(x, t)}{2}. \tag{2.7}$$

Let $A_1^-(x, t)$, $F_1^-(x)$, $\alpha_1^-(x, t)$ and $\beta_1^+(x, t)$ respectively be the complex conjugate functions of $A_1^+(x, t)$, $F_1^+(x)$, $\alpha_1^+(x, t)$ and $\beta_1^-(x, t)$. It is easy to find the relation

$$A_1^\pm(x, t) = F_1^\mp(x)\alpha_1^\pm(x, t) + F_1^\pm(x)\beta_1^\mp(x, t), \qquad t \geq x, \quad x \in \mathbb{R}. \tag{2.8}$$

One can prove that the functions $\alpha_1^\pm(x, t)$ and $\beta_1^\mp(x, t)$ belong to the class $\mathscr{A}_1$ and are twice continuously differentiable. Certain useful bounds can be obtained for the partial derivatives (see [2]).

Inserting (2.8) in the equation (I.2.20) and using (I.2.21) and the condition $z_1'(\infty) = 0$, we find the differential equation

$$z_1' = 2i\alpha_1^+(x, x)\exp(iz_1) - 2i\alpha_1^-(x, x)\exp(-iz_1) - 2i\beta_1^+(x, x) + 2i\beta_1^-(x, x), \qquad x \in \mathbb{R}, \tag{2.9}$$

with the condition

$$z_1(\infty) = 0. \tag{2.10}$$

This equation admits a unique solution in the space of real functions differentiable for $x \in \mathbb{R}$. This solution is given by the limit as $n \to \infty$ of the sequence

$$z_{1,n+1}(x) = \int_\infty^x f(t, z_{1,n}(t))dt, \; z_{1,0}(x) = 0 \qquad (x \in \mathbb{R}, \quad n \in \mathbb{N}). \tag{2.11}$$

One can prove without difficulty that the function $A_1^+(x, t)$ defined by (2.8), with $F_1^+(x)$ obtained from $z_1(x) by$ (I.2.21), belongs to the class $\mathscr{A}_1$ and is twice continuously differentiable for $t \geq x$, $x \in \mathbb{R}$. It follows from our study that this pair $(F_1^+(x), A_1^+(x, t))$ is the unique solution of S_1.

2.3. **The pair** $(U_1(x), Q_1(x))$

From the pair $(F_1^+(x, A_1^+(x, t))$, the solution of S_1, we define a pair $(U_1(x), Q_1(x))$ by the formulas (I.2.16). It can be proved that this pair $(U_1(x), Q_1(x))$ satisfies the conditions D_1^+ and D_2^+.

Let us prove that the function $f_1^{\pm}(k, x)$ defined from $(F_1^+(x), A_1^+(x, t))$ by formula (I.2.10) { where $F_1^-(x)$ and $A_1^-(x, t)$ are respectively the complex conjugates of $F_1^+(x)$ and $A_1^+(x, t)$ } is the « Jost solution at $+\infty$ » of the differential equation (1.4) associated with the pair $(U_1(x), Q_1(x))$. We consider the function $a_1^{\pm}(x, t)$ defined as

$$a_1^{\pm}(x, t) = \frac{\partial^2 A_1^{\pm}}{\partial x^2}(x, t) - \frac{\partial^2 A_1^{\pm}}{\partial t^2}(x, t) \pm 2iQ_1(x)\frac{\partial A_1^{\pm}}{\partial t}(x, t), \qquad t \geq x, \quad x \in \mathbb{R}. \quad (2.12)$$

Applying the operator « $\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} + 2iQ_1(x)\frac{\partial}{\partial t}$ » to both sides of the equation (I.2.19) we find, by differentiating under the integral sign and integrating by parts, that $a_1^+(x, t)$ is the solution of the equation obtained by replacing $F_1^+(x)$ by $f_1^+(x)$ in (I.2.19). We therefore have from paragraph 2.2

$$a_1^{\pm}(x, t) = f_1^{\mp}(x)\alpha_1^{\pm}(x, t) + f_1^{\pm}(x)\beta_1^{\mp}(x, t), \qquad t \geq x, \quad x \in \mathbb{R}. \quad (2.13)$$

Now using (I.2.16) and (2.8) to express the R. H. S. of (2.13) in another form we find the equation

$$a_1^{\pm}(x, t) = U(x)A_1^{\pm}(x, t), \qquad t \geq x, \quad x \in \mathbb{R}, \quad (2.14)$$

which is nothing but the equation (I.4.4) occuring in the theorem in section 4 of I. It can be proved that the other conditions for the application of this theorem hold. Hence the desired result.

Let us give the following equation obtained by writing that the Jost solutions $f_1^+(0, x)$ and $f_1^-(0, x)$ are equal:

$$F_1^+(x) + \int_x^{\infty} A_1^+(x, t)dt = F_1^-(x) + \int_x^{\infty} A_1^-(x, t)dt, \qquad x \in \mathbb{R}. \quad (2.15)$$

Using (2.8), (2.15) can be written as

$$[F_1^+(x)]^2\left(1 - \int_x^{\infty} [\alpha_1^-(x, t) - \beta_1^-(x, t)]dt\right) = 1 - \int_x^{\infty} [\alpha_1^+(x, t) - \beta_1^+(x, t)]dt, \qquad x \in \mathbb{R}. \quad (2.16)$$

Formula (2.16) allows us to determine $[F_1^+(x)]^2$ for the values of x which do not cancel the second factor of the L. H. S. This is clearly true at least for x sufficiently large. In general we do not know the position of the zeros of this quantity. This is the reason why in the system S_1 of our inversion procedure we have used equation (I.2.20) and not equation (2.15) as the coupling condition between $F_1^+(x)$ and $A_1^+(x, t)$.

2.4. The pair $(U_2(x), Q_2(x))$

It is clear that the above study can be transposed to the case where we start from system S_2 instead of S_1. System S_2 thus has a unique solution $(F_2^+(x), A_2^+(x, t))$. From this solution we define a pair $(U_2(x), Q_2(x))$ by the formulas (I.2.17). The pair $(U_2(x), Q_2(x))$ satisfies the conditions D_1^- and D_2^-. The function $f_2^\pm(k, x)$ defined from $(F_2^+(x), A_2^+(x, t))$ by formula (I.2.11) $\{$ where $F_2^-(x) = \overline{F_2^+(x)}$ and $A_2^-(x, t) = \overline{A_2^+(x, t)}$ $\}$ is the « Jost solution at $-\infty$ » of the differential equation (1.4) associated with the pair $(U_2(x), Q_2(x))$.

3. THE SCATTERING MATRIX ASSOCIATED WITH THE PAIR $(U_2(x), Q_2(x))$ AND IDENTITY OF THE PAIRS $(U_1(x), Q_1(x))$ AND $(U_2(x), Q_2(x))$

We have explained in the introduction that at this point in our study we can assert that there is at most one pair in $\mathscr{V}$ which admits a given function in $\mathscr{S}$ as its scattering matrix. In this section we seek to solve the existence question for the inverse scattering problem. The proof of existence is interconnected with the identity of the two pairs $(U_1(x), Q_1(x))$ and $(U_2(x), Q_2(x))$ constructed in section 2. Precisely we will prove that these two pairs are identical, that this unique pair belongs to $\mathscr{V}$ and admits as its scattering matrix the function

$$S_\varepsilon^+(k) = \begin{pmatrix} \varepsilon s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & \varepsilon s_{22}^+(k) \end{pmatrix} \qquad (k \in \mathbb{R})$$

where ε is equal to « $+1$ » or « -1 », i. e. the scattering matrix is the input element $S^+(k)$ $(k \in \mathbb{R})$ of $\mathscr{S}$ or the element $S_{-1}^+(k)$ $(k \in \mathbb{R})$ of $\mathscr{S}$ deduced from it by inverting the sign of the diagonal coefficients. Our method of solution is explained in paragraph 3.2 and developed in the following paragraphs. Before this, we introduce in paragraph 3.1 some new quantities which will be very useful. We also give a relation which will be of fundamental importance in our method.

3.1. The functions $g_1^\pm(k, x)$, $g_2^\pm(k, x)$, $\mathscr{F}_1^\pm(k, x)$, $\mathscr{G}_1^\pm(k, x)$ and a fundamental relation

From the given function $S^+(k)$ $(k \in \mathbb{R})$ in $\mathscr{S}$, we construct another function $S^-(k) = [s_{ij}^-(k)]$ $(i = 1, 2 ; j = 1, 2)$ $(k \in \mathbb{R})$ by setting

$$s_{ij}^-(k) = \overline{s_{ij}^+(-k)}, \qquad k \in \mathbb{R}. \tag{3.1}$$

We have the relation

$$S^{\pm}(k)^{t}S^{\mp}(-k) = I, \qquad k \in \mathbb{R}. \tag{3.2}$$

It is easy to see that the function $S^{-}(k)$ ($k \in \mathbb{R}$) also belongs to $\mathscr{S}$ and that we have the relations

$$s_{11}^{-}(k) = \overline{s_{11}^{+}(-\bar{k})}, \qquad \operatorname{Im} k \geq 0, \tag{3.3}$$

$$s_{11}^{\pm}(k) = F_1^{\pm} + 0(k^{-1}) \qquad (|k| \to \infty, \operatorname{Im} k \geq 0), \tag{3.4}$$

where F_1^{+} is defined by the formula (I.2.5) and F_1^{-} is defined as the complex conjugate of F_1^{+}. Furthermore the case $s_{11}^{+}(0) = 0$ is equivalent to the case $s_{11}^{-}(0) = 0$, and the numbers L, L_1 and L_2 which occur in the condition 4) of the class $\mathscr{S}$ are the same for the functions $S^{+}(k)$ and $S^{-}(k)$.

Now we define the following functions

$$g_1^{\pm}(k, x) = s_{21}^{\pm}(k) f_1^{\mp}(-k, x) + f_1^{\pm}(k, x), \qquad k \in \mathbb{R}, \tag{3.5}$$

$$g_2^{\pm}(k, x) = s_{12}^{\pm}(k) f_2^{\mp}(-k, x) + f_2^{\pm}(k, x), \qquad k \in \mathbb{R}, \tag{3.6}$$

$$\mathscr{F}_1^{\pm}(k, x) = g_2^{\mp}(-k, x)[s_{11}^{\mp}(-k)]^{-1}, \qquad k \in \mathbb{R}^*, \tag{3.7}$$

$$\mathscr{G}_1^{\pm}(k, x) = s_{11}^{\pm}(k) f_2^{\mp}(-k, x), \qquad k \in \mathbb{R}. \tag{3.8}$$

These functions are continuous. Furthermore the function $\mathscr{F}_1^{\pm}(k, x)$ ($k \in \mathbb{R}^*$) admits a continuous extension $\mathscr{F}_1^{\pm}(k, x)$ ($k \in \mathbb{R}$) and we have the equality

$$\mathscr{F}_1^{+}(0, x) = \mathscr{F}_1^{-}(0, x), \qquad x \in \mathbb{R}. \tag{3.9}$$

This can be seen from the identity

$$\mathscr{F}_1^{\mp}(-k, x) = \frac{s_{12}^{\pm}(k) + 1}{k} \frac{k}{s_{11}^{\pm}(k)} f_2^{\mp}(-k, x) + \frac{f_2^{\pm}(k, x) - f_2^{\mp}(-k, x)}{k} \frac{k}{s_{11}^{\pm}(k)}, \qquad k \in \mathbb{R}^*, \tag{3.10}$$

and from formulas (I.2.2) and (I.2.4), and the analogues for $f_2^{\pm}(k, x)$ of formulas (I.4.7) and (I.4.8).

Let us give the following algebraic relation which will be of fundamental importance in the next paragraphs:

$$s_{21}^{+}(k)\mathscr{F}_1^{-}(-k, x) + \mathscr{F}_1^{+}(k, x) = \mathscr{G}_1^{+}(k, x), \qquad x \in \mathbb{R}, \quad k \in \mathbb{R}. \tag{3.11}$$

3.2. **The method**

In this paragraph we first show that we will achieve the purpose set at the beginning of this section if we prove that there exist a real number A and a number ε equal to « 1 » or « − 1 » such that

$$\mathscr{F}_1^{+}(k, x) = \varepsilon f_1^{+}(k, x), \qquad k \in \mathbb{R}, \quad x \geq A. \tag{3.12}$$

Then we point out the steps that we will follow in the next paragraphs to prove this equality. Using (3.12) in the fundamental relation (3.11), and recalling (3.5), we obtain the equality

$$\varepsilon \mathscr{G}_1^+(k, x) = g_1^+(k, x), \qquad k \in \mathbb{R}, \quad x \geq \mathrm{A}, \tag{3.13}$$

which, because of (3.8) and (3.5), may also be written in the form

$$\varepsilon s_{11}^+(k) f_2^-(-k, x) = s_{21}^+(k) f_1^-(-k, x) + f_1^+(k, x), \qquad k \in \mathbb{R}, \quad x \geq \mathrm{A}. \tag{3.14}$$

Let us consider the function $\psi_2^+(k, x)$ defined as

$$\psi_2^+(k, x) = \varepsilon s_{11}^+(k) f_2^-(-k, x), \qquad k \in \mathbb{R}, \quad x \in \mathbb{R}. \tag{3.15}$$

From (3.15) $\psi_2^+(k, x)$ is clearly a solution of the differential equation (1.4) associated with the pair $(\mathrm{U}_2(x), \mathrm{Q}_2(x))$ for $x \in \mathbb{R}$ and $k \in \mathbb{R}$. From (3.14) it is also a solution of the equation (1.4) associated with the pair $(\mathrm{U}_1(x), \mathrm{Q}_1(x))$ for $x \geq \mathrm{A}$ and $k \in \mathbb{R}$. We conclude that the pairs $(\mathrm{U}_1(x), \mathrm{Q}_1(x))$ and $(\mathrm{U}_2(x), \mathrm{Q}_2(x))$ coincide for x sufficiently large and therefore that $(\mathrm{U}_2(x), \mathrm{Q}_2(x))$ satisfies assumptions D_1 and D_2. Looking at the asymptotic behaviour of the solution $\psi_2^+(k, x)$ as $x \to \infty$ and $x \to -\infty$, we find that the pair $(\mathrm{U}_2(x), \mathrm{Q}_2(x))$ admits $\mathrm{S}_\varepsilon^+(k)$ as its scattering matrix and belongs to the class $\mathscr{V}$. To show that $(\mathrm{U}_1(x), \mathrm{Q}_1(x))$ and $(\mathrm{U}_2(x), \mathrm{Q}_2(x))$ are equal for all real x, we first notice that our inverse problem admits a solution which is $(\mathrm{U}_2(x), \mathrm{Q}_2(x))$ if $\mathrm{S}_\varepsilon^+(k)$ $(k \in \mathbb{R})$ is the input function. We have explained in the introduction that this solution must coincide with the pair constructed from solution of S_1. This pair is nothing but $(\mathrm{U}_1(x), \mathrm{Q}_1(x))$ since the input functions $\mathrm{S}^+(k)$ $(k \in \mathbb{R})$ and $\mathrm{S}_\varepsilon^+(k)$ $(k \in \mathbb{R})$ lead to the same system S_1.

The proof of equality (3.12) is the subject of the next paragraphs. In 3.3, $\mathscr{F}_1^+(k, x)$ is shown to admit a representation formula similar to that (I.2.10) of $f_1^+(k, x)$ with functions $\mathrm{F}_1^+(x)$ and $\mathrm{A}_1^+(x, t)$ replaced by certain functions $\tilde{\mathrm{F}}_1^+(x)$ and $\tilde{\mathrm{A}}_1^+(x, t)$. A representation formula similar to that (I.2.11) of $f_2^+(k, x)$ is also derived for $\mathscr{G}_1^+(k, x)$. In 3.4 we use the fundamental relation (3.11) to show that the pair $(\tilde{\mathrm{F}}_1^+(x), \tilde{\mathrm{A}}_1^+(x, t))$ is the solution of the integral equation (I.2.19). This allows us to conclude that $\tilde{\mathrm{A}}_1^+(x, t)$ depends on $\tilde{\mathrm{F}}_1^+(x)$ in the same way that $\mathrm{A}_1^+(x, t)$ depends on $\mathrm{F}_1^+(x)$ in formula (2.8). Then using the fact that the functions $\mathscr{F}_1^+(0, x)$ and $\mathscr{F}_1^-(0, x)$ are equal for $x \in \mathbb{R}$ as do the functions $f_1^+(0, x)$ and $f_1^-(0, x)$, we prove that there exist a real number A and a number ε equal to « 1 » or « -1 » such that $\tilde{\mathrm{F}}_1^+(x)$ and $\varepsilon \mathrm{F}_1^+(x)$ are equal for $x \geq \mathrm{A}$. Hence we find that $\tilde{\mathrm{A}}_1^+(x, t)$ and $\varepsilon \mathrm{A}_1^+(x, t)$ are equal for $x \geq \mathrm{A}$ and finally we obtain the equality (3.12).

3.3. Representation formulas for the functions $\mathscr{F}_1^+(k, x)$ and $\mathscr{G}_1^+(k, x)$; the functions $\tilde{F}_1^+(x)$ and $\tilde{A}_1^+(x, t)$

Let us first derive a transformation formula for the function $g_2^+(k, x)$. Using theorems on Fourier transforms we see that $g_2^+(k, x)$ may be written in the following form, for $x \in \mathbb{R}$ and a. e. (almost every) $k \in \mathbb{R}$:

$$g_2^+(k, x) = F_2^+(x) \exp(ikx) + \underset{n\to\infty}{\mathrm{l.i.m.}} \int_{-n}^{n} B_2^+(x, t) \exp(ikt)dt, \quad (3.16)$$

where $B_2^+(x, t)$ is for real x a square integrable function in t, such that

$$B_2^+(x, t) = A_2^+(x, t) - F_2^-(x) r_2^+(x + t) - \int_{-\infty}^{x} r_2^+(u + t) A_2^-(x, u)du, \quad \text{for a. e.} \quad t \leq x. \quad (3.17)$$

Therefore from equation (I.2.22), $B_2^+(x, t) = 0$ for a. e. $t \leq x$. $B_2^-(x, t)$ being the complex conjugate function of $B_2^+(x, t)$ we can write, for $x \in \mathbb{R}$, a. e. $k \in \mathbb{R}$,

$$g_2^\pm(k, x) = F_2^\pm(x) \exp(ikx) + \underset{n\to\infty}{\mathrm{l.i.m.}} \int_{x}^{n} B_2^\pm(x, t) \exp(ikt)dt. \quad (3.18)$$

Applying a Titchmarsh theorem recalled in the appendix, we conclude that, x being any fixed real number, the function

$$\text{« } g_2^\pm(k, x) \exp(-ikx) - F_2^\pm(x) \text{ »} \qquad (k \in \mathbb{R})$$

belongs to the class $\mathscr{C}$ defined in this appendix. Using the definition of the class $\mathscr{C}$ and theorem 1 of the appendix we find that the function $g_2^\pm(k, x)$ $(k \in \mathbb{R})$ admits a continuous extension $g_2^\pm(k, x)$ $(\mathrm{Im}\, k \geq 0)$ which is analytic for $\mathrm{Im}\, k > 0$ and such that

$$g_2^\pm(k, x) \exp(-ikx) = 0(1) \qquad (\mathrm{Im}\, k \geq 0). \quad (3.19)$$

Applying the Phragmen-Lindelöf theorem we can also prove that

$$g_2^\pm(k, x) \exp(-ikx) - F_2^\pm(x) = 0(k^{-1}) \qquad (|k| \to \infty, \mathrm{Im}\, k \geq 0). \quad (3.20)$$

Using the formula (3.7) it is now easy to see that the function $\mathscr{F}_1^\pm(k, x)$ $(k \in \mathbb{R})$ admits a continuous extension $\mathscr{F}_1^\pm(k, x)$ $(\mathrm{Im}\, k \leq 0)$ which is analytic for $\mathrm{Im}\, k < 0$ (the case $s_{11}^+(0) = 0$ requires special investigation and use of theorem 2 of the appendix). On the other hand, using (3.20) and (3.4), we can prove that

$$\mathscr{F}_1^\mp(-k, x) \exp(-ikx) - F_2^\pm(x) F_1^\mp = 0(k^{-1}) \qquad (|k| \to \infty, \mathrm{Im}\, k \geq 0). \quad (3.21)$$

With the help of (3.21) it is easy to prove that the function

$$\text{« } \mathscr{F}_1^\mp(-k, x) \exp(-ikx) - F_2^\pm(x) F_1^\mp \text{ »} \qquad (k \in \mathbb{R})$$

belongs to the class $\mathscr{C}$. Therefore, from Titchmarsh's theorem, there exists a function $\tilde{A}_1^{\pm}(x, t)$ defined for $x \in \mathbb{R}$, a. e. $t \geq x$, and square integrable in t for $t \geq x$ such that

$$\mathscr{F}_1^{\pm}(k, x) = \tilde{F}_1^{\pm}(x) \exp(-ikx) + \underset{n \to \infty}{\text{l. i. m.}} \int_x^n \tilde{A}_1^{\pm}(x, t) \exp(-ikt) dt, \quad \text{for} \quad x \in \mathbb{R}, \quad \text{a. e.} \quad k \in \mathbb{R}, \tag{3.22}$$

where we have set $\tilde{F}_1^{\pm}(x) = F_1^{\pm} F_2^{\mp}(x)$. Clearly the functions $\tilde{A}_1^{+}(x, t)$ and $\tilde{A}_1^{-}(x, t)$, $\tilde{F}_1^{+}(x)$ and $\tilde{F}_1^{-}(x)$ are respectively conjugate.

It is clear from formula (3.8) that, for fixed x, the function $\mathscr{G}_1^{\pm}(k, x)$ $(k \in \mathbb{R})$ admits a continuous extension $\mathscr{G}_1^{\pm}(k, x)$ $(\operatorname{Im} k \geq 0)$ which is analytic for $\operatorname{Im} k > 0$ and such that

$$\mathscr{G}_1^{\pm}(k, x) \exp(ikx) - F_1^{\pm} F_2^{\mp}(x) = 0(k^{-1}) \qquad (|k| \to \infty, \operatorname{Im} k \geq 0). \tag{3.23}$$

As a consequence the function « $\mathscr{G}_1^{\pm}(k, x) \exp(ikx) - F_1^{\pm} F_2^{\mp}(x)$ » $(k \in \mathbb{R})$ belongs to the class $\mathscr{C}$. Therefore, from Titchmarsh's theorem, there exists a function $\tilde{B}_1^{\pm}(x, t)$ defined for $x \in \mathbb{R}$, a. e. $t \leq x$, and square integrable in t for $t \leq x$ such that

$$\mathscr{G}_1^{\pm}(k, x) = \tilde{F}_1^{\pm}(x) \exp(-ikx) + \underset{n \to \infty}{\text{l. i. m.}} \int_{-n}^x \tilde{B}_1^{\pm}(x, t) \exp(-ikt) dt, \quad \text{for} \quad x \in \mathbb{R}, \quad \text{a. e.} \quad k \in \mathbb{R}. \tag{3.24}$$

Clearly the functions $\tilde{B}_1^{+}(x, t)$ and $\tilde{B}_1^{-}(x, t)$ are conjugate.

3.4. Relation between $f_1^{+}(k, x)$ and $\mathscr{F}_1^{+}(k, x)$

Let us insert the representation formulas (I.2.8), (3.22) and (3.24) in the fundamental relation (3.11). Using properties of Fourier transforms we find that the pair $(\tilde{F}_1^{+}(x), \tilde{A}_1^{+}(x, t))$ satisfies the integral equation (I.2.19) for $x \in \mathbb{R}$, a. e. $t \geq x$. We know that $\tilde{A}_1^{+}(x, t)$ is a square integrable function whose Fourier transform is essentially bounded in $\mathbb{R}$. So we can use the lemma of paragraph 2.1 and results of paragraph 2.2 to assert that $\tilde{A}_1^{\pm}(x, t)$ can be chosen in such a way that for every pair (x, t) such that $t \geq x$, $x \in \mathbb{R}$, we have

$$\tilde{A}_1^{\pm}(x, t) = \tilde{F}_1^{\mp}(x) \alpha_1^{\pm}(x, t) + \tilde{F}_1^{\pm}(x) \beta_1^{\mp}(x, t). \tag{3.25}$$

So, for fixed x, the function $\tilde{A}_1^{\pm}(x, t)$ is integrable for $t \geq x$. Hence we can replace (3.22) by the following representation formula valid for every real x and every real k:

$$\mathscr{F}_1^{\pm}(k, x) = \tilde{F}_1^{\pm}(x) \exp(-ikx) + \int_x^{\infty} \tilde{A}_1^{\pm}(x, t) \exp(-ikt) dt. \tag{3.26}$$

Thanks to (3.26) the equality (3.9) may be expressed as

$$\tilde{F}_1^+(x) + \int_x^\infty \tilde{A}_1^+(x, t)dt = \tilde{F}_1^-(x) + \int_x^\infty \tilde{A}_1^-(x, t)dt, \qquad x \in \mathbb{R}. \quad (3.27)$$

Using (3.25) and the fact that $\tilde{F}_1^+(x)\tilde{F}_1^-(x) = 1$, we obtain from (3.27)

$$[\tilde{F}_1^+(x)]^2\left(1 - \int_x^\infty [\alpha_1^-(x, t) - \beta_1^-(x, t)]dt\right)$$
$$= 1 - \int_x^\infty [\alpha_1^+(x, t) - \beta_1^+(x, t)]dt, \qquad x \in \mathbb{R}. \quad (3.28)$$

Since a number A exists such that the quantity

$$« 1 - \int_x^\infty [\alpha_1^-(x, t) - \beta_1^-(x, t)]dt »$$

does not vanish for $x \geq A$, comparison of formulas (2.16) and (3.28) yields the equality of $[\tilde{F}_1^+(x)]^2$ and $[F_1^+(x)]^2$ for $x \geq A$. Therefore a number ε equal to « 1 » or « -1 » exists such that $\tilde{F}_1^\pm(x)$ is equal to $\varepsilon F_1^\pm(x)$ for $x \geq A$. Because of (2.8) and (3.25) $\tilde{A}_1^+(x, t)$ is equal to $\varepsilon A_1^+(x, t)$ for $t \geq x \geq A$. Then using formulas (3.26) and (I.2.10) we obtain the equality (3.12).

4. THE INVERSE REFLECTION PROBLEM

In this section we are given a function $s_{21}^+(k)$ $(k \in \mathbb{R})$ belonging to the class $\mathscr{R}$ and we consider the inverse reflection problem. Before investigating this inverse problem let us note that if $s_{21}^+(k)$ $(k \in \mathbb{R})$ belongs to $\mathscr{R}$ and is therefore the (2,1) element of a function

$$S^+(k) = \begin{pmatrix} s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & s_{22}^+(k) \end{pmatrix} \qquad (k \in \mathbb{R})$$

which belongs to $\mathscr{S}$, it is also the (2,1) element of the function

$$S_{-1}^+(k) = \begin{pmatrix} -\, s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & -\, s_{22}^+(k) \end{pmatrix} \qquad (k \in \mathbb{R})$$

which also belongs to $\mathscr{S}$. It is also interesting to note that if $s_{11}^+(0) = 0$, we can prove in a rather direct way that there is no other function in $\mathscr{S}$ whose (2,1) element is $s_{21}^+(k)$ $(k \in \mathbb{R})$. Indeed the following relations:

$$s_{11}^+(k) = F_1^+ \exp\left\{\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Log}(1 - |s_{21}^+(k')|^2)}{k' - k}dk'\right\}, \quad \operatorname{Im} k > 0, \quad (4.1)$$

$$s_{11}^+(k) = \lim_{\varepsilon \to 0} s_{11}^+(k + i\varepsilon), \qquad k \in \mathbb{R}, \quad (4.2)$$

show that $s_{11}^{+}(k)$ $(k \in \mathbb{R})$ is determined by $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ except for a multiplicative constant of modulus one. Since $s_{11}^{+}(0) = 0$, we can use the formula (I.2.2) where L is a purely imaginary number. Then we find that $s_{11}^{+}(k)$ is determined by $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ except for the sign. On the other hand the relation

$$s_{12}^{+}(k) = - \overline{s_{21}^{+}(k)} s_{11}^{+}(k) [\overline{s_{11}^{+}(k)}]^{-1}, \qquad k \in \mathbb{R}^*, \tag{4.3}$$

which follows from the unitarity of $S^{+}(k)$, shows that $s_{12}^{+}(k \in \mathbb{R})$ is determined by $s_{21}^{+}(k)$ $(k \in \mathbb{R})$. Hence the desired result.

Now applying the results of sections 1, 2 and 3 to one of the functions $S^{+}(k)$ $(k \in \mathbb{R})$ in $\mathscr{S}$ whose (2,1) element is the input function $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ in $\mathscr{R}$, we conclude that there exists a pair (U(x), Q(x)) in $\mathscr{V}$ which admits $S^{+}(k)$ $(k \in \mathbb{R})$ or $S_{-1}^{+}(k)$ $(k \in \mathbb{R})$ as its scattering matrix. In any case this pair admits the input function $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ as its reflection coefficient to the right. On the other hand, we notice that the argument of uniqueness given in section 1 for the solution of the inverse scattering problem can be developed in a similar way here since $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ is the only element of $S^{+}(k)$ $(k \in \mathbb{R})$ which occurs in the system S_1. So the inverse reflection problem has a unique solution (U(x), Q(x)) in $\mathscr{V}$ for any given function in $\mathscr{R}$. (U(x), Q(x)) can be obtained by solving S_1 and using formulas (I.2.16).

Note that it is clear from our study that the scattering matrix $S^{+}(k)$ $(k \in \mathbb{R})$ associated with a pair in $\mathscr{V}$ is completely determined by the reflection coefficient to the right $s_{21}^{+}(k)$ $(k \in \mathbb{R})$. We also find that any function $s_{21}^{+}(k)$ $(k \in \mathbb{R})$ belonging to $\mathscr{R}$ is the (2,1) element of only two functions in $\mathscr{S}$: if $S^{+}(k)$ $(k \in \mathbb{R})$ is one of them, the other one is $S_{-1}^{+}(k)$ $(k \in \mathbb{R})$. This last result has been proved more directly above in the case $s_{11}^{+}(0) = 0$.

5. A PARTICULAR CASE

Let us consider the subclass $\mathscr{V}_0$ of pairs (U(x), Q(x)) in $\mathscr{V}$ which satisfy the condition $Q(x) = 0$ $(x \in \mathbb{R})$. In this case each quantity indexed as « + » coincides with the corresponding quantity indexed as « − ». It is clear that the scattering matrix associated with a pair in $\mathscr{V}$ belongs to the subclass $\mathscr{S}_0$ of functions $S^{+}(k)$ $(k \in \mathbb{R})$ in $\mathscr{S}$ which satisfy the conditions $F_1^{+} = 1$ and

$$s_{ij}^{+}(k) = \overline{s_{ij}^{+}(-k)} \qquad (i = 1, 2\,; j = 1, 2) \qquad (k \in \mathbb{R})\,. \tag{5.1}$$

These conditions (5.1) are clearly equivalent to the condition $S^{+}(k) = S^{-}(k)$ $(k \in \mathbb{R})$, or to the condition $S^{+}(-k) = [{}^{t}S^{+}(k)]^{-1}$ $(k \in \mathbb{R})$. So in this case we see that the scattering matrix $S^{+}(k)$ $(k \in \mathbb{R})$ is completely determined by its « physical part » $S^{+}(k)$ $(k > 0)$, and therefore our inverse scattering problem coincides with the « physical » inverse scattering problem associated with (1.1) and (1.2).

Conversely, if we are given a function $S^+(k)$ ($k \in \mathbb{R}$) in $\mathscr{S}_0$, we know from our study in the above sections that there is a pair in $\mathscr{V}$,

$$(U_1(x), Q_1(x)) = (U_2(x), Q_2(x))$$

which admits $S_\varepsilon^+(k)$ ($k \in \mathbb{R}$) as its scattering matrix. Besides it is easy to see that here the functions $r_1^+(t)$, $a_{1,1}^+(x, t)$, $ia_{1,2}^+(x, t)$, $\alpha_1^+(x, t)\,(= \alpha_1^-(x, t))$, $\beta_1^+(x, t)\,(= \beta_1^-(x, t))$ are real. The differential equation (2.9) becomes

$$z_1' = -\, 4\alpha_1^+(x, x) \sin z_1 \,. \tag{5.2}$$

The only solution of (5.2) satisfying the condition (2.10) is $z_1(x) = 0$. The pair $(U_1(x), Q_1(x))$ constructed by solving S_1 is therefore given as

$$Q_1(x) = 0,\ U_1(x) = -\, 2\frac{d}{dx} A_1^+(x, x)\,, \qquad x \in \mathbb{R}\,, \tag{5.3}$$

where $A_1^+(x, t)$ $\big(= a_{1,1}^+(x, t)\big)$ is the solution of the integral equation

$$A_1^+(x, t) = r_1^+(x + t) + \int_x^\infty r_1^+(u + t) A_1^+(x, u) du\,, \qquad t \geq x\,, \quad x \in \mathbb{R} \tag{5.4}$$

(note that a similar result is obtained if we consider S_2 instead of S_1). So $(U_1(x), Q_1(x))$ belongs to $\mathscr{V}_0$. On the other hand since clearly $\tilde{F}_1^+(x) = F_1^+(x) = 1$, we have $\varepsilon = 1$, and so the pair $(U_1(x), Q_1(x))$ admits the input function $S^+(k)$ ($k \in \mathbb{R}$) in $\mathscr{S}_0$ as its scattering matrix.

So we can assert that a necessary and sufficient condition for a function $S^+(k)$ ($k \in \mathbb{R}$) to be the scattering matrix associated with a pair $(U(x), Q(x))$ in $\mathscr{V}_0$ is that $S^+(k)$ ($k \in \mathbb{R}$) belongs to $\mathscr{S}_0$. (We remark that in the case $s_{11}^+(0) = 0$, contrary to Faddeev, we have to specify the asymptotic behaviour of the coefficients $s_{ij}^+(k)$ as $k \to 0$.) This pair is the only one in $\mathscr{V}$—and therefore in $\mathscr{V}_0$—admitting this function as its scattering matrix. Note that it would be easy to consider also the inverse reflection problem. Note also that formula (4.1) with $F_1^+ = 1$ shows directly here that the scattering matrix $S^+(k)$ ($k \in \mathbb{R}$) associated with a pair in $\mathscr{V}_0$ is completely determined by the reflection coefficient to the right $s_{21}^+(k)$ ($k \in \mathbb{R}$).

6. THE INVERSE PROBLEM FOR THE ONE-DIMENSIONAL KLEIN-GORDON EQUATION OF ZERO MASS WITH A STATIC POTENTIAL

With the additional assumption

$$U(x) = -\, Q^2(x)\,, \qquad x \in \mathbb{R}\,, \tag{6.1}$$

the formulas (1.4) and (1.5) for the index « + », represent, for $k > 0$, the one-dimensional Klein-Gordon equation for a particle of zero mass and of energy k subject to a static potential $Q(x)$; (1.4) and (1.5) for the index « $-$ » describe the correspondent antiparticle. Let $\mathscr{Q}$ be the set of potentials $Q(x)$ which satisfy assumptions D_2 and D_3 ; if $Q(x)$ belongs to $\mathscr{Q}$, $(U(x), Q(x))$—with $U(x)$ defined by (6.1)—belongs to $\mathscr{V}$. Let us note that assumption D_3 which expresses the hypothesis that there is no bound state, is not physically restrictive here. The scattering matrix $S^+(k)$ ($k \in \mathbb{R}$) associated with $(U(x), Q(x))$ is here physically observable since, for $k > 0$, it describes the scattering of the particle, and, for $k < 0$, it describes the scattering of the antiparticle. Let us call $\mathscr{S}_{KG}$ the subclass of functions $S^+(k)$ ($k \in \mathbb{R}$) in $\mathscr{S}$ which satisfy the following condition, where $z_1(x)$ is the solution of the differential equation (2.9) with condition (2.10) and $f_1^+(x)$ is the function defined by (I.2.18):

$$4f_1^+(x) + [z_1'(x)]^2 F_1^+(x) = 0, \qquad x \in \mathbb{R}. \tag{6.2}$$

It is clear from our study that a necessary and sufficient condition for one of the functions

$$S^+(k) = \begin{pmatrix} s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & s_{22}^+(k) \end{pmatrix} \qquad (k \in \mathbb{R})$$

and

$$S_{-1}^+(k) = \begin{pmatrix} -\, s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & -\, s_{22}^+(k) \end{pmatrix} \qquad (k \in \mathbb{R})$$

to be the scattering matrix associated with a potential $Q(x)$ in $\mathscr{Q}$ is that $S^+(k)$ ($k \in \mathbb{R}$) belongs to the class $\mathscr{S}_{KG}$. This potential is the only one in $\mathscr{Q}$ admitting this function as its scattering matrix. Note that it would be easy to consider also the inverse reflection problem. To end we remark that it would be desirable to find a condition more direct than (6.2) on the input function $S^+(k)$ ($k \in \mathbb{R}$) of the inverse scattering problem.

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APPENDIX

SOME THEOREMS ON FUNCTIONS ANALYTIC IN THE UPPER HALF-PLANE

In this appendix we recall a Titchmarsh theorem and we state two theorems which are used in paragraph 3.3 (Proofs of these two theorems can be found in [2]).

We shall say that a complex function $\Phi(k)$ ($k \in \mathbb{R}$) defined a. e. in $\mathbb{R}$ and square integrable in $\mathbb{R}$, belongs to the class $\mathscr{C}$ if there exists a complex function $\Phi(k')$ analytic for $\operatorname{Im} k' > 0$ satisfying the following conditions:

1 — there exists a positive number K such that for every $h > 0$

$$\int_{-\infty}^{\infty} |\Phi(k + ih)|^2 dk < \mathrm{K}\,; \tag{A.1}$$

2 — for a. e. real k, $\Phi(k + ih) \to \Phi(k)$ as $h \to 0$ (k being fixed).

The interest of the class $\mathscr{C}$ is explained by the following theorem which is often used in paragraph 3.3:

THEOREM (Titchmarsh [9], theorem 95, p. 128). — *Let $\Phi(k)$ be a complex function defined for a. e. real k and square integrable in $\mathbb{R}$. Then the two following conditions are equivalent:*

a) *$\Phi(k)$ belongs to the class $\mathscr{C}$;*

b) *the function $\varphi(t)$ defined for a. e. real t by*

$$\varphi(t) = (2\pi)^{-1} \underset{n\to\infty}{\mathrm{l.i.m.}} \int_{-n}^{n} \Phi(k) \exp(-ikt)dk\,, \tag{A.2}$$

is equal to zero for a. e. real $t < 0$.

Now, we state the two following theorems:

THEOREM 1. — *Let $\Phi(k)$ ($k \in \mathbb{R}$) be a function belonging to the class $\mathscr{C}$ which is essentially bounded in $\mathbb{R}$ by a number M (i. e. we have $|\Phi(k)| \leq \mathrm{M}$ a. e. for $k \in \mathbb{R}$). Then we have*

1 — $$|\Phi(k')| \leq \mathrm{M}\,, \qquad \operatorname{Im} k' > 0\,; \tag{A.3}$$

2 — *if $\Phi(k)$ ($k \in \mathbb{R}$) is continuous at the point $k_0 \in \mathbb{R}$, then $\Phi(k') \to \Phi(k_0)$ as $k' \to k_0$* ($\operatorname{Im} k' > 0$).

In particular if $\Phi(k)$ ($k \in \mathbb{R}$) belongs to $\mathscr{C}$, is continuous for every real k, and is bounded in $\mathbb{R}$ by M, we conclude that it admits an extension $\Phi(k)$ ($\operatorname{Im} k \geq 0$) which is continuous for $\operatorname{Im} k \geq 0$, bounded in $\operatorname{Im} k \geq 0$ by M and analytic for $\operatorname{Im} k > 0$.

THEOREM 2. — *Let $\Phi(k)$ ($\operatorname{Im} k \geq 0$) be a function continuous and bounded for $\operatorname{Im} k \geq 0$ and analytic for $\operatorname{Im} k > 0$. If a complex number l exists such that*

$$\lim_{|k|\to 0,\, k\in\mathbb{R}^*} \frac{\Phi(k)}{k} = l\,, \tag{A.4}$$

then we also have

$$\lim_{|k|\to 0,\, \operatorname{Im} k \geq 0 - \{0\}} \frac{\Phi(k)}{k} = l\,. \tag{A.5}$$

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