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**The inverse problem  
for the one-dimensional schrödinger equation  
with an energy-dependent potential. I (\*)**

by

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**ABSTRACT.** — The one-dimensional Schrödinger equation

$$y^{+''} + [k^2 - V^+(k, x)]y^+ = 0, \quad x \in \mathbb{R},$$

is considered when the potential  $V^+(k, x)$  depends on the energy  $k^2$  in the following way:  $V^+(k, x) = U(x) + 2kQ(x)$ ;  $(U(x), Q(x))$  belongs to a large class  $\mathcal{V}$  of pairs of real potentials admitting no bound state. To each pair in  $\mathcal{V}$  is associated a  $2 \times 2$  matrix-valued function, the « scattering

matrix »  $S^+(k) = \begin{pmatrix} s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & s_{22}^+(k) \end{pmatrix}$  ( $k \in \mathbb{R}$ ), for which  $S^+(k)$  ( $k > 0$ ) represents

the « physical part » in the scattering problem associated with the Schrödinger equation. The complex function  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) is the « reflection coefficient to the right ». It is proved that  $S^+(k)$  ( $k \in \mathbb{R}$ ) belongs to a certain class  $\mathcal{S}$  and that  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) belongs to a certain class  $\mathcal{R}$ . On the other hand, two systems  $S_1$  and  $S_2$  of differential and integral equations are derived connecting quantities related to  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) and  $s_{12}^+(k)$  ( $k \in \mathbb{R}$ ) with quantities related to  $(U(x), Q(x))$ . In a following paper, starting from these equations, we will study existence and uniqueness for pairs in  $\mathcal{V}$ ,

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given the scattering matrix in  $\mathcal{S}$ -i. e. existence and uniqueness for the « inverse scattering problem »—, and existence and uniqueness for pairs in  $\mathcal{V}$ , given the reflection coefficient to the right in  $\mathcal{R}$ -i. e. existence and uniqueness for the « inverse reflection problem »—.

RÉSUMÉ. — On considère l'équation de Schrödinger à une dimension  $y^{+''} + [k^2 - V^+(k, x)]y^+ = 0$ ,  $x \in \mathbb{R}$ , dans le cas où le potentiel  $V^+(k, x)$  dépend de l'énergie  $k^2$  de la façon suivante :  $V^+(k, x) = U(x) + 2kQ(x)$ ;  $(U(x), Q(x))$  appartient à une vaste classe  $\mathcal{V}$  de couples de potentiels réels n'admettant pas d'état lié. A chaque couple de  $\mathcal{V}$  est associée une fonction à valeurs dans l'espace des matrices  $2 \times 2$ , la « matrice de diffusion »

$$S^+(k) = \begin{pmatrix} s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & s_{22}^+(k) \end{pmatrix} \quad (k \in \mathbb{R});$$

$S^+(k)$  ( $k > 0$ ) représente la « partie physique » dans le problème de la diffusion associé à l'équation de Schrödinger. La fonction complexe  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) est le « coefficient de réflexion à droite ». On démontre que  $S^+(k)$  ( $k \in \mathbb{R}$ ) appartient à une certaine classe  $\mathcal{S}$  et que  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) appartient à une certaine classe  $\mathcal{R}$ . On établit d'autre part deux systèmes d'équations différentielles et intégrales  $S_1$  et  $S_2$  qui relient des quantités déduites de  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) et  $s_{12}^+(k)$  ( $k \in \mathbb{R}$ ) à des quantités déduites de  $(U(x), Q(x))$ . Dans un article suivant on partira de ces équations pour étudier le « problème inverse de la diffusion » (c'est-à-dire l'existence et l'unicité de  $(U(x), Q(x))$  dans  $\mathcal{V}$  quand on se donne  $S^+(k)$  ( $k \in \mathbb{R}$ ) dans  $\mathcal{S}$ ) et le « problème inverse de la réflexion » (c'est-à-dire l'existence et l'unicité de  $(U(x), Q(x))$  dans  $\mathcal{V}$  quand on se donne  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) dans  $\mathcal{R}$ ).

## 1. INTRODUCTION

Let us consider the scattering problem for the one-dimensional Schrödinger equation

$$y^{+''} + [E - V^+(E, x)]y^+ = 0, \quad x \in \mathbb{R}, \quad (1.1)$$

where the potential  $V^+(E, x)$  depends on the energy  $E$  in the following simple way

$$V^+(E, x) = U(x) + 2\sqrt{E}Q(x), \quad x \in \mathbb{R}, \quad E \in \mathbb{C}, \quad (1.2)$$

$$\sqrt{E} = -|E|^{\frac{1}{2}} \exp\left(\frac{i}{2} \text{Arg } E\right), \quad 0 < \text{Arg } E \leq 2\pi. \quad (1.3)$$

This problem is of interest not only for its own sake, but also because there are other scattering problems in Physics which can be reduced to it. (See [1] where such a reduction is done for scattering problems in absorb-

ing media occurring in transmission line theory, electromagnetism and elasticity theory.) It is useful to consider both equations

$$y^{\pm''} + [k^2 - V^{\pm}(k, x)]y^{\pm} = 0, \quad x \in \mathbb{R}, \quad (1.4)$$

$$V^{\pm}(k, x) = U(x) \pm 2kQ(x), \quad x \in \mathbb{R}, \quad k \in \mathbb{C}. \quad (1.5)$$

Indeed, if we set  $k = \sqrt{E}$  ( $E \in \mathbb{C}$ ), we see that for the index « + » formulas (1.4) and (1.5) for  $\text{Im } k < 0$  or  $k > 0$  reduce to (1.1) and (1.2). For a large class of pairs of potentials  $(U(x), Q(x))$ , we introduce in section 3 the « Jost solution at  $+\infty$  »  $f_1^{\pm}(k, x)$  ( $\text{Im } k \leq 0$ ) and the « Jost solution at  $-\infty$  »  $f_2^{\pm}(k, x)$  ( $\text{Im } k \leq 0$ ) by the asymptotic conditions

$$\lim_{x \rightarrow \infty} f_1^{\pm}(k, x) \exp(ikx) = 1, \quad \lim_{x \rightarrow -\infty} f_2^{\pm}(k, x) \exp(-ikx) = 1. \quad (1.6)$$

Then we can prove that, for  $k \in \mathbb{R}^* (= \mathbb{R} - \{0\})$ , there exist two solutions of (1.4),  $\psi_1^{\pm}(k, x)$  and  $\psi_2^{\pm}(k, x)$ , having the following asymptotic forms as  $x \rightarrow \infty$  and  $x \rightarrow -\infty$ .

$$\begin{aligned} \psi_1^{\pm}(k, x) &= s_{12}^{\pm}(k)e^{-ikx} + e^{ikx} + o(1) \quad (x \rightarrow -\infty); \\ \psi_1^{\pm}(k, x) &= s_{11}^{\pm}(k)e^{ikx} + o(1) \quad (x \rightarrow \infty); \end{aligned} \quad (1.7)$$

$$\begin{aligned} \psi_2^{\pm}(k, x) &= s_{22}^{\pm}(k)e^{-ikx} + o(1) \quad (x \rightarrow -\infty); \\ \psi_2^{\pm}(k, x) &= s_{21}^{\pm}(k)e^{ikx} + e^{-ikx} + o(1) \quad (x \rightarrow \infty). \end{aligned} \quad (1.8)$$

The complex function  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ )—resp.  $s_{12}^+(k)$  ( $k \in \mathbb{R}$ )—will be called the « reflection coefficient to the right »—resp. « to the left »—associated with the pair  $(U(x), Q(x))$ . The complex function  $s_{11}^+(k)$  ( $k \in \mathbb{R}$ ) will be called the « transmission coefficient » associated with the pair  $(U(x), Q(x))$  (note that  $s_{11}^+(k) = s_{22}^+(k)$ ). The functions  $s_{21}^-(k)$  ( $k \in \mathbb{R}$ ),  $s_{12}^-(k)$  ( $k \in \mathbb{R}$ ) and  $s_{11}^-(k)$  ( $k \in \mathbb{R}$ ) are so the reflection and transmission coefficients associated with the pair  $(U(x), -Q(x))$ . We set

$$S^{\pm}(k) = \begin{pmatrix} s_{11}^{\pm}(k) & s_{21}^{\pm}(k) \\ s_{12}^{\pm}(k) & s_{22}^{\pm}(k) \end{pmatrix}, \quad k \in \mathbb{R}. \quad (1.9)$$

The  $2 \times 2$  matrix-valued function  $S^+(k)$  ( $k \in \mathbb{R}$ ) will be called the « scattering matrix » associated with the pair  $(U(x), Q(x))$ . The function  $S^-(k)$  ( $k \in \mathbb{R}$ ) is so the scattering matrix associated with the pair  $(U(x), -Q(x))$ . Clearly only the functions  $s_{21}^+(k)$  ( $k > 0$ ),  $s_{12}^+(k)$  ( $k > 0$ ),  $s_{11}^+(k)$  ( $k > 0$ ) and  $S^+(k)$  ( $k > 0$ ) have a physical meaning in the scattering problem associated with equations (1.1) and (1.2). Each of these functions will be referred as the « physical part » of the corresponding function defined for every  $k \in \mathbb{R}$ .

Now we consider a large class  $\mathcal{V}$  of pairs of real potentials  $(U(x), Q(x))$  admitting no bound state and we pose the following « inverse scattering problem »: having given a function  $S^+(k)$  ( $k \in \mathbb{R}$ ) does there exist a pair  $(U(x), Q(x))$  belonging to  $\mathcal{V}$  which admits the input function  $S^+(k)$  ( $k \in \mathbb{R}$ )

as its scattering matrix? If so, is this pair unique? Since the elements  $s_{ij}^+(k)$  ( $i = 1, 2; j = 1, 2$ ) of  $S^+(k)$  are not independent, we are also led to pose in a similar way the « *inverse reflection problem* », in which only  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ), the reflection coefficient to the right, is given — we obtain an analogous problem if we are given  $s_{12}^+(k)$  ( $k \in \mathbb{R}$ ), the reflection coefficient to the left, instead of  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) —. The investigation of the inverse scattering — resp. reflection — problem is of obvious interest for the « *physical* » inverse scattering — resp. reflection — problem in which only the « *physical part* » of the scattering matrix, i. e.  $S^+(k)$  ( $k > 0$ ) — resp. of the reflection coefficient to the right, i. e.  $s_{21}^+(k)$  ( $k > 0$ ) — is given; the part  $S^+(k)$  ( $k \leq 0$ ) — resp.  $s_{21}^+(k)$  ( $k \leq 0$ ) — then plays the role of a parameter in general.

If  $Q(x) = 0$ , the functions  $s_{ij}^+(k)$  and  $s_{ij}^+(-k)$  ( $k \in \mathbb{R}$ ) are complex conjugate. So the scattering matrix  $S^+(k)$  ( $k \in \mathbb{R}$ ) — resp. the reflection coefficient to the right  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) — is completely determined by its « *physical part* »  $S^+(k)$  ( $k > 0$ ) — resp.  $s_{21}^+(k)$  ( $k > 0$ ) —. The inverse scattering and reflection problems in this case have been solved by Kay [2], Kay and Moses [3], and Faddeev [4] (they used a method similar to the Marchenko method [5], in the inverse scattering problem for the radial version ( $x \geq 0$ ) of equation (1.1) with  $Q(x) = 0$ ). In this case one can prove in a rather direct way that the scattering matrix is completely determined by the reflection coefficient to the right, so that it is easy to go from the inverse scattering problem to the inverse reflection problem.

In this paper, and in a following one, referred to as II, we present a method for solving the inverse scattering and reflection problems when  $Q(x)$  is not necessarily the zero function. A more detailed version of our work can be found in [6]. We make two comments on our method. On the one hand it is a generalization of the method of Kay and Moses, and Faddeev; we tend to follow Faddeev's paper. On the other hand, it is formally similar to the method that we developed in recent works [7] [8] [9] for the radial version of equation (1.1) but there are several important technical differences. The aim of this paper is to prepare the investigation of the inverse problems in II. In the frame of the scattering theory for a pair  $(U(x), Q(x))$  in a certain class  $\mathcal{V}$ , we determine a class  $\mathcal{S}$  of  $2 \times 2$  matrix-valued functions to which the scattering matrix  $S^+(k)$  ( $k \in \mathbb{R}$ ) must belong, and a class  $\mathcal{R}$  of complex functions to which the reflection coefficient to the right  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) must belong. We also derive two systems  $S_1$  and  $S_2$  of differential and integral equations which connect quantities related to  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) and  $s_{12}^+(k)$  ( $k \in \mathbb{R}$ ) with quantities related to  $(U(x), Q(x))$ . This study will give us a way in II to tackle the existence question for the inverse scattering problem by choosing the input function  $S^+(k)$  ( $k \in \mathbb{R}$ ) in the class  $\mathcal{S}$  and by then using the solution of the systems of equations as the starting point of the inversion procedure. The fact that these equations are necessarily satisfied when  $S^+(k)$  ( $k \in \mathbb{R}$ ) is the scattering matrix associated with a

pair  $(U(x), Q(x))$  in  $\mathcal{V}$ , will also be used in II to investigate the uniqueness question. We shall see that the study of the inverse scattering problem can be easily adjusted to that of the inverse reflection problem by choosing the input function  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) in the class  $\mathcal{R}$ . We shall then prove, in an indirect way, that the scattering matrix  $S^+(k)$  ( $k \in \mathbb{R}$ ) is completely determined by the reflection coefficient to the right  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ). In Section 2 we state more precisely the principal results of this paper and refer to the appropriate sections for the proofs.

## 2. PRINCIPAL RESULTS

We are interested in the class  $\mathcal{V}$  of pairs of functions  $(U(x), Q(x))$  which satisfy the following conditions  $D_1$ ,  $D_2$  and  $D_3$ :

$D_1$ :  $U(x)$  ( $x \in \mathbb{R}$ ) is real, continuously differentiable, and  $x^2U(x)$  and  $xU'(x)$  are integrable in  $\mathbb{R}$ .

$D_2$ :  $Q(x)$  ( $x \in \mathbb{R}$ ) is real, twice continuously differentiable, goes to zero as  $|x| \rightarrow \infty$ , and  $x^2Q'(x)$  and  $xQ''(x)$  are integrable in  $\mathbb{R}$ .

$D_3$ : The function  $c_{12}^+(k) = (2ik)^{-1}W[f_1^+(k, x), f_2^+(k, x)]$  has no zero for  $\text{Im } k < 0$ , i. e. there is no bound state for equation (1.1) (we denote by  $W[u, v]$  the wronskian of two functions  $u$  and  $v$ ).

Collecting our results in sections 3,5 and 6 we can state that the scattering matrix associated with a pair in  $\mathcal{V}$  belongs to the class  $\mathcal{S}$  of  $2 \times 2$  matrix-valued functions

$$S^+(k) = \begin{pmatrix} s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & s_{22}^+(k) \end{pmatrix} \quad (k \in \mathbb{R})$$

satisfying the following conditions 1, 2, 3, 4, 5 and 6:

1) 
$$s_{11}^+(k) = s_{22}^+(k), \quad k \in \mathbb{R}; \tag{2.1}$$

2) the  $2 \times 2$  matrix  $S^+(k)$  is unitary for every  $k \in \mathbb{R}$ ;

3) the functions  $s_{11}^+(k)$ ,  $s_{21}^+(k)$  and  $s_{12}^+(k)$  ( $k \in \mathbb{R}$ ) are continuous, and the function  $s_{11}^+(k)$  ( $k \in \mathbb{R}$ ) admits a continuous extension  $s_{11}^+(k)$  ( $\text{Im } k \geq 0$ ) which is analytic for  $\text{Im } k > 0$  and such that  $s_{11}^+(k) \neq 0$  for  $\text{Im } k \geq 0 - \{0\}$  (note that such an extension is defined uniquely);

4) if  $s_{11}^+(0) = 0$  there exist a non zero purely imaginary number  $L$  and two purely imaginary numbers  $L_1$  and  $L_2$  such that

$$\lim_{|k| \rightarrow 0, \text{Im } k \geq 0 - \{0\}} \frac{s_{11}^+(k)}{k} = L, \tag{2.2}$$

$$\lim_{|k| \rightarrow 0, k \in \mathbb{R}^*} \frac{s_{21}^+(k) + 1}{k} = L_1, \tag{2.3}$$

$$\lim_{|k| \rightarrow 0, k \in \mathbb{R}^*} \frac{s_{12}^+(k) + 1}{k} = L_2; \tag{2.4}$$

5) there exists a complex number of modulus one  $F_1^+$  such that

$$s_{11}^+(k) = F_1^+ + o\left(\frac{1}{k}\right) \quad (|k| \rightarrow \infty, \operatorname{Im} k \geq 0), \quad (2.5)$$

$$s_{21}^+(k) = o\left(\frac{1}{k}\right) \quad (|k| \rightarrow \infty, k \in \mathbb{R}), \quad (2.6)$$

$$s_{12}^+(k) = o\left(\frac{1}{k}\right) \quad (|k| \rightarrow \infty, k \in \mathbb{R}); \quad (2.7)$$

6) the functions  $r_1^+(t)$  and  $r_2^+(t)$  defined as

$$r_1^+(t) = -\frac{1}{2\pi} \text{l. i. m.}_{n \rightarrow \infty} \int_{-n}^n s_{21}^+(k) e^{ikt} dk, \quad t \in \mathbb{R}, \quad (2.8)$$

$$r_2^+(t) = -\frac{1}{2\pi} \text{l. i. m.}_{n \rightarrow \infty} \int_{-n}^n s_{12}^+(k) e^{-ikt} dk, \quad t \in \mathbb{R}, \quad (2.9)$$

where l. i. m. stands for « limit in mean », and which are therefore square integrable in  $\mathbb{R}$ , are twice continuously differentiable for  $t \in \mathbb{R}$ ; furthermore,  $x_0$  being any real number, the functions  $t^2 r_1^{+\prime\prime}(t)$  and  $tr_1^{+\prime\prime}(t)$  are integrable in the interval  $[x_0, \infty[$  and the functions  $t^2 r_2^{+\prime\prime}(t)$  and  $tr_2^{+\prime\prime}(t)$  are integrable in the interval  $]-\infty, x_0]$ .

The reflection coefficient to the right associated with a pair in  $\mathcal{V}$  belongs to the class  $\mathcal{R}$  of complex functions  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) which satisfy the following condition:  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) is the (2, 1) element — i. e. the element at the intersection of the second column and of the first row — of a  $2 \times 2$  matrix-valued function

$$S^+(k) = \begin{pmatrix} s_{11}^+(k) & s_{21}^+(k) \\ s_{12}^+(k) & s_{22}^+(k) \end{pmatrix} \quad (k \in \mathbb{R})$$

belonging to the class  $\mathcal{S}$ .

Note that  $S^+(k)$  ( $k \in \mathbb{R}$ ) is not defined uniquely by this condition (we investigate this point in II).

In section 4 we prove that the Jost solutions  $f_1^\pm(k, x)$  and  $f_2^\pm(k, x)$  admit the following representation for  $\operatorname{Im} k \leq 0$ ,  $x \in \mathbb{R}$ ,

$$f_1^\pm(k, x) = F_1^\pm(x) \exp(-ikx) + \int_x^\infty A_1^\pm(x, t) \exp(-ikt) dt, \quad (2.10)$$

$$f_2^\pm(k, x) = F_2^\pm(x) \exp(ikx) + \int_{-\infty}^x A_2^\pm(x, t) \exp(ikt) dt, \quad (2.11)$$

where

$$F_1^\pm(x) = \exp\left(\mp i \int_x^\infty Q(t) dt\right), \quad F_2^\pm(x) = \exp\left(\mp i \int_{-\infty}^x Q(t) dt\right), \quad (2.12)$$

and where  $A_1^\pm(x, t)$  belongs to the class  $\mathcal{A}_1$  and  $A_2^\pm(x, t)$  belongs to the class  $\mathcal{A}_2$ .  $\mathcal{A}_1$  is the class of complex functions  $A_1^\pm(x, t)$  defined for  $t \geq x$ ,  $x \in \mathbb{R}$ , continuous with respect to  $(x, t)$ , and admitting the following bound for any given real number  $x_0$ :

$$|A_1^\pm(x, t)| \leq C_{x_0} \sigma_1 \left( \frac{x+t}{2} \right), \quad t \geq x \geq x_0, \quad (2.13)$$

where  $C_{x_0}$  is, for given  $x_0$ , a general positive constant, and  $\sigma_1(x)$  ( $x \in \mathbb{R}$ ) is a general non-increasing positive function, integrable on  $[x_0, \infty[$  for any  $x_0$ .  $\mathcal{A}_2$  is the class of complex functions  $A_2^\pm(x, t)$  defined for  $t \leq x$ ,  $x \in \mathbb{R}$ , continuous with respect to  $(x, t)$ , and admitting the following bound for any given real number  $x_0$ :

$$|A_2^\pm(x, t)| \leq C_{x_0} \sigma_2 \left( \frac{x+t}{2} \right), \quad t \leq x \leq x_0, \quad (2.14)$$

where  $\sigma_2(x)$  is a general non-decreasing positive function, integrable on  $] -\infty, x_0]$  for any  $x_0$ .  $U(x)$  and  $Q(x)$  being real we have

$$\overline{F_m^+(x)} = F_m^-(x), \quad \overline{A_m^+(x, t)} = A_m^-(x, t) \quad (m = 1, 2). \quad (2.15)$$

$(U(x), Q(x))$  can be easily obtained from each of the two pairs  $(F_1^+(x), A_1^+(x, t))$  and  $(F_2^+(x), A_2^+(x, t))$  by formulas

$$Q(x) = \mp i F_1^{\pm\prime}(x) [F_1^\pm(x)]^{-1}, \quad U(x) = f_1^\pm(x) [F_1^\pm(x)]^{-1}, \quad x \in \mathbb{R}, \quad (2.16)$$

$$Q(x) = \pm i F_2^{\pm\prime}(x) [F_2^\pm(x)]^{-1}, \quad U(x) = f_2^\pm(x) [F_2^\pm(x)]^{-1}, \quad x \in \mathbb{R}, \quad (2.17)$$

where

$$f_m^\pm(x) = F_m^{\pm\prime\prime}(x) - 2 \frac{d}{dx} A_m^\pm(x, x) + 2 F_m^{\pm\prime}(x) [F_m^\pm(x)]^{-1} A_m^\pm(x, x) \quad (m = 1, 2). \quad (2.18)$$

In section 6, we prove that  $(F_1^+(x), A_1^+(x, t))$  is a solution of the following system  $S_1$  of equations

$$A_1^+(x, t) = \overline{F_1^+(x)} r_1^+(x+t) + \int_x^\infty r_1^+(t+u) \overline{A_1^+(x, u)} du, \quad t \geq x, \quad x \in \mathbb{R}, \quad (2.19)$$

$$f_1^+(x) \overline{F_1^+(x)} = \overline{f_1^+(x)} F_1^+(x), \quad x \in \mathbb{R}, \quad (2.20)$$

where  $A_1^+(x, t)$  belongs to the class  $\mathcal{A}_1$  and is twice continuously differentiable for  $t \geq x$ ,  $x \in \mathbb{R}$ , and  $F_1^+(x)$  can be written in the form

$$F_1^+(x) = \exp \left( -\frac{i}{2} z_1(x) \right), \quad x \in \mathbb{R}, \quad (2.21)$$

where  $z_1(x)$  is a three times continuously differentiable real function for  $x \in \mathbb{R}$  such that  $z_1(\infty) = 0$ ,  $z_1'(\infty) = 0$ . So the system  $S_1$  connects  $r_1(t)$  ( $t \in \mathbb{R}$ )—which is obtained from  $s_{21}^+(k)$  ( $k \in \mathbb{R}$ ) by formula (2.8)—



with  $(F_1^+(x), A_1^+(x, t))$ —from which we can obtain  $(U(x), Q(x))$  by formulas (2.16). Similarly we prove that  $(F_2^+(x), A_2^+(x, t))$  is a solution of the following system  $S_2$  of equations

$$A_2^+(x, t) = \overline{F_2^+(x)} r_2^+(x+t) + \int_{-\infty}^x r_2^+(t+u) \overline{A_2^+(x, u)} du, \quad t \leq x, \quad x \in \mathbb{R}, \quad (2.22)$$

$$f_2^+(x) \overline{F_2^+(x)} = \overline{f_2^+(x)} F_2^+(x), \quad x \in \mathbb{R}, \quad (2.23)$$

where  $A_2^+(x, t)$  belongs to the class  $\mathcal{A}_2$  and is twice continuously differentiable for  $t \leq x$ ,  $x \in \mathbb{R}$ , and  $F_2^+(x)$  can be written in the form

$$F_2^+(x) = \exp\left(-\frac{i}{2} z_2(x)\right), \quad x \in \mathbb{R}, \quad (2.24)$$

where  $z_2(x)$  is a three times continuously differentiable real function for  $x \in \mathbb{R}$  such that  $z_2(-\infty) = 0$ ,  $z_2'(-\infty) = 0$ . So the system  $S_2$  connects  $r_2(t)$  ( $t \in \mathbb{R}$ )—which is obtained from  $s_{12}^+(k)$  ( $k \in \mathbb{R}$ ) by formula (2.9)—with  $(F_2^+(x), A_2^+(x, t))$ —from which we can obtain  $(U(x), Q(x))$  by formulas (2.17). In II we shall investigate the solution of systems  $S_1$  and  $S_2$  when we are given a function  $S^+(k)$  ( $k \in \mathbb{R}$ ) in  $\mathcal{S}$ .

### 3. THE SOLUTIONS $f_1^\pm(k, x)$ AND $f_2^\pm(k, x)$ AND THE SCATTERING MATRIX

In this section we are given a pair  $(U(x), Q(x))$  satisfying conditions  $D_1$  and  $D_2$ . The Jost solutions  $f_1^\pm(k, x)$  and  $f_2^\pm(k, x)$  of (1.4) are defined equivalently as the solutions in the class of functions continuous for real  $x$  of the following integral equations:

$$f_1^\pm(k, x) = e^{-ikx} + \int_x^\infty \frac{\sin k(y-x)}{k} V^\pm(k, y) f_1^\pm(k, y) dy, \quad (3.1)$$

$$f_2^\pm(k, x) = e^{ikx} + \int_{-\infty}^x \frac{\sin k(x-y)}{k} V^\pm(k, y) f_2^\pm(k, y) dy. \quad (3.2)$$

$f_1^\pm(k, x)$  and  $f_2^\pm(k, x)$  are (for fixed  $x$ ) defined and continuous for  $\text{Im } k \leq 0$ , analytic for  $\text{Im } k < 0$  and obey the bounds

$$|f_1^\pm(k, x)| \leq \exp(-bx) \exp(2N_1(x)), \quad \text{Im } k \leq 0, \quad (3.3)$$

$$|f_2^\pm(k, x)| \leq \exp(-bx) \exp(2N_2(x)), \quad \text{Im } k \leq 0, \quad (3.4)$$

where

$$N_1(x) = \int_x^\infty [(y-x)|U(y)| + 2|Q(y)|] dy, \quad (3.5)$$

$$N_2(x) = \int_{-\infty}^x [(x-y)|U(y)| + 2|Q(y)|] dy. \quad (3.6)$$

We have the complex conjugate relations

$$f_m^\mp(k, x) = \overline{f_m^\pm(-\bar{k}, x)}, \quad \text{Im } k \leq 0 \quad (m = 1, 2). \quad (3.7)$$

We remark that the solution  $f_1^\pm(k, x)$  (resp.  $f_2^\pm(k, x)$ ) still exists and has the same properties if we only make the following weaker assumptions  $D_1^+$  and  $D_2^+$  (resp.  $D_1^-$  and  $D_2^-$ ):

**ASSUMPTION  $D_1^+$**  (resp.  $D_1^-$ ). —  $U(x)$  is real, continuously differentiable for  $x \in \mathbb{R}$  and the functions  $x^2U(x)$  and  $xU'(x)$  are integrable in the interval  $[x_0, \infty[$  (resp.  $]-\infty, x_0]$ ).

**ASSUMPTION  $D_2^+$**  (resp.  $D_2^-$ ). —  $Q(x)$  is real, twice continuously differentiable for  $x \in \mathbb{R}$ ,  $Q(\infty) = 0$  (resp.  $Q(-\infty) = 0$ ) and the functions  $x^2Q'(x)$  and  $xQ''(x)$  are integrable in the interval  $[x_0, \infty[$  (resp.  $]-\infty, x_0]$ ).

For  $k \in \mathbb{R}^*$ ,  $f_1^\pm(k, x)$  and  $f_1^\mp(-k, x)$ — resp.  $f_2^\pm(k, x)$  and  $f_2^\mp(-k, x)$ — form a fundamental system of solutions of (1.4). So for  $x \in \mathbb{R}$  and  $k \in \mathbb{R}^*$  we have the relations

$$f_2^\pm(k, x) = c_{11}^\pm(k)f_1^\pm(k, x) + c_{12}^\pm(k)f_1^\mp(-k, x), \quad (3.8)$$

$$f_1^\pm(k, x) = c_{22}^\pm(k)f_2^\pm(k, x) + c_{21}^\pm(k)f_2^\mp(-k, x), \quad (3.9)$$

where

$$c_{12}^\pm(k) = c_{21}^\pm(k) = \frac{1}{2ik} W[f_1^\pm(k, x), f_2^\pm(k, x)], \quad (3.10)$$

$$c_{11}^\pm(k) = -c_{22}^\mp(-k) = \frac{1}{2ik} W[f_2^\pm(k, x), f_1^\mp(-k, x)]. \quad (3.11)$$

It follows from (3.8) and (3.9) that, for  $k \in \mathbb{R}^*$ , two solutions of (1.4) exist,  $\psi_1^\pm(k, x)$  and  $\psi_2^\pm(k, x)$  satisfying conditions (1.7) and (1.8). They are given by

$$\begin{aligned} \psi_1^\pm(k, x) &= f_1^\mp(-k, x)[c_{21}^\mp(-k)]^{-1}, \\ \psi_2^\pm(k, x) &= f_2^\mp(-k, x)[c_{12}^\mp(-k)]^{-1}, \end{aligned} \quad (3.12)$$

and the reflection and transmission coefficients are given by

$$s_{11}^\pm(k) = s_{22}^\pm(k) = [c_{21}^\mp(-k)]^{-1}, \quad k \in \mathbb{R}^*, \quad (3.13)$$

$$\begin{aligned} s_{12}^\pm(k) &= c_{22}^\mp(-k)[c_{21}^\mp(-k)]^{-1}, \\ s_{21}^\pm(k) &= c_{11}^\mp(-k)[c_{12}^\mp(-k)]^{-1}, \end{aligned} \quad k \in \mathbb{R}^*. \quad (3.14)$$

For  $k \in \mathbb{R}^*$  we can easily prove the following relations

$$s_{ij}^-(k) = \overline{s_{ij}^+(-k)} \quad (i = 1, 2; j = 1, 2), \quad (3.15)$$

$$S^\pm(k)S^\mp(-k) = I, \quad (3.16)$$

where «  $t$  » means transposed and  $I$  is the  $2 \times 2$  identity matrix. From (3.15) and (3.16) we conclude that  $S^\pm(k)$  is unitary.

#### 4. REPRESENTATION FORMULAS FOR $f_1^\pm(k, x)$ AND $f_2^\pm(k, x)$

We sketch the proof of the representation formula (2.10) for  $f_1^\pm(k, x)$ .  $(U(x), Q(x))$  is supposed to satisfy conditions  $D_1^+$  and  $D_2^+$ . Replacing  $f_1^\pm(k, x)$  by its representation (2.10) in the integral equation (3.1) and then using properties of Fourier transforms, we obtain the first formula in (2.12) and the integral equation

$$\begin{aligned} A_1^\pm(x, t) = & \frac{1}{2} \int_{\frac{x+t}{2}}^{\infty} F_1^\pm(y)U(y)dy + \frac{1}{2} \int_x^{\frac{x+t}{2}} U(y)dy \int_{t+x-y}^{t+y-x} A_1^\pm(y, u)du \\ & + \frac{1}{2} \int_{\frac{x+t}{2}}^{\infty} U(y)dy \int_y^{t+y-x} A_1^\pm(y, u)du \pm \frac{i}{2} F_1^\pm\left(\frac{x+t}{2}\right)Q\left(\frac{x+t}{2}\right) \\ & \mp i \int_x^{\infty} Q(y)A_1^\pm(y, t+y-x)dy \\ & \pm i \int_x^{\frac{x+t}{2}} Q(y)A_1^\pm(y, t+x-y)dy, \quad t \geq x, \quad x \in \mathbb{R}. \end{aligned} \quad (4.1)$$

Using the Neumann series expansion of (4.1) we find that (4.1) admits a unique solution  $A_1^\pm(x, t)$  in the class  $\mathcal{A}_1$ . We then obtain the bound

$$|A_1^\pm(x, t)| \leq \frac{1}{2} \sigma_{1v}\left(\frac{x+t}{2}\right) \exp(N_1(x)), \quad t \geq x, \quad x \in \mathbb{R}, \quad (4.2)$$

where

$$\sigma_{1v}(x) = \int_x^{\infty} (|U(y)| + |Q'(y)|)dy, \quad x \in \mathbb{R}. \quad (4.3)$$

Differentiating the Neumann series twice one can prove without difficulty that  $A_1^\pm(x, t)$  is twice continuously differentiable for  $t \geq x$ ,  $x \in \mathbb{R}$ . Certain bounds can be derived for the partial derivatives (see [6]).

Now we differentiate both sides of the integral equation (4.1) twice to obtain the partial differential equation

$$\frac{\partial^2 A_1^\pm}{\partial x^2}(x, t) - \frac{\partial^2 A_1^\pm}{\partial t^2}(x, t) - U(x)A_1^\pm(x, t) \pm 2iQ(x) \frac{\partial A_1^\pm}{\partial t}(x, t) = 0, \quad t \geq x, \quad x \in \mathbb{R}. \quad (4.4)$$

Differentiation of equation (4.1) for  $t = x$  yields the following differential equation (identical with the second equation in (2.16)):

$$U(x) = f_1^\pm(x)[F_1^\pm(x)]^{-1}, \quad x \in \mathbb{R}, \quad (4.5)$$

where  $f_1^\pm(x)$  is defined in (2.18). Conversely we have the following theorem which will be used in II:

**THEOREM.** — Suppose that  $(U(x), Q(x))$  satisfies conditions  $D_1^+$  and  $D_2^+$  and let  $A_1^\pm(x, t)$  be a twice continuously differentiable function belonging to the class  $\mathcal{A}_1$  such that the partial differential equation (4.4) and the condition (4.5) are satisfied and also the condition

$$\lim_{N \rightarrow \infty} \left( \sup_{t+x \geq N} \left| \frac{\partial A_1^\pm}{\partial x}(x, t) \right| \right) = \lim_{N \rightarrow \infty} \left( \sup_{t+x \geq N} \left| \frac{\partial A_1^\pm}{\partial t}(x, t) \right| \right) = 0. \quad (4.6)$$

Then the function  $f_1^\pm(k, x)$  defined from  $(F_1^\pm(x), A_1^\pm(x, t))$  (where  $F_1^\pm(x)$  is given by (2.12)) by formula (2.10), is the Jost solution at  $\infty$  of the differential equation (1.4).

Using (2.10) and the trivial equality

$$f_1^\pm(0, x) = f_1^\mp(0, x), \quad (4.7)$$

one can easily obtain the following result which will be used in II: if  $(U(x), Q(x))$  satisfies  $D_1^+$  and  $D_2^+$ , there exists a purely imaginary number  $m$  such that,  $x$  being any given real number,

$$k^{-1} [f_1^\pm(k, x) - f_1^\mp(-k, x)] \rightarrow m \quad \text{as } |k| \rightarrow 0 \quad (k \in \mathbb{R}^*). \quad (4.8)$$

Clearly all the above results have analogues when  $f_1^\pm(k, x)$  is replaced by  $f_2^\pm(k, x)$  and conditions  $D_1^+$  and  $D_2^+$  by  $D_1^-$  and  $D_2^-$ . In particular (2.10) is replaced by (2.11) and the bound (4.2) by

$$|A_2^\pm(x, t)| \leq \frac{1}{2} \sigma_{2v} \left( \frac{x+t}{2} \right) \exp(N_2(x)), \quad t \leq x, \quad x \in \mathbb{R}, \quad (4.9)$$

where

$$\sigma_{2v}(x) = \int_{-\infty}^x (|U(y)| + |Q'(y)|) dy, \quad x \in \mathbb{R}. \quad (4.10)$$

The complex conjugate relations (2.15) follow readily from the reality of  $U(x)$  and  $Q(x)$ .

### 5. SOME PROPERTIES OF THE SCATTERING MATRIX

We only assume for the moment that  $(U(x), Q(x))$  satisfies conditions  $D_1$  and  $D_2$ . Let us write (2.10) and (2.11) in the slightly different form

$$f_1^\pm(k, x) = \exp(-ikx) h_1^\pm(k, x), \quad f_2^\pm(k, x) = \exp(ikx) h_2^\pm(k, x), \quad (5.1)$$

where

$$h_1^\pm(k, x) = F_1^\pm(x) + \int_0^\infty A_1^\pm(x, x+u) \exp(-iku) du, \quad x \in \mathbb{R}, \quad \text{Im } k \leq 0, \quad (5.2)$$

$$h_2^\pm(k, x) = F_2^\pm(x) + \int_{-\infty}^0 A_2^\pm(x, x+u) \exp(iku) du, \quad x \in \mathbb{R}, \quad \text{Im } k \leq 0. \quad (5.3)$$

Using (5.2), (5.3) and properties of functions  $A_1^\pm(x, t)$  and  $A_2^\pm(x, t)$ , it is easy to prove that the functions  $h_1^\pm(k, x)$  and  $h_2^\pm(k, x)$  are differentiable with respect to  $x$  and that these derivatives  $h_1^{\pm'}(k, x)$  and  $h_2^{\pm'}(k, x)$  are, for fixed  $x$ , continuous for  $\text{Im } k \leq 0$  and analytic for  $\text{Im } k < 0$ . With notations (5.1) formulas (3.10) and (3.11) can be written in the following form for  $k \in \mathbb{R}^*$ :

$$c_{12}^\pm(k) = h_1^\pm(k, 0)h_2^\pm(k, 0) + (2ik)^{-1} [h_1^\pm(k, 0)h_2^{\pm'}(k, 0) - h_1^{\pm'}(k, 0)h_2^\pm(k, 0)], \quad (5.4)$$

$$c_{11}^\pm(k) = (2ik)^{-1} [h_1^{\mp'}(-k, 0)h_2^\pm(k, 0) - h_1^\mp(-k, 0)h_2^{\mp'}(k, 0)]. \quad (5.5)$$

We see from these formulas that the functions  $c_{11}^\pm(k)$  ( $k \in \mathbb{R}^*$ ) and  $c_{12}^\pm(k)$  ( $k \in \mathbb{R}^*$ ) are continuous and that the function  $c_{12}^\pm(k)$  ( $k \in \mathbb{R}^*$ ) admits through formula (5.4) a unique continuous extension  $c_{12}^\pm(k)$  ( $\text{Im } k \leq 0 - \{0\}$ ) which is analytic for  $\text{Im } k < 0$ .

Now we assume that  $(U(x), Q(x))$  satisfies the conditions  $D_1, D_2$  and  $D_3$  of the class  $\mathcal{V}$ . Note that condition  $D_3$  is equivalent to  $c_{12}^\pm(k) \neq 0$  for  $\text{Im } k \leq 0 - \{0\}$ . (For the proof use the fact that the functions  $c_{12}^\pm(k)$  and  $c_{12}^\mp(-\bar{k})$  are complex conjugates for  $\text{Im } k \leq 0 - \{0\}$ , and the fact that  $|c_{12}^\pm(k)|$  is greater than 1 for  $k \in \mathbb{R}^*$  because of the unitarity of  $S^\pm(k)$ .) It is easy to see that  $D_3$  corresponds to there being no « bound state » for equation (1.1), i. e. no square integrable solution. Using formulas (5.4) and (5.5) we can prove straightforwardly (see [6]) that  $S^\pm(k)$  ( $k \in \mathbb{R}^*$ ) can be continuously extended to  $k = 0$  and that  $S^\pm(k)$  ( $k \in \mathbb{R}$ ) satisfies the conditions 1) through 4) of the class  $\mathcal{V}$ .

Still using formulas (5.4) and (5.5) we can prove (see [6]) that  $S^\pm(k)$  ( $k \in \mathbb{R}$ ) satisfies the condition 5) of the class  $\mathcal{V}$ . We remark that here we use a technique different from Faddeev's. This is due to the fact that we have found weaker bounds (see formulas (4.2), (4.3), (3.5), (4.9), (4.10) and (3.6)) than Faddeev's (formula (1.11) of his paper) and we have been unable to verify his.

### 6. DERIVATION OF SYSTEMS $S_1$ AND $S_2$

In this section we assume that  $(U(x), Q(x))$  belongs to the class  $\mathcal{V}$ . First we derive equation (2.19). We start from the relation (3.8) written in the form

$$s_{11}^\pm(k)f_2^\mp(-k, x) = s_{21}^\pm(k)f_1^\mp(-k, x) + f_1^\pm(k, x), \quad x \in \mathbb{R}, \quad k \in \mathbb{R}. \quad (6.1)$$

From (6.1) it follows that the following square integrable functions  $A_x^\pm(k)$  and  $B_x^\pm(k)$  are equal for  $k \in \mathbb{R}$  ( $x$  is any fixed real number).

$$A_x^\pm(k) = s_{11}^\pm(k)f_2^\mp(-k, x) - F_1^\pm(x)e^{-ikx} = [s_{11}^\pm(k) - F_1^\pm(-\infty)]f_2^\mp(-k, x) + F_1^\pm(-\infty)[f_2^\mp(-k, x) - F_2^\mp(x)e^{-ikx}], \quad (6.2)$$

$$B_x^\pm(k) = s_{21}^\pm(k)[f_1^\mp(-k, x) - F_1^\mp(x)e^{ikx}] + F_1^\mp(x)e^{ikx}s_{21}^\pm(k) + [f_1^\pm(k, x) - F_1^\pm(x)e^{-ikx}]. \quad (6.3)$$

Let us evaluate their Fourier transforms  $a_x^\pm(t)$  and  $b_x^\pm(t)$  for  $t \geq x$ . The function  $A_x^\pm(k)$  is continuous for  $\text{Im } k \geq 0$ , analytic for  $\text{Im } k > 0$ , and admits the bound

$$A_x^\pm(k) = \exp(-ikx)O(k^{-1}) \quad (|k| \rightarrow \infty, \text{Im } k \geq 0). \quad (6.4)$$

From the Cauchy theorem the integral  $\int_\Gamma A_x^\pm(k) \exp(ikt)dk$  is equal to zero along the closed path  $\Gamma$  contained in the upper half of the complex  $k$ -plane and consisting of the segment  $[-R, R]$  and of the half circle  $|k| = R$ . Thanks to (6.4) we can apply a Jordan lemma to prove that the integral along the half circle vanishes for  $t > x$  as  $R \rightarrow \infty$ . Hence  $a_x^\pm(t) = 0$  a. e. (almost everywhere) for  $t \geq x$ . On the other hand we evaluate  $b_x^\pm(t)$  with the help of formula (2.10) and of formula (2.8) which defines the function  $r_1^\pm(t)$ . Using the equality of  $a_x^\pm(t)$  and  $b_x^\pm(t)$  a. e. for  $t \geq x$  we find the equation

$$A_1^\pm(x, t) = F_1^\mp(x)r_1^\pm(x+t) + \int_x^\infty r_1^\pm(t+u)A_1^\mp(x, u)du, \quad \text{a. e. for } t \geq x, \quad x \in \mathbb{R}. \quad (6.5)$$

We easily see that the function  $r_1^\pm(t)$  which has been defined a. e. for  $t \in \mathbb{R}$  by formula (2.8) can be chosen continuous for all real  $t$  and such that the equation (6.5) be valid for every  $t \geq x, x \in \mathbb{R}$ . Furthermore the functions  $A_1^+(x, t)$  and  $A_1^-(x, t), F_1^+(x)$  and  $F_1^-(x), r_1^+(t)$  and  $r_1^-(t)$  being respectively complex conjugates, we find that the equations (6.5) for the upper and lower indices are complex conjugates. They are therefore equivalent. Hence the integral equation (2.19). Similarly starting from relation (3.9) instead of relation (3.8) and defining the function  $r_2^\pm(t)$  by formula (2.9), we derive the integral equation (2.22). Equations (2.20) and (2.23) are easily deduced from (2.16) and (2.17). Hence the systems  $S_1$  and  $S_2$ .

From equations (2.19) and (2.22) we can deduce new properties of  $S^\pm(k) (k \in \mathbb{R})$ . Using equations (2.19) and (2.22) for  $t = x$  and certain bounds for  $A_1^\pm(x, t)$  and  $A_2^\pm(x, t)$  and their first and second partial derivatives, it is tedious but not difficult to prove that  $S^\pm(k) (k \in \mathbb{R})$  satisfies the condition 6) of the class  $\mathcal{S}$ .

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