ANNALES DE L'I. H. P., SECTION A

T. H. DATE

On relativistic magnetofluids

Annales de l'I. H. P., section A, tome 24, n° 4 (1976), p. 417-427
<http://www.numdam.org/item?id=AIHPA_1976__24_4_417_0>

Ann. Inst. Henri Poincaré,
Vol. XXIV, n° 4, 1976, p. 417-427.

Section A :
Physique théorique.

On relativistic magnetofluids

by

T. H. DATE

Department of Mathematics, Shivaji University,
Kolhapur-416004 (M. S.). India

ABSTRACT. — The stress-energy-momentum tensor for thermally conducting, viscous, compressible fluid with infinite electrical conductivity and constant magnetic permeability is constructed. Several consequences of the relativistic magnetohydrodynamic field equations are derived. « Maxwell-like » equations for the gravitational field in a magnetofluid are obtained and the expressions for the refractive index and the ray shear of a null gravitational field are computed.

INTRODUCTION

The classical magnetohydrodynamics has been applied with considerable success to the astronomical systems like magnetic variable stars, sun-spots, and spiral arms [*1*]-[*3*]. However, on astronomical scale the gravitational attractions far exceed electromagnetic attractions and repulsions. How to formulate a theory incorporating the intense gravitational fields which are inevitably present in the astronomical system? Precisely to meet this demand the theory of relativistic magnetohydrodynamics (RMHD) has come into existence.

The genesis of RMHD is in Minkowski's electrodynamics of moving bodies. The significant contributions to RMHD are due to Coburn [*4*], Taub [*5*] and Greenberg [*6*]. On giving an elegant account of the RMHD field equations, Lichnerowicz [*7*] has established their existence and uniqueness of solutions. His RMHD field equations are used by Yodzis [*8*] to infer the magnetic effect in galactic cosmogony, gravitational collapse, and pulsar theory and by Date [9] to study the local behaviour of congruences in self-gravitating magnetofluids. The solutions of Lichnerowicz's field equations are found by Date [10]-[11] and interpreted as a class of non-uniform cosmological models filled with irrotational and shearfree thermodynamical perfect fluid with infinite electrical conductivity and constant magnetic permeability. Bray [12] has obtained an exact solution of Lichnerowicz's RMHD field equations by assuming axial symmetry. Moreover, he [13]-[14] has obtained Godel type of universes filled with magnetofluid. Shaha [15] has extended definite material schemes to magnetohydrodynamics.

In this article, we propose to study the consequences of the RMHD field equations by modifying the stress-energy-momentum tensor in Ref. [7] for thermally conducting, viscous, compressible fluid with infinite electrical conductivity and constant magnetic permeability. The field equations (Einstein equations, Maxwell equations, equations connecting thermodynamical variables) are used to study some consequences. Analogous to the derivation of Maxwell equations for propagation of electromagnetic field in matter, « Maxwell-like » equations for gravitational field in magnetofluid are derived by using the theory developed in Ref. [16]-[21]. The behaviour of gravitational radiation in the universe filled with magnetofluid is investigated and the propagation of a null gravitational field is studied.

The purpose of this article is two fold (*i*) it is of course of pure theoretical interest to extend the theory of relativistic hydrodynamics to that of RMHD (*ii*) it is observed that some astronomical objects like neutron stars possess very strong magnetic field of the order $10^{12} - 10^{13}$ G and very high electrical conductivity [22]. A thermally conducting, viscous, compressible fluid with infinite electrical conductivity and constant magnetic permeability befits theoretical considerations pertaining to such astronomical objects.

1. PRELIMINARIES

1.1. Geometry

Arbitrary co-ordinates x^α are used in a four-dimensional Riemannian manifold V_4. The metric is

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \qquad (\alpha, \beta = 1, 2, 3, 4) \tag{1.1}$$

where $g_{\alpha\beta}$ are the gravitational potentials. Signature of the metric *is* $(-, -, -, +)$. The time-like curves are

$$x^\alpha = x^\alpha(a^i, s), \qquad (i = 1, 2, 3) \tag{1.2}$$

where a^i are Lagrangian co-ordinates of fluid element and s is a parameter along the world line. The unit 4-velocity vector tangential to the world-line is

(1.3) $$u^{\alpha} = \frac{dx^{\alpha}}{ds} \qquad (a^i \text{ fixed}),$$

with $u^{\alpha}u_{\alpha} = +1$.

The space-metric or projection operator is

(1.4) $$p_{\alpha\beta} = g_{\alpha\beta} - u_{\alpha}u_{\beta}.$$

The covariant derivative of u^{α} is decomposed as

(1.5) $$u_{\alpha;\beta} = \theta p_{\alpha\beta} + \dot{u}_{\alpha}u_{\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta},$$

where the scalar $\theta = 1/3u^{\alpha}{}_{;\alpha}$ is the expansion, $\dot{u}^{\alpha} = u^{\alpha}{}_{;\beta}u^{\beta}$ is the acceleration vector, $\sigma_{\alpha\beta} = u_{(\alpha;\beta)} - \dot{u}_{(\alpha}u_{\beta)} - \theta p_{\alpha\beta}$ is the shear tensor and $\omega_{\alpha\beta} = u_{[\alpha;\beta]} - \dot{u}_{[\alpha}u_{\beta]}$ is the rotation tensor. The vorticity vector ω^{α} is defined as

$$\omega^{\alpha} = \frac{1}{2}\eta^{\alpha\beta\gamma\delta}u_{\beta}u_{\gamma;\delta},$$

where $\eta^{\alpha\beta\gamma\delta}$ is the Levi-Civita permutation tensor. The magnitudes of $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are given by

$$\sigma^2 = \frac{1}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta}, \qquad \omega^2 = \frac{1}{2}\omega^{\alpha\beta}\omega_{\alpha\beta}.$$

Here semicolon indicates covariant differentiation, round brackets around suffixes denote symmetrization and square brackets around suffixes denote anti-symmetrization. Units are such that k, the gravitational constant and c, the velocity of light are 1.

The equation of a space-like curve is

(1.6) $$x^{\alpha} = x^{\alpha}(\eta^i, \xi), \qquad (i = 1, 2, 3)$$

where η^i takes constant values for a particular curve and ξ is the parameter along the space-like curve. The unit vector tangent to the space-like curve is given by

(1.7) $$n^{\alpha} = dx^{\alpha}/d\xi, \qquad (\eta^i \text{ fixed})$$

with $n^{\alpha}n_{\alpha} = -1$.

The expansion parameter Ⓗ associated with space-like curve is defined as

(1.8) $$Ⓗ = \frac{1}{2}(n^{\alpha}{}_{;\alpha} - n_{\alpha;\beta}u^{\alpha}u^{\beta}).$$

1.2. **The stress-energy-momentum tensor of the fluid**

A stress-energy-momentum tensor for a thermally conducting, viscous, compressible fluid has a general form [21]

(1.9) $$\underset{(m)}{T_{\alpha\beta}} = (\rho + p)u_{\alpha}u_{\beta} - pg_{\alpha\beta} + \nu\sigma_{\alpha\beta} + q_{\alpha}u_{\beta} + q_{\beta}u_{\alpha},$$

where ρ is the matter energy density of the fluid, p is the isotropic pressure of the fluid, ν ($\geqslant 0$) is the coefficient of shear viscosity and q^α is the heat flux vector.

The matter energy density ρ is connected with the proper matter density ρ_0 and the internal energy density ε by [7]:

$$\rho = \rho_0(1 + \varepsilon). \tag{1.10}$$

The equations connecting thermodynamical variables are

$$\mathrm{T}dS = d\varepsilon + pd(1/\rho_0), \tag{1.11}$$

$$\mathrm{S}^\alpha = \rho_0 \mathrm{S} u^\alpha + q^\alpha/\mathrm{T}, \tag{1.12}$$

$$q^\alpha = \mathrm{K}(\mathrm{T}_{,\beta} - \mathrm{T}\dot{u}_\beta)p^{\alpha\beta}, \tag{1.13}$$

where T is the rest temperature, S is the specific entropy, S^α is the entropy flux vector and K is the heat conduction coefficient.

1.3. Stress-energy-momentum tensor of the magnetofluid

An asymmetric stress-energy-momentum tensor of electromagnetic field was given by Minkowski [7]:

$$\underset{(\mathrm{em})}{\mathrm{T}_{\alpha\beta}} = \frac{1}{4} g_{\alpha\beta}\mathscr{H}_{\gamma\delta}\mathscr{G}^{\gamma\delta} - \mathscr{H}_{\alpha\gamma}\mathscr{G}^\gamma{}_\beta, \tag{1.14}$$

where $\mathscr{H}_{\alpha\beta}$ is the skew symmetric electric field-magnetic induction tensor and $\mathscr{G}_{\alpha\beta}$ is the skew symmetric magnetic field-electric induction tensor. The tensor $\underset{(\mathrm{em})}{\mathrm{T}_{\alpha\beta}}$ in terms of the electric field vector e^α, the magnetic field vector h^α, the electric induction vector d^α, the magnetic induction vector b^α and the vectors v^α, w^α corresponding to energy flux and momentum flux of the electromagnetic field respectively, is of the form

$$\underset{(\mathrm{em})}{\mathrm{T}_{\alpha\beta}} = \left(\frac{1}{2} g_{\alpha\beta} - u_\alpha u_\beta\right)(e_\gamma d^\gamma + h_\gamma b^\gamma) - (e_\alpha d_\beta + h_\alpha b_\beta) - (u_\alpha v_\beta + u_\beta w_\alpha). \tag{1.15}$$

Under the assumptions of infinite electrical conductivity and constant magnetic permeability $\underset{(\mathrm{em})}{\mathrm{T}_{\alpha\beta}}$ reduces to [7].

$$\mathrm{M}_{\alpha\beta} = \mu\left[\left(\frac{1}{2} g_{\alpha\beta} - u_\alpha u_\beta\right)|\bar{h}|^2 - h_\alpha h_\beta\right], \tag{1.16}$$

where μ is the constant magnetic permeability,

$$|\bar{h}|^2 = -h^\alpha h_\alpha \qquad \text{and} \qquad u^\alpha h_\alpha = 0.$$

The total stress-energy-momentum tensor for thermally conducting, viscous, compressible fluid with infinite electrical conductivity and constant magnetic permeability is the sum of $\underset{(m)}{T_{\alpha\beta}}$ and $M_{\alpha\beta}$:

$$(1.17)\quad T_{\alpha\beta} = (\rho + p + \mu\,|\,\bar{h}\,|^2)u_\alpha u_\beta - \left(p + \frac{1}{2}\mu\,|\,\bar{h}\,|^2\right)g_{\alpha\beta} + \nu\sigma_{\alpha\beta} + q_\alpha u_\beta + q_\beta u_\alpha - \mu h_\alpha h_\beta\,.$$

For thermodynamical perfect fluid with infinite electric conductivity and constant magnetic permeability ($\nu = q^\alpha = 0$) we get Lichnerowicz's [7] tensor

$$(1.18)\quad \underset{(L)}{T_{\alpha\beta}} = (\rho + p + \mu\,|\,\bar{h}\,|^2)u_\alpha u_\beta - \left(p + \frac{1}{2}\mu\,|\,\bar{h}\,|^2\right)g_{\alpha\beta} - \mu h_\alpha h_\beta\,.$$

1.4. The field equations

The field equations of relativistic magnetohydrodynamics are the Einstein equations

$$(1.19)\qquad R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = -\,T_{\alpha\beta}\,,$$

where $T_{\alpha\beta}$ is given by (1.17) and the Maxwell equations

$$(1.20)\qquad (u^\alpha h^\beta - u^\beta h^\alpha)_{;\beta} = 0\,.$$

Remark. — Taub [5] has derived the field equations for self-gravitating charged fluid with constant electric permitivity and constant magnetic permeability by using variational principle. His equations reduce to (1.19) and (1.20) under the assumption of infinite electrical conductivity.

1.5. Free gravitational field

The Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ can be algebraically separated into the Ricci tensor $R_{\alpha\beta}$ and the Weyl tensor $C_{\alpha\beta\gamma\delta}$ as

$$(1.21)\qquad R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - g_{\alpha[\beta}R_{\gamma]\delta} - g_{\beta[\gamma}R_{\delta]\alpha} - 1/3Rg_{\alpha[\gamma}g_{\delta]\beta}\,.$$

The Weyl tensor represents the free gravitational field. It can be decomposed into « electric » and « magnetic » components [16]:

$$(1.22)\qquad E_{\alpha\beta} = C_{\alpha\gamma\beta\delta}u^\gamma u^\delta\,,$$

$$(1.23)\qquad H_{\alpha\beta} = \frac{1}{2}C_\alpha{}^{\mu\nu\delta}\eta_{\nu\delta\beta\varepsilon}u_\mu u^\varepsilon\,,$$

satisfying the properties

$$E_{\alpha\beta} = E_{(\alpha\beta)}, \qquad H_{\alpha\beta} = H_{(\alpha\beta)},$$
$$E^{\alpha}_{\alpha} = H^{\alpha}_{\alpha} = E_{\alpha\beta}u^{\beta} = H_{\alpha\beta}u^{\beta} = 0.$$

The matter current $J_{\alpha\beta\gamma}$ is defined as [19].

$$J_{\alpha\beta\gamma} = C_{\alpha\beta\gamma}{}^{\delta}{}_{;\delta}\,. \tag{1.24}$$

2. THE MAXWELL EQUATIONS

We now study some consequences of the Maxwell equations (1.20). It is shown by Khade [23] that the equations (1.20) can be identified as material transport laws. Magnetic field vector h^{α} being space-like, it can be written as

$$h^{\alpha} = |\bar{h}|\, n^{\alpha}, \qquad (n^{\alpha}n_{\alpha} = -1) \tag{2.1}$$

where n^{α} is a unit space-like vector.

THEOREM 2.1. — *Congruences of magnetic lines are expansion-free if and only if the magnitude of the magnetic field conserves along the lines of force.*

From Maxwell equations (1.20), we get

$$u^{\alpha}{}_{;\beta}h^{\beta} + u^{\alpha}h^{\beta}{}_{;\beta} - u_{;}{}^{\beta}{}_{\beta}h^{\alpha} - u^{\beta}h^{\alpha}{}_{;\beta} = 0. \tag{2.2}$$

Contracting (2.2) with u_{α}, we have

$$h^{\alpha}{}_{;\alpha} - h_{\alpha;\beta}u^{\alpha}u^{\beta} = 0. \tag{2.3}$$

On substituting (2.1) and using (1.8), equation (2.3) reduces to

$$\text{Ⓗ} + \frac{1}{2}(\log|\bar{h}|)_{,\alpha}n^{\alpha} = 0. \tag{2.4}$$

Therefore, we must have

$$\text{Ⓗ} = 0 \Leftrightarrow (|\bar{h}|)_{,\alpha}n^{\alpha} = 0. \tag{2.5}$$

Thus, the proof of the theorem is complete.

THEOREM 2.2. — *For a Born-rigid flow of the magnetofluid, the magnitude of the magnetic field is conserved along the world-line.*

On using the kinematical parameters associated with time-like congruences, Maxwell equation (2.2) can be written in the form

$$(\sigma_{\alpha\beta} + \omega_{\alpha\beta})h^{\beta} + h^{\beta}{}_{;\beta}u_{\alpha} - 2\theta h_{\alpha} - h_{\alpha;\beta}u^{\beta} = 0. \tag{2.6}$$

Transvecting with h^{α}, (2.6) produces

$$\sigma_{\alpha\beta}h^{\alpha}h^{\beta} + 2\theta|\bar{h}|^{2} + \frac{1}{2}(|h|^{2})_{,\beta}u^{\beta}\,. \tag{2.7}$$

When the flow is Born-rigid ($\sigma_{\alpha\beta} = \theta = 0$), we get

$$(2.8)\qquad (|\bar{h}|)_{,\beta}u^{\beta} = 0 .$$

Thus, the magnitude of the magnetic field vector is conserved along the world line when $\theta = \sigma_{\alpha\beta} = 0$.

THEOREM 2.3. — *The magnetic field in the magnetofluid is divergence-free if and only if the acceleration and the magnetic field vector are orthogonal to each other.*

Since u^{α} and h^{α} are orthogonal, equation (2.3) yields

$$(2.9)\qquad h^{\alpha}{}_{;\alpha} + h^{\alpha}\dot{u}_{\alpha} = 0 .$$

Hence, we have

$$(2.10)\qquad h^{\alpha}{}_{;\alpha} = 0 \Leftrightarrow h^{\alpha}\dot{u}_{\alpha} = 0 ,$$

and the proof is complete.

3. HEAT TRANSFER EQUATION

From the equation of conservation $T^{\alpha\beta}_{;\beta} = 0$, we shall develop some differential identities which govern the behaviour of the magnetofluid. For magnetofluid characterized by (1.17), equation of conservation is

$$(3.1)\quad (\rho + p + \mu|\bar{h}|^2)\dot{u}^{\alpha} + (\rho + p + \mu|\bar{h}|^2)(\dot{u}^{\alpha} + 3\theta u^{\alpha}) - \left(p + \frac{1}{2}\mu|\bar{h}|^2\right)_{;\beta} g^{\alpha\beta} + \nu\sigma^{\alpha\beta}_{;\beta} + 4\theta q^{\alpha} + q^{\beta}_{;\beta}u^{\alpha} + \dot{q}^{\alpha} + (\omega^{\alpha\beta} + \sigma^{\alpha\beta})q_{\beta} - \mu(|\bar{h}|^2)_{;\beta}n^{\alpha}n^{\beta} - \mu|\bar{h}|^2(\dot{n}^{\alpha} + 3\phi n^{\alpha}) = 0 ,$$

where $\dot{n}^{\alpha} = n^{\alpha}{}_{;\beta}u^{\beta}$ and $\phi = 1/3n^{\beta}_{;\beta}$.
Contracting (3.1) by u_{α} and $p_{\beta\gamma}$ and using Maxwell equations (1.20), we get

$$(3.2)\qquad \dot{\rho} + 3\theta(\rho + p) + 2\nu\sigma^2 + q^{\alpha}_{;\alpha} + q^{\alpha}\dot{u}_{\alpha} = 0 ,$$

$$(3.3)\quad (\rho + p + \mu|\bar{h}|^2)\dot{u}_{\gamma} - \left(p + \frac{1}{2}\mu|\bar{h}|^2\right)_{;\beta} p^{\beta}_{\gamma} + \frac{1}{2}(|\bar{h}|^2)_{,\beta}n^{\beta}n_{\gamma} + \nu(\sigma^{\beta}_{\gamma;\beta} + 2\sigma^2 u_{\gamma}) + 4\theta q_{\gamma} + q^{\alpha}p_{\alpha\gamma} + (\omega_{\beta\gamma} + \sigma_{\beta\gamma})q^{\beta} = 0 .$$

It is interesting to observe that the same equation (3.2) holds true even for thermally conducting, viscous, compressible fluid [*21*]. While in equation (3.3) magnetic field is explicitly present. On substituting equations (1.10) and (1.11) in equation (3.2) we get the heat transfer equation:

$$(3.4)\qquad (\rho_0 u^{\alpha})_{;\alpha} + T\rho_0 S_{,\alpha}u^{\alpha} = 2\nu\sigma^2 + \dot{u}^{\alpha}q_{\alpha} - q^{\alpha}{}_{;\alpha} .$$

If we assume that the matter density is conserved i. e. $(\rho_0 u^\alpha)_{;\alpha} = 0$, and use (1.12) the heat transfer equation reduces to

$$S^\alpha{}_{,\alpha} = 1/T(2\nu\sigma^2 + 1/KT\,|\bar{q}|^2), \tag{3.5}$$

where $|\bar{q}|^2 = -q^\alpha q_\alpha$.

Thus, for thermally conducting, viscous, compressible magnetofluid the entropy generation is always positive.

Remark (*1*). — It is to be noted that the magnetic field is not explicitly present in the entropy generation equation. Moreover, the entropy generation is not only due to heat flux but also due to viscosity.

Remark (*2*). — It is shown by Date [*24*] that for variable magnetic permeability, magnetic field is explicitly present in the heat transfer equation.

Remark (*3*). — For thermodynamical perfect fluid with infinite electric conductivity and constant magnetic permeability, Lichnerowicz's [*7*] results can be recovered.

4. GRAVITATIONAL FIELD IN PERFECT MAGNETOFLUID

For perfect magnetofluid ($\nu = q^\alpha = 0$) equations (3.2) and (3.3) reduce to

$$\dot{\rho} + 3\theta(\rho + p) = 0, \tag{4.1}$$

$$(\rho + p + \mu|\bar{h}|^2)u_\alpha - \perp\left(p + \frac{1}{2}\mu|\bar{h}|^2\right)_{,\alpha} - \mu|\bar{h}|^2 n_{\alpha;\beta}u^\beta = 0, \tag{4.2}$$

where $\perp$ indicates projection by $p_{\alpha\beta}$ orthogonal to u_α. Only kinematical parameters associated with time-like congruences appear in equations (4.1) and (4.2) are expansion and acceleration. In Szekeres' [*19*] sense, these equations represent the « inert part » of the gravitational field in the perfect magnetofluid. The « active part » of the gravitational field in the perfect magnetofluid can be found by observing the propagation of the free gravitational field. This part occurs in the matter current $J_{\alpha\beta\gamma}$ given by

$$\begin{aligned} J_{\alpha\beta\gamma} = {} & \rho_{,[\alpha}u_{\beta]}u_\gamma \\ & + \frac{1}{3}\perp\rho_{,[\beta}g_{\alpha]\gamma} + (\rho + p + \mu|\bar{h}|^2)\times(\omega_{\beta\alpha}u_\gamma + \sigma_{\gamma[\alpha}u_{\beta]} + \omega_{\gamma[\alpha}u_{\beta]}) \\ & + \frac{1}{2}\mu(|\bar{h}|^2)_{,[\alpha}u_{\beta]}u_\gamma + \mu|\bar{h}|^2\theta g_{\gamma[\alpha}u_{\beta]} + \frac{1}{2}\mu(|\bar{h}|^2)_{,[\beta}g_{\alpha]\gamma} \\ & + \frac{1}{2}\mu\{h_{\beta;\delta u_\alpha} - h_{\alpha;\delta \dot{u}_\beta}\}h^\delta u_\gamma + \mu h^\delta_{;\delta}h_{[\beta}u_{\alpha]u\gamma} + \mu h_{\gamma;[\beta}h_{\alpha]} + \mu h_\gamma h_{[\alpha;\beta]}\,. \end{aligned} \tag{4.3}$$

Using the decompositions in (1.21)-(1.23), the field equations analogous to Maxwell equations for the free gravitational field in perfect magnetofluid are as follows:

$$(4.4)\quad 3H_{\alpha\beta}\omega^{\beta} + p_{\alpha}{}^{\beta}p^{\gamma\delta}E_{\beta\gamma;\delta} - \eta_{\alpha\beta\gamma\delta}H^{\delta\varepsilon}\sigma_{\varepsilon}{}^{\gamma}u^{\beta} = -\frac{1}{3}\rho_{,\beta}p_{\alpha}{}^{\beta} + \frac{1}{2}\mu h_{\alpha;\beta}h^{\beta} - \frac{1}{2}\mu h_{\beta;\gamma}h^{\gamma}u^{\beta}u_{\alpha}\,,$$

$$(4.5)\quad \perp\dot{H}_{\alpha\beta} - p^{\nu}{}_{(\alpha}\eta_{\beta)\gamma\delta\varepsilon}u^{\gamma}E^{\delta;\varepsilon}_{\nu} + \theta H_{\alpha\beta} - H^{\gamma}{}_{(\alpha}\omega_{\beta)\gamma} - H^{\gamma}{}_{(\alpha}\sigma_{\beta)\gamma} - \eta_{\alpha\gamma\delta\varepsilon}\eta_{\beta\mu\nu\tau}u^{\gamma}u^{\mu}\sigma^{\delta\nu}H^{\varepsilon\tau} + 2H^{\delta}{}_{(\alpha}\eta_{\beta)\gamma\delta\varepsilon}u^{\gamma}\dot{u}^{\varepsilon} = -\mu(h^{\gamma}h^{\delta})^{;\varepsilon}p_{\gamma(\alpha}\eta_{\beta)\nu\varepsilon\delta}u^{\nu}\,.$$

$$(4.6)\quad p_{\alpha}{}^{\beta}H_{\beta\gamma;\delta}h^{\gamma\delta} - 3E_{\alpha\beta}\omega^{\beta} - \eta_{\alpha\beta\gamma\delta}u^{\beta}\sigma^{\gamma}_{\varepsilon}E^{\delta\varepsilon} = -(\rho + p + \mu|\bar{h}|^2)\omega_{\alpha} + \mu h_{\gamma}h_{\varepsilon;\beta}\eta_{\alpha\nu}{}^{\gamma\beta}u^{\nu}u^{\varepsilon}\,,$$

$$(4.7)\quad \perp\dot{E}_{\alpha\beta} + p^{\nu}{}_{(\alpha}\eta_{\beta)\gamma\delta\varepsilon}u^{\gamma}H^{\nu;\varepsilon} + \theta E_{\alpha\beta} - E^{\gamma}{}_{(\alpha}\omega_{\beta)\gamma} - E^{\gamma}{}_{(\alpha}\sigma_{\beta)\gamma} - \eta_{\alpha\gamma\delta\varepsilon}\eta_{\beta\mu\nu\tau}u^{\gamma}u^{\mu}\sigma^{\delta\nu}E^{\varepsilon\tau} + 2H^{\delta}{}_{(\alpha}\eta_{\beta)\gamma\delta\varepsilon}u^{\gamma}\dot{u}^{\varepsilon} = -\frac{1}{2}(\rho + p + \mu|\bar{h}|^2)\sigma_{\alpha\beta} - \frac{1}{4}\mu|\bar{h}|^2{}_{;\gamma}p_{\alpha\beta}u^{\gamma} - \frac{1}{2}\mu h_{\delta;\gamma}h_{\alpha}u^{\gamma}p^{\delta}_{\beta} - \frac{1}{6}\mu|\bar{h}|^2\theta p_{\alpha\beta} - \frac{1}{2}\mu(h_{\delta;\gamma} - h_{\gamma;\delta})h_{\beta}u^{\gamma}p^{\delta}_{\alpha}\,.$$

In absence of the magnetic field, equations (4.4) to (4.7) reduce to Hawking's [18] equations for perfect fluid. Comparison with his equations shows that the magnetic field on the right of the equations (4.4) to (4.7) produces disturbance in the gravitational radiation. If the undisturbed state is conformally flat ($C_{\alpha\beta\gamma\delta} = 0$) then the equations (4.4) to (4.7) become

$$(4.8)\quad \frac{1}{3}\rho_{,\beta}p^{\beta}_{\alpha} + \frac{1}{2}\mu h_{\alpha;\beta}u^{\beta} - \frac{1}{2}\mu h_{\beta;\gamma}u_{\alpha}u^{\beta}h^{\gamma} = 0\,,$$

$$(4.9)\quad (h^{\gamma}h^{\delta})^{;\varepsilon}p_{\gamma(\alpha}\eta_{\beta)\nu\varepsilon\delta}u^{\nu} = 0\,,$$

$$(4.10)\quad (\rho + p + \mu|\bar{h}|^2)\omega_{\alpha} - \mu h_{\gamma}h_{\varepsilon;\delta}\eta_{\alpha\nu}{}^{\gamma\delta}u^{\nu}u^{\varepsilon} = 0\,,$$

$$(4.11)\quad \frac{1}{2}(\rho + p + \mu|\bar{h}|^2)\sigma_{\alpha\beta} + \frac{1}{4}\mu(|\bar{h}|^2)^{\cdot}p_{\alpha\beta} + \frac{1}{6}\mu|\bar{h}|^2\theta p_{\alpha\beta} + \frac{1}{2}\mu h_{\delta;\gamma}p^{\delta}_{\beta}h_{\alpha}u^{\gamma} + \mu h_{[\gamma;\delta]}h_{\beta}u^{\gamma}p^{\delta}_{\alpha} = 0\,.$$

For uniform magnetic field ($h_{\alpha;\beta} = 0$), these equations produce

$$(4.12)\quad \omega_{\alpha\beta} = \sigma_{\alpha\beta} = 0\,,$$

$$(4.13)\quad \rho_{,\beta}p^{\alpha\beta} = \theta_{,\beta}p^{\alpha\beta} = 0\,.$$

When the equation of state is $p = p(\rho)$, we get

$$(4.14)\quad p_{,\beta}p^{\alpha\beta} = \dot{u}^{\alpha} = 0\,.$$

Thus, for uniform magnetic field, the universe filled with perfect magnetofluid is spatially homogeneous and isotropic.

5. PROPAGATION OF A NULL GRAVITATIONAL FIELD IN A PERFECT MAGNETOFLUID

Following Szekeres [19], we write the tetrad in terms of a Vierbein $(u^\alpha, s^\alpha, g^\alpha, f^\alpha)$

$$k^\alpha = 1/\sqrt{2}(u^\alpha + s^\alpha), \qquad t^\alpha = 1/\sqrt{2}(g^\alpha + if^\alpha),$$
$$m^\alpha = 1/\sqrt{2}(u^\alpha - s^\alpha), \qquad \bar{t}^\alpha = 1/\sqrt{2}(g^\alpha - if^\alpha).$$

In a null gravitational field, we choose k^α pointing along the ray propagation. The Weyl tensor takes the form

$$\text{(5.1)} \qquad C_{\alpha\beta\gamma\delta} = 2C(k_{[\alpha}g_{\beta]}k_{[\gamma}g_{\delta]} - k_{[\alpha}f_{\beta]}k_{[\gamma}f_{\delta]}),$$

where C is a real constant. The ray shear Σ and refraction $\mathscr{R}$ of the gravitational field are of the forms

$$\text{(5.2)} \qquad \Sigma = k_{\alpha;\beta}\bar{t}^\alpha\bar{t}^\beta,$$

$$\text{(5.3)} \qquad \mathscr{R} = k_{\alpha;\beta}\bar{t}^\alpha k^\beta = \mathscr{R}_1 + i\mathscr{R}_2 .$$

From equation (5.1) we have

$$\text{(5.4)} \qquad k^\gamma C_{\alpha\beta\gamma}{}^\delta{}_{;\delta} + k^\gamma{}_{;\delta}C_{\alpha\beta\gamma}{}^\delta = 0,$$

$$\text{(5.5)} \qquad k^\alpha C_{\alpha\beta\gamma}{}^\delta{}_{;\delta} + k^\alpha{}_{;\delta}C_{\alpha\beta\gamma}{}^\delta = 0 .$$

On substituting equations (4.3), (5.1) to (5.3) in equations (5.4), (5.5) we get

$$\begin{aligned}
\text{(5.6)} \quad & \frac{1}{2}\rho_{,[\alpha}u_{\beta]} - \frac{1}{3}\perp \rho_{,[\alpha}k_{\beta]} + (\rho + p + \mu\,|\,\bar{h}\,|^2)(\sigma_{\gamma[\alpha}u_{\beta]}s^\gamma + \omega_{\gamma[\alpha}u_{\beta]}s^\gamma - \omega_{\alpha\beta}) \\
& + \frac{1}{2}\mu(\,|\,\bar{h}\,|^2)_{,[\alpha}u_{\beta]} + \mu\,|\,\bar{h}\,|^2\theta k_{[\alpha}u_{\beta]} + \frac{1}{2}\mu(\,|\,\bar{h}\,|^2)_{,[\beta}k_{\alpha]} + \frac{1}{2}\mu\,\{\,h_{\beta;\delta}u_\alpha - h_{\alpha;\delta}u_\beta\,\}\,h^\delta \\
& + \mu h^\delta{}_{;\delta}h_{[\beta}u_{\alpha]} + \mu h_{\gamma;[\beta}h_{\alpha]}s^\gamma + \mu h_\gamma h_{[\alpha;\beta]}s^\gamma \\
& + C\mathscr{R}_1/\sqrt{2}(k_\beta e_\alpha - k_\alpha e_\beta) + C\mathscr{R}_2/\sqrt{2}(k_\beta f_\alpha - k_\alpha f_\beta) = 0,
\end{aligned}$$

$$\begin{aligned}
\text{(5.7)} \quad & \frac{1}{2}\rho_{,\delta}k^\delta u_\alpha u_\beta - \frac{1}{2}\rho_{,\alpha}u_\beta + \frac{1}{6}\perp \rho_{,\alpha}u_\beta - \frac{1}{6}\perp \rho_{,\delta}k^\delta g_{\alpha\beta} \\
& + (\rho + p + \mu\,|\,\bar{h}\,|^2)\left(\omega_{\alpha\gamma}k^\gamma u_\beta + \frac{1}{2}\sigma_{\beta\gamma}u_\alpha k^\gamma - \frac{1}{2}\sigma_{\alpha\beta} + \frac{1}{2}\omega_{\beta\gamma}u_\alpha k^\gamma - \frac{1}{2}\omega_{\beta\alpha}\right) \\
& + \frac{1}{4}\mu(\,|\,\bar{h}\,|^2)_{,\gamma}k^\gamma u_\alpha u_\beta - \frac{1}{4}\mu(\,|\,\bar{h}\,|^2)_{,\alpha}u_\beta + \frac{1}{2}\mu\,|\,\bar{h}\,|^2\theta u_\alpha k_\beta - \frac{1}{2}\mu\,|\,\bar{h}\,|^2\theta g_{\alpha\beta} \\
& + \frac{1}{2}\mu(\,|\,\bar{h}\,|^2)_{,\alpha}k_\beta - \frac{1}{2}\mu(\,|\,\bar{h}\,|^2)_{,\gamma}k^\gamma g_{\alpha\beta} + \frac{1}{2}\mu\,\{\,h_{\alpha;\gamma} - h_{\delta;\gamma}u_\alpha k^\delta\,\}\,h^\gamma u_\beta \\
& + \frac{1}{2}\mu h^\delta_{;\delta}\,\{\,h_\alpha - h_\gamma k^\gamma u_\alpha\,\}\,u_\beta + \frac{1}{2}\mu(h_{\beta;\alpha}h_\gamma k^\gamma - h_{\beta;\gamma}h_\alpha k^\gamma) + \mu h_{[\gamma;\alpha]}h_\beta k^\gamma \\
& - C\Sigma k_\alpha k_\beta - \frac{C\mathscr{R}_1}{\sqrt{2}}k_\alpha e_\beta - \frac{C\mathscr{R}_2}{\sqrt{2}}k_\alpha f_\beta = 0 .
\end{aligned}$$

In absence of the magnetic field, from equations (5.6) and (5.7) we get the following equations, as found by Kundt and Trumper [17]:

$$(\rho + p)\sigma = \sqrt{3}C\Sigma, \tag{5.8}$$

$$(\rho + p)\omega = \sqrt{2}C\,|\mathscr{R}|. \tag{5.9}$$

Moreover, we observe that due to magnetic field, the expressions for Σ and $\mathscr{R}$ are not explicit as in equations (5.8) and (5.9) and a pure null wave does not propagate along shear-free null geodesics even if the magnetofluid is non-rotating and non-shearing.

ACKNOWLEDGMENT

The author is grateful to Dr. L. Radhakrishna for valuable discussion and to Professor J. V. Narlikar for remarks on an earlier version of this paper.

REFERENCES

[1] H. Alfven, *Cosmical Electrodynamics*. Clarendon Press, Oxford, 1950.
[2] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*. Clarendon Press, Oxford, 1961.
[3] T. G. Cowling, *Reports on Progress in Physics*, Vol. **XXV**, 1962.
[4] N. Coburn, *Relativistic Fluid Mechanics and Magnetohydrodynamics*. Academic Press, New York, 1963.
[5] A. H. Taub, *Colloques Internationaux*, n° **184**, 1970, p. 189.
[6] P. J. Greenberg, *Astrophys. J.*, t. **164**, 1971, p. 589.
[7] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics*. W. A. Benjamin, New York, 1967.
[8] P. Yodzis, *Phys. Rev.*, **D 3**, 1971, p. 2941.
[9] T. H. Date, *J. Shivaji Univ.*, t. **5**, 1972, p. 123.
[10] T. H. Date, *Curr. Sci.*, t. **42**, 1973, p. 15.
[11] T. H. Date, *Indian J. Pure and Appl. Maths*, t. **4**, 1973, p. 612.
[12] M. Bray, *C. R. Acad. Sci. Paris*, t. **272**, 1971, p. 694.
[13] M. Bray, *C. R. Acad. Sci. Paris*, t. **274**, 1972, p. 809.
[14] M. Bray, *C. R. Acad. Sci. Paris*, t. **274**, 1972, p. 874.
[15] R. R. Shaha, *Ann. Inst. Henri Poincaré*, t. **XX**, n° 2, 1974, p. 189.
[16] P. Jordan, J. Ehlers and W. Kundt, *Ahb. Akad. Wiss. Mainz*, No 7, 1960.
[17] W. Kundt and M. Trumper, *Ahb. Akad. Wiss. Mainz*, No 12, 1962.
[18] S. W. Hawking, *Astrophys. J.*, t. **145**, 1966, p. 544.
[19] P. Szekeres, *Ph. D. Thesis*. Univ. of London, 1964.
[20] P. Szekeres, *Relativity and Gravitation*. Gordon and Breach, New York, 1971.
[21] G. F. R. Ellis, *General Relativity and Cosmology*. Academic Press, New York, 1971.
[22] M. Ruderman, *Annual Review of Astronomy and Astrophysics*, t. **10**, 1972.
[23] V. D. Khade, *Ph. D. Thesis*. Shivaji University, 1974.
[24] T. H. Date, *Ph. D. Thesis*. Shivaji University, 1973.

(Manuscrit reçu le 3 octobre 1975)