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Quantization in terms of local Heisenberg systems

by

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ABSTRACT. — Quantization in terms of representations of structures called local Heisenberg systems is considered. These structures are defined by triplets $(\mathfrak{A}, \underline{g}, \delta)$ where $\mathfrak{A}$ is a topological $*$-algebra, $\underline{g}$ a Lie-algebra and δ a map from $\underline{g}$ into Der $(\mathfrak{A})$. By considering $\underline{g}$-annihilating generalized states on $\mathfrak{A}$ it is possible to quantize systems that are not treatable with systems of imprimitivity. Applications to non-relativistic and relativistic systems are considered.

1. INTRODUCTION

Let $M = \mathbb{R}^3$ be the Euclidean manifold on which an elementary particle $\mathscr{A}$ without spin moves, and let Σ be the σ-algebra of Borel sets on $\mathbb{R}^3$. The real 6-dimensional Euclidean Lie group E(3) of (transitive) motions on M induce automorphisms α(E3)) in Σ by

$$(1.1) \qquad (\Sigma, E(3)) \in (\Delta, (R, \bar{a})) \rightarrow \alpha_{R,\bar{a}}(\Delta) = \Delta_{R,\bar{a}} \in \Sigma$$

where R and $\bar{a}$ are rotations and translations respectively and $\Delta_{R,\bar{a}} = \{ R\bar{x} + \bar{a} ; \bar{x} \in \Delta \}$. Let $\mathscr{H}$ be a separable Hilbert space and E a map from Σ onto a family $\mathscr{F} = \{ E(\Delta) \}_{\Delta \in \Sigma}$ of orthogonal projectors in $\mathscr{H}$. The automorphisms α(E(3)) are represented in $\mathscr{H}$ by a group of unitary operators U(E(3)) satisfying

$$(1.2) \qquad U(R, \bar{a})E(\Delta)U^+(R, \bar{a}) = E(\Delta_{(R,\bar{a})^{-1}})$$

The triplet $(\mathscr{F}, \mathrm{U}(\mathrm{E}(3)), \mathscr{H})$ is a canonical system of imprimitivity for $\mathscr{A}$ [1]. In particular, the projectors $\mathscr{F}$ define an abelian group $\mathrm{V}(\mathrm{T}^3)$ of unitary operators by

$$\mathrm{V}(\bar{b}) = \int_{\mathbb{R}^3} e^{i\bar{b}\cdot\bar{x}} d\mathrm{E}(x) \qquad \bar{b} \in \mathbb{R}^3 \tag{1.3}$$

Defining $\mathrm{U}(\bar{a}) = \mathrm{U}(\mathbb{1}, \bar{a})$ the set $\{ \mathrm{U}(\bar{a}), \mathrm{V}(\bar{b}) \}$ generates a unitary representation $\mathrm{U}(\mathrm{H}_7)$ of the real 7-dimensional Heisenberg Lie group H_7, satisfying

$$\mathrm{U}(\bar{a})\mathrm{V}(\bar{b}) = e^{i\bar{a}\bar{b}\hbar}\mathrm{V}(\bar{b})\mathrm{U}(\bar{a}) \tag{1.4}$$

which is the Weyl form [2] of the canonical commutation relations. The class of unitarily equivalent, unitary irreducible representations of H_7 is unique up to the scale factor $\hbar$ [3] [4]. The skew-adjoint generators

$$\bar{\mathrm{P}} = \frac{d\mathrm{U}}{d\bar{a}}(0) \qquad \text{and} \qquad \bar{\mathrm{X}} = \frac{d\mathrm{V}}{d\bar{b}}(0)$$

satisfy on the Gårding domain $\mathscr{D}_{\mathrm{G}}$ of $\mathrm{U}(\mathrm{H}_7)$ the commutation relations

$$[\mathrm{P}_k, \mathrm{X}_j] = \delta_{kj} i\hbar \mathrm{I}\,; \qquad [\mathrm{P}_j, \mathrm{P}_k] = [\mathrm{X}_j, \mathrm{X}_k] = 0\,; \qquad k, j = 1, 2, 3 \tag{1.5}$$

of the Heisenberg algebra $\underline{h}_7$.

This method of quantization was developed by Wightman [5] and Mackey [6] and is closely related to the « quantum logics » of proposition calculus. Unfortunately the use of imprimitivity systems is restricted to the case when there is a group of transitive motions on the manifold M. In cases where $\mathrm{M} \neq \mathbb{R}^3$, expressing constraints of the system, this is not the case, and one would like to generalize the scheme.

One way to do this, is to observe that the system of imprimitivity can be cast into the form of a « Heisenberg system » studied among others by Segal [7] and Dixmier [8].

Hence if $\mathfrak{A}$ is the concrete abelian C*-algebra generated by $\mathrm{U}(\mathrm{T}^3)$ or the projectors $\mathscr{F}$, then $\mathrm{U}(\mathrm{E}(3))$ acts as a group of automorphisms of $\mathfrak{A}$. Denoting by α' the map from $\mathrm{E}(3)$ into $*$-Aut $(\mathfrak{A})$ defined by

$$\alpha' : (\mathfrak{A}, \mathrm{E}(3)) \ni (f, g) \rightarrow \mathrm{U}(g) f \mathrm{U}^+(g) \in \mathfrak{A}$$

then the triplet $(\mathfrak{A}, \mathrm{E}(3), \alpha')$ is a Heisenberg system [8].

In the present paper we shall discuss quantization in terms of structures that we call local Heisenberg systems. These structures are one type of generalizations of Heisenberg systems and hence of the systems of imprimitivity. It turns out that the structure of the representations of local Heisenberg systems is very rich, and generates Lie algebra representations, the enveloping algebras of which describe the considered systems by means of symmetric operators in a Hilbert space. Interesting applications can be found in the realm of relativistic quantum mechanics, where realizations of relativistic Lie algebras (Lie algebras containing the Poincaré Lie algebra) on manifolds combining geometrical external, and internal degrees of freedom are able to describe hadron like structures.

In section 2 we collect some properties of local Heisenberg systems and their representations useful for the applications to quantization which is outlined in section 3, both for non-relativistic and relativistic systems. Section 4 contains a discussion of some problems connected with the generalized quantization outlined in section 3 and the use of symmetric but not self-adjoint operators as observables.

2. LOCAL HEISENBERG SYSTEMS

Let $\mathfrak{A}$ be a topological $*$-algebra over $\mathbb{C}$ and let $\underline{g}$ be real Lie algebra.

DEFINITION. — A *local Heisenberg system* (lHs) is a triplet $(\mathfrak{A}, \underline{g}, \delta)$ where $\mathfrak{A}$ and $\underline{g}$ are as above and $\delta : \underline{g} \to *\text{-Der}(\mathfrak{A})$ is a Lie algebra homomorphism into the $*$-derivations of $\mathfrak{A}$ (i. e. $\delta_x f^* = (\delta_x f)^*$ for any $f \in \mathfrak{A}$, $x \in \underline{g}$, where δ_x denotes the derivation of $\mathfrak{A}$ corresponding to x).

Let $\mathscr{H}$ be a separable complex Hilbert space. A *representation* (rep) $(\Pi, \mathrm{T}, \mathscr{H})$ of a lHs $(\mathfrak{A}, \underline{g}, \delta)$ is an algebra morphism $\Pi : \mathfrak{A} \to \mathscr{L}(\mathscr{H})$ and a Lie algebra homomorphism $\mathrm{T} : \underline{g} \to \mathscr{L}(\mathscr{H})$ such that

$$\Pi(f)^* = \Pi(f^*)$$

$$[\mathrm{T}(x), \Pi(f)] = \Pi(\delta_x f) \qquad f \in \mathfrak{A}\,; \qquad x \in \underline{g},$$

on some dense invariant domain $\mathrm{D} \subset \mathscr{H}$. Suppose that there is a (generalized) vector ε in the (anti-) dual $\tilde{\mathrm{D}}$ of D, such that $\Pi(\mathfrak{A})\varepsilon$ is a nontrivial subspace of $\mathscr{H}$, and $\mathrm{T}(x)\varepsilon = 0$, then $\mathscr{H}$ carries a rep of $\underline{g}$ by the operators $\mathrm{T}(\underline{g})$:

$$\mathrm{T}(x)\Pi(f)\varepsilon = [\mathrm{T}(x), \Pi(f)]\varepsilon = \Pi(\delta_x f)\varepsilon$$

$$[\mathrm{T}(x), \mathrm{T}(y)]\Pi(f)\varepsilon = \Pi((\delta_x\delta_y - \delta_y\delta_x)f)\varepsilon = \Pi(\delta_{[x,y]}f)\varepsilon = \mathrm{T}([x, y])\Pi(f)\varepsilon$$

If also $\mathrm{T}^*(x)\varepsilon = 0$, then the operators $\mathrm{T}(x)$, $x \in \underline{g}$ are skew-symmetric.

A standard way of studying reps of lHs: s is by means of the GNS construction relative to (generalized) states on $\mathfrak{A}$.

The sufficient condition $\mathrm{T}^*(x)\varepsilon = \mathrm{T}(x)\varepsilon = 0$ for skew-symmetric operators cannot in general be relaxed if one wants to ensure skew-symmetry, as simple examples show.

The condition implies $(\varepsilon, [\mathrm{T}(x), \Pi(f)]\varepsilon) = (\varepsilon, \Pi(\delta_x f)\varepsilon) = 0$. This leads to the conclusion that in order always to ensure the operators $\mathrm{T}(\underline{g})$ to be skew-symmetric, the generalized states ω on $\mathfrak{A}$ should satisfy $\omega(\delta_x f) = 0$.

Denote by $\mathfrak{A}'^+$ the positive continuous linear functionals on $\mathfrak{A}$. Let

$\mathscr{L}_{\underline{g}}$ be the sub-space of $\mathfrak{A}$ generated by elements of the form $\delta_x f$, $f \in \mathfrak{A}$, $x \in \underline{g}$. Then

$$\mathscr{L}_{\underline{g}}^{\perp} = \{ \omega \in \mathfrak{A}' \,;\, f \in \mathscr{L}_{\underline{g}} \Rightarrow \omega(f) = 0 \}$$

is the space of g-annihilating continuous linear forms on $\mathfrak{A}$. The set $\mathscr{L}_{\underline{g}}^{\perp} \cap \mathfrak{A}'^{+}$ is the set of g-annihilating generalized states $\mathscr{E}_{\underline{g}}(\mathfrak{A})$ on $\mathfrak{A}$. If δ'_x denotes the linear operator on the dual $\mathfrak{A}'$ of $\mathfrak{A}$, such that if $f \in \mathfrak{A}$, $\omega \in \mathfrak{A}'^{+}$, then

$$(\delta'_x \omega)(f) = -\,\omega(\delta_x f) \tag{2.1}$$

An alternative characterization of $\mathscr{E}_{\underline{g}}(\mathfrak{A})$ is then

$$\mathscr{E}_{\underline{g}}(\mathfrak{A}) = \{ \omega \in \mathfrak{A}'^{+} \,;\, \delta'_x \omega(\cdot) = 0, \quad \forall x \in \underline{g} \}$$

If $\mathfrak{A}$ contains a unit element e ; $ef = fe$, $\forall f \in \mathfrak{A}$ and $\omega(e) = 1$, then ω is a state on $\mathfrak{A}$.

Most of our algebras will not have a unity, but only an approximate one, $\{ e_\alpha \}_{\alpha \in \mathrm{A}}$ where A is some directed index set. In this case the GNS construction takes the following form.

Let $\omega \in \mathfrak{A}'^{+}$ and define the set N_ω by

$$\mathrm{N}_\omega = \{ f \in \mathfrak{A} \,;\, \omega(f^* f) = 0 \}$$

Clearly N_ω is a left ideal of $\mathfrak{A}$ and $\mathfrak{A}/\mathrm{N}_\omega$ is a pre-Hilbert space, where the scalar product is defined by

$$(\varepsilon_\omega(f), \varepsilon_\omega(g)) = \omega(f^* g)$$

with ε_ω being the canonical map from $\mathfrak{A}$ onto $\mathfrak{A}/\mathrm{N}_\omega$. Put $\overline{\mathfrak{A}/\mathrm{N}_\omega} = \mathscr{H}_\omega$. Then $\mathfrak{A}$ has a canonical representation $\Pi_\omega(\mathfrak{A})$ in $\mathscr{L}(\mathscr{H}_\omega)$ by the definition

$$\Pi_\omega(f)\varepsilon_\omega(g) = \varepsilon_\omega(fg) \qquad \forall f, \qquad g \in \mathfrak{A}\,.$$

Thus we only lack a cyclic vector for $\Pi_\omega(f)$ in $\mathscr{H}_\omega$, ($\lim\limits_{\alpha \in \mathrm{A}} \varepsilon_\omega(e_\alpha)$ does not exist as a vector in $\mathscr{H}_\omega$, where $\{ e_\alpha \}_{\alpha \in \mathrm{A}}$ is the approximate unity).

Now, let $\mathfrak{A}_1$ be a topological $*$-algebra containing $\mathfrak{A}$ as a topological dense $*$-ideal.

DEFINITION. — Put $\mathrm{M}_{\mathrm{L}}(\mathfrak{A}) = \{ m \in \mathfrak{A}_1 \,;\, mf, m^* f \in \mathfrak{A}, \ \forall f \in \mathfrak{A} \}$ and $\mathrm{M}_{\mathrm{R}}(\mathfrak{A}) = \{ m \in \mathfrak{A}_1 \,;\, fm, fm^* \in \mathfrak{A}, \ \forall f \in \mathfrak{A} \}$.

A multiplier of $\mathfrak{A}$ is an element in $\mathrm{M}(\mathfrak{A}) = \mathrm{M}_{\mathrm{L}}(\mathfrak{A}) \cap \mathrm{M}_{\mathrm{R}}(\mathfrak{A})$. Obviously $\mathrm{M}(\mathfrak{A}) \supset \mathfrak{A}$ and $\mathrm{M}(\mathfrak{A})$ is a $*$-subalgebra of $\mathfrak{A}_1$. We shall always understand that $\mathrm{M}(\mathfrak{A})$ shall contain the unit element of $\mathfrak{A}$ defined by $\lim\limits_{\alpha \in \mathrm{A}} e_\alpha = e$ as a multiplier.

LEMMA 1. — Let $f \in \mathrm{N}_\omega$. Then $mf \in \mathrm{N}_\omega$ for $m \in \mathrm{M}_{\mathrm{L}}(\mathfrak{A})$.

Proof. — $f \in \mathrm{N}_\omega \Rightarrow \omega(f^* f) = 0$. Thus

$$|\,\omega(f^* m^* m f)\,|^2 \leq \omega(f^* m^* m m^* m f)\omega(f^* f) = 0\,.$$

Hence $mf \in \mathrm{N}_\omega$. Q. E. D.

From Lemma 1 follows that N_ω is a left multiplier ideal.

We can now extend $\Pi_\omega(\mathfrak{A})$ to elements in $M(\mathfrak{A})$ in the following way.

Let $\Phi = \varepsilon_\omega(\mathfrak{A})$ be the dense subspace of $\mathscr{H}_\omega$ defined by the canonical map, and Φ' its (anti-)dual space. Clearly $\varepsilon_\omega(e) \in \Phi'$. In fact, for any $m \in M(\mathfrak{A})$ we have that $\varepsilon_\omega(m)$, defined by $(\varepsilon_\omega(e), \varepsilon_\omega(m^*f)) = (\varepsilon_\omega(m), \varepsilon_\omega(f)) = \omega(m^*f)$, belongs to Φ'. We can therefore define a rep $\Pi'_\omega(M(\mathfrak{A}))$ by

$$(\varepsilon_\omega(f), \Pi'_\omega(m)\varepsilon_\omega(g)) = (\varepsilon_\omega(f), \varepsilon_\omega(mg)) \qquad f, g \in \mathfrak{A} m \in M(\mathfrak{A})$$

Since $\mathfrak{A} \subset M(\mathfrak{A})$, Π'_ω is an extension of Π_ω. This extension will again be denoted Π_ω.

THEOREM 1. — Let $(\mathfrak{A}, \underline{g}, \delta)$ be a lHs with a multiplier algebra $M(\mathfrak{A})$ containing the identity $e: ef = fe = f, \forall f \in \mathfrak{A}$. If $\omega \in \mathscr{E}_{\underline{g}}$ and $\{ \mathscr{H}_\omega, \Pi_\omega \}$ is the corresponding GNS rep of $\mathfrak{A}$, then there exists a unique faithful rep $T_\omega(\underline{g})$ of $\underline{g}$ in $\mathscr{H}_\omega$, such that for $\forall x \in \underline{g}$, $f \in \mathfrak{A}$ we have

$$[T_\omega(x), \Pi_\omega(f)] = \Pi_\omega(\delta_x f) \tag{2.2}$$

$$(\varepsilon_\omega(f), T_\omega(x)\varepsilon_\omega(e)) = 0 \tag{2.3}$$

The representation is continuous from $\underline{g}$ to $\mathscr{L}(\mathscr{H}_\omega)$ and every $T_\omega(x)$, $x \in \underline{g}$ is skew-symmetric on $\Pi_\omega(\mathfrak{A})\varepsilon_\omega(e)$. $\varepsilon_\omega(e)$ is a cyclic generalized vector for $\mathscr{H}_\omega$.

Proof. — 1. Clearly $\Pi_\omega(\mathfrak{A})\varepsilon_\omega(e) = \Phi$ is dense, and $\varepsilon_\omega(e)$ is cyclic.

2. Let $x \in \underline{g}$ and define the operator $T_\omega(x)$ on Φ by

$$T_\omega(x)\Pi_\omega(f)\varepsilon_\omega(e) = T_\omega(x)\varepsilon_\omega(f) = \varepsilon_\omega(\delta_x f). \tag{2.4}$$

The linearity of $T_\omega(x)$ is obvious, as well as the fact that Φ is an invariant domain for $T_\omega(\underline{g})$.

That $T_\omega(\underline{g})$ gives a rep of g follows from the continuity of $\delta(\underline{g})$.

To prove that $T_\omega(x)$ is skew-symmetric let $f, h \in \mathfrak{A}$ and consider

$$\begin{aligned}\omega(\delta_x(f^*h)) &= \omega((\delta_x f^*)h + f^*\delta_x h) \\ &= \omega((\delta_x f)^*h + f^*\delta_x h) = (\varepsilon_\omega(\delta_x f), \varepsilon_\omega(h)) + (\varepsilon_\omega(f), \varepsilon_\omega(\delta_x h))\end{aligned} \tag{2.5}$$

But since $\delta'_x\omega(.) = 0$ we get from (2.4) and (2.5) that

$$(T_\omega(x)\varepsilon_\omega(f), \varepsilon_\omega(h)) = -(\varepsilon_\omega(f), T_\omega(x)\varepsilon_\omega(h)). \qquad \text{Q. E. D.}$$

The action of any $\delta \in \operatorname{Der}(\mathfrak{A})$ on elements of $M(\mathfrak{A})$ can be defined by

$$(\delta m)f = \delta(mf) - m(\delta f) \tag{2.6}$$

This definition extends the action of $T_\omega(\underline{g})$ onto Φ'. Thus from (2.6) applied to m^* we get

$$(\varepsilon_\omega(\delta_x m), \varepsilon_\omega(f)) = (T_\omega(x)\varepsilon_\omega(m), \varepsilon(f)) = -(\varepsilon(m), T_\omega(x)\varepsilon(f)). \quad f \in \mathfrak{A}. \tag{2.7}$$

Multipliers of C*-algebras have previously been studied in references [*10*] and [*11*].

Example. — By a *Heisenberg system* (Hs) we mean a triplet $(\mathfrak{A}, G, \alpha)$ where $\mathfrak{A}$ is a topological $*$-algebra, G is a Lie group and α is an injection of G into Aut $(\mathfrak{A})$. Define $\alpha'(g), g \in G$ by the relation $\alpha'(g^{-1})\omega(f) = \omega(\alpha(g)f)$, $\forall f \in \mathfrak{A}$, $\omega \in \mathfrak{A}'^{+}$. The G-invariant states satisfy $\alpha'(g)\omega(.) = \omega(.)$. If the map α is differentiable, then $(\mathfrak{A}, \underline{g}, d\alpha)$, where $\underline{g}$ is the Lie algebra of G, is a lHs, every one-parameter sub-group e^{xt} of G defining a derivation δ_x by the definition

$$\delta_x f = \frac{d}{dt}(\alpha(e^{xt})f)_{t=0} .$$

If ω is G-invariant then for any $x \in \underline{g}$ we have

$$\omega(\delta_x f) = \frac{d}{dt}(\omega(\alpha(e^{xt})f))_{t=0} = \frac{d}{dt}(\omega(f))_{t=0} = 0 \qquad f \in \mathfrak{A} .$$

The study of lHs's is therefore a generalization of the study of Hs's. Let G be a Lie group with Lie algebra $\underline{g}$.

DEFINITION. — Let $(\mathfrak{A}, \underline{g}, \delta)$ be a lHs. $\delta(\underline{g})$ will be said to be integrable if it is obtained as the differential $d\alpha(\underline{g})$ of a Lie group of automorphismes $\alpha(G)$ in a Hs $(\mathfrak{A}, G, \alpha)$.

DEFINITION. — Let $(\mathfrak{A}, \underline{g}, \delta)$ be a lHs and $\omega \in \mathscr{E}\underline{g}(\mathfrak{A})$. The rep $T_\omega(\underline{g})$ of $\underline{g}$ in $\mathscr{K}_\omega$ is said to be integrable if it is obtained as the differential $dU_\omega(\underline{g})$ of a unitary rep $U_\omega(G)$ of G in $\mathscr{K}_\omega$. $U_\omega(G)$ then defines a Lie-group of $*$-automorphisms $\alpha(G)$ such that $(\mathfrak{A}, G, \alpha)$ is a Hs with rep $H_\omega(\mathfrak{A})$ in $\mathscr{K}_\omega$ and ω is G-invariant. The question of integrability is intimately related to the concept of analytic vectors for Lie algebra reps in Hilbert spaces, for which we refer to [*12*] and [*13*]. For $\mathfrak{A}$ the following definition is useful.

DEFINITION. — $f \in \mathfrak{A}$ is analytic for $\delta \in \mathrm{Der}\,(\mathfrak{A})$ if

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^n f < \infty \qquad \text{some} \qquad t > 0$$

in the topology of $\mathfrak{A}$.

The subset of analytic elments of δ will be denoted $\mathfrak{A}_\delta^\omega$.

Let ω be a generalized $\underline{g}$-annihilating state on $(\mathfrak{A}, \underline{g}, \delta)$. Then $\varepsilon_\omega(f)$ is an analytic vector for $T_\omega(x)$, $x \in \underline{g}$, if

$$\| T_\omega^n(x)\varepsilon_\omega(f) \| \leq ct^n n! , \qquad \text{some} \qquad t > 0, \qquad c \in \mathbb{R}_+ .$$

This implies

$$\omega(((i\delta_x)^n f)^*(i\delta_x)^n f) \leq c^2 t^2 (n!)^2 \tag{2.8}$$

DÉFINITION. — Let $\mathfrak{A}$ be a locally convex linear topological vectorspace. Then p is an algebra semi-norm for $\mathfrak{A}$ if $p(fh) \leq p(f)p(h)$, $f, h \in \mathfrak{A}$ [14], and $p(f^*f) \leq p^2(f)$. If $\mathrm{P} = \{ p_\alpha \}$ is a family of algebra semi-norms for $\mathfrak{A}$, then (2.8) is satisfied provided that for any $p_\alpha \in \mathrm{P}$

$$p_\alpha(\delta_x^n f) \leq c' t^n n\,!\,, \qquad \forall f \in \mathfrak{A}\,.$$

PROPOSITION 1. — If $(\mathfrak{A}, \underline{g}, \delta)$ is a lHs with $\mathfrak{A}$ locally convex with an algebra semi-norm family P and $\omega \in \mathscr{E}_{\underline{g}}$, then δ_x $(x \in \underline{g})$ having a dense set $\mathfrak{A}^{\omega}_{\delta_x}$ in $\mathfrak{A} \Rightarrow \mathrm{T}_\omega(x)$ integrable in $\mathscr{H}_\omega$.

Proof. — $\omega \in \mathscr{E}_{\underline{g}} \Rightarrow \mathrm{T}_\omega(x)\varepsilon_\omega(e) = 0$.

By assumption $\displaystyle\sum_{n=0}^{\infty} \frac{t^n}{n\,!}\, p_\alpha(\delta^n f) < \infty$ some $t > 0$, $p_\alpha \in \mathrm{P}$, $f \in \mathfrak{A}^{\omega}_{\delta_x}$. Now

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{s^n}{n\,!} \| \mathrm{T}_\omega(x)^n \varepsilon_\omega(f) \| &= \sum_{n=0}^{\infty} \frac{s^n}{n\,!} \| \varepsilon_\omega(\delta^n f) \| = \sum_{n=0}^{\infty} \frac{s^n}{n\,!} (\varepsilon_\omega(\delta^n f), \varepsilon_\omega(\delta^n f))^{\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} \frac{s^n}{n\,!} \{ \omega((\delta^n f)^* \delta^n f) \}^{\frac{1}{2}} \leq \sum_{n=0}^{\infty} \frac{s^n}{n\,!} c \cdot p_\alpha^{\frac{1}{2}}(\delta^n f^* \delta^n f) \\ &\leq c \sum_{n=0}^{\infty} \frac{s^n}{n\,!}\, p_\alpha(\delta^n f) < \infty \qquad \text{for} \qquad s \leq t^{-1}, \qquad c \in \mathbb{R}^+\,.\end{aligned}$$

Thus $\varepsilon_\omega(f)$ is an analytic vector for $\mathrm{T}_\omega(x)$, $\forall f \in \mathfrak{A}^{\omega}_{\delta_x}$. This implies that the closure $\overline{\mathrm{T}}_\omega(x)$ of $\mathrm{T}_\omega(x)$ is skew-adjoint and generates a one-parameter unitary symmetry group $\exp t\overline{\mathrm{T}}_x$ satisfying

$$(\varepsilon_\omega(f), \exp t\overline{\mathrm{T}}_x \varepsilon_\omega(e)) = (\varepsilon_\omega(f), \varepsilon_\omega(e)) \qquad \forall f \in \mathfrak{A}\,.$$

It should be noted that not every $\varepsilon_\omega(f) \in \mathscr{H}_\omega$ is cyclic for $\Pi_\omega(\mathfrak{A})$! In the following we shall study the case when $\mathfrak{A}$ is abelian. If ω is extremal then $\Pi_\omega(\mathfrak{A})$ is irreducible in $\mathscr{H}_\omega$ with $\varepsilon_\omega(e)$ as a generalized cyclic vector.

LEMMA 1. — Let $\mathfrak{A}$ be abelian and $\omega \in \mathscr{E}_{\underline{g}}(\mathfrak{A})$. Then, if ω is extremal and $\delta(\underline{g})$ faithful, $\underline{g}$ is trivial.

Proof. — Let ω be extremal and g-annihilating, and $\Pi_\omega(\mathfrak{A})$ be the rep. of $\mathfrak{A}$ in $\mathscr{H}_\omega$. Since $\Pi_\omega(\mathfrak{A})$ is irreducible in $\mathscr{H}_\omega$ and $\overline{\Pi_\omega(\mathfrak{A})\varepsilon_\omega(e)} = \mathscr{H}_\omega$, $\mathscr{H}_\omega$ is one-dimensional.

a) Let $\mathrm{T}_\omega(\underline{g})$ be the rep of $\underline{g}$ in $\mathscr{H}_\omega$. If $\mathrm{T}_\omega(x) \in \Pi_\omega(\mathfrak{A})$, $x \in \underline{g}$ then δ_x is an inner derivation, which since $\mathfrak{A}$ is abelian, is impossible unless $\delta(\underline{g})$ is trivial. Thus $\delta(\underline{g}) \in \mathrm{Der}\,(\mathfrak{A})/\mathrm{Int}\,(\mathfrak{A})$.

b) Let $\mathrm{T}_\omega(x)$ be an outer derivation of $\Pi_\omega(\mathfrak{A})$.

From (2.3) we have

$$T_\omega(x)\Pi_\omega(\mathfrak{A})\varepsilon_\omega(e) = \Pi_\omega(\delta_x\mathfrak{A})\varepsilon_\omega(e) = \varepsilon_\omega(\delta_x\mathfrak{A})$$

Since $\Pi_\omega(\mathfrak{A})$ is abelian and irreducible, $\mathcal{H}_\omega$ is one-dimensional

$$\Rightarrow T_\omega(x) = \lambda_x I_\omega \Rightarrow T_\omega(x)\Pi_\omega(\mathfrak{A})\varepsilon_\omega(e) = \Pi_\omega(\mathfrak{A})T_\omega(x)\varepsilon_\omega(e) = 0$$

according to (2.3). Hence $\lambda_x = 0 \Rightarrow \delta_x = 0$, $x \in \underline{g}$ and $\delta(\underline{g})$ is trivial. If $\delta(\underline{g})$ is faithful this implies that $\underline{g}$ is trivial.

We shall therefore study only reducible reps of $\mathfrak{A}$ given by mixed generalized states $\omega \in \mathscr{E}_{\underline{g}}$.

When $\mathfrak{A}$ is an abelian topological $*$-algebra, then it can be written in the form $\mathfrak{A} = \mathfrak{A}_R + i\mathfrak{A}_R$ where any $f \in \mathfrak{A}_R$ satisfies $f^* = f$.

THEOREM 2. — Let $\mathfrak{A}$ be an abelian locally convex $*$-algebra with algebra seminorms $P = \{ p_\alpha \}_{\alpha \in A}$ and ω a generalized state in $\mathscr{E}_{\underline{g}}$. Then $\overline{\Pi_\omega(f)}$ is self-adjoint in $\mathcal{H}_\omega$ for any $f \in \mathfrak{A}_R$.

Proof. — Let $\varepsilon_\omega(g) \in \mathcal{H}_\omega$, $g \in \mathfrak{A}$ be an element in $\varepsilon_\omega(\mathfrak{A})$. Consider the series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \| \Pi_\omega^n(f)\varepsilon_\omega(g) \|$$

Now

$$\| \Pi_\omega^n(f)\varepsilon_\omega(g) \| = \| \varepsilon_\omega(f^n g) \| = \sqrt{\omega((f^n g)^* f^n g)}$$
$$= \sqrt{\omega(g^* g (f^* f)^n)} \leq \sqrt{c^2 \cdot p_\alpha(g^* g (f^* f)^n)} \leq c \cdot p_\alpha^{\frac{1}{2}}(g^* g) p_\alpha^{n/2}(f^* f)$$

Thus

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \| \Pi_\omega^n(f)\varepsilon_\omega(g) \| \leq c \sum_{n=0}^{\infty} \frac{t^n}{n!} p_\alpha^{n/2}(f^* f) p_\alpha^{1/2}(g^* g)$$
$$= c p_\alpha^{1/2}(g^* g) \sum_{n=0}^{\infty} \frac{t^n}{n!} p_\alpha^{n/2}(f^* f) = c \cdot p_\alpha^{1/2}(g^* g) \exp(t p_\alpha^{1/2}(f^* f))$$

This shows that every $\varepsilon_\omega(g)$, $g \in \mathfrak{A}$ is an analytic vector for $\Pi_\omega(f)$, $f \in \mathfrak{A}$. Hence $\Pi_\omega(f)$ is essentially self-adjoint on $\varepsilon_\omega(\mathfrak{A})$ for any $f \in \mathfrak{A}_R$ [*12*].

We shall need also a more specific result.

Let M be a real manifold and $\mathfrak{A}$ an algebra of C^∞, functions on M with compact support, with algebra semi-norms $P = \{ p_\alpha \}$ of the usual supremum type.

PROPOSITION 2. — Let $\mathfrak{A}$ be as above and $\omega \in \mathfrak{A}'^+$ be a generalized state on $\mathfrak{A}$. Then the powers of x, $x \in M$ are represented by self-adjoint operators in $\mathcal{H}_\omega$.

Proof. — Let $\Pi_\omega(\mathfrak{A})$ be the rep of $\mathfrak{A}$ in $\mathcal{H}_\omega$. For any x^m, $x \in M$ we have

$x^m f \in \mathfrak{A}$ if $f \in \mathfrak{A}$. Then define $\Pi'_\omega(x^m)\Pi_\omega(f) = \Pi'_\omega(x)^m\Pi_\omega(f) = \Pi_\omega(x^m f)$. The operator $\Pi'_\omega(x)$ is a linear operator in $\mathcal{H}_\omega$ with domain $\Pi_\omega(\mathfrak{A})\varepsilon_\omega(e)$. If Σ is the σ-algebra of Borel sets in M, then all f : s in $\mathfrak{A}$ with compact support in Σ generate $\mathfrak{A}$. Let $f \in \mathfrak{A}$ with compact support in Σ. For any such f there exists a positive number L such that $p_\alpha(x^n f) \leq L^n p_\alpha(f)$. Then

$$|| \Pi'_\omega(x^n)^m \varepsilon_\omega(f) || \leq c L^{mn} p_\alpha(f) \leq c L^{mn} m\,!\, p_\alpha(f)$$

This shows that $\Pi'_\omega(x^m)$ is essentially self-adjoint on $\Pi_\omega(\mathfrak{A})\varepsilon_\omega(e)$ and can be extended to a self-adjoint operator in $\mathcal{H}_\omega$.

Finally we give the notions of irreducibility useful for reps of lHs.

Let $\Pi(\mathfrak{A})'$ denote the commutant of $\Pi(\mathfrak{A})$. Then the rep $(\Pi(\mathfrak{A}), T(\underline{g}), \mathcal{H})$ of the lHs $(\mathfrak{A}, \underline{g}, \delta)$ is *weakly (strongly) Schur-irreducible* [9] if the only operators $B \in (\Pi(\mathfrak{A}))'$ that commute strongly (weakly) with $T(\underline{g})$ on D are the multiples $\lambda\mathbb{1}$, $\lambda \in \mathbb{C}$ of the unit operator in $\mathcal{H}$. In order for strong commutativity to hold it is necessary that each $T(x)$, $x \in \underline{g}$ is essentially skew-adjoint on D.

THEOREM 3. — Let $\mathfrak{A} = \mathcal{F}(\mathfrak{M})$ be an abelian, locally convex topological $*$-algebra of functions on the real (analytic) manifold $\mathfrak{M}$, G a real finite-dimensional Lie group with Lie algebra $\underline{g}$, and $(\Pi_\omega(\mathfrak{A}), T_\omega(\underline{g}), \mathcal{H}_\omega)$ a weakly Schur-irreducible rep of the lHs $(\mathfrak{A}, \underline{g}, \delta)$ in $\mathcal{H}$, obtained from $\omega \in \mathcal{E}_{\underline{g}}$ by the GNS construction. Then if $(\Pi_\omega(\mathfrak{A}), T_\omega(\underline{g}), \mathcal{H}_\omega)$ is integrable, there is a corresponding rep $(\Pi_\omega(\mathfrak{A}), U_\omega(G), \mathcal{H}_\omega)$ of the Hs $(\mathfrak{A}, G, \alpha)$ that defines a (topologically) irreducible system of imprimitivity based on $\mathfrak{M}$.

Proof. — The weak Schur-irreducibility implies that $\mathcal{H}_\omega$ carries a rep of $\underline{g}$ by skew-symmetric operators $T(\underline{g})$. This rep is integrable to a unitary rep U(G) of G. The U(G) : s define by $U_\omega(g)\Pi_\omega(f)U_\omega^+(g) = \Pi_\omega(\alpha_g f)$, $g \in G$ a Lie group $\alpha(G)$ of automorphisms, with generators $\delta(\underline{g})$. $(\mathfrak{A}, G, \alpha)$ is the Hs with rep $(\Pi_\omega(\mathfrak{A}), U_\omega(G), \mathcal{H}_\omega)$.

Let $\mathfrak{A}_R$ be the real elements in $\mathfrak{A} = \mathfrak{A}_R + i\mathfrak{A}_R$. The operators $\Pi_\omega(\mathfrak{A}_R)$ are self-adjoint according to Theorem 2 and admit a spectral decomposition $\{ E(\Delta) \}_{\Delta\in\Sigma}$ where $(\mathfrak{M}, \Sigma)$ is the Borel measurable space on $\mathfrak{M}$. From the action

$$U_\omega(e^{xt})\Pi_\omega(f_R)U_\omega^+(e^{xt}) = \Pi_\omega(\alpha(e^{xt})f_R) = \Pi_\omega(f_{R,\gamma(xt)}) \qquad x \in \underline{g}, \qquad t \in \mathbb{R},$$

where $\gamma(xt)$ is the induced action of G on $\mathfrak{M}$ defined by

$$(\alpha(e^{xt})f_R)(m) = f_{R,\gamma(xt)}(m) = f_R(\gamma^{-1}(xt)m), \qquad m \in \mathfrak{M},$$

we deduce that

$$U_\omega(e^{xt})E(\Delta)U_\omega^+(e^{xt}) = E_{\gamma(xt)}(\Delta) = E(\gamma^{-1}(xt)\Delta)$$

where

$$\gamma(xt)\Delta = \{ \gamma(xt)m,\ m \in \Delta \}$$

Now $\{ E(\Delta) \}'_{\Delta\in\Sigma} = \Pi_\omega(\mathfrak{A})'$ since strong commutativity implies commutativity of the spectral families of the self-adjoint operators. Since furthermore the operators U(G) are bounded, we have that

$$\{ E(\Delta) \}'_{\Delta\in\Sigma} \cap U(G)' = \{ \lambda I, \lambda \in \mathbb{C} \}$$

which is equivalent to topological irreducibility of the imprimitivity system $(\{ E(\Delta) \}, U_\omega(G), \mathscr{K}_\omega)$. The irreducibility implies that the induced action of G on $\mathfrak{M}$ is transitive.

3. QUANTIZATION

At the root of quantum mechanics lies the Born interpretation of probability density. If $\mathfrak{M}$ is the manifold on which the density is to be defined, one could introduce a set of « primitive observables » for a system $\mathscr{A}$, corresponding to localization measurements on $\mathfrak{M}$.

The algebra of real primitive observables is embedded in a complex function algebra $\mathscr{F}(\mathfrak{M})$ on $\mathfrak{M}$ equipped with a family $\mathscr{P}$ of algebra seminorms.

Let Σ be the σ-algebra of Borel sets Δ on $\mathfrak{M}$. A function $f \in \mathscr{F}(\mathfrak{M})$ with support $\bar{\Delta}$ localizes the system $\mathscr{A}$ in Δ with probability density f^*f. If ω is a suitable generalized state on $\mathscr{F}(\mathfrak{M})$ i. e. a positive, σ-finite measure on $(\mathfrak{M}, \Sigma)$, then $\omega(f^*f)$ is the probability of finding the system localized by f.

The observables in $(\mathscr{F}(\mathfrak{M}))$ will be « a complete set of commuting observables » for $\mathscr{A}$, if the generalized state $\omega \in \mathscr{F}(\mathfrak{M})'^{+}$ is such that the Hilbert state $\mathscr{K}_\omega$ has a direct integral decomposition into one-dimensional subspaces corresponding to extremal generalized states on $\mathscr{F}(\mathfrak{M})$.

The kinematical transformations of the measuring device of $\mathscr{A}$ determine a Lie algebra $\underline{g}$ of generators of these transformations. This Lie algebra acts on $\mathscr{F}(\mathfrak{M})$ by a map $\delta : \underline{g} \to *\text{-Der}(\mathscr{F}(\mathfrak{M}))$. If $\delta(\underline{g})$ is integrable on $\mathscr{F}(\mathfrak{M})$, then $\alpha(G)$ will be a Lie group of automorphisms of $\mathscr{F}(\mathfrak{M})$, ($\underline{g}$ being the Lie algebra of G). Choosing ω $\underline{g}$-annihilating, $\alpha(G)$ will have a unitary rep $U_\omega(G)$ in $\mathscr{K}_\omega$, leaving the probabilities invariant.

According to Theorem 3, the rep $(\Pi_\omega(\mathscr{F}(\mathfrak{M})), U_\omega(G), \mathscr{K}_\omega)$ of the Hs $(\mathscr{F}(\mathfrak{M}), G, \alpha)$ determines a system $\{ (E_\omega(\Delta))_{\Delta\in\Sigma}, U_\omega(G), \mathscr{K}_\omega \}$ of imprimitivity for $\mathscr{A}$ which is irreducible if the rep $(\Pi_\omega(\mathscr{F}(\mathfrak{M})), T_\omega(\underline{g}), \mathscr{K}_\omega)$ is weakly Schur-irreducible.

Irreducible systems are associated with elementary systems. In case $\delta(\underline{g})$ is not integrable the rep $(\Pi_\omega(\mathscr{F}(\mathfrak{M}), T_\omega(\underline{g}), \mathscr{K}_\omega)$ can be considered as a generalization of the imprimitivity system for $\mathscr{A}$. Again an elementary system will be associated with a weakly (or strongly) Schur-irreducible rep of $(\mathscr{F}(\mathfrak{M}), \underline{g}, \delta)$.

In the following we shall illustrate how this method of quantization works in specific cases both for non-relativistic and relativistic systems.

3.1 « Non »-relativistic systems

For a « non »-relativistic system $\mathscr{A}$, the primitive observables is taken to correspond to localization measurements (position) in configuration space $M = \mathbb{R}^3$.

Let $\mathscr{S}_n$ be the nuclear $*$-algebra of functions $f \in \mathscr{S}(\mathbb{R}^n)$ with ordinary pointwise multiplication as the composition law and complex conjugation as the involution. The family $P = \{ p_{\alpha\beta} \}$ of semi-norms on $\mathscr{S}_n$ can be taken as

$$p_{\alpha\beta}(f) = \sup_{\substack{x \in \mathbb{R}^n \\ |i| \leq \alpha \\ |j| \leq \beta}} 2^{|j|} \, | x^i \partial^j f | \qquad f \in \mathscr{S}_n \tag{3.1}$$

with ordinary multi-index notation. These semi-norms are algebra semi-norms.

For a non-relativistic elementary particle without spin the primitive observables are the real elements of $\mathscr{S}_3$.

$\mathscr{S}_3$ contains the important sub-algebra $\mathscr{D}_3$, composed of the functions $f \in \mathscr{S}_3$ which have compact support.

The observables in $\mathscr{S}_3$ do not exhaust the complete set $\mathscr{C}$ of commuting observables for $\mathscr{A}$: we must add a number of observables called super-selection observables. $\mathscr{C}$ for a « non »-relativistic elementary particle is then composed of (the real elements of) $\mathscr{S}_3$ and the polynomial algebra of the super-selection observables such as mass m, Baryonic number B, electric charge Q, etc. These are mapped into the set $\{ \mathfrak{A}, \mathfrak{M}(\mathfrak{A}) \}$.

By definition the super-selection observables commute with the observables on $\mathscr{A}$. For any $\omega \in \mathfrak{A}^+$ we shall therefore have a decomposition of $\mathscr{H}_\omega$ into subspaces labelled by the eigen-values of the super-selection observables. In the following we shall neglect super-selection observables, other than m.

The generalized states $\{ \omega \}$ on $\mathscr{S}_3$ are the positive tempered distributions $\mathscr{S}'(\mathbb{R}^3)^+$. For any one of these, $\mathscr{H}_\omega$ can be written as $L^2(\mathbb{R}^3, d\mu)$ where μ is a tempered (positive) measure. The one-dimensional subspaces of $\mathscr{H}_\omega$, carrying the irreps of $\mathscr{S}_3$ are the generalized eigenvectors of position in $\mathscr{S}'(\mathbb{R}^3)$. The Euclidean Lie algebra $\underline{e}(3)$ of translations and rotations in $\mathbb{R}^3$ is mapped into $*$-Der $(\mathscr{S}_3)$ by

$$\delta_{\bar{p}} f(\bar{x}) = - \frac{\partial}{\partial \bar{x}} f(\bar{x}) \qquad f(\bar{x}) \in \mathscr{S}_3 \tag{3.2a}$$

$$\delta_{\bar{L}} f(\bar{x}) = - \bar{x} \times \frac{\partial}{\partial \bar{x}} f(\bar{x}) \qquad \bar{p}, \bar{L} \in \underline{e}(3) \tag{3.2b}$$

The triplet $(\mathscr{S}_3, \underline{e}(3), \delta)$ so defined is a lHs. It can be imbedded in a Hs

($\mathcal{S}_3$, E(3), α) since $\delta(\underline{e}(3))$ is integrable to α(E(3)). The $\underline{e}(3)$-annihilating states give rise to reps where μ satisfies

$$(3.3) \qquad \left(\frac{\partial}{\partial x_k}\mu\right)(\cdot) = 0 \qquad k = 1, 2, 3$$

μ is then the Lebesgue measure on $\mathbb{R}^3$.

The rep $\Pi_\omega(\mathcal{S}_3)$ is thus the natural imbedding of $\mathcal{S}(\mathbb{R}^3)$ in $L^2(\mathbb{R}^3, d^3x)$ with the same composition and involution as before. The operators $\overline{\Pi_\omega(f)}$, $f^* = f$; $f \in \mathcal{S}_3$ are self-adjoint according to Theorem 2. The natural multiplier algebra $O_M(\mathbb{R}^3)$ of $\mathcal{S}_3$ contains the multipliers

$$ix_k : f(\bar{x}) \to -ix_k f(\bar{x}) \qquad \text{and} \qquad i : f(x) \to if(x).$$

These close with the derivations $\delta_{\bar{p}}$ under commutation to the Heisenberg algebra $\underline{h}_7$:

$$(3.4) \qquad [-ix_k, -ix_l]f(\bar{x}) = 0 ; \qquad [\delta_{p_k}, \delta_{p_l}]f(\bar{x}) = 0$$
$$[\delta_{p_k}, -ix_l]f(\bar{x}) = i\delta_{kl}f(\bar{x}) \qquad f(\bar{x}) \in \mathcal{S}_3$$

Denote $T_\omega(\delta_{\bar{p}}) = -i\bar{P}$ and $\Pi_\omega(-i\bar{x}) = -i\bar{X}$. Now $\Pi_\omega(\mathcal{S}_3)\varepsilon_\omega(1)$ is dense in $L^2(\mathbb{R}^3, d^3x)$. By Proposition 2, the $\bar{X}$ operators are essentially self-adjoint on $\Pi_\omega(\mathcal{D}_3)\varepsilon_\omega(1)$, hence also on the invariant domain $\Pi_\omega(\mathcal{S}_3)\varepsilon_\omega(1)$. The same holds for the $\bar{P}$: *s* on $\Pi_\omega(\mathcal{S}_3)\varepsilon_\omega(1)$. The operators $-i\bar{P}$, $-i\bar{X}$ and iI satisfy the commutation relations of a rep of $\underline{h}_7$ on $\Pi_\omega(\mathcal{S}_3)\varepsilon_\omega(1)$ by means of essentially skew-adjoint operators. This rep is *weakly* Schur-irreducible i. e. the only bounded operators that commute with the spectral projectors of $\bar{P}$, $\bar{X}$ and I on their domains of definition, are the multiples λI, $\lambda \in \mathbb{C}$ of the unit operator in $\mathcal{H}_\omega$. This rep of $\underline{h}_7$ has a common dense domain of analytic vectors for $\bar{P}$ and $\bar{X}$ in $\Pi_\omega(\mathcal{S}_3)\varepsilon_\omega(1)$ and can be integrated to a unitary rep of the Heisenberg group H_7. This is the ordinary Schrödinger rep of H_7 in the Weyl form which determines a unique equivalence class of unitary irreps up to a scale factor.

In $\mathcal{H}_\omega$ the rep of $\underline{e}(3)$ by means of the operators $i\bar{P}$ and $i\bar{L}$ is likewise integrable (since $\delta(\underline{e}(3))$ is integrable on $\mathcal{S}_3$).

Dynamics is introduced in the ordinary way by letting the operators ($\bar{X}$, $\bar{P}$) become time-dependent (thus also, $\Pi_\omega(\mathcal{S}_3)$). This time-dependence should preserve the algebraic structure $\underline{h}_7$.

Rather than elaborating on this point we shall indicate in which way the use of lHs : *s* can take care of more general cases.

Instead of taking the manifold $M = \mathbb{R}^3$, consider the following examples

a) $M_a = T^3 = \{ \bar{x} \in \mathbb{R}^3 ; 0 \le x_i \le 2\pi \}$

b) $M_b = (\mathbb{R}^3_+) = \{ \bar{x} \in \mathbb{R}^3 ; x_i \ge 0 \}$

c) $M_c = \mathbb{R}^3 \backslash S^3 = \{ \bar{x} \in \mathbb{R}^3 ; x_1^2 + x_2^2 + x_3^2 \ge a \ge 0 \}$

and let there be an elementary spinless particle moving on these manifolds.

The algebra $F(M_\alpha)$ are composed of functions vanishing together with all their derivatives on all boundaries. For these algebras the algebra semi-norms in (3.1) can be used, provided the functions vanish together with all their derivates on the finite boundaries.

If Σ_α is the σ-algebra of Borel sets on M_α ; $\alpha = a, b, c$, then the C^∞-functions with compact support on $\{ \bar{\Delta} \}$; $\Delta \in \Sigma_\alpha$ generate dense sub-algebras of the $F(M_\alpha)$: *s*. Hence due to Proposition 2, the $\Pi_\omega(x_i)$: *s* will be self-adjoint on $L^2(M_\alpha, d\mu_\alpha)$.

In all cases the derivations corresponding to the kinematical Lie algebra $\underline{e}(3)$ can be defined on the $F(M_\alpha)$: *s* as in equations (3.2*a*-*b*). This derivation algebra is not integrable on any of the $F(M_\alpha)$: *s*, however.

Consider reps $\Pi_{\omega,a}$ of the lHs : *s* $(F(M_\alpha, \underline{e}(3), \delta)$ by $\underline{e}(3)$-annihilating generalized states on $F(M_\alpha)$. The corresponding Hilbert space generated by the GNS construction is $\mathcal{H}_{\omega,a} = L^2(M_\alpha, d^3x)$ where d^3x is the Lebesgue measure on M_α. By Theorem 1 there are reps $T_{\omega,\alpha}(\underline{e}(3)))$ of $\underline{e}(3)$ by means of skew-symmetric operators in $\mathcal{H}_{\omega,\alpha}$.

The following situations occur

a) In $L^2(T^3, d^3x)$ the operators $\bar{P} = -i\frac{\partial}{\partial \bar{x}}$ and $\bar{L} = -i\bar{x} \times \frac{\partial}{\partial \bar{x}}$ are symmetric on $\Pi_\omega(F(T^3))\varepsilon_\omega(x(T^3))$, which is an invariant dense domain for this rep of $\underline{e}(3)$. The same domain is the largest invariant dense domain for the rep of $\underline{h}_7$ generated by the algebra $\underline{h}_7$ defined as in (3.4) with $f \in F(M_a)$. $\bar{P}$ has defect indices (1,1) on this domain but can be extended to a self-adjoint operator on the domain of periodic (Lebesgue) differentiable functions on T^3. The rep of $\underline{h}_7$ is *weakly* Schur-irreducible (as is the rep of $(F(M_a)\underline{e}(3), \delta))$ but not integrable to a rep of H_7, due to lack of common dense domain of analytic vectors for $\bar{P}$ and $\bar{X} = \Pi_\omega(x)$. The operators $\bar{L}$ are self-adjoint on the same domain as $\bar{P}$. Lack of common invariant domain of analytic vectors makes this rep of $\underline{so}(3)$ non-integrable [*15*]. The rep $T_\omega(\underline{e}(3))$ is accordingly non-integrable.

b) In $L^2((\mathbb{R}_+)^3, d^3x)$ the operators $\bar{P}$ are symmetric with defect indices (1,0). The rep of $\underline{h}_7$ in this space is not weakly Schur-irreducible, since the operators $\bar{P}$ don't have any orthogonal projectors in $\mathcal{H}_\omega$. It is not integrable to a rep of H_7. The rep of $(F(M_a), \underline{e}(3), \delta)$ is not weakly Schur-irreducible. The operators $\bar{L}$ have properties as in *a*) and neither $T_\omega(\underline{so}(3))$ nor $T_\omega(\underline{e}(3))$ is integrable.

c) In $L^2(\mathbb{R}^3S^3, d^3x)$ the operators $\bar{P}$ are self-adjoint and the rep of $\underline{h}_7$ is weakly Schur-irreducible, but not integrable to a rep of H_7. The operators $\bar{L}$ have self-adjoint extensions and the rep $T_\omega(\underline{so}(3))$ is integrable to a unitary rep of SO(3). The rep $T_\omega(\underline{e}(3))$ is non-integrable however.

In each of the cases *a*)-*c*) there is a mechanics defined by the observable algebra spanned by reps of $\underline{e}(3) \,\oplus\, (\underline{t}^3 \oplus \mathbb{1}) \approx \underline{so}(3) \,\oplus\, \underline{h}_7$.

The symmetry between $\bar{P}$ and $\bar{X}$ which exist in the ordinary case $M = \mathbb{R}^3$ is lost when $M \neq \mathbb{R}^3$. There is, however, nothing to prevent us from using the momentum space for M, provided this is meaningful for the problem. $\bar{P}$ is then self-adjoint and $\bar{X} = i\frac{\partial}{\partial \bar{p}}$ will be only symmetric in general.

The Casimir operator for $T_\omega(\underline{e}(3))$ is $\bar{P}^2$. If $T_\omega(\underline{e}(3))$ is invariant under the time-development we can expand the states according to irreps of $\underline{e}(3)$ labelled by the eigenvalues of $\bar{P}^2$.

3.2 Systems with spin

The measureability of the spin of a free lepton is a delicate matter [16]. Primarily the spin degree of freedom shows up in the degeneracy of the eigenstate of the observables, corresponding to the fact that quantum mechanical spin is not a classical concept in general. The Hilbert space has the form $\mathscr{K} = \mathscr{K}_\omega \hat{\otimes} \mathscr{K}_v$ where $\mathscr{K}_v$ is the carrier space of a rep of the spin algebra $\underline{su}(2)$. This situation can only be achieved by considering $\mathfrak{A} = F(M)$ to be embedded into a larger algebra $\tilde{\mathfrak{A}}$ such that $\tilde{\mathfrak{A}}/\mathfrak{A} = \mathscr{E}(\underline{su}(2))$ where $\mathscr{E}(\underline{su}(2))$ is the complexification of the enveloping algebra of $\underline{su}(2)$. A state on $\tilde{\mathfrak{A}}$ then decomposes into a family of identical states when restricted to $\mathfrak{A}$.

The kinematical algebra $\underline{e}(3)$ can be mapped into Der $(\tilde{\mathfrak{A}})$ by the definition

$$\delta_{\bar{p}} f(\bar{x}, \bar{s}) = - \frac{\partial}{\partial \bar{x}} f(\bar{x}, \bar{s})$$

$$\delta_{\bar{J}} f(\bar{x}, \bar{s}) = - \bar{x} \times \frac{\partial}{\partial \bar{x}} f(\bar{x}, \bar{s}) + [\bar{s}, f(\bar{x}, \bar{s})]$$

where $[s_i, s_j] = \varepsilon_{ijk} s_k$.

If ω is a state on $\tilde{\mathfrak{A}}$, then the $\underline{e}(3)$-annihilating states give rise to the trivial rep of $\underline{su}(2)$ thus for these states the GNS construction gives an unfaithful rep of $\tilde{\mathfrak{A}}$ coinciding with a rep of $\mathfrak{A}$. A faithful rep of $\tilde{\mathfrak{A}}$ is therefore given in $\mathscr{K}_\omega \hat{\otimes} \mathscr{K}_v$, where ω is the $\underline{e}(3)$ annihilating state only when restricted to $\mathfrak{A}$, and $\mathscr{K}_v$ is a carrier space of a rep of $\underline{su}(2)$. If we insist upon a rep by skew symmetric operators in $\mathscr{K}_v$, then it is either non-decomposable or a direct sum of the ordinary irreps D^j ; $j \in \frac{1}{2} \mathbb{N}$ of SU(2) [15].

If the particle has extension, then the spin can be introduced by extending the primitive observables $\mathscr{S}_3$ to $\mathscr{S}_3 \dot{\otimes} \mathscr{D}_3(\Omega)$, where $\mathscr{D}_3(\Omega)$ is the algebra of C^∞-functions on

$$\Omega = - \{ \alpha, \beta, \gamma ; 0 \leq \alpha \leq 2\pi, \ 0 \leq \beta \leq \pi, \ 0 \leq \gamma \leq 2\pi \},$$

the space of Euler angles, where composition is ordinary point-wise multiplication, involution is complex conjugation and the algebra semi-norm is given by

$$p_k = \sup_{\substack{x \in \Omega \\ |j| \leq k}} 2^{|j|} | \partial^j f |$$

The derivations

$$\delta_s^1\ f(\alpha\beta\gamma) = \left(\cos\alpha\cot\beta\frac{\partial}{\partial\alpha} + \sin\alpha\frac{\partial}{\partial\beta} - \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right)f(\alpha\beta\gamma)$$

$$\delta_s^2\ f(\alpha\beta\gamma) = \left(\sin\alpha\cot\beta\frac{\partial}{\partial\alpha} - \cos\alpha\frac{\partial}{\partial\beta} - \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right)f(\alpha\beta\gamma)$$

$$\delta_s^3\ f(\alpha\beta\gamma) = -\frac{\partial}{\partial\alpha}f(\alpha\beta\gamma)$$

define a map δ from $\underline{su}(2)$ into Der $(\mathscr{D}_3(\Omega))$.

In a natural manner the derivations $\delta_{\bar{J}} = \delta_{\bar{L}} \hat{\otimes} 1 + 1 \hat{\otimes} \delta_{\bar{s}}$ define a map δ' from $\underline{su}(2)$ into Der $(\mathscr{S}_3 \hat{\otimes} \mathscr{D}_3(\Omega))$ which combines with $\delta_{\bar{p}}f = -\frac{\partial}{\partial\bar{x}}f$; $f \in \mathscr{S}_3 \hat{\otimes} \mathscr{D}_3(\Omega)$ to $\delta'(\underline{e}(3))$. An $\underline{e}(3)$-annihilating generalized state ω on $\mathscr{S}_3 \hat{\otimes} \mathscr{D}_3(\Omega)$ gives $\mathscr{K}_\omega = L^2(\mathbb{R}^3, d^3x) \hat{\otimes} L^2(\Omega, d\Omega)$ where $d\Omega = \sin\beta\, d\alpha d\beta d\gamma$.

The symmetric operators $\bar{S} = iT_\omega(\bar{J})\,|_{L^2(\Omega,d\Omega)}$ have self-adjoint extensions on functions periodic on the boundaries of Ω and generate an integrable rep of $\underline{su}(2)$. $L^2(\Omega, d\Omega)$ decomposes into a direct sum of carrier spaces of irreps of SU(2), but these are not stable under $\Pi_\omega(\mathscr{D}_3(\Omega))$.

The irreps of the lHs $(\mathscr{S}_3 \hat{\otimes} \mathscr{D}_3(\Omega), \underline{e}(3), \delta)$ in $\mathscr{K}_\omega$ thus describe systems with rotational bands. This situation is typical for extended objects like molecules and nuclei, and is suggestive for hadrons (Regge trajectories) indicating that hadrons also might be extended objects.

For leptons we have so far not seen any rotational levels, and the spin for these particles seems to be an intrinsic property. If these particles are described by the Euler angles, then one has to postulate that the operators $\Pi_\omega(f(\alpha, \beta, \gamma))$ are not observables in $\mathscr{K}_\omega$, and that the superposition principle is valid only in sectors with fixed eigenvalues of the super-selection observable $\bar{S}^2$.

For particles with spin, the Casimir operators characterizing the irreps of $T_\omega(\underline{e}(3))$ are $\bar{P}^2$, $\bar{S}^2$ and $\bar{P}\cdot\bar{S}$, where $\bar{P}\cdot\bar{S}/\sqrt{\bar{P}^2}$ is the helicity operator.

3.3 **Relativistic systems**

A. Configuration space representation

Let $M = \mathbb{R}^4$ and the real functions of $\mathscr{S}_4$ correspond to the primitive observables describing localization in space-time for an elementary relativistic particle.

The algebra $\mathfrak{P} = \underline{t}^4 \oplus \underline{so}(3,1)$ is mapped into Der $(\mathscr{S}_4)$ by

$$\delta_{p^\mu} f(x) = \partial^\mu f(x)$$
$$\delta_{M^{\mu\nu}} f(x) = (x^\mu\partial^\nu - x^\nu\partial^\mu)f(x), \qquad f(x) \in \mathscr{S}_4$$

Since space-time is homogeneous and isotropic we can integrate $\mathfrak{P}$ to the automorphisms $\alpha(\mathscr{P}_+^\uparrow)$ where $\mathscr{P}_+^\uparrow = T^4 \otimes SO_0(3,1)$ is the restricted Poincaré group.

The Lebesgue measure on $\mathbb{R}^4$ is $\mathfrak{P}$-annihilating. Thus the lHs $(\mathscr{S}_4, \mathfrak{P}, \delta)$ can be embedded in the Hs $(\mathscr{S}_4, \mathscr{P}_+^\uparrow, \alpha)$ which carries in $\mathscr{K}_\omega = L^2(\mathbb{R}^4, dx^4)$ a rep of $\underline{h}_9$, the covariant Heisenberg algebra, by means of the operators $iT_\omega(\delta_{p\mu}) = P^\mu$ $\Pi_\omega(x^\nu) = X^\nu$ and $\mathbb{1}$ on the domain $\varepsilon_\omega(\mathscr{S}_4)$. On the same domain $U_\omega(\mathscr{P}_+^\uparrow)$ is differentiable and its differential $dU_\omega(\mathscr{P}_+^\uparrow)$ coincides with $T_\omega(\mathfrak{P})$. The group $(U(1) \otimes T^4) \otimes \mathscr{P}_+^\uparrow \approx H_9 \otimes SO_0(3,1)$ was studied in [*17*].

Introducing the time-parameter τ, corresponding to proper world-time (historical time) leads, in an analogous way as for the non-relativistic elementary particle, to the dynamics of Horwitz and Piron [*18*]. Spin degrees of freedom can be introduced by considering $\tilde{\mathscr{S}}_4 = \mathscr{S}_4 \otimes \mathscr{E}$ $(\underline{sl}(2, \mathbb{C}))$ and representing the lHs $(\mathscr{S}_4, \underline{t}^4 \oplus \underline{sl}(2, \mathbb{C}), \delta)$ in $\mathscr{K} = \mathscr{K}_\omega \hat{\otimes} \mathscr{K}_\nu$, where $\mathscr{K}_\nu$ carries an irrep of $\underline{sl}(2, \mathbb{C})$.

B. Momentum space representation

In many situations in particle physics one is not concerned with the position of the particles but rather with their velocities along rays in space-time and with their masses (and super-selection quantum numbers). This situation is typical for scattering experiments. It seems then worthwhile to investigate this situation *per se*.

Introduce the primitive observables $f(\bar{\Delta})$, $\Delta \in (M, \Sigma)$ where $M = \mathbb{R}^4$ is the momentum space (always neglecting super-selection observables). The corresponding algebra is again $\mathscr{S}_4$. Define the derivations $\delta(\underline{so}(3,1))$ by

$$\delta_{M^{\mu\nu}} f(p) = \left(p^\mu \frac{\partial}{\partial p_\nu} - p^\nu \frac{\partial}{\partial p_\mu} \right) f(p) ; \qquad f(p) \in \mathscr{S}_4$$

The triplet $(\mathscr{S}_4, \underline{so}(3,1), \delta)$ is a lHs. It can be represented in $\mathscr{K}_\omega = L^2(\mathbb{R}^4, d\mu_\omega)$ where

$$\mu_\omega(\Delta) = \int_{-\infty}^{\infty} \int_\Delta d\rho(m^2) d^4p \delta(p^2 - m^2) + c \int_\Delta d^4p \delta^{(4)}(p)$$

is a $\underline{so}(3,1)$-annihilating measure. The rep $\Pi_\omega(\mathscr{S}_4)$ of $\mathscr{S}_4$ is generated by the natural embedding of $\mathscr{S}(\mathbb{R}^4)$ into $\mathscr{K}_\omega$. For a free particle $p^2 = M^2$; $p^0 > 0$. We then take $d\mu_\omega = \theta(p^0)\delta(p^2 - M^2)d^4p$. The ensuing rep is then isomorphic to the algebra $\mathscr{S}_3$ in $\mathscr{K}_{\omega'} = L^2(\mathbb{R}^3, (2(\bar{p}^2 + M^2))^{-\frac{1}{2}} d^3p)$ with the generators of $T_\omega(\underline{so}(3,1))$ given by

$$M^{ij} f(\bar{p}) = \left(p^i \frac{\partial}{\partial p_j} - p^j \frac{\partial}{\partial p_i} \right) f(\bar{p})$$

$$M^{0i} f(\bar{p}) = \sqrt{\bar{p}^2 + M^2} \frac{\partial}{\partial p_i} f(\bar{p}) \qquad f(\bar{p}) \in \mathscr{S}(\mathbb{R}^3)$$

Together with the multipliers

$$\begin{aligned} P^0 f(\bar{p}) &= \sqrt{\bar{p}^2 + M^2} f(\bar{p}) \\ \bar{P} f(\bar{p}) &= \bar{p} f(\bar{p}) \end{aligned}$$

the $M^{\mu\nu}$: s close to a rep of the Poincaré algebra $\mathfrak{P} = \underline{t}^4 \oplus so(3,1)$ which is integrable to a rep of $\mathscr{P}_+^\uparrow$.

Spin is introduced by considering the algebra $\tilde{\mathscr{S}}_3 = \mathscr{S}_3 \otimes \mathscr{E}(su(2))$ and the derivations

$$\delta_{\bar{J}} f(\bar{p}, \bar{s}) = \bar{p} \times \frac{\partial}{\partial \bar{p}} f(\bar{p}, \bar{s}) + [\bar{s}, f(\bar{p}, \bar{s})]$$

$$\delta_{\bar{K}} f(\bar{p}, \bar{s}) = \sqrt{\bar{p}^2 + M^2} \frac{\partial}{\partial \bar{p}} f(\bar{p}, \bar{s}) - (\sqrt{\bar{p}^2 + M^2} + M)^{-1} \bar{p} \times [\bar{s}, f(\bar{p}, \bar{s})]$$

where $J^i = \varepsilon^{ijk} M^{jk}$ and $K^i = M^{0i}$. The lHs $(\tilde{\mathscr{S}}_3, \underline{so}(3,1), \delta)$ is represented in $\mathscr{K}_\omega = L^2(\mathbb{R}^3, (2\sqrt{\bar{p}^2 + M^2})^{-1} d^3p \,\hat{\otimes}\, \mathscr{K}_v$ where $\mathscr{K}_v$ is the carrier space of a rep of $\underline{su}(2)$. Together with the operators P^0 and $\bar{P}$, previously defined, the operators $\bar{J}$ and $\bar{K}$ close to a rep of $\mathfrak{P}$, given earlier in [9]. If $\mathscr{K}_v$ carries an irrep of $\underline{su}(2)$, then the vectors in $\mathscr{K}_v$ satisfy

$$\begin{aligned} P_\mu P^\mu f_\omega &= M^2 f_\omega, \qquad\qquad f_\omega \in \mathscr{K}_\omega \\ W_\mu W^\mu f_\mu &= -\, M^2 s(s+1) f_\omega \end{aligned}$$

where $W_\mu = \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma$. If $iT_\omega(\bar{J})$ and $iT_\omega(\bar{K})$ are symmetric, then $\mathscr{K}_v$ is finite-dimensional and $s \in \frac{1}{2}\mathbb{N}$. In this case the irreps are integrable to the real mass, positive energy unitary irreps $[M, s]$ of $\mathscr{P}_+^\uparrow$ given by Wigner [19].

Reps of the Weyl algebra $\underline{\omega} = \underline{t} \oplus \mathfrak{P}$.

The following example illustrates the use of lHs to treat relativistic systems with mass-spectra.

Let $T^2 = \{ u, v \,;\, 0 \leq u \leq 2\pi,\ 0 \leq v \leq 2\pi \}$ and $\mathscr{D}_2$ be the algebra of C_0^∞-functions on T^2 equipped with the algebra semi-norms

$$P_{\alpha,\beta}(f) = \sup_{\substack{u,v \in T^2 \\ j \leq \alpha \\ k \leq \beta}} 2^{|j+k|} \, | \partial_u^j \partial_v^k f \, |$$

Put $\mathfrak{A} = \mathscr{S}_4 \otimes \mathscr{D}_2$ and define the derivations $\delta(\underline{\omega})$ by

$$\begin{aligned} \delta_{M^{\mu\nu}} f(p, u, v) &= \left(p^\mu \frac{\partial}{\partial p_\nu} - p^\nu \frac{\partial}{\partial p_\mu} \right) f(p, u, v) \\ \delta_{P^\mu} f(p, u, v) &= p^\mu (\partial_u + \partial_v) f(p, u, v) \\ \delta_D f(p, u, v) &= (u \partial_v + v \partial_u) f(p, u, v) \end{aligned}$$

The lHs $(\mathfrak{A}, \underline{\omega}, \delta)$ so defined can be represented in $\mathcal{H} = L^2(\mathbb{R}^4, T^2; d\mu, dq)$. Taking $d\mu = \theta(p^0)\delta(p^2 - M^2)d^4p$ gives a rep Π of $\mathfrak{A}$ that is given by $\mathcal{S}_3(p) \otimes \mathcal{D}_2$ with the rep $T(\underline{\omega})$ given by

$$
\begin{aligned}
i\mathrm{T}(\bar{\mathrm{J}})f &= \bar{\mathrm{J}}f = i\left(\bar{p} \times \frac{\partial}{\partial \bar{p}}\right) f(\bar{p}, u, v) \\
i\mathrm{T}(\bar{\mathrm{K}})f &= \bar{\mathrm{K}}f = i\sqrt{p^2 + \mathrm{M}^2}\,\frac{\partial}{\partial \bar{p}} f(\bar{p}, u, v) \\
i\mathrm{T}(\mathrm{P}^0)f &= \mathrm{P}^0 f = i\sqrt{\bar{p}^2 + \mathrm{M}^2}(\partial_u + \partial_v) f(\bar{p}, u, v) \\
i\mathrm{T}(\bar{\mathrm{P}})f &= \bar{\mathrm{P}}f = i\bar{p}(\partial_u + \partial_v) f(\bar{p}, u, v) \\
i\mathrm{T}(\mathrm{D})f &= \mathrm{D}f = i(u\partial_v + v\partial_u) f(\bar{p}, u, v), \qquad f(\bar{p}, u, v) \in \mathcal{S}(\mathbb{R}^3) \otimes \mathcal{D}(\mathrm{T}^2)
\end{aligned}
$$

The operator $P_\mu P^\mu$ has the spectrum $\{ m^2 ; m^2 = M^2(k + l)^2 ; k, l \in \mathbb{Z} \}$ on the domain $D(P^2) = \{ f(\bar{p}, u, v) \in \mathcal{H} ; f(\bar{p}, u, v)$ periodic on T^2; $\| (\partial_u + \partial_v)^2 f \|_{\mathcal{H}} < \infty \}$.

The operator D has also a self-adjoint extension. There is, however, no common invariant domain of analytic vectors for D and P^μ, hence no integrability of $T(\underline{\omega})$.

This rep of $T(\underline{\omega})$ can be extended to include spin degrees of freedom but is not weakly Schur-irreducible.

Weakly Schur-irreducible reps of $\underline{\omega}$ with the same mass spectrum as above (although with different multiplicity) can be obtained as in [*20*] if one relaxes the condition of a rep in terms of symmetric operators.

4. DISCUSSION

The examples of quantization by means of reps of lHs : *s* given above should be enough to show the potentialities of the scheme. Below part of the physics involved in describing observables with symmetric operators will be discussed.

From an algebraic point of view quantization in terms of reps of lHs : *s* is an exercice in induced reps of Lie algebras, which have been studied i. a. by Dixmier [*21*]. Our approach emphasizes topological aspects.

The weakly (or strongly) Schur-irreducible reps of the lHs $(\mathfrak{A}, \underline{g}, \delta)$ are in many cases connected to reps of finite-dimensional Lie algebras due to the fact that $\mathfrak{A}$ is generated by an abelian finite-dimensional real Lie algebra $\delta_{\mathfrak{A}}$ contained in $\mathfrak{M}(\mathfrak{A})$. The quantum mechanical system $\mathcal{A}$ is in such cases characterized by the Lie algebra structure $\underline{A} = \delta_{\mathfrak{A}} \otimes \underline{g}$ and the time-development generator. The reps of $\underline{A}$ pertinent to a description of $\mathcal{A}$ as an elementary system, are the weakly (or strongly) Schur-irreducible reps of $\underline{A}$. An interesting but unsolved problem is in which way various notions of irreducibility reflect themselves in the concept of elementarity for a system.

The irreps of $\underline{A}$ or $(\mathfrak{A}, \underline{g}, \delta)$ are obtained by suitable choice of the state on $\mathfrak{A}$, in practice (when $\mathfrak{A}$ is abelian) by certain positive measures on a measurable space $(\mathfrak{M}, \Sigma)$. We have demanded that this state should annihilate $\delta(\underline{g})$ when $\mathfrak{A}$ is abelian. One would then like to know whether such a state (measure) can always be found. This seems to be the case for a large number of Lie algebras $\underline{g}$, but we have no answer for the general case.

Also a classification of reps of lHs : s would be desirable, but even when $\underline{A}$ is finite-dimensional this seems to be difficult since there are plenty of non-decomposable reps. A type classification a la von Neumann-Murray might, however, be attempted, e. g. by considering the W*-algebras generated by the reps $(\Pi(\mathfrak{A}), T(\underline{g}), \mathscr{K})$.

We shall now discuss the use of symmetric, non-selfadjoint operators for the description of observables of quantum phenomena. In favour of such operators the following arguments could, e. g., be advanced.

1) Quantum mechanics essentially always deals with closed systems. This is an idealization which is violated already by the contact between the measuring apparatus and the system. It must be considered as a weakeness of the conventional theory not to be stable under small perturbations from this, idealized, situation of describing isolated systems. In order to describe open systems non-selfadjoint symmetric and dissipative operators come into play, being generators of contractive semi-groups, etc., and the time development might in general even be non-Hamiltonian.

2) Already canonical quantization leads to ambiguities, since e. g. the use of spherical coordinates for a free particle, renders the radial momentum symmetric, without selfadjoint extensions inside the Hilbert space. Why not still try to interpret the radial momentum as an observable?

3) Only commuting operators can be measured simultaneously with arbitrary precision. Taking such a « complete set $\mathscr{C}$ of commuting observables » defined on a common dense domain D, one might ask why those observables that do not commute with $\mathscr{C}$ should be essentially selfadjoint on D, since they are anyhow incompatible with those in $\mathscr{C}$. In continuation of this argument one might well ask why they should be essentially selfadjoint at all.

For the cases discussed in this work symmetric operators are considered. Any such operator A, densely defined, can be extended outside the given Hilbert space $\mathscr{K}$ to a selfadjoint operator A^+ in a Hilbert space $\mathscr{K}^+ \supset \mathscr{K}$ [22]. By projecting back to the original space $\mathscr{K}$ one obtains a rep of A in the form

$$(4.1)\qquad (f, Ag) = \int_{-\infty}^{+\infty} \lambda(f, dB(\lambda)g) \qquad f, g \in D_A$$

where $\{ B(\lambda) \}_{\lambda \in \mathbb{R}}$ is a so called « generalized spectral family », satisfying

1) $B(\lambda) \leqslant B(\mu)$ for $\lambda < \mu$.

2) $B(\lambda - 0) = B(\lambda)$.
3) $B(\lambda) \to 0, \lambda \to -\infty$; $B(\lambda) \to I, \lambda \to +\infty$.

Unless A is selfadjoint in $\mathscr{H}$, the $B(\lambda)$: *s* are not orthogonal and are thus not in general projections. The $B(\lambda)$: *s* are of the form $P^+E(\lambda)P^+$, where P^+ is the projection in $\mathscr{H}^+$ on $\mathscr{H}$ and $\{ E(\lambda) \}$ is the spectral family of the self-adjoint extension A^+ in $\mathscr{H}^+$. This extension is unique if A is maximally symmetric.

Since the $B(\lambda)$: *s* are not projections, they do not generate a rep of the ordinary quantum logics of yes-no propositions. This means that with respect to the operator A (as an observable) we cannot get complete information of the system. Thus if Δ is a Borel set on $\mathbb{R}$, then the question $B(\Delta) = \int_\Delta dB(\lambda)$ applied to a normalized vector $f \in \mathscr{H}$, does not prepare f to be within or outside Δ with respect to A after the measurement, since for any such f, $\| B(\Delta)f \| < 1$ and consequently $\| B(\Delta)B(\Delta)f \| < \| B(\Delta)f \|$.

$\{ B(\lambda) \}$ is a family of contraction mappings, the only fixed point of which is the zero vector in $\mathscr{H}$. Repetition of the question therefore does not improve the preparation. An attempt to illustrate the physics connected with this phenomenon is made in the discussion of case *b*) of 3.1 below.

Let Δ and $\tilde{\Delta}$ be two Borel sets in $\mathbb{R}$ such that $\Delta \cup \tilde{\Delta} = \mathbb{R}$ and $\Delta \cap \tilde{\Delta} = \emptyset$. The two operators $B(\Delta)$ and $B(\tilde{\Delta})$ will then both be positive and selfadjoint. The spectral projections of $B(\Delta)$ and $B(\tilde{\Delta})$ do give a complete set of questions of the two operators respectively. In fact we have

$$B(\Delta) + B(\tilde{\Delta}) = 1$$

hence

$$[B(\Delta), B(\tilde{\Delta})] = 0$$

Thus $B(\Delta)$ and $B(\tilde{\Delta})$ are compatible questions. However, for arbitrary Borel sets Δ and Δ' the corresponding operators $B(\Delta)$ and $B(\Delta')$ do not in general commute, but since they are observables in the ordinary sense, they satisfy the usual uncertainty relations on vectors in $\mathscr{H}$.

A general feature of the quantization of systems *a*, *b* and *c* in section 3.1 by means of reps of lHs : *s* is that the symmetry between $\bar{P}$ and $\bar{X}$ is broken. In the case $\mathfrak{M} = \mathbb{R}^3$ this symmetry is connected to the unitarity of the Fourier transformation in $L^2(\mathbb{R}^3, d^3x)$, and questions regarding position or momentum can be asked with the same right, although not at the same time according to the idea of complementarity. However, in case $\mathfrak{M} \neq \mathbb{R}^3$ as in the examples *a*, *b* and *c* of 3.1, questions pertaining at localization are distinguished and can be asked, whereas questions of momentum will depend on the possible boundary conditions that can be put on the respective manifolds. This seems to be intuitively connected to the possibility of a geometrical arrangement for momentum measurements in the manifolds.

In all cases let us consider a particle of mass $\frac{1}{2}$ moving in the manifolds. The Hamiltonian is then $H = \bar{P}^2$.

Case a. — The momentum of the particle is measured on the plane waves with periodic boundary conditions. This domain is not invariant under the operators $\bar{X}$, which explains why there is no contradiction with the Heisenberg uncertainity relations. On the dense subdomain of vectors in $C_0^\infty(T^3)$ physics goes on as if there were no boundaries. Trouble only comes when the states describe particles that hit the boundaries, since the arrangements for momentum measurements are then affected by the geometry.

Case b. — On the domain

$$D(\bar{P}^2) = \{ f \in L^2((\mathbb{R}_+)^3, d^3x) ; f'(0) = f''(0) = 0 ; \quad f'(x), f''(x) \in L^2((\mathbb{R}_+)^3, d^3x) \}$$

$H = \hat{\bar{P}}^2$ and $\bar{P}$ commute ($\hat{}$ means closure). On this domain the state functions vanish fast enough so that the dynamics is unaffected by the walls. This is of course extreemely important, since only when the particle is located away from the walls is the momentum conserved (Energy is always conserved). In fact, considering the walls as infinitely heavy one might well have elastic collisions without energy loss. Any arrangement put up to measure the momentum at the walls must take up (either some energy or) some momentum and transport it away from the particle, or one cannot get information from that part of the measurement apparatus (say the second of two slits). Insisting upon that the measurement process should not affect the boundary conditions results in a lack of information whether the particle passed the second slit or not and makes the question « does the particle have momentum in $\Delta \subset \mathbb{R}^3$ » impossible to answer with « yes » or « no ». There is a certain probability that the particle did have momentum in Δ before the measurement, but it will not have this momentum after the measurement, since it will have hit the wall and subsequently changed its momentum. It is in this case intuitively as well as formally clear that the modulus of $\bar{P}$ can be measured, being in fact $|\bar{P}| = \sqrt{H}$ which is a selfadjoint operator. On the normalized eigenstates f of H we have $(f, \bar{P}f) = 0$ but $(f, |\bar{P}| f) = \sqrt{E}$.

As we mentioned in 3.1, the rep of the lHs describing the system is not weakly Schur-irreducible, since $\bar{P}$ does not have any selfadjoint extensions in $\mathscr{H}_\omega$ (It is certainly neither strongly Schur-irreducible). In cases like this, with maximally symmetric operators we suggest a different notion of irreducibility.

Let $(\Pi(\mathfrak{A}), T(\underline{g}), \mathscr{H})$ be a rep of a lHs $(\mathfrak{A}, \underline{g}, \delta)$ where $T(\underline{g})$ is an algebra of maximally symmetric operators on a common dense domain D. Let $\mathscr{H}^+$ be the minimal extension space of in which $T^+(\underline{g})$ is represented by essentially skewadjoint operators.

DÉFINITION. — $(\Pi(\mathfrak{A}), T(\underline{g}), \mathscr{H})$ is quasi-weakly Schur-irreducible if the only bounded operators $B \in \Pi^+(\mathfrak{A})'$ in $\mathscr{H}^+$ that commute with the set $T^+(\underline{g})$ are multiples of the unit operator in $\mathscr{H}^+$. The rep of the lHs in case *b*) is quasi-weakly Schur-irreducible.

We conjecture that the physically more appealing definition of a quasi-weakly Schur-irreducible rep of a lHs which demands that the only bounded operators $B \in \Pi(\mathfrak{A})'$ that commute with the generalized spectral families of $T(\underline{g})$ are the multiples of the unit operator in $\mathscr{H}$ is equivalent to the one given above.

Case c. — The situation closely resembles that of case *a*), only that we now have rotational invariance around the origin.

From the above discussion the following simple rule seems to emerge.

Given that all questions i. e. all measurements regarding an observable can be asked (performed) with unique result, this observable should have a selfadjoint representation and conversely, if it has a selfadjoint representation, any question regarding this observable can be asked, with unique answer. If all measurements cannot be performed, then it should not have a selfadjoint representation and conversely.

An alternative examination of this point, avoiding the possible critisism that $\bar{P} = -i\nabla$ is not necessarily a dynamically consistent definition, could be made in the following way for case *b*).

Let $\mathscr{S}_3$ be the observable algebra on $(\mathbb{R}_+)^3$, and let a particle of mass $\frac{1}{2}$ move in $(\mathbb{R}_+)^3$. If, as is often stated [*16*], all measurements can be reduced to position measurements, it is enough to give a rep of $\mathscr{S}_3$ and the time-development of $\dot{\bar{x}}(t)$. The momentum of the particle can then be derived by defining it as $\frac{1}{2}\dot{\bar{x}}(t)$. Knowing the algebra of the $\bar{x}(t)$: *s* implies that $\bar{P} = \frac{1}{2}\dot{\bar{x}}(t)$ can be calculated by taking the limits

$$\bar{P}_+ = \frac{1}{2}\lim_{\varepsilon\to 0_+}\frac{\bar{X}(t+\varepsilon) - \bar{X}(t)}{\varepsilon}$$

$$\bar{P}_- = \frac{1}{2}\lim_{\varepsilon\to 0_+}\frac{\bar{X}(t-\varepsilon) - \bar{X}(t)}{\varepsilon}$$

on certain well behaved state functions. If $\bar{P}_+ = \bar{P}_-$ on a sufficiently nice set of functions of $\bar{X}$, then $\bar{P}$ will be a good operator with respect to which any question can be asked with unique answer. If the set is too small to give unique answer to all questions then $\bar{P}_+ \neq \bar{P}_-$ for some states and $\bar{P}$ should not be self-adjoint. But it could still be used on the nice states! Taking $H = -\Delta$, which has self-adjoint extension in $L^2((\mathbb{R}_+)^3, d^3x)$ the result is that

$$\bar{X}(t) = \bar{X} - 2it\frac{2}{\partial\bar{X}} \qquad (\bar{X} = \bar{X}(0))$$

and

$$\bar{P}_+ f(\bar{X}) = - i\nabla f(\bar{X})$$
$$\bar{P}_- f(\bar{X}) = - i\nabla f(\bar{X})$$

Hence $\bar{P}_+ = \bar{P}_- = \bar{P}$ on all states $f(\bar{X})$ differentiable on $(\mathbb{R}_+)^3$. This is true only for those states that vanish, with vanishing derivative at the origin. We thus reach the same conclusion as before, namely that questions pertaining at momentum measurements can not be asked on states where the particle is hitting the walls after the measurement. If the question should be possible to ask arbitrarily many times, then obviously we get the condition that $f(\bar{X})$ is C^∞ and $f^{(n)}(0) = 0$, $\forall n \in \mathbb{N}$. It should be mentioned, that the rather primitive analysis made here of the use of symmetric non-selfadjoint operators as observables is no substitute for an incorporation of these operators (and hopefully even dissipative ones) in a formal theory of measurements.

Finally some remarks on the use of lHs : *s* in relativistic physics. The Lorentz invariant boundary conditions in $\mathbb{R}^4$ are probably not of much interest to particle physics. However, the idea that constituents of elementary particles are confined to a small volume is in direct accordance with the type of constraints imposed in the example of the Weyl algebra $\underline{w}$. This way of mixing internal and external degrees of freedom seems very promising, giving a well defined mathematical structure to such systems.

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