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Convergence and accuracy of approximation methods in general relativity. The time-independent case

by

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ABSTRACT. — A special class of iterative approximation methods for solving Einstein's field equations of general relativity is investigated, being based on weak field assumptions. In these methods the first approximation may be the linearized theory but must not. In each step of approximation the equations of motion for the energy tensor and usual inhomogenuous wave equations must be solved. In this paper the time-independent case is discussed. We prove the convergence to rigorous solutions under sufficient conditions, which are to be imposed on the first approximation. Essentially these conditions are: sufficient weakness, Hölder-continuous differentiability and sufficiently decreasing behaviour in space-like infinity. The conditions admit the application to matter-field space as well as to systems with vacuum regions (e. g. with isolated material sources). The regarded space-times are homöomorph to the Minkowski-space. Explicit estimations are derived for the deviation of obtained approximations (e. g. the linearized theory) from rigorous solutions.

RÉSUMÉ. — Nous étudions une classe spéciale des méthodes itératives approximatives pour résoudre les équations Einsteiniennes de champs de la relativité générale fondées sur des suppositions faibles du champ. Dans ces méthodes la première approximation peut être la théorie linéaire mais ne le doit pas. Ce sont les équations de mouvement pour le tenseur d'énergie et les équations d'ondes ordinaires et inhomogènes qui doivent être résolues dans chaque pas d'approximation. Dans cet article on discute le cas d'indépendance du temps. Nous éprouvons la convergence aux solutions exactes sous des conditions suffisantes qui doivent être imposées à la première approximation. Essentiellement ces conditions sont : faiblesse suffisante, différentiabilité continuante de Hölder et une conduite diminuante suffisamment dans l'infinité d'espace. Ces conditions admettent l'application aux systèmes pleins de matière et aussi aux systèmes avec des régions du vide (par exemple avec des sources matérielles isolées). Les espaces-temps considérées sont homöomorphes à l'espace-temps de Minkowski. Des estimations explicites sont dérivées pour la déviation des approximations obtenues (par exemple la théorie linéaire) des solutions exactes.

1. Introduction

Approximation procedures and approximate solutions for Einstein's field equations of General Relativity have always been subject to controversies. Up to now the situation concerning the convergence and accuracy of these procedures could not be clarified in a satisfactory way. In this paper we start the investigation of a wide class of approximation procedures and approximate results based on *weak field calculations.* Explicitely we shall discuss those approximation methods which in each step of approximation yield usual inhomogenous wave equations. The first approximation may be the *linearized* field equations but must not. We shall give the mathematical justification for this class of approximation methods (proof of convergence to rigorous solutions of the field equations and explicit estimations of error). Especially we shall be able to estimate the maximum error of the first approximation (linearized theory).

We emphasize that on the contrary to other mathematical investigations by Fischer and Marsden, Choquet-Bruhat *et al.* (see e. g. [*1*] [*6*]), concerning existence, uniqueness and continuous dependence on the Cauchy data, our main intention is to get criteria for the validity and accuracy of the usually applicated weak-field approximation procedures. Our methods shall be as elementary as possible. In this paper they are based on estimations of upper limits for Poisson's integral.

Subject of this paper is the *time-independent* case:

In part A we discuss some systems of nonlinear partial differential equations, which in many respects have a similar structure as the Einstein equations. The linear part of these equations is the Laplace operator. In the *first* example (chapter 2, 3) the nonlinear part contains partial derivatives up to first order only. In the *second* example (chapter 4) the results are extended to nonlinearities containing second derivatives. We present in detail the applied techniques for the *pure iteration* method, where in each step the complete $(n-1)$-th approximation is used for calculating the n-th approximation. The iteration is treated with the fix-point theorem for contractions.

In part B the methods and results of part A are applied for the time-independent case of Einstein equations. We investigate these equations in special coordinates satisfying a De-Donder condition (see (5,10)), in which the linear part is the usual Laplace operator. Throughout part B *time-independence* is related to this special choice of coordinates. We show the existence of solutions of the *equations of motion*, which have to be solved in each step of approximation. We prove that the pure iteration converges to rigorous (global) solutions of the field equations in case the inhomogenety (e. g. the energy-tensor) in the first approximation is sufficiently weak, Hölder-continuously differentiable and sufficiently decreasing in space-like infinity (R^{-4}). These conditions of convergence admit the application to globally *matter-filled systems* as well as to systems with *vacuum regions* (e. g. with *isolated material sources*).

Finally a possible extension of our results to time-dependent (radiating) systems is mentioned. Under the restriction that the system is time dependent for a finite time only, a proof of convergence for iterative approximations should be possible with essentially the same methods as for the time-independent case.

In the appendix the essential properties of Poisson's integral and its first and second derivatives concerning boundedness and asymptotic behavior at infinity are derived. We mention that in a recent paper of Choquet-Bruhat and Deser [6] somewhat weaker properties of asymptotic behavior have been used.

Usually, in practical approximate calculations a *perturbational* approach is performed, where all occuring quantities are decomposed according to their constituents of different orders in a perturbation parameter. The convergence of the perturbation procedure may be proved under essentially the same conditions. The results will be published elsewhere. But the iteration leads to sharper and more transparent estimations of error. The maximum error of an obtained (weak-field) approximation may be estimated with the results for the iteration in this paper.

A. SPECIAL NON-LINEAR SYSTEMS SIMILAR TO THE EINSTEIN EQUATIONS

2. Systems with a nonlinearity containing partial derivatives up to first order

We first investigate a simple type of systems of non-linear partial differential equations, from which we shall get insight to more complicated systems such as Einstein's field equations of General Relativity. We consider systems where in the nonlinearity occur partial derivatives up to first order only, e. g.:

$$\Delta\psi_\mu = -(\mathrm{T}_\mu + \mathrm{N}_\mu) = -\tau_\mu, \qquad \mathrm{T}_\mu = \mathrm{T}_\mu(x^a)$$
$$\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right), \qquad \mu, \alpha_i = 1, 2, \ldots, q \tag{2,1}$$

with (1)

$$\mathrm{N}_\mu = [\psi_{\alpha_1|a}\cdot\psi_{\alpha_2|b}\cdot(1+\psi_{\alpha_3})\ldots(1+\psi_{\alpha_m})]_\mu \tag{2,1a}$$

From (2,1) we can go over to the integral equations

$$\psi_\mu(x^a) = \frac{1}{4\pi}\iiint_{\mathrm{V}} \frac{\tau_\mu(x^{a'})}{r}\, d^3x' \tag{2,2}$$

where

$$r = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}}$$

and V is the entire 3-dimensional space.

On (2,2) can be based iterative approximation procedures for solving (2,1). We begin with a first approximation $\overset{(1)}{\tau}_\mu$ of the « sources » τ_μ, for example $\overset{(1)}{\tau}_\mu = \mathrm{T}_\mu$ (2), which is put into (2,2) in order to calculate a first approximation $\overset{(1)}{\psi}_\mu$ of the potentials ψ_μ. From $\overset{(1)}{\psi}_\mu$ we determine with (2,1*a*) the second approximation $\overset{(2)}{\tau}_\mu$ and so on.

In the usual *perturbational approach* a small perturbation parameter ε is introduced. The occuring quantities are decomposed according to their constituents of different orders of magnitude. The n-th order of magnitude is then given by the sum of all contributions, in which the factor ε^n occurs.

An alternative procedure is the *pure iteration* where in each step the complete $(n-1)$-approximation is taken for calculating the n-th approximation. Then the $\overset{(n)}{\mathrm{N}}_\mu$ are defined by:

$$\overset{(n)}{\mathrm{N}}_\mu = (\overset{(n-1)}{g}_{\alpha_1|a}\cdot \overset{(n-1)}{g}_{\alpha_2|b}\cdot \overset{(n-1)}{g}_{\alpha_3}\cdots \overset{(n-1)}{g}_{\alpha_m})_\mu, \qquad g_\mu =: 1 + \psi_\mu \tag{2,3}$$

and

$$\overset{(n)}{\psi}_\mu = \frac{1}{4\pi}\iiint_{\mathrm{V}} \frac{\overset{(n)}{\tau}(x^{a'})}{r}\, d^3x', \qquad \overset{(n)}{\tau}_\mu = \mathrm{T}_\mu + \overset{(n)}{\mathrm{N}}_\mu \tag{2,4}$$

In this paper only the iteration is regarded. A method of proving the convergence of the perturbational approach will be published elsewhere.

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(1) $_{|a} = \dfrac{\partial}{\partial x^a}$, latin indices: $a, b, \ldots = 1, 2, 3$; except $i, k = 1, 2, \ldots, m$.

(2) (n) denotes the n-th degree of approximation within an approximation procedure.

3. Convergence of the iteration

We will prove that the integral operator

$$I_\mu(\varphi_{\alpha_i}, T_\mu) = \frac{1}{4\pi} \iiint_V \frac{(n_\mu + T_\mu)}{r} d^3x' \tag{3,1}$$

with

$$n_\mu = [\varphi_{\alpha_1|a} \cdot \varphi_{\alpha_2|b} \cdot (1 + \varphi_{\alpha_3}) \dots (1 + \varphi_{\alpha_m})]_\mu$$

is contracting.

DÉFINITION. — Let M be a complete metrical space in which the distance $\rho(f_i, f_k)$ between elements f_i, f_k of M is defined. Let A be an operator in M. Then A is *contracting* on the domain of definition $D_A \subset M$, if D_A is closed, $A(f) \in D_A$ for $f \in D_A$ and if

$$\rho(A(f_i), A(f_k)) \leqslant a\rho(f_i, f_k), \qquad 0 \leqslant \alpha < 1 \tag{3,2}$$

for all f_i, $f_k \in D_A$.

M is given by the space of continuous real functions $f_\mu(x^a)$, $x^a \in V$, $\lim_{R\to\infty} f_\mu = 0$ with the max-Norm

$$\| f_\mu \| = \max_{\substack{x^a \in V \\ \mu = 1,2,\dots,q}} | f_\mu |$$

A distance ρ between two functions $\underset{(i)}{f_\mu}$, $\underset{(k)}{f_\mu} \in M$ may be defined by the norm of the difference $\underset{(i)}{f_\mu} - \underset{(k)}{f_\mu}$:

$$\rho(\underset{(i)}{f_\mu}, \underset{(k)}{f_\mu}) = \| \underset{(i)}{f_\mu} - \underset{(k)}{f_\mu} \| \tag{3,4}$$

a) I_μ IS A CONTRACTING OPERATOR

The domain of definition D_I of I_μ with

$$I_\mu(\varphi_{\alpha_i}, T_\mu) \in D_I, \qquad \varphi_{\alpha_i} \in D_I \tag{3,5}$$

is given by the set of differentiable functions φ_μ in M with

$$| \varphi_\mu | \leqslant \left\{ \vartheta, \vartheta \cdot \left(\frac{R_0}{R}\right) \right. \tag{3,6a}$$

$$| R_0 \cdot \varphi_{\mu|a} | \leqslant \left\{ \vartheta, \vartheta \cdot \left(\frac{R_0}{R}\right)^2 \right. \tag{3,6b}$$

in case the inhomogenity T_μ satisfies

$$| R_0^2 \cdot T_\mu | \leqslant \left\{ \varepsilon_0, \varepsilon_0 \cdot \left(\frac{R_0}{R}\right)^4 \right. \tag{3,7}$$

ϑ, ε_0 in (3,6) and (3,7) must be chosen sufficiently small. ϑ is dependent on ε_0. We get a condition on ϑ and ε_0 with the use of the upper limits (A,5) and (A,13) on Poisson's integral and its first derivatives. From (3,1) and (3,6) follows:

$$| R_0^2 \cdot n_\mu(\varphi_{\alpha_i}) | \leqslant \vartheta^2(1 + \vartheta)^{m-2} \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^4 \right. \tag{3,8}$$

Hence with (3,7), (A,5) and (A,13):

$$| I_\mu(\varphi_{\alpha_i}, T_\mu) | \leqslant (\varepsilon_0 + \vartheta^2(1 + \vartheta)^{m-2}) \cdot \left\{ 1, \frac{4}{3}\left(\frac{R_0}{R}\right) \right. \tag{3,9a}$$

$$| R_0 \cdot I_{\mu|a} | \leqslant (\varepsilon_0 + \vartheta^2(1 + \vartheta)^{m-2}) \cdot \left\{ \frac{4}{3}, \frac{5}{2}\left(\frac{R_0}{R}\right)^2 \right. \tag{3,9b}$$

Then by comparison of (3,9) and (3,6) the condition (3,5) yields

$$\frac{5}{2}(\varepsilon_0 + \vartheta^2(1 + \vartheta)^{m-2}) \leqslant \vartheta$$

or

$$\varepsilon_0 \leqslant g(\vartheta), \qquad g(\vartheta) = \frac{2}{5}\vartheta\left(1 - \frac{5}{2}\vartheta(1 + \vartheta)^{m-2}\right) \tag{3,10}$$

In Fig. 1. we have drawn the qualitative shape of $g(\vartheta)$ in the relevant interval $0 \leqslant \vartheta < \vartheta_0$.

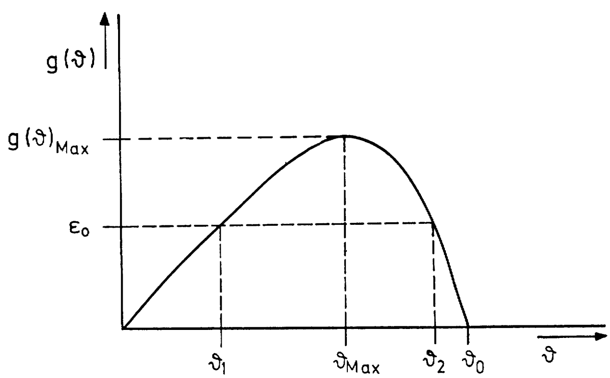

FIG. 1. — Qualitative shape of $g(\vartheta)$.

Relation (3,10) can only be satisfied for $\varepsilon_0 > 0$, if $\varepsilon_0 < g(\vartheta_{\text{Max}})$. In case this condition is fulfilled, $\vartheta(\varepsilon_0)$ lies in the interval $(\vartheta_1, \vartheta_2)$. The maximum possible domain of definition D_I with (3,5) is reached for $\varepsilon_0 = 0$, i. e. $T_\mu = 0$. Qualitatively ϑ becomes smaller with growing m. By differentiation of (3,10) we find that $\vartheta_{\text{Max}}(m)$ is given by the equation

$$\vartheta_{\text{Max}} \cdot (1 + \vartheta_{\text{Max}})^{m-3} \cdot (2 + m \cdot \vartheta_{\text{Max}}) = \frac{2}{5} \tag{3,11}$$

Now we derive from (3,2) for ϑ the condition of contraction for I_μ. For two functions $\underset{(i)}{\varphi_\mu}$ and $\underset{(k)}{\varphi_\mu}$ defined by (3,6), i. e. with a norm $\leqslant \vartheta$, we get from (3,1) and (3,9), that (3,2) is satisfied, if the following relation is valid:

$$\rho(\mathrm{I}_\mu(\underset{(j)}{\varphi_{\alpha_i}}, \mathrm{T}_\mu), \mathrm{I}_\mu(\underset{(k)}{\varphi_{\alpha_i}}, \mathrm{T}_\mu)) \leqslant \frac{5}{2}(2\vartheta + m\vartheta^2)(1+\vartheta)^{m-3}\cdot\rho(\underset{(j)}{\varphi_\mu}, \underset{(k)}{\varphi_\mu}) \leqslant a\cdot\rho(\underset{(j)}{\varphi_\mu}, \underset{(k)}{\varphi_\mu})$$

Hence the condition of contraction for I_μ is

$$\frac{5}{2}(2\vartheta + m\vartheta^2)(1+\vartheta)^{m-3} \leqslant a < 1 \qquad (3,12)$$

Comparison with (3,11) shows that (3,12) means

$$\vartheta < \vartheta_{\mathrm{Max}} \qquad (3,12a)$$

With (3,12) and (3,10) we have worked out the essential conditions on ϑ and ε_0 in (3,6) and (3,7) for the operator I_μ to be contracting. Using the fix-point theorem for contractions the above results yield.

THEOREM I. — The iteration converges to a unique solution of the integral equations (2,2), if one begins with a first approximation $\overset{(1)}{\tau}_\mu(\overset{(0)}{\psi}_{\alpha_i}, \mathrm{T}_\mu)$, where the function $\overset{(0)}{\psi}_\mu$, and the inhomogenity T_μ satisfy (3,6) and (3,7) with ε_0, ϑ restricted by (3,10), (3,12) respectively. When I_μ is contracting, one can always take as a first approximation the *linear* approximation $\overset{(1)}{\tau}_\mu = \mathrm{T}_\mu$, $\overset{(0)}{\psi}_\mu = 0$.

We restrict the domain of definition $\mathrm{D_I}$ to functions for which additionally exist *second* derivatives with

$$|\,\mathrm{R}_0^2\cdot\varphi_{\mu|a|b}\,| \leqslant \vartheta\cdot\left\{1, \left(\frac{\mathrm{R}_0}{\mathrm{R}}\right)^3\right. \qquad (3,13)$$

and first derivatives of T_μ with:

$$|\,\mathrm{R}_0^3\cdot\mathrm{T}_{\mu|a}\,| \leqslant \varepsilon_0\cdot\left\{1, \left(\frac{\mathrm{R}_0}{\mathrm{R}}\right)^4\right. \qquad (3,14)$$

Upper limits for the second derivatives of I_μ then are with (A,19) (where we have to set $\mu = 1$):

$$|\,\mathrm{R}_0^2\cdot\mathrm{I}_{\mu|a|b}\,| \leqslant 8[\varepsilon_0 + (2\vartheta^2 + m\vartheta^3)\cdot(1+\vartheta)^{m-3}]\cdot\left\{1, \left(\frac{\mathrm{R}_0}{\mathrm{R}}\right)^3\right. \qquad (3,15)$$

The domain of definition is now instead of (3,10) limited by the sharper conditions:

$$\varepsilon_0 \leqslant \bar{g}(\vartheta),\ \bar{g}(\vartheta) = \left[\frac{1}{8}\,\vartheta - (2\vartheta^2 + m\vartheta^3)(1+\vartheta)^{m-3}\right] \qquad (3,16)$$

and

$$\bar{\vartheta}_{\mathrm{Max}}\cdot(1+\bar{\vartheta}_{\mathrm{Max}})^{m-4}\cdot\left[4 + (5m-2)\bar{\vartheta}_{\mathrm{Max}} + m^2\bar{\vartheta}_{\mathrm{Max}}^2\right] = \frac{1}{8} \qquad (3,17)$$

With (3,16), (3,17) holds the extension of Theorem I.

THEOREM II. — If additionally to (3,6) and (3,7) $\overset{(0)}{\psi}$ and T_μ satisfy the conditions (3,13) and (3,14) with ε_0 restricted by (3,16) and $\vartheta \leqslant \bar{\vartheta}_{\mathrm{Max}}$ in (3,17), then the iteration even converges to a unique solution of the system of partial differential equations (2,1).

b) REGION OF CONVERGENCE

In practical applications it will be advantageous to begin with the linear approximation of (2,1) and (2,2)

$$\overset{(1)}{\psi}_\mu(x^a) = \frac{1}{4\pi} \iiint_V \frac{T_\mu(x^{a'})}{r} d^3x' \tag{3,18}$$

We discuss only the case, where the conditions of Theorem II are fulfilled, i. e. the procedure converges to a solution of the differential equations (2,1) as well as of the integral equations (2,2). We define a *region of convergence* D by the condition, that for all positive $\varepsilon_0 < D$ the iteration with $\overset{(1)}{\psi}_\mu$ given by (3,18) converges. From (3,16), (3,17) follows, that D is given by

$$D = \bar{g}(\bar{\vartheta}_{\mathrm{Max}}) \tag{3,19}$$

The evaluation of (3,19) for large m yields an asymptotic behaviour of $\bar{\vartheta}_{\mathrm{Max}}$ (and hence of D) weaker than m^{-1}.

Oviously a majorant of the convergent series expansion for ψ_μ

$$\psi_\mu = \overset{(1)}{\psi}_\mu + (\overset{(2)}{\psi}_\mu - \overset{(1)}{\psi}_\mu) + (\overset{(3)}{\psi}_\mu - \overset{(2)}{\psi}_\mu) + \dots$$

is given by the geometric series

$$F = \frac{5}{2}\varepsilon_0 \sum_{\nu=0}^{\infty} a^\nu = \frac{5}{2}\varepsilon_0\left(\frac{1}{1-a}\right) \tag{3,20}$$

where a is the factor of convergence determined in (3,12).

An upper bound for the maximum error of the n-th approximation is then

$$\frac{5}{2}\varepsilon_0 \cdot \left[\left(\frac{1}{1-a}\right) - \sum_{\nu=0}^{n-1} a^\nu\right] = \frac{5}{2}\frac{a^n}{1-a} \tag{3,21}$$

a is determined as follows (see Fig. 1 and (3,16), (3,17). First we solve the equation

$$\varepsilon_0 = g(\vartheta) \tag{3,22}$$

If $\varepsilon_0 < D$ (3,22) has two different real solutions $\vartheta_{1,2} > 0$ with $\vartheta_1 < \vartheta_{\mathrm{Max}} < \vartheta_2$. ϑ_1 is the smallest value of ϑ, for which the condition (3,16) is satisfied. Therefore according to (3,12)

$$a = \frac{5}{2}(2\vartheta_1 + m\vartheta_1^2) \cdot (1 + \vartheta_1)^{m-3} \tag{3,23}$$

By (3,21) also the error of the first derivatives of the n-th approximation is determined. For calculation of the (greater) error of the second derivatives we obviously have to replace in (3,20), (3,21) a by $\bar{a}$ with

$$\bar{a} = 8\vartheta_1 \cdot (4 + (5m - 2) \cdot \vartheta_1 + m^2 \vartheta_1^2) \cdot (1 + \vartheta_1)^{m-4} \tag{3,24}$$

and the maximum error of the second derivatives of the n-th approximation is:

$$8\varepsilon_0 \cdot \frac{\bar{a}^n}{1 - \bar{a}} \tag{3,25}$$

4. Systems with a nonlinearity containing partial derivatives up to second order

We extend our considerations to systems of differential equations similar to (2,1)

$$\Delta \psi_\mu = \mathrm{T}_\mu + \mathrm{N}_\mu = \tau_\mu \tag{4,1}$$

but with a nonlinearity containing partial derivatives of second order, e. g.

$$\mathrm{N}_\mu = [\psi_{\alpha_1|a|b} \cdot \psi_{\alpha_2} \cdot (1 + \psi_{\alpha_3}) \ldots (1 + \psi_{\alpha_m})]_\mu \tag{4,1a}$$

As above in (2,2) we can go over to the integral representation (3)

$$\psi_\mu(x^a) = \iiint_V \frac{\tau_\mu(x^{a'})}{r} d^3x' \tag{4,2}$$

and base on (4,2) iterative approximation procedures in complete analogy to chapter 2. The integral operator I_μ, defined by

$$\begin{aligned} \mathrm{I}_\mu(\varphi_{\alpha_i}, \mathrm{T}_\mu) &= \frac{1}{4\pi} \iiint_V \frac{(n_\mu + \mathrm{T}_\mu)}{r} d^3x' \\ n_\mu &= [\varphi_{\alpha_1|a|b} \cdot \varphi_{\alpha_2} \cdot (1 + \varphi_{\alpha_3}) \ldots (1 + \varphi_{\alpha_m})]_\mu \end{aligned} \tag{4,3}$$

contains second derivatives of φ_α. The essential problem now is to impose restrictions on φ_α, which on the other hand are satisfied by the operator $\mathrm{I}_\mu(\varphi_\alpha, \mathrm{T}_\mu)$ as well. In the appendices A3 and A4 we have solved this problem essentially. For a source function τ_μ, satisfying a (modified) Hölder-condition (A,15) and being integrable, limited and sufficiently decreasing in the infinity (A,2), the second derivatives of Poisson's integral satisfy the conditions imposed on τ_μ apart from a factor.

Applying this result to (4,3) we get with (A,5), (A,13), (A,19), (A,20).

(3) We use the same notation as above, but with τ_μ, N_μ, T_μ ... now defined by (4,1a)-(4,3).

LEMMA I. — Let φ_α, T_μ be functions satisfying [4]

$$\left.\begin{aligned} |\varphi_\alpha| &\leqslant \vartheta \cdot \left\{ 1, \left(\frac{R_0}{R}\right) \right. \\ |R_0 \cdot \varphi_{\alpha|a}| &\leqslant \vartheta \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^2 \right. \\ |R_0^2 \cdot \varphi_{\alpha|a|b}| &\leqslant \vartheta \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^3 \right. \end{aligned}\right\} \tag{4,4a}$$

$$|R_0^2 \cdot (\varphi_{\alpha|b|c}(x^a + \Delta x^a) - \varphi_{\alpha|b|c}(x^a))| \leqslant \vartheta \cdot \left(\frac{\Delta R}{R_0}\right)^\mu \cdot \left\{ 1, \left(\frac{R_0}{\tilde{R}}\right)^3 \right. \tag{4,4b}$$

and

$$|R_0^2 \cdot T_\mu| \leqslant \varepsilon_0 \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^4 \right. \tag{4,4c}$$

$$|R_0^2 \cdot (T_\mu(x^a + \Delta x^a) - T_\mu(x^a))| \leqslant \varepsilon_0 \cdot \left(\frac{\Delta R}{R_0}\right)^\mu \cdot \left\{ 1, \left(\frac{R_0}{\tilde{R}}\right)^4 \right. \tag{4,4d}$$

Then for $\tau_\mu = n_\mu + T_\mu$ holds with $\gamma_1 = \varepsilon_0 + \vartheta^2(1 + \vartheta)^{m-2}$ and $\gamma_2 = \varepsilon_0 + (2\vartheta^2 + m\vartheta^3)(1 + \vartheta)^{m-3}$

$$|R_0^2 \cdot \tau_\mu| \leqslant \gamma_1 \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^4 \right. \tag{5,5a}$$

$$|R_0^2 \cdot (\tau_\mu(x^a + \Delta x^a) - \tau_\mu(x^a))| \leqslant \gamma_2 \cdot \left(\frac{\Delta R}{R_0}\right)^\mu \cdot \left\{ 1, \left(\frac{R_0}{\tilde{R}}\right)^4 \right. \tag{5,5b}$$

and I_μ satisfies

$$|I_\mu| \leqslant \gamma_1 \left\{ 1, \frac{4}{3}\left(\frac{R_0}{R}\right) \right. \tag{5,6a}$$

$$|R_0 \cdot I_{\mu|a}| \leqslant \gamma_1 \left\{ \frac{4}{3}, \frac{5}{2}\left(\frac{R_0}{R}\right)^2 \right. \tag{5,6b}$$

$$|R_0^2 \cdot I_{\mu|a|b}| \leqslant \gamma_2 \cdot \left\{ \left(1 + \frac{1}{\mu}\cdot\frac{4}{3}\right), \left(6 + \frac{2}{\mu}\right)\left(\frac{R_0}{R}\right)^3 \right. \tag{4,6c}$$

$$|R_0^2 \cdot (I_{\mu|b|c}(x^a + \Delta x^a) - I_{\mu|b|c}(x^a))| \leqslant \gamma_2 \cdot M(\mu) \cdot \left(\frac{\Delta R}{R_0}\right)^\mu \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^3 \right. \tag{4,6d}$$

where $M(\mu) = H$ in (A,20*a*).

With Lemma I the methods of chapter 3 may easily be transferred to differential and integral equations (4,1), (4,2) respectively. The result is:

THEOREM III. — The integral operator $I_\mu(\varphi_\alpha, T_\mu)$, defined by (4,3) is a

[4] For definition of Δx^a, ΔR, $\tilde{R}$, μ see (A,15).

contraction for functions φ_α and the inhomogenity T_μ satisfying (4,4) and ε_0, ϑ limited by the conditions

$$M_{(\mu)} \cdot (\varepsilon_0 + (2\vartheta^2 + m\vartheta^3)(1 + \vartheta)^{m-3}) < \vartheta \tag{4,7a}$$

and $\vartheta \leqslant \tilde{\vartheta}_{\text{Max}}$ with

$$\frac{1}{M_{(\mu)}} = (4\tilde{\vartheta}_{\text{Max}} + (5m - 2)\tilde{\vartheta}_{\text{Max}}^2 + m^2\tilde{\vartheta}_{\text{Max}}^2)(1 + \vartheta_{\text{Max}})^{m-4} \tag{4,7b}$$

The evaluation of (4,7) and the calculation of a factor of convergence $\tilde{a}$ may be done analogously as in the preceding chapter.

B. THE EINSTEIN EQUATIONS

We apply the methods and results of part A for the time-independent case of Einstein's field equations. The main difference between the systems discussed in part A, and the Einstein equations is the fact that in the latter case approximation procedures cannot be performed with a given inhomogenity. On the contrary, in each step of approximation simultaneously differential equations of motion must be solved, which determine the inhomogenity in the same degree of accuracy as the gravitational potentials.

5. **Integral representation of Einstein's field equations**

Einstein's field equations of general relativity are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{1}{2} T_{\mu\nu} \tag{5,1}$$

($R_{\mu\nu}$: Ricci-tensor, R : Ricci-scalar, $g_{\mu\nu}$: metrical tensor, $G_{\mu\nu}$: Einstein-tensor), where we have set Einstein's gravitational constant $K = 1/2$. $T_{\mu\nu}$: energy-momentum-stress-tensor of matter. Throughout part B greek and latin indices range and sum over $1, \ldots, 4, 1, \ldots, 3$ respectively. Signature is chosen to be -2. Explicitely $R_{\mu\nu}$ is given by

$$R_{\mu\nu} = \Gamma^\kappa_{\kappa\mu|\nu} - \Gamma^\kappa_{\mu\nu|\kappa} + \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\kappa\mu} - \Gamma^\kappa_{\mu\nu}\Gamma^\rho_{\kappa\rho} \tag{5,2a}$$

with the Christoffel-symbols

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\kappa}(g_{\beta\kappa|\gamma} + g_{\gamma\kappa|\beta} - g_{\beta\gamma|\kappa}) \tag{5,2b}$$

The contravariant components $g^{\alpha\beta}$ of the metrical tensor are the inverse matrix elements of $g_{\alpha\beta}$:

$$g^{\alpha\beta} = \frac{adj_{\alpha\beta}}{\det g}, \qquad g_{\alpha\nu}g^{\nu\beta} = g^\beta_\alpha = \vartheta^\beta_\alpha \tag{5,2c}$$

If we set

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} + \bar{\gamma}^{\mu\nu} \tag{5,3}$$

with

$$\left.\begin{matrix}\eta_{\mu\nu}\\ \eta^{\mu\nu}\end{matrix}\right\} = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ 0 & & -1 & \\ & & & +1 \end{pmatrix} \tag{5,3a}$$

and define

$$^{|\lambda} = {}_{|\kappa}\eta^{\kappa\lambda} \tag{5,4}$$

we can separate from $G_{\mu\nu}$ the linear differential expression

$$\frac{1}{2}L_{\mu\nu} = \frac{1}{2}(-g_{\lambda\nu|\mu}{}^{|\lambda} - g_{\mu\lambda}{}^{|\lambda}{}_{|\nu} + g_{\mu\nu|\lambda}{}^{|\lambda}) + \frac{1}{2}(g_{\alpha\beta}\eta^{\alpha\beta})_{|\mu|\nu} - \frac{1}{2}((g_{\alpha\beta}\eta^{\alpha\beta})_{|\lambda}{}^{|\lambda} - g_{\kappa\lambda}{}^{|\kappa|\lambda})\eta_{\mu\nu} \tag{5,5}$$

We write the field equations (6,1) in the form

$$L_{\mu\nu} = -(T_{\mu\nu} + N_{\mu\nu}) = -\tau_{\mu\nu} \tag{5,6}$$

where

$$\frac{1}{2}N_{\mu\nu} = G_{\mu\nu} - \frac{1}{2}L_{\mu\nu} \tag{5,6a}$$

according to (5,2) and (5,5) consists of a sum of constituents, which are nonlinear in the metrical tensor and its first and second derivatives, similarly to the quantities N_μ in (2,1*a*) and (4,1*a*). Schematically (indices omitted):

$$N_{\mu\nu} \approx 600 \cdot \gamma \cdot \gamma_{|a|b} \cdot (1 + 0(\gamma)) + 648 \cdot \gamma_{|\alpha} \cdot \gamma_{|b} \cdot (1 + 0(\gamma)) \tag{5,6b}$$

As easily may be verified, the linear part $L_{\mu\nu}$, in case it is differentiable, satisfies the « conversation laws »

$$L_{\mu\nu}{}^{|\nu} = 0 \tag{5,7}$$

Hence modulo the field equations holds

$$\tau_{\mu\nu}{}^{|\nu} = 0 \tag{5,8}$$

The (non-tensorial) « conservation laws » (5,8) are equivalent to the covariant equations of motion for the matter tensor $T_{\mu\nu}$:

$$T_\mu{}^\nu{}_{||\nu} = 0 \tag{5,8a}$$

$_{||\nu}$: covariant derivative with respect to x^ν.

If we now set

$$\psi_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\gamma_{\alpha\beta}\eta^{\alpha\beta}\eta_{\mu\nu} \tag{5,9}$$

and restrict the free choice of coordinates by the coordinate condition (De Donder condition)

$$\psi_{\mu\nu}{}^{|\nu} = 0 \tag{5,10}$$

the field equations (5,6) become ($\Box\psi_{\mu\nu} = L_{\mu\nu}$ in these coordinates)

$$\Box\psi_{\mu\nu} = -\tau_{\mu\nu}, \qquad \Box = -{}_{|\lambda}{}^{|\lambda} = \Delta - \frac{\partial^2}{\partial x^{4^2}} \tag{5,11}$$

For the derivation of (5,11) is used continuity of the second derivatives of $g_{\mu\nu}$, which makes possible the exchange of first and second derivatives. By restriction to the retarded Green's function we get from (5,11) the *coordinate-dependent* integral representation of the field equations

$$\psi_{\mu\nu} = \frac{1}{4\pi}\iiint_V \frac{\tau_{\mu\nu}\left(x^{a'}, t - \frac{r}{c}\right)}{r} d^3x' \tag{5,12}$$

6. Approximation procedures

a) Iterative approximation methods in the time-independent case

In the *time-independent* case the field equations (5,11), (5,12) are

$$\Delta\psi_{\mu\nu} = -\tau_{\mu\nu} \tag{6,1a}$$

$$\psi_{\mu\nu}(x^a) = \frac{1}{4\pi}\iiint_V \frac{\tau_{\mu\nu}(x^{a'})}{r} d^3x' \tag{6,1b}$$

On (6,1*b*) iterative approximation methods can be based similarly as described in chapter 2. One begins with a first approximation $\overset{(1)}{\tau}_{\mu\nu}$ of the « sources » $\tau_{\mu\nu}$, satisfying the equations of motion $\overset{(1)}{\tau}_{\mu\nu}{}^{|\nu} = 0$. In case the first approximation is the linearized theory, $\overset{(1)}{\tau}_{\mu\nu}$ is the energy-momentum-stress tensor $\overset{(1)}{T}_{\mu\nu}$ in special-relativistic approximation (without taking into account gravitational interactions). In general $\overset{(n)}{N}_{\mu\nu}(\overset{(n-1)}{g}_{\alpha\beta})$ determine the equations of motion in the n-th approximation. After having solved them the $\overset{(n)}{\psi}_{\mu\nu}$ are calculated as Poisson's integrals over $\overset{(n)}{\tau}_{\mu\nu}$.

b) Consistency of approximation procedures

First, because the equations of motion are of third order in the gravitational potentials, the existence of third partial derivatives of the n-th approximation $\overset{(n)}{\psi}_{\mu\nu}$ must be guaranteed. In addition to conditions as (4,4*a*) and (4,4*b*) we shall postulate $\overset{(1)}{\tau}_{\mu\nu}$ to be Hölder-continuously differentiable. Then it is easy to show by application of Gauss' law that the third derivatives of $\overset{(1)}{\psi}_{\mu\nu}$ exist and again satisfy a Hölder-condition. By recurrent application of this result to the higher approximations the existence of the third derivatives on each degree of approximation is guaranteed, in case we find in each step a solution $\overset{(n)}{T}_{\mu\nu}$ of the equations of motion, which is Hölder-continuously differentiable as well.

The existence of $\overset{(n)}{T}_{\mu\nu}$ satisfying the equations of motion (5,8) is a *second* condition of consistency. The equations of motion for $\overset{(n)}{T}_{\mu\nu}$ are according to (5,8)

$$\overset{(n)}{T}_{\mu\nu}{}^{|\nu} = - \overset{(n)}{N}_{\mu\nu}{}^{|\nu} \tag{6,2}$$

If they are fulfilled in the $(n - 1)$-th approximation, then the solution of (6,2) has the form

$$(\overset{(n)}{T}_{\mu\nu} - \overset{(n-1)}{T}_{\mu\nu}) = \underset{n}{h}_{\mu\nu} - (\overset{(n)}{N}_{\mu\nu} - \overset{(n-1)}{N}_{\mu\nu}) \tag{6,3}$$

where $\underset{n}{h}_{\mu\nu}$ is of the order of magnitude of the difference $(\overset{(n)}{N}_{\mu\nu} - \overset{(n-1)}{N}_{\mu\nu})$ and satisfies the homogenous equations

$$\underset{n}{h}_{\mu\nu}{}^{|\nu} = 0 \tag{6,4}$$

Non-trivial solutions of (6,4) may easily be constructed. Hence also the existence of $\overset{(n)}{T}_{\mu\nu}$ is guaranteed, the deviation of which from $\overset{(n-1)}{T}_{\mu\nu}$ has the desired order of magnitude. Naturally from the described arbitrary construction $\overset{(n)}{T}_{\mu\nu}$ cannot be expected to satisfy certain additional equations of state (special model of matter). The existence of a $\overset{(n)}{T}_{\mu\nu}$ under special equations of state is a problem, which we do not intend to solve in this paper.

A *third* problem arises, if we want to treat the field equations for isolated material sources. In the *vacuum* region the equations of motion (5,8) yield the conditions

$$N_{\mu\nu}{}^{|\nu} = 0 \tag{6,5a}$$

Because the $\overset{(n)}{N}_{\mu\nu}$ are uniquely constructed by the $\overset{(n-1)}{g}_{\mu\nu}$ in $(n - 1)$-th approximation, equations (6,5*a*) yield *integrability conditions*. An alternative form of the integrability conditions to be imposed on the *second derivatives* is

$$\Delta\psi_{\mu\nu} = L_{\mu\nu} \tag{6,5b}$$

Below we shall show that indeed equations (6,5*b*) are satisfied in the required degree of accuracy, if they hold up to the $(n - 1)$-th approximation.

7. Proof of convergence

We will proceed in the following two steps:

First we shall omit the problem of the equations of motion and prove with the methods developed in part A that the relevant integral operator

$$I_{\mu\nu}(\varphi_{\alpha\beta}, T_{\mu\nu}) = \frac{1}{4\pi} \iiint_V \frac{(N_{\mu\nu} + T_{\mu\nu})}{r} d^3x' \tag{7,1}$$

is contracting for a *given* fixed inhomogenity $T_{\mu\nu}$.

Second we shall discuss the equations of motion and the integrability conditions (6,5) in vacuo.

a) CONTRACTION OF $I_{\mu\nu}$

Regard the metrical space M of normable functions $f_{\mu\nu}(x^a)$, $x^a \in V$ with the max-norm

$$\| f_{\mu\nu} \| = \max_{\substack{x^a \in V \\ \mu,\nu = 1,2,3,4}} (| f_{\mu\nu} |, | R_0 f_{\mu\nu|a} |, | R_0^2 f_{\mu\nu|a|b} |) \tag{7,2}$$

A distance ρ between two functions is defined by

$$\rho(\underset{(i)}{f_{\mu\nu}}, \underset{(k)}{f_{\mu\nu}}) = \| \underset{(i)}{f_{\mu\nu}} - \underset{(k)}{f_{\mu\nu}} \| \tag{7,3}$$

We determine the domain of definition D_I of $I_{\mu\nu}$ with

$$I_{\mu\nu}(\varphi_{\alpha\beta}, T_{\mu\nu}) \in D_I, \qquad \varphi_{\mu\nu} \in D_I \tag{7,4}$$

In case the inhomogenity $T_{\mu\nu}$ satisfies

$$| R_0^2 \cdot T_{\mu\nu} | \leqslant \varepsilon_0 \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^4 \right. \tag{7,5a}$$

and the Hölder-condition

$$| R_0^2 \cdot (T_{\mu\nu}(x^a + \Delta x^a) - T_{\mu\nu}(x^a)) | \leqslant \varepsilon_0 \cdot \left(\frac{\Delta R}{R_0}\right)^{\mu} \left\{ 1, \left(\frac{R_0}{R}\right)^4 \right. \tag{7,5b}$$

a complete D_I is given by the functions $\varphi_{\mu\nu}$ in M with.

i) *Sufficient limitedness:*

$$| \varphi_{\mu\nu} | \leqslant \vartheta \cdot \left\{ 1, \left(\frac{R_0}{R}\right) \right. \tag{7,6a}$$

$$| R_0 \cdot \varphi_{\mu\nu|a} | \leqslant \vartheta \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^2 \right. \tag{7,6b}$$

$$| R_0^2 \cdot \varphi_{\mu\nu|a|b} | \leqslant \vartheta \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^3 \right. \tag{7,6c}$$

ii) *Hölder continuity of the second derivatives* $(0 < \mu \leqslant 1)$:

$$| R_0^2 \cdot (\varphi_{\mu\nu|b|c}(x^a + \Delta x^a) - \varphi_{\mu\nu|b|c}(x^a)) | \leqslant \vartheta \cdot \left(\frac{\Delta R}{R_0}\right)^{\mu} \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^3 \right. \tag{7,7}$$

iii) *Existence of Hölder-continuous third derivatives.*

For sufficiently small ϑ (this assumption is justified below, see (7,13)) a study of the schematic structure of $N_{\mu\nu}$ according to (5,6*b*) yields:

$$| R_0^2 \cdot \tau_{\mu\nu} | < (\varepsilon_0 + 2 \cdot 10^3 \vartheta^2) \cdot \left\{ 1, \left(\frac{R_0}{R}\right)^4 \right. \tag{7,8}$$

$$| R_0^2 \cdot (\tau_{\mu\nu}(x^a + \Delta x^a) - \tau_{\mu\nu}(x^a)) | < (\varepsilon_0 + 4 \cdot 10^3 \vartheta^2) \cdot \left(\frac{\Delta R}{R_0}\right)^\mu \cdot \left\{ 1, \left(\frac{R_0}{\tilde{R}}\right)^4 \right. \tag{7,9}$$

With (A,5), (A,13), (A,19), (A,33) for $I_{\mu\nu}(\varphi_{\alpha\beta}, T_{\mu\nu})$ we get.

LEMMA II. — Let $\varphi_{\alpha\beta}$, $T_{\mu\nu}$ be functions satisfying (7,5)-(7,9). Then for the integral operator $I_{\mu\nu}$, defined by (7,1), hold the relations

$$| I_{\mu\nu} | < (\varepsilon_0 + 2 \cdot 10^3 \vartheta^2) \cdot \left\{ 1, \frac{4}{3}\left(\frac{R_0}{R}\right) \right. \tag{7,10a}$$

$$| R_0 \cdot I_{\mu\nu|a} | < (\varepsilon_0 + 2 \cdot 10^3 \vartheta^2) \cdot \left\{ \frac{4}{3}, \frac{5}{2}\left(\frac{R_0}{R}\right)^2 \right. \tag{7,10b}$$

$$| R_0^2 \cdot I_{\mu\nu|a|b} | < (\varepsilon_0 + 4 \cdot 10^3 \vartheta^2) \cdot \left\{ \left(1 + \frac{2}{\mu}\right), \left(6 + \frac{2}{\mu}\right)\left(\frac{R_0}{R}\right)^3 \right. \tag{7,10c}$$

$$| R_0^2 \cdot (I_{\mu\nu|b|c}(x^a + \Delta x^a) - I_{\mu\nu|b|c}(x^a) | < M(\mu) \cdot (\varepsilon_0 + 4 \cdot 10^3 \vartheta^2) \cdot \left(\frac{\Delta R}{R_0}\right)^\mu \cdot \left\{ 1, \left(\frac{R_0}{\tilde{R}}\right)^3 \right. \tag{7,11}$$

with $M(\mu) = H$ in (A,20*a*).

From Lemma II follows that with (7,4) ε_0, ϑ are restricted by the relation

$$\varepsilon_0 \leqslant \hat{g}(\mu, \vartheta), \qquad \hat{g}(\mu, \vartheta) = \frac{1}{M(\mu)} \cdot \vartheta - 4 \cdot 10^3 \vartheta^2 \tag{7,12}$$

The qualitative shape of $\hat{g}(\mu, \vartheta)$ is the same as for $g(\vartheta)$ in Fig. 1.

The maximum possible value of ε_0 is given by

$$\varepsilon_0 = \hat{g}(\mu, \vartheta_{\text{Max}}), \qquad M(\mu)^{-1} \cdot \frac{1}{8} \cdot 10^{-3} = \vartheta_{\text{Max}} \tag{7,13}$$

(7,12), (7,13) depend on the special value of μ (strength of Hölder-continuity). e. g. for $\mu = \frac{1}{4}$

$$\varepsilon_0 \leqslant \frac{1}{4} 10^{-1} \vartheta - 4 \cdot 10^3 \vartheta^2 \tag{7,12a}$$

$$\varepsilon_{0_{\text{Max}}} \approx 5 \cdot 10^{-8} \tag{7,13a}$$

The condition of contraction of the operator $I_{\mu\nu}$

$$\rho(I_{\mu\nu}(\underset{(i)}{\varphi_{\alpha\beta}}, T_{\mu\nu}), I_{\mu\nu}(\underset{(k)}{\varphi_{\alpha\beta}}, T_{\mu\nu})) \leqslant a\rho(\underset{(i)}{\varphi_{\alpha\beta}}, \underset{(k)}{\varphi_{\alpha\beta}}) \tag{7,14}$$

is with (7,3) satisfied for

$$\vartheta \leqslant \vartheta_{\text{Max}} \tag{7,15}$$

The relations (7,12), (7,13) and (7,15) yield

THEOREM IV. — The integral operator $I_{\mu\nu}(\varphi_{\alpha\beta}, T_{\mu\nu})$ defined in (7,1) is a contraction for functions $\varphi_{\alpha\beta}$ and the inhomogenity $T_{\mu\nu}$ satisfying (7,6), (7,7), (7,5) respectively, and ε_0, ϑ restricted by (7,12), (7,13).

We additionally consider the difference

$$\underset{(i,k)}{D} I_{\mu\nu} = I_{\mu\nu|a|b}(\underset{(i)}{\varphi_{\alpha\beta}}, T_{\mu\nu}) - I_{\mu\nu|a|b}(\underset{(k)}{\varphi_{\alpha\beta}}, T_{\mu\nu}) . \tag{7,16a}$$

If for

$$\underset{(i,k)}{D} \varphi_{\mu\nu} = \underset{(i)}{\varphi_{\mu\nu|a|b}} - \underset{(k)}{\varphi_{\mu\nu|a|b}} \tag{7,16b}$$

of functions in D_I holds with $\| \underset{(i)}{\varphi_{\mu\nu}} - \underset{(k)}{\varphi_{\mu\nu}} \| \leqslant A$

$$| R_0^2 \cdot (\underset{(i,k)}{D} \varphi_{\mu\nu}(x^a + \Delta x^a) - \underset{(i,k)}{D} \varphi_{\mu\nu}(x^a)) | \leqslant A \cdot \left(\frac{\Delta R}{R_0}\right)^{\mu} \cdot \left\{ 1, \left(\frac{R_0}{\tilde{R}}\right)^3 \right. \tag{7,17a}$$

then for

$$\underset{(i,k)}{D} N_{\mu\nu} = N_{\mu\nu}(\underset{(i)}{\varphi_{\alpha\beta}}) - N_{\mu\nu}(\underset{(k)}{\varphi_{\alpha\beta}}) \tag{7,16c}$$

one gets with (5,6b) and $A < \vartheta$

$$| R_0^2 \cdot (\underset{(i,k)}{D} N_{\mu\nu}(x^a + \Delta x^a) - \underset{(i,k)}{D} N_{\mu\nu}(x^a)) | \leqslant 4 \cdot 10^3 \cdot \vartheta \cdot A . \left(\frac{\Delta R}{R_0}\right)^{\mu} \cdot \left\{ 1, \left(\frac{R_0}{\tilde{R}}\right)^4 \right. \tag{7,17b}$$

hence with (A,20)

$$| R_0^2 \cdot (\underset{(i,k)}{D} I_{\mu\nu}(x^a + \Delta x^a) - \underset{(i,k)}{D} I_{\mu\nu}(x^a)) | \leqslant \bar{a} \cdot A \cdot \left(\frac{\Delta R}{R_0}\right)^{\mu} \cdot \left\{ 1, \left(\frac{R_0}{\tilde{R}}\right)^3 \right. \tag{7,18}$$

where $0 \leqslant \bar{a} \leqslant M(\mu) \cdot 8 \cdot 10^3 \vartheta$ and for $\vartheta < \vartheta_{\text{Max}} : \bar{a} < 1$ according to (7,13). We shall need the relation (7,18) below in (7,23b).

b) EQUATIONS OF MOTION AND INTEGRABILITY CONDITIONS

Before application of Theorem IV we must take into account that in the field equations (5,11), (6,2a) respectively the integral operator $I_{\mu\nu}$ is not $\gamma_{\mu\nu}$, but a linear combination (5,9) of $\gamma_{\mu\nu}$. Hence we must weaken the above estimations by a factor 2. Then the iteration converges to a solution of (6,2). But this solution must not be a solution of Einstein's field equations! This would be valid if $\Delta\psi_{\mu\nu}$ is the linear differential form $L_{\mu\nu}$ (5,6) and therefore, if $\psi_{\mu\nu}$ satisfies the De-Donder condition (5,10). Because we cannot hope, that (by higher inspiration) we are able to choose $T_{\mu\nu}$ so, that the limiting value $\psi_{\mu\nu}$ satisfies the De-Donder condition, we have to begin with a first approximation $\overset{(1)}{T}_{\mu\nu}$ (or $\overset{(1)}{\tau}_{\mu\nu}$) and then solve in each step of approximation the equations of motion (5,8). Above we have shown the existence of solutions with the difference $(\overset{(n)}{T}_{\mu\nu} - \overset{(n-1)}{T}_{\mu\nu})$ between two successive approximations in a desired order of magnitude. Especially we can choose $| R_0^2 \cdot (\overset{(n)}{T}_{\mu\nu} - \overset{(n-1)}{T}_{\mu\nu}) |$ to be not greater than $| R_0^2 \cdot (\overset{(n)}{N}_{\mu\nu} - \overset{(n-1)}{N}_{\mu\nu}) |$. Then obviously the estimations (7,12)-(7,15) guarantee convergence if we correct them once more by a factor 2.

For systems with *isolated* material sources the integrability conditions in vacuo (6,5) must be satisfied. From the Bianchi-identities $G_{\mu\nu||}{}^{\nu} = 0$ follows:

$$\overset{(n+1)}{N}_{\mu\nu}{}^{|\nu} = [(\Gamma^{\rho}_{\mu\kappa}G_{\rho\nu} + \Gamma^{\rho}_{\nu\kappa}G_{\mu\rho})g^{\kappa\nu} - G_{\mu\nu|\kappa}\gamma^{\kappa\nu}]_{(\overset{(n)}{g}_{\alpha\beta})} \tag{7,19}$$

At first sight one could try to show that for $n \to \infty$ holds $\lim_{n\to\infty} \overset{(n)}{N}_{\mu\nu}{}^{|\nu} = 0$

But because $\overset{(n+1)}{N}_{\mu\nu}{}^{|\nu}$ contains third derivatives of $\overset{(n)}{g}_{\alpha\beta}$, the imposed properties *i*), *ii*), *iii*) are too weak for estimations of $\overset{(n+1)}{N}_{\mu\nu}{}^{|\nu}$. Hence we show the validity of the alternative form of integrability conditions (6,5*b*) and regard the difference

$$\overset{(n)}{H}_{\mu\nu} = R_0^2 \cdot (\Delta\overset{(n)}{\psi}_{\mu\nu} - \overset{(n)}{L}_{\mu\nu}) \tag{7,20}$$

We assume in the n-th approximation $\overset{(n)}{H}_{\mu\nu}$ to satisfy

$$| \overset{(n)}{H}_{\mu\nu} | \leqslant q^{n+1} \tag{7,21a}$$

$$| \overset{(n)}{H}_{\mu\nu}(x^a + \Delta x^a) - \overset{(n)}{H}_{\mu\nu}(x^a) | \leqslant q^{n+1} \cdot \left(\frac{\Delta R}{R_0}\right)^{\mu} \tag{7,21b}$$

where q is a suitable constant with $a < q < 1$. We shall prove that from (7,21) corresponding relations follows for $\overset{(n+1)}{H}_{\mu\nu}$ and so by induction $\lim_{n\to\infty} \overset{(n)}{H}_{\mu\nu} = 0$.

First, from (7,21*a*) we get with (5,1), (5,6*a*) in *vacuo* $(\Delta\overset{(n)}{\psi}_{\mu\nu} + \overset{(n)}{N}_{\mu\nu} = 0)$:

$$| R_0^2 \cdot G_{\mu\nu}(\overset{(n)}{g}_{\alpha\beta}) | = \left| \frac{1}{2}[- \overset{(n)}{H}_{\mu\nu} + (\overset{(n+1)}{N}_{\mu\nu} - \overset{(n)}{N}_{\mu\nu})] \cdot R_0^2 \right|$$
$$\leqslant \frac{1}{2}q^{n+1} + \frac{1}{2}| R_0^2 \cdot (\overset{(n+1)}{N}_{\mu\nu} - \overset{(n)}{N}_{\mu\nu}) | \tag{7,22}$$

The magnitude of the second term on the right-hand side of this relation is determined by the distance $\rho(\overset{(n)}{g}_{\mu\nu}, \overset{(n-1)}{g}_{\mu\nu}) \leqslant 2\vartheta a^{n-1}$.

Regarding (5,6*b*), (7,8) we see that:

$$| R_0^2 \cdot (\overset{(n+1)}{N}_{\mu\nu} - \overset{(n)}{N}_{\mu\nu}) | \leqslant 16 \cdot 10^3 \vartheta^2 \cdot a^{n-1}$$

The right-hand side may be estimated with ϑ_{Max} according to (7,13) to be $< \vartheta \cdot a^{n-1} < \vartheta \cdot q^{n-1}$. Because $\vartheta \leqslant 10^{-5}$ we can achieve (for suitable $q<1$) $\vartheta < q^2$. Hence in vacuo:

$$| G_{\mu\nu}(\overset{(n)}{g}_{\alpha\beta}) \cdot R_0^2 | < q^{n+1} \tag{7,23a}$$

Assuming the equations of motion to be solved in the $(n + 1)$-th approximation, we can estimate the divergence of Poisson's integral over $\overset{(n+1)}{\tau}_{\mu\nu}$. We get from (7,23*a*) and (7,19) with

$$G_{\mu\nu|\kappa}\gamma^{\kappa\nu} = (G_{\mu\nu}\gamma^{\kappa\nu})_{|\kappa} - (G_{\mu\nu}\gamma^{\kappa\nu})_{|\nu}$$

for suitable $q < 1$:

$$\left| R_0 \cdot \left(\iiint_V \frac{\overset{(n+1)}{\tau}_{\mu\nu}}{r} d^3x' \right)^{|\nu} \right|$$

$$= | R_0 \cdot \overset{(n+1)}{\psi}_{\mu\nu}{}^{|\nu} | < 10^3 \cdot \vartheta_{\mathrm{Max}} \cdot q^{n+1} < 10^{-1} \cdot q^{n+2} \quad (7{,}24a)$$

The relation (7,24*a*) has been obtained by rough estimations being performed similarly as those leading to (5,6*b*) and with the use of (A,13).

The Hölder-continuity of $\overset{(n)}{H}_{\mu\nu}$ is given by (7,21*b*). The Hölder-continuity of the difference $| \overset{(n+1)}{N}_{\mu\nu} - \overset{(n)}{N}_{\mu\nu} |$ is determined by the Hölder continuity of the difference $(\overset{(n)}{g}_{\alpha\beta} - \overset{(n-1)}{g}_{\alpha\beta})$ and its first and second derivatives. With the use of (7,18) and for suitable $\bar{a} < q < 1$ one can easily derive from (7,22)

$$| R_0^2[G_{\mu\nu}(\overset{(n)}{g}_{\alpha\beta}, x^a + \Delta x^a) - G_{\mu\nu}(\overset{(n)}{g}_{\alpha\beta}, x^a)] | < q^{n+1}\left(\frac{\Delta R}{R_0}\right)^{\mu} \quad (7{,}23b)$$

From (7,23*b*) follows similarly as (7,24*a*) with the use of (A,19) and (A,33):

$$| R_0^2 \cdot \overset{(n+1)}{\psi}_{\mu\nu}{}^{|\nu}{}_{|\kappa} | < 10^{-1} \cdot q^{n+2} \quad (7{,}24b)$$

and

$$| R_0^2 \cdot [\overset{(n+1)}{\psi}_{\mu\nu}{}^{|\nu}{}_{|\kappa}(x^a + \Delta x^a) - \overset{(n+1)}{\psi}_{\mu\nu}{}^{|\nu}{}_{|\kappa}(x^a)] | < 10^{-1} \cdot q^{n+2}\left(\frac{\Delta R}{R_0}\right)^{\mu} \quad (7{,}24c)$$

(7,24*a*-*c*) put into (5,5) yields immediately

$$| \overset{(n+1)}{H}_{\mu\nu} | \leqslant q^{n+2}, \qquad | \overset{(n+1)}{H}_{\mu\nu}(x^a + \Delta x^a) - \overset{(n+1)}{H}_{\mu\nu}(x^a) | \leqslant q^{n+2}\left(\frac{\Delta R}{R_0}\right)^{\mu}$$

Hence by induction (the validity for $n = 1$ is obvious) we get

THEOREM V. — If the pure iteration converges and the equations of motion are solved on each step of approximation, then holds throughout in V

$$\lim_{n\to\infty} \overset{(n)}{\psi}_{\mu\nu}{}^{|\nu} = \psi_{\mu\nu}{}^{|\nu} = 0, \qquad \lim_{n\to\infty} \Delta\overset{(n)}{\psi}_{\mu\nu} = L_{\mu\nu} \quad (7{,}25)$$

and the pure iteration converges to an exact solution of Einstein's field equations.

8. Final results and possibility of extension to time-dependent systems

As shown above the estimations (7,12), (7,13) must be corrected by a factor 4. With theorems IV, V the results for the time-independent case of

Einstein-equations are e. g. for $\mu = \frac{1}{4}$: The iteration of the field equations (6,2) converges, if the first approximation $\overset{(1)}{\tau}_{\mu\nu}$ (in case the first approximation is the linearized theory: $\overset{(1)}{\tau}_{\mu\nu} = \overset{(1)}{T}_{\mu\nu}$) satisfies, apart from certain continuous properties

$$| \overset{(1)}{\tau}_{\mu\nu} \cdot R_0^2 | < \left\{ 10^{-8}, 10^{-8}\left(\frac{R_0}{R}\right)^4 \right. \tag{8,1}$$

The maximum possible deviation of metric from flat space is

$$\vartheta_{\text{Max}} \simeq 8 \cdot 10^{-7} \tag{8,2}$$

The factor of convergence a is according to the definition of distance (7,3) and with (7,10) and (7,14) (weakened by a factor 4) determined by:

$$a \simeq 3 \cdot 10^5 \, \vartheta$$

where the minimal ϑ is given by ε_0 through

$$\vartheta \simeq \frac{1}{6} \cdot 10^{-5} - \sqrt{\frac{1}{36} \cdot 10^{-10} - \frac{1}{4} 10^{-3} \varepsilon_0} \tag{8,3}$$

In each step of the iteration from conditions *i*)-*iii*) the existence of the differential equations of motion is guaranteed, but not for the rigorous solution as limiting value of the iteration, because the existence of third derivatives of $g_{\mu\nu}$ cannot be proven. Instead of the differential equations weaker (coordinate-dependent !) integral conservation laws are satisfied. If we demand the validity and existence of the differential equations $T_{\mu\nu}{}^{||\nu} = 0$, we have to replace *iii*) by sharper conditions, essentially the existence of sufficiently small upper limits for the first Hölder-continuous derivatives of $\overset{(1)}{\tau}_{\mu\nu}$.

The procedure admits the application to systems with isolated material sources. At the transition from a matter-filled region to vacuum the energy tensor $T_{\mu\nu}$ must be Hölder-continuous.

We finally emphasize that the present methods and results may be applied also for *time-independent* systems. We give a short outline on the main ideas which probably make possible the extension to a class of time-dependent (radiating) systems. The condition of time-independence on the first approximation $\overset{(1)}{\tau}_{\mu\nu}$ may be weakened by admittance of time-dependence in a finite characteristic region $R \leqslant R_0$ for times $t \geqslant t_0$, i. e. :

$$\begin{aligned} &\overset{(1)}{\tau}_{\mu\nu|4} = 0 \quad \text{for} \quad t < t_0, \quad \text{for} \quad (t \geqslant t_0, \quad R > R_0) \\ &\overset{(1)}{\tau}_{\mu\nu|4} \neq 0 \qquad\qquad\qquad\quad \text{for} \quad (t \geqslant t_0, \quad R \leqslant R_0) \end{aligned} \tag{8,4}$$

(8,4) means that the system goes over to a time-dependent radiating state at $t = t_0$. The iterative approximation is now based on the retarded integrals (5,12). But because of the retardation the metric in the region

$$R > R_s(t) = c \cdot (t - t_0) + R_0 \tag{8,5}$$

is completely determined by the state of the system for $t < t_0$, i. e. time-independent. Hence the essential points of the above investigations concerning the behaviour in the space-like infinity can be applied.

The exact mathematical treatment of iterations of these time-dependent systems, mainly concerning *a*) the formulation of continuity conditions at the boundary between time-dependent and-independent regions and *b*) the estimation of the time-dependent contributions to the retarded integrals in the finite region $R \leqslant R_s$ will be subject to future work.

APPENDIX

In order to avoid that this appendix becomes too voluminous, we had to omit the presentation of a number of lengthy calculations, especially in (A,3) and (A,4).

A 1. Upper limits for Poisson's integral

We consider Poisson's integral

$$\psi = \frac{1}{4\pi} \iiint_V \frac{\tau(x^{a'})}{r} d^3x' \tag{A,1}$$

where $r = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{\frac{1}{2}}$ and V is the entire (infinite) 3-dimensional space. The integrable « source » function is assumed to be absolutely limited and to decrease sufficiently in the infinity ([5]):

$$| R_0^2 \cdot \tau | \leqslant \left\{ 1, \left(\frac{R_0}{R}\right)^4 \right. \tag{A,2}$$

$R = (x^2 + y^2 + z^2)^{\frac{1}{2}}$. R_0 is a quantity characteristic of the linear extension of the system. From (A,2) we get the relation

$$\left| \iiint_V \frac{\tau}{r} d^3x' \right| \leqslant \frac{1}{R_0^2}\left[\iiint_{0 \leqslant R' \leqslant R_0} \frac{1}{r} d^3x' + R_0^4 \iiint_{R_0 \leqslant R' \leqslant \infty} \frac{1}{R'^4} \cdot \frac{1}{r} d^3x' \right] \tag{A,3}$$

The calculation of the right-hand integrals yields:

$$\iiint_{0 \leqslant R' \leqslant R_0} \frac{1}{r} d^3x' = \begin{cases} 2\pi\left(R_0^2 - \frac{1}{3}R^2\right) & \text{for} \quad R \leqslant R_0 \\ \frac{4}{3}\pi R_0^3 \cdot \frac{1}{R} & \text{for} \quad R \geqslant R_0 \end{cases} \tag{A,4a}$$

$$\iiint_{R_0 \leqslant R' \leqslant \infty} \frac{1}{R'^4} \cdot \frac{1}{r} d^3x' = \begin{cases} \frac{2\pi}{R_0^2} & \text{for} \quad R \leqslant R_0 \\ 4\pi \frac{1}{RR_0} - \frac{2\pi}{R^2} & \text{for} \quad R \geqslant R_0 \end{cases} \tag{A,4b}$$

Hence with (A,1), (A,3) and (A,4) the potential ψ satisfies the relations:

$$| \psi | \leqslant \left\{ 1, \frac{4}{3}\frac{R_0}{R} \right. \tag{A,5}$$

([5]) Instead of $\left(\frac{R_0}{R}\right)^4$ we could have operated throughout this paper with a behaviour of τ proportional to $\left(\frac{R_0}{R}\right)^{3+\alpha}$, $\alpha > 0$. We have restricted to the case $\alpha = 1$ in order to avoid additional confusion which is caused by introduction of a further parameter.

A 2. The first derivatives

We now determine upper limits for the first partial derivatives of ψ:

$$\psi_{|a} = \frac{1}{4\pi} \cdot \left[\iiint_{V-V_e} \tau \frac{(x^a - x^{a'})}{r^3} d^3x' + \iiint_{V_e} \tau \cdot \frac{(x^a - x^{a'})}{r^3} d^3x' \right] \quad \text{(A,6)}$$

Where V_e is a sphere (radius e) with the center point $\{ x^a \}$. For $e \to 0$ the second integral vanishes. Hence we have to consider only the first integral with $e \to 0$. In spherical polar coordinates R, ϑ, φ $(R', \vartheta', \varphi')$ with $\vartheta = 0$ holds

$$| R_0 \cdot \psi_{|a} | \leqslant \frac{1}{2R_0} \int_0^\infty \int_0^\pi | \tau | \frac{R'^2}{r^2} dR' \sin \vartheta' d\vartheta' \quad \text{(A,7)}$$

For source points $\{ x^a \}$ with $R > R_0$ we get

$$| R_0 \cdot \psi_{|a} | \leqslant \frac{1}{2R_0} \int_0^{R_0} \int_0^\pi \frac{R'^2}{r^2} dR' \sin \vartheta' d\vartheta' + \frac{R_0^3}{2} \lim_{e \to 0} \left[\int_{R_0}^{R-e} \int_0^\pi \frac{1}{R'^2 r^2} dR' \sin \vartheta' d\vartheta' + \int_{R_0+e}^\infty \int_0^\pi {}^{\cdot}/. \right] \quad \text{(A,8)}$$

The evaluation of the first integral in (A,8) yields:

$$\frac{1}{2} \int_0^{R_0} \int_0^\pi \frac{R'^2}{r^2} dR' \sin \vartheta' d\vartheta' = \frac{1}{2R_0 R} \int_0^{R_0} R' \ln \left(\frac{R + R'}{R - R'} \right) dR'$$
$$= \left(\frac{R_0}{R} \right)^2 \sum_{\nu=1}^\infty \left(\frac{1}{4\nu^2 - 1} \right) \cdot \left(\frac{R_0}{R} \right)^{2\nu - 2} < \frac{1}{2} \left(\frac{R_0}{R} \right)^2 \quad \text{(A,9)}$$

because of $\frac{R_0}{R} < 1$ and $\sum_{\nu=1}^\infty \frac{1}{4\nu^2 - 1} = \frac{1}{2}$. For the last two integrals in (A,8) we get similarly:

$$\frac{R_0^3}{2} \lim_{e \to 0} [{}^{\cdot}/.] = \frac{R_0^3}{2R} \cdot \left(\frac{4}{R_0 R} - \frac{4}{R^2} \sum_{\nu=1}^\infty \left(\frac{1}{4\nu^2 - 1} \right) \left(\frac{R_0}{R} \right)^\nu \right) < 2 \left(\frac{R_0}{R} \right)^2 \quad \text{(A,10)}$$

(A,9) and (A,10) together yield an upper limit for the first derivatives of ψ in case $R > R_0$:

$$| R_0 \cdot \psi_{|a} | < \frac{5}{2} \left(\frac{R_0}{R} \right)^2 \quad \text{(A,11)}$$

For $R \leqslant R_0$ holds with (A,2):

$$| R_0 \cdot \psi_{|a} | \leqslant \frac{1}{R_0} \int_0^{R_0} dR' + \int_0^\infty \frac{R_0^3}{R'^4} dR' = \frac{4}{3} \quad \text{(A,12)}$$

Hence we get with (A,11) and (A,12) the following upper limits:

$$| R_0 \cdot \psi_{|a} | < \left\{ \frac{4}{3}, \frac{5}{2} \left(\frac{R_0}{R} \right)^2 \right. \quad \text{(A,13)}$$

For the determination of upper limits of the potentials ψ and their first derivatives we have used essentially the conditions (A,2). For estimations of higher derivatives we need further conditions on the source function τ, which simultaneously guarantee the existence of these derivatives.

A 3. The second derivatives

We now consider the second derivatives

$$\psi_{|a|b} = \frac{1}{4\pi}\left(\iiint \frac{\tau(x^{a'})}{r} d^3x'\right)_{|a|b} \tag{A,14}$$

Additionally to (A,2) we assume the source function $\tau(x, y, z)$ to satisfy the following (modified) Hölder condition ([6])

$$| \mathrm{R}_0^2 \cdot [\tau(x^a + \Delta x^a) - \tau(x^a)] | \leqslant \left(\frac{\Delta \mathrm{R}}{\mathrm{R}_0}\right)^{\mu} \cdot \left\{ 1, \left(\frac{\mathrm{R}_0}{\tilde{\mathrm{R}}}\right)^4 \right. \tag{A,15}$$

$$0 < \mu \leqslant 1, \qquad \Delta \mathrm{R} = (\Delta x^2 + \Delta y^2 + \Delta z^2)^{\frac{1}{2}} \leqslant \mathrm{R}_0$$

$$\tilde{\mathrm{R}} = \min \left[((x + \Delta x^2) + (y + \Delta y)^2 + (z + \Delta z)^2)^{\frac{1}{2}}, \mathrm{R}\right]$$

For estimation of the second derivatives we decompose as follows

$$\psi_{|a|b} = \frac{1}{4\pi}\left[\iiint_{\mathrm{V_B}} \frac{\tau(x^{a'})}{r} d^3x' + \iiint_{\mathrm{V-V_B}} \frac{\tau(x^{a'})}{r} d^3x'\right]_{|a|b} \tag{A,16}$$

where $\mathrm{V_B}$ is a sphere of radius R_0, enclosing the field point $\{ x^a \}$. We omit the explicite calculation of the (maximum) contribution of the region $(\mathrm{V} - \mathrm{V_B})$. This contribution may be determined with the use of (A,2) only. The calculations are lengthy but similar to those in appendix A,2.

For estimation of the contribution of $\mathrm{V_B}$ we first regard field points $\{ x^a \}$ with $\mathrm{R} \leqslant \mathrm{R}_0$. We introduce coordinates $\{ \overset{*}{x}{}^a \}$ with the origin at $\{ x^a \}$. Then holds (see e. g. [7]):

$$\left(\iiint_{\mathrm{V_B}} \frac{\tau}{r} d^3x'\right)_{|a|b} = \frac{4\pi}{3} \tau(x^a) \cdot \vartheta^{ab} + \left(\iiint_{\mathrm{V_B}} \frac{(\tau(x^{a'}) - \tau(x^a))}{r} d^3x'\right)_{|a|b} \tag{A,17}$$

In the right-hand integral in (A,17) differentiation and integration may be interchanged [8]. Thus from (A,17) follows with regard to (A,15) and (A,2)

$$\left| \frac{\mathrm{R}_0^2}{4\pi} \cdot \left(\iiint_{\mathrm{V_B}} \frac{\tau}{r} d^3x'\right)_{|a|b} \right| \leqslant \frac{1}{3} + 2\int_0^{\mathrm{R}_0} \left(\frac{\overset{*}{\mathrm{R}}}{\mathrm{R}_0}\right)^{\mu} d\mathrm{R}^* = \frac{1}{3} + \frac{2}{\mu} \tag{A,18a}$$

Similarly for field points $\{ x^a \}$ with $\mathrm{R} \geqslant \mathrm{R}_0$ we get with the use of the second condition in (A,15)

$$\left| \frac{\mathrm{R}_0^2}{4\pi} \cdot \left(\iiint_{\mathrm{V_B}} \frac{\tau}{r} d^3x'\right)_{|a|b} \right| < \left(\frac{1}{3} + \frac{2}{\mu}\right)\left(\frac{\mathrm{R}_0}{\mathrm{R}}\right)^3 \tag{A,18b}$$

Taking into account the additional contributions of the region $(\mathrm{V} - \mathrm{V_B})$, the resulting relations for the second derivatives are

$$| \mathrm{R}_0^2 \cdot \psi_{|a|b} | < \left\{ \left(1 + \frac{2}{\mu}\right), \left(6 + \frac{2}{\mu}\right)\left(\frac{\mathrm{R}_0}{\mathrm{R}}\right)^3 \right. \tag{A,19}$$

([6]) See Hölder [6] (1882). The second relation in (A,15) guarantees a sufficiently decreasing behaviour of the Hölder-continuous source function in space-like infinity.

A 4. Hölder-continuity of the second derivatives

The second derivatives (A,14) satisfy a Hölder condition corresponding to (A,15) *with the same exponent* μ in case $0 < \mu < 1$:

$$| R_0^2 \cdot [\psi_{|b|c}(x^a + \Delta x^a) - \psi_{|b|c}(x^a)] | \leqslant H \cdot \left(\frac{\Delta R}{R_0}\right)^\mu \cdot \left\{ 1, \left(\frac{R_0}{\tilde{R}}\right)^4 \right. \qquad (A,20)$$

The proof of Hölder continuity of $\psi_{|a|b}$ with the same μ may be found e. g. in [*10*]. We have done the explicit estimation of H by rather lengthy and trouble some calculations, the presentation of which we will omit. The essential point is the estimation of a finite source region V_R enclosing the two field points $\{ x^a \}$ and $\{ x^a + \Delta x^a \}$. The contribution of the remaining region $V - V_R$ to the Poisson-integral (A,1) is a harmonic potential function of class C^∞, the third derivatives of which may be estimated by lengthy but elementary calculations similar to those in A,1, A,2 with the use of (A,2) only. The final result for H is

$$H < 26 + \frac{18\mu^2 + 2 - \mu}{\mu(1 - \mu)}, \qquad 0 < \mu < 1 \qquad (A,20a)$$

$$H < 40 \qquad \text{for} \qquad \mu = \frac{1}{4}$$

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