ANNALES DE L'I. H. P., SECTION A

A. M. PERELOMOV

Schrödinger equation spectrum and Korteweg-De Vries type invariants

Annales de l'I. H. P., section A, tome 24, nº 2 (1976), p. 161-164

<http://www.numdam.org/item?id=AIHPA_1976__24_2_161_0>

Ann. Inst. Henri Poincaré,
Vol. XXIV, n° 2, 1976, p. 161-164.

Section A :
Physique théorique.

Schrödinger equation spectrum and Korteweg-de Vries type invariants

by

A. M. PERELOMOV
Institute for Theoretical and Experimental Physics, Moscow, USSR

Let us consider the Schrödinger equation on a compact locally euclidean manifold M of dimension d

$$L\psi_n = \lambda_n \psi_n, \qquad L = -\Delta + u(x) \tag{1}$$

Here Δ is the Laplace operator on M and the potential $u(x)$ is a real smooth function.

Let $Z(\tau)$ be the partition function (sometimes it is called theta function of the operator L):

$$Z(\tau) = Sp(e^{-\tau L}) = \sum_n e^{-\lambda_n \tau} \tag{2}$$

The well known problem of the asymptotic behaviour of the spectrum $\{\lambda_n\}$ at $n \to \infty$ at the first stage is reduced to finding the asymptotic behaviour of $Z(\tau)$ at $\tau \to 0$

$$Z(\tau) \simeq (4\pi\tau)^{-\frac{d}{2}}(1 + a_1\tau + a_2\tau^2 + \dots) \tag{3}$$

Note that if the potential $u(x)$ equals zero, then

$$Z(\tau) = Z_0(\tau) \sim (4\pi\tau)^{-\frac{d}{2}}(1 + O(\exp(-c\tau^{-1}))).$$

Analogously, if the manifold M is noneuclidean, then the function $Z(\tau)$ has an expansion of the type (3) and the explicit form of several coefficients is known (see refs [1]-[3]). However, general rules of constructing these coefficients are still unclear.

In this note, recurrence formulæ are obtained for the coefficients a_K in the euclidean case, from which in the one-dimensional case follows the expression of a_K via the well known invariants of the Korteweg-de Vries (K*d*V) equation, studied in detail in refs [*4*]-[*9*]. Note that although for the one-dimensional case the connection of the Schrödinger and Korteweg-de Vries equations was often used, the case of the purely discrete spectrum of interest has not been considered in those papers.

Let us now turn to obtain the recurrence relations. Note first that $Z(\tau) = \int G(x, x; \tau)dx$ where $G(x, y; \tau)$ is the Green function of the « heat-conduction » equation

$$G_\tau(x, y; \tau) + LG(x, y; \tau) = \delta(\tau)\delta(x - y) \tag{4}$$

Let us turn to the new function $F(x, y; \tau) = G_0^{-1}G$ where G_0 is the Green function of eq. (4) with a potential equal zero

$$G_0 \sim (4\pi\tau)^{-\frac{d}{2}} \exp[-(x - y)^2/4\tau], \quad \tau \to 0.$$

For the function F at $\tau \to 0$ we obtain the following equation, which is correct up to terms of order $\exp(-c\tau^{-1})$, $c > 0$

$$F_\tau(x, y; \tau) + \tau^{-1}(x - y)\nabla_x F(x, y; \tau) - \Delta_x F(x, y; \tau) + u(x)F(x, y; \tau) = 0 \tag{5}$$

Here $\nabla_x F$ is the gradient of the function F. For the function F at $\tau \to 0$ the asymptotic expansion $F \sim (1 + F_1\tau + F_2\tau^2 + \dots)$ is valid. From eq. (5) follows the recurrence relation

$$F_n(x + \xi, x) = \int_0^1 [\Delta_x - u(x)]F_{n-1}(x + t\xi, y = x)t^{n-1}dt \tag{6}$$

$$F_n(x, x) = \frac{1}{n}[(\Delta_x - u(x))F_{n-1}(x, y = x)] \tag{7}$$

From this it follows in particular that the quantities $F_n(x, x)$ are polynomials in u and derivatives of u:

$$F_n = \frac{(-1)^n}{n!}u^n + \dots$$

where the rest of the terms contain the derivatives of u. Note that

$$a_K = \int F_K(x, x)dx$$

and hence we may neglect in $F_K(x, x)$ terms which are a total divergence and thus do not contribute to a_K. Therefore

$$a_K = \frac{(-1)^K}{k!}\int P_K(u, u_i, u_{ij} \dots)dx, \qquad \int_M dx = 1 \tag{8}$$

where $P_K = u^K + \ldots$ is some polynomial which can be determined using the recurrence relations (6). Let us give the expressions for the first four polynomials

$$P_1 = u, \quad P_2 = u^2, \quad P_3 = u^3 + \frac{1}{2}(\nabla u)^2, \quad P_4 = u^4 - u^2\Delta u + \frac{1}{5}(\Delta u)^2 \qquad (9)$$

These polynomials are the generalization of invariants of the Korteweg-de Vries (one-dimensional) equation.

$$u_t = 6uu_x - u_{xxx} \qquad (10)$$

This interesting and important equation has been studied in a number of papers. Of them let us mention the papers [4]-[9].

For us the following properties are important.

Let $\{ u(x, t) \}$ be the one-parametric family of potentials satisfying the K*d*V equation. Then the Schrödinger equation spectrum $\{ \lambda_n \}$ is t independent. Hence the coefficients a_K are invariants of the K*d*V equation. It is also known that the K*d*V equation on the circle as well as the K*d*V equation on the straight line for rapidly decreasing initial conditions admits an infinite set of conserved integrals of the type

$$I_n[u] = \int \tilde{P}_n(u, u_x, \ldots)dx$$

where $\check{P}_n(u, u_x, \ldots)$ is a polynomial in u and space derivatives of u containing the term u^n. The explicit expression for the first eleven $\check{P}_n$ is given in refs [4] [5]. The papers [5] and [7] give the algorithm for their determination. In paper [8] a simple recurrence relation for the densities $\check{P}_n$ is found.

Now in view of the uniqueness of the polynomial invariants for the K*d*V equation (see refs [4]-[7]) is follows that we may consider $P_n = \tilde{P}_n$. Therefore, for the case when the manifold M is a circle, we prove

THEOREM. — The coefficient a_K in (3) is given by

$$a_K = \frac{(-1)^K}{k!} I_k[u] \qquad (12)$$

Remark. — The quantities $F_K(x, x)$ locally depend on the potential $u(x)$. Hence these relations are valid also for the Schrödinger equation on a noncompact locally euclidean manifold.

REFERENCES

[1] M. KAC, *Amer. Math. Monthly*, t. **73**, 1966, p. 1-23.
[2] H. P. MCKEAN, I. M. SINGER, *Jour. Diff. Geometry*, t. **1**, 1967, p. 43-69.
[3] M. ATIYAH, R. BOTT, V. K. PATODI, *Invent. Math.*, t. **19**, 1973, p. 279-330.
[4] C. S. GARDNER, J. M. GREENE, M. D. KRUSKAL, R. M. MIURA, *Phys. Rev. Let.*, t. **19**, 1967, p. 1095-1097.

[5] R. M. Miura, C. S. Gardner, M. D. Kruskal, *J. Math. Phys.*, t. **9**, 1968, p. 1204-1209.
[6] M. D. Kruskal, R. M. Miura, C. S. Gardner, N. J. Zabusky, *J. Math. Phys.*, t. **11**, 1970, p. 952-960.
[7] P. D. Lax, *Comm. Pure Appl. Math.*, t. **21**, 1968, p. 467-490.
[8] L. D. Faddeev, V. E. Zakharov, *Funct. Anal. Appl.*, t. **5**, 1971, p. 280.
[9] S. P. Novikov, *Funct. Anal. Appl.*, t. **8**, n° 3, 1974, p. 54-66 (in Russian).

(Manuscrit reçu le 9 juin 1975)